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SOLVABILITY OF HILFER FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS AT RESONANCE IN $\mathbb{R}^M$

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Abstract In this paper, the solvability of a class of nonlinear Hilfer fractional differential equations boundary value problems is considered at resonance in $\mathbb{R}^m$. The interesting point is that Hilfer is a more general differential operator that contains both the Riemann-Liouville and the Caputo derivative, and the dimension of the kernel of the fractional differential operator with Rimman-stieltjes integral boundary condition can take any value in $\{1, 2, \cdots, m\}$. By applying Mawhin's coincidence degree theory, the existence result of solutions is obtained.

Keywords Hilfer fractional differential equation, Mawhin's coincidence degree theory, resonance, Rimman-stieltjes integral.

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1. Introduction

In this paper, we consider the following Hilfer fractional differential equations boundary value problems at resonance in $\mathbb{R}^m$:

$$\begin{cases} D_{0+}^{\alpha,\beta}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1,\beta}u(t)\right), \ t \in [0,1], \\ u(0) = D_{0+}^{\gamma-2}u(0) = \cdots = D_{0+}^{\gamma-n+1}u(0) = \theta, u(1) = A\int_0^1 u(t)dh(t), \end{cases} \tag{1.1}$$

where $n-1 < \alpha \leq n$, $0 \leq \beta \leq 1$, $\gamma = \alpha + n\beta - \alpha\beta$, $n \geq 2$, $n \in \mathbb{N}$, θ is the zero vector in $\mathbb{R}^m$, A is m-order nonzero square matrices, $h(t)$ is a function of bounded variation, $h'(t)$ is bounded almost everywhere on $[0,1]$, $D_{0+}^{\alpha,\beta}$ is Hilfer fractional derivative of order α and type β, and $f : [0,1] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies Carathéodory, that is,
(i) $f(\cdot, u, v)$ is measurable on [0,1] for all $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$,
(ii) $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^m \times \mathbb{R}^m$ for almost every $t \in [0,1]$,
(iii) The function $m_R(t) = sup\{|f(t,u,v)| : (u,v) \in R\}$ is Lebesgue integrable on $0 \leq t \leq 1$ for all compact set $R \subset \mathbb{R}^m \times \mathbb{R}^m$, where $|f| = \max\{|f_i|, i = 1, 2\cdots, m\}$.

Fractional differential equations are increasingly used in various fields to solve practical problems, such as physics, chemistry, engineering and so on [3, 15, 22–25].

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A large number of results are obtained on the existence of solutions to boundary value problems of Hilfer fractional differential equations [1,4,10,13,14,16,17,21,27]. It is well known that the problem (1.1) is a generalization of elliptic differential equations on smooth surfaces [2]. M. Benchohra et al. [7] considered the existence and uniqueness of the solution to the problem (1.1) by using Banach contraction principle and Krasnoselskii's fixed point theorem. Furthermore, Z. Bouazza et al. [6] also considered the problem (1.1) when $\beta = 1$ and established the existence result. A. Hasanen et al. [11] considered the following three-dimensional system of multi-point boundary value problem:

$$\begin{cases} D^{\iota}\varpi(z) = f_1(z, \varpi(z), \rho(z), \varrho(z)), \\ D^{\gamma}\varpi(z) = f_2(z, \varpi(z), \rho(z), \varrho(z)), \\ D^{\chi}\varpi(z) = f_3(z, \varpi(z), \rho(z), \varrho(z)), \\ \varpi(0) = \nu_1(\varpi), \varpi(1) = \eta_1\varpi(\xi_1), \\ \rho(0) = \nu_2(\rho), \rho(1) = \eta_2\rho(\xi_2), \\ \varrho(0) = \nu_3(\varrho), \varrho(1) = \eta_3\varrho(\xi_3), \end{cases}$$

where $\iota, \gamma, \chi \in (1, 2]$, $z \in [0, 1]$, $\eta_1, \eta_2, \eta_3 \in (0, 1)$. Moreover, K. O. Ezekiel et al. [18] established the following multipoint boundary value problem with two-dimensional kernel at resonance:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, u(t), D^{\alpha-3}_{0+}u(t), D^{\alpha-2}_{0+}u(t), D^{\alpha-1}_{0+}u(t), \\ u(0) = D^{\alpha-3}_{0+}u(0) = 0,\ D^{\alpha-2}_{0+}u(0) = \sum_{i=1}^{m} \mu_i D^{\alpha-2}_{0+}u(\xi_i), \\ D^{\alpha-1}_{0+}u(+\infty) = \int_0^{\eta} D^{\alpha-2}_{0+}u(t)dh(t), \end{cases}$$

where $t \in (0, +\infty)$, $h(t)$ is a continuous and bounded variation function on $(0, +\infty)$.

In recent years, there has been some related research on resonance boundary value problems of fractional differential equations in $\mathbb{R}^m$ [12,19,26]. P. D. Phung et al. [20] studied the following second-order three-point boundary value problems in $\mathbb{R}^m$:

$$\begin{cases} u'' = f(t, u(t), u'(t)),\ t \in (0, 1), \\ u'(0) = \theta,\ u(1) = Au(\eta), \end{cases}$$

where θ is an m-order zero vector, the matrix A satisfies one of the condition: $A^2 = A$ or $A^2 = I$. F. D Ge et al. [9] concerned the following fractional three-point boundary value problems in $\mathbb{R}^m$:

$$\begin{cases} D^{\alpha}_{0+}x(t) = f(t, x(t), D^{\alpha-1}_{0+}x(t)),\ 1 < \alpha \le 2,\ t \in (0, 1), \\ x(0) = \theta,\ D^{\alpha-1}_{0+}x(1) = AD^{\alpha-1}_{0+}x(\xi), \end{cases}$$

where θ is an n-order zero vector, the matrix A satisfies one of the condition: $A^2 = A$ or $A^2 = I$. The author extends the order from integer order to fractional order and obtains the existence result of the solution by using Mawhin's coincidence degree theory. Feng et al. [8] used similar methods to study the following four-point boundary value problems in $\mathbb{R}^m$:

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) = f(t, u(t), {}^C D_{0+}^{\alpha-1} u(t)),\ t \in (0,1), \\ u(0) = Bu(\xi),\ u(1) = Cu(\eta), \end{cases}$$

where $0 < \eta, \xi < 1$, $1 < \alpha \leq 2$, B, C are two n-order nonzero square matrices. In [8, 9, 12, 19, 20, 26], the variable u is an n-dimensional vector function, and the kernel dimension can take any value in $\{1, 2, \cdots, n\}$.

However, we found that there are still some unresolved issues in $\mathbb{R}^m$. Firstly, the derivative operators in references [8,9,12,19,20,26] have not been unified. Therefore, it is imperative to mention that the Hilfer fractional differential system considered in the problem (1.1) is a more general form. For instance, the Hilfer fractional differential system in (1.1) corresponds to (i) the Riemann-Liouville fractional differential system for $\beta = 0$; (ii) the Caputo fractional differential system when $\beta = 1$. Secondly, the order of the derivative operator is limited. Therefore, the order was extended from $1 < \alpha \leq 2$ to $n-1 < \alpha \leq n$ and an interesting new Rimman-stieltjes boundary condition was used. In addition, the use of Moore-Penrose generalized inverse matrix and their properties eliminates the restriction on matrix A.

2. Preliminaries

Definition 2.1. ([16]) Let X and Y be real Banach spaces. Linear operator $L : domL \subset X \to Y$ to be a Fredholm operator of index zero if
(A_1) ImL is a closed subset of Y;
(A_2) $dim\ KerL = codim\ ImL < +\infty$.

If L satisfies (A_1) and (A_2), there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL$, $KerQ = ImL$, $X = KerL \oplus KerP$, $Y = ImL \oplus ImQ$. It follows that $L|_{domL \cap KerP} : domL \cap KerP \to ImL$ is invertible. We denote the inverse of $L|_{domL \cap KerP}$ by $K_p : ImL \to domL \cap KerP$.

Definition 2.2. ([16]) If Ω is an open bounded subset of X, and $domL \cap \Omega \neq \phi$, the map $N : X \to Y$ will be called L-compact on Ω if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N(\bar{\Omega})$ is completely continuous.

Lemma 2.1. (Theorem 2.4, [16]) *Let $L : domL \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ be $L-$compact on $\bar{\Omega}$. Suppose the following conditions are satisfied:*
(1) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(domL \backslash KerL) \cap \partial\Omega] \times (0, 1)$;
(2) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$;
(3) $deg(JQN|_{KerL}, \Omega \cap KerL, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $ImL = KerQ$, and $J : ImQ \to KerL$ is an isomorphism.
Then the equation $Lu = Nu$ has at least one solution in $domL \cap \bar{\Omega}$.

Definition 2.3. ([14]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

provided the right side is pointwise on $(0,+\infty)$.

Definition 2.4. ([14]) The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $y:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}y(s)ds,$$

where $n=[\alpha]+1$, $[\alpha]$ denotes the integer part of number α, and this derivative is called the right side is pointwise defined on $(0,+\infty)$.

Definition 2.5. ([14]) The left-sided Hilfer fractional derivative of order α and type β for a function $y:(0,+\infty)\to\mathbb{R}$ is given by

$$D_{a+}^{\alpha,\beta}y(t)=I_{a+}^{\beta(n-\alpha)}\frac{d^n}{dt^n}(I_{a+}^{(1-\beta)(n-\alpha)}y)(t),\ n-1<\alpha<n,\ 0\le\beta\le 1.$$

Lemma 2.2. (Corollary 2.1, [14]) *Let $\alpha>0$, if $y\in C(0,1)\bigcap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha}y(t)=0$$

has a unique solution

$$y(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_i\in\mathbb{R}, i=1,2,\cdots,n$, $n=[\alpha]+1$.

Lemma 2.3. (Lemma 2.4, [14]) *If $f\in L(0,1)$, $\alpha>0$, $\beta>0$, then*

$$D_{a\pm}^{\alpha}I_{a\pm}^{\alpha}y(t)=y(t).$$

Lemma 2.4. (Lemma 2.5, [14]) *Let $\alpha>0$, $n=[\alpha]+1$, if $y\in L_1(0,1)$ and $I_{0+}^{n-\alpha}y\in AC^n[0,1]$, then the following holds*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}y(t)=y(t)-\sum_{j=1}^{n}\frac{(I_{0+}^{n-\alpha}y(t))^{(n-j)}\big|_{t=0}}{\Gamma(\alpha-j+1)}t^{\alpha-j}.$$

Lemma 2.5. ([5]) *T^+ be the Moore-Penreose pseudoinverse matrix of T, meaning the matrix satisfying*
(j_1) $T^+TT^+=T^+$,
(j_2) $TT^+T=T$,
(j_3) $ImT^+T=ImT$,
(j_4) $Im(I-T^+T)=KerT$.

Lemma 2.6. (Property 2.1, [14]) *If $\alpha>0$, $\nu>-1$, then the following holds*

$$D_{0+}^{\alpha}t^{\nu}=\frac{\Gamma(\nu+1)}{\Gamma(n+\nu-\alpha+1)}\frac{d^n}{dt^n}(t^{n+\nu-\alpha}),$$

where $n=[\alpha]+1$.

In order to study the boundary value problem (1.1). We defined two spaces $X=\left\{u\middle|u,D_{0+}^{\alpha-1,\beta}u\in\left([0,1],\mathbb{R}^m\right)\right\}$ with the norms $\|u\|=\max\left\{\|u\|_\infty,\|D_{0+}^{\alpha-1,\beta}u\|_\infty\right\}$, where $\|\cdot\|_\infty = \max\limits_{1\le i\le n}\max\limits_{t\in[0,1]} \mid u_i(t) \mid$ and $Y = L^1\Big([0,1],\mathbb{R}^m\Big)$ with the norm $\|y\|_1 = \max\int_0^1 |y_i(s)|ds$.

In this paper, we will always suppose that the following conditions hold:
(H_1) $det(I - A\int_0^1 t^{\gamma-1}dh(t)) = 0$,
(H_2) $\int_0^1 (t^{\gamma-1} - t^{\alpha})dh(t) \neq 0$.

Define operators $L : domL \subset X \to Y$ and $N : X \to Y$ as follows

$$\begin{aligned} &Lu = D_{0+}^{\alpha,\beta}u(t),\ u \in domL,\\ &Nu = f\left(t, u(t), D_{0+}^{\alpha-1,\beta}u(t)\right),\ u \in X, \end{aligned}$$

where

$$\begin{aligned} domL =&\Big\{u\Big|u \in X, D_{0+}^{\alpha,\beta}u \in Y, u(0) = D_{0+}^{\gamma-2}u(0) = \cdots = D_{0+}^{\gamma-n+1}u(0) = \theta,\\ &u(1) = A\int_0^1 u(t)dh(t)\Big\}. \end{aligned}$$

Then the problem(1.1) is equivalent to $Lu = Nu,\ u \in domL$.

3. Main results

Lemma 3.1. *Suppose (H_1) holds, then $L : domL \subset X \to Y$ is a Fredholm operator of index zero.*

Proof. It is easy to get that

$$KerL = \Big\{u \in domL\Big|u(t) = ct^{\gamma-1}, c \in KerT\Big\}.$$

Now, we prove

$$ImL = \big\{y \in Y|\phi y(t) \in ImT\big\},$$

where $\phi y(t) : Y \to \mathbb{R}^m$ is a continuous linear operator defined by

$$\phi(y) = I_{0+}^{\alpha}y(1) - A\int_0^1 I_{0+}^{\alpha}y(t)dh(t). \tag{3.1}$$

Let $|h'(t)| < M$, a.e. $t \in [0,1]$. For $y_1, y_2 \in Y$, if $\|y_1 - y_2\|_1 < \delta$, then

$$\begin{aligned} &|\phi(y_1) - \phi(y_2)|\\ =&|I_{0+}^{\alpha}y_1(1) - I_{0+}^{\alpha}y_2(1) - A\int_0^1 I_{0+}^{\alpha}y_1(t)dh(t) + A\int_0^1 I_{0+}^{\alpha}y_2(t)dh(t)|\\ \le&|I_{0+}^{\alpha}y_1(1) - I_{0+}^{\alpha}y_2(1)| + \|A\|_\infty\left|\int_0^1 I_{0+}^{\alpha}[y_2(t) - y_1(t)]dh(t)\right|\\ \le&\frac{1}{\Gamma(\alpha)}\left|\int_0^1 (1-s)^{\alpha-1}(y_1(s) - y_2(s))ds\right|\\ &+\frac{\|A\|_\infty}{\Gamma(\alpha)}\left|\int_0^1\int_0^t (t-s)^{\alpha-1}(y_2(s) - y_1(s))h'(s)dsdt\right| \end{aligned}$$

$$\leq \frac{1}{\Gamma(\alpha)}\|y_1 - y_2\|_1 + \frac{\|A\|_\infty M}{\Gamma(\alpha)}\|y_1 - y_2\|_1,$$

where $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$. Therefore, the operator ϕ is continuous.

In fact, for any $y \in ImL$, there exists a function $u \in domL$ such that $D_{0+}^{\alpha,\beta}u(t) = y(t)$. By Definition 2.5 and Lemma 2.4, we obtain $u(t) = I_{0+}^{\alpha}y(t)+c_1t^{\gamma-1}+c_2t^{\gamma-2}+\cdots+c_nt^{\gamma-n}$. Since $u(0) = D_{0+}^{\gamma-2}u(0) = \cdots = D_{0+}^{\gamma-n+1}u(0) = \theta$, we can get

$$u(t) = I_{0+}^{\alpha}y(t) + c_1t^{\gamma-1}.$$

And then from $u(1) = A\int_0^1 u(t)dh(t)$, we can get

$$I_{0+}^{\alpha}y(t)|_{t=1} - A\int_0^1 I_{0+}^{\alpha}y(t)dh(t) = -Tc_1,\ c_1 \in \mathbb{R}^m,$$

which means that $\phi y(t) \in ImT$. Consequently, $ImL \subseteq \{y \in Y|\phi y(t) \in ImT\}$.

On the other hand, if $y \in Y$ satisfies $\phi y(t) \in ImT$, there exist a constant ξ such that $\phi y(t) = -T\xi$. Let $u(t) = I_{0+}^{\alpha}y(t) + \xi t^{\gamma-1}$. It is easy to prove that u satisfies the boundary conditions of the problem (1.1), and we have $Lu = y(t)$. Then $ImL \supseteq \{y \in Y|\phi y(t) \in ImT\}$.

In summary, we get

$$ImL = \{y \in Y|\phi y(t) \in ImT\}.$$

Define the operator $Q : Y \to Y$ by

$$Qy = G(I - TT^+)\phi(y) := C, \tag{3.2}$$

where $G = \frac{\Gamma(\alpha+1)\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1 (t^{\gamma-1}-t^{\alpha})dh(t)}$.

For $y \in Y$, $t \in [0, 1]$

$$\begin{aligned}
Q^2y &= G(I - TT^+)\phi(C)\\
&= G(I - TT^+)\Big[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}C\ ds\\
&\quad - A\int_0^1 \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}C\ dsdh(t)\Big]\\
&= \frac{G}{\Gamma(\alpha+1)}(I - TT^+)\big(I - A\int_0^1 t^{\alpha}dh(t)\big)C\\
&= \frac{G}{\Gamma(\alpha+1)}\Big[(I - TT^+) - \frac{1}{\int_0^1 t^{\gamma-1}dh(t)}(I - TT^+)\int_0^1 t^{\alpha}dh(t)\Big]C\\
&= G\ \frac{\int_0^1 t^{\gamma-1}dh(t) - \int_0^1 t^{\alpha}dh(t)}{\Gamma(\alpha+1)\int_0^1 t^{\gamma-1}dh(t)}(I - TT^+)C\\
&= (I - TT^+)Qy\\
&= Qy.
\end{aligned}$$

Actually, $(I - TT^+)A = \frac{1}{\int_0^1 t^{\gamma-1}dh(t)}(I - TT^+)$.

Hence, Q is a linear projection operator. Obviously, $ImL = KerQ$. For $y \in Y$, we can set $y = (y - Qy) + Qy$. By $(y - Qy) \in KerQ = ImL$, $Qy \in ImQ$, we can get $Y = ImL + ImQ$. It follows from $y \in ImL \cap ImQ$ that $y \in ImL = KerQ$ and $y = Qy$, then $y = \theta$. Hence $Y = ImL \oplus ImQ$. It is obvious that $codim\ ImL = dim\ ImQ = dim\ KerL$. Thus, L is a Fredholm operator of index zero. □

Define the operator $P : X \to X$ by

$$Pu(t) = (I - T^+T)\frac{D_{0+}^{\gamma-1}u(0)}{\Gamma(\gamma)}t^{\gamma-1}. \tag{3.3}$$

It is easy to get $P^2u = Pu$ and $ImP = KerL$. Clearly $X = KerL \oplus KerP$. So $P : X \to X$ is a projector.

Lemma 3.2. *Define a linear operator $K_p : ImL \to domL \cap KerP$,*

$$K_py(t) = I_{0+}^{\alpha}y(t) - T^+\phi y(t)t^{\gamma-1}. \tag{3.4}$$

Then $K_p = (L|_{domL \bigcap KerP})^{-1}$.

Proof. For $y \in ImL$, we have $\phi(y) \in ImT$ which means that $\phi(y) = T\xi$, then

$$K_Py(0) = D_{0+}^{\gamma-2}K_Py(0) = \cdots = D_{0+}^{\gamma-n+1}K_Py(0) = \theta$$

and

$$\begin{aligned}
&K_Py(t)|_{t=1} - A\int_0^1 K_Py(t)dh(t)\\
=&I_{0+}^{\alpha}y(t)|_{t=1} - T^+\phi(y) - A\int_0^1 \left[I_{0+}^{\alpha}y(t) - T^+\phi(y)t^{\gamma-1}\right]dh(t)\\
=&I_{0+}^{\alpha}y(t)|_{t=1} - A\int_0^1 I_{0+}^{\alpha}y(t)dh(t) - \left[I - A\int_0^1 t^{\gamma-1}dh(t)\right]T^+\phi(y)\\
=&\phi(y) - TT^+\phi(y)\\
=&(I - TT^+)\phi(y)\\
=&(I - TT^+)T\xi\\
=&\theta.
\end{aligned}$$

Thus, $K_Py \in domL$. It follows from (3.2) and (3.4)

$$\begin{aligned}
PK_Py =&(I - TT^+)\frac{D_{0+}^{\gamma-1}K_Py(t)|_{t=0}}{\Gamma(\gamma)}t^{\gamma-1}\\
=&(I - TT^+)\frac{D_{0+}^{\gamma-1}[I_{0+}^{\alpha}y(t)]|_{t=0}}{\Gamma(\gamma)}t^{\gamma-1}\\
=&\theta.
\end{aligned}$$

Thus, $K_Py \in KerP$. It is easy to prove $K_py \in domL \bigcap KerP$ and we have

$$\begin{aligned}
LK_py(t) &= D_{0+}^{\alpha,\beta}\left[I_{0+}^{\alpha}y(t) - T^+\phi y(t)t^{\gamma-1}\right]\\
&= I_{0+}^{\beta(n-\alpha)}D_{0+}^{\gamma}I_{0+}^{\alpha}y(t) - I_{0+}^{\beta(n-\alpha)}D_{0+}^{\gamma}T^+\phi y(t)t^{\gamma-1}\\
&= I_{0+}^{\beta(n-\alpha)}D_{0+}^{\beta(n-\alpha)}y(t)
\end{aligned}$$

$$= y(t).$$

On the other hand, for $u \in domL \cap KerP$, we have

$$\begin{aligned}
&K_pLu(t)\\
=&I^{\alpha}_{0+}D^{\alpha,\beta}_{0+}u(t) - t^{\gamma-1}T^{+}\left[I^{\alpha}_{0+}D^{\alpha,\beta}_{0^+}u(t)|_{t=1} - A\int_0^1 I^{\alpha}_{0+}D^{\alpha,\beta}_{0+}u(t)dh(t)\right]\\
=&I^{\gamma}_{0+}D^{\gamma}_{0+}u(t) - t^{\gamma-1}T^{+}\left[I^{\gamma}_{0+}D^{\gamma}_{0+}u(t)|_{t=1} - A\int_0^1 I^{\gamma}_{0+}D^{\gamma}_{0+}u(t)dh(t)\right]\\
=&u(t) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1} - \frac{D^{\gamma-2}_{0+}u(0)}{\Gamma(\gamma-1)}t^{\gamma-2} - \cdots - \frac{I^{n-\gamma}_{0+}u(0)}{\Gamma(\gamma-n+1)}t^{\gamma-n}\\
&- t^{\gamma-1}T^{+}\Bigg[u(1) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)} - \frac{D^{\gamma-2}_{0+}u(0)}{\Gamma(\gamma-1)} - \cdots - \frac{I^{n-\gamma}_{0+}u(0)}{\Gamma(\gamma-n+1)}\\
&- A\int_0^1 u(t) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1} - \frac{D^{\gamma-2}_{0+}u(0)}{\Gamma(\gamma-1)}t^{\gamma-2} - \cdots - \frac{I^{n-\gamma}_{0+}u(0)}{\Gamma(\gamma-n+1)}t^{\gamma-n}dh(t)\Bigg]\\
=&u(t) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1}\\
&- t^{\gamma-1}T^{+}\left[u(1) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)} - A\int_0^1 u(t)dh(t) + A\int_0^1 \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1}dh(t)\right]\\
=&u(t) - \frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1} + t^{\gamma-1}T^{+}\left[I - A\int_0^1 t^{\gamma-1}dh(t)\right]\frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}\\
=&u(t) - (I - T^{+}T)\frac{D^{\gamma-1}_{0+}u(0)}{\Gamma(\gamma)}t^{\gamma-1}\\
=&u(t).
\end{aligned}$$

That means $K_p = \left(L|_{domL\bigcap KerP}\right)^{-1}$. □

Lemma 3.3. *Assume $\Omega \subset X$ is an open bounded subset and $domL \cap \overline{\Omega} \neq \phi$, then N is $L-$compact on $\overline{\Omega}$.*

Proof. Let $\Omega \subset X$ is an open bounded subset. By the hypothesis (iii) on the function f, there exists a function $m_R(t) \in L^1[0,1]$ such that for all $u \in \Omega$,

$$|Nu(t)| = |f(t, u(t), D^{\alpha-1,\beta}_{0+}u(t))| \leq m_R(t), t \in [0,1].$$

And then we shall prove that $K_p(I-Q)Nu$ is completely continuous. It follows from (3.1) definition of ϕ that

$$|\phi(Nu)| = \left|I^{\alpha}_{0+} - A\int_0^1 I^1_0 Nu(t)dh(t)\right| \leq \frac{1}{\Gamma(\alpha)}\left(1+ \parallel A \parallel_\infty \left|\int_0^1 dh(t)\right|\right) \parallel m_R \parallel_1 .$$

Combing with (3.3), one has

$$\left|QNu(t)\right| = \left|\frac{\Gamma(\alpha+1)\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1 (t^{\gamma-1} - t^{\alpha})dh(t)}(I - TT^{+})\phi(Nu)\right|$$

$$
\begin{aligned}
&=\left|\frac{\Gamma(\alpha+1)\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}(I-TT^+)\Big[I^{\alpha}_{0+}Nu(1)-A\int_0^1 I^{\alpha}_{0+}Nu(t)dh(t)\Big]\right|\\
&\le\left|\frac{\alpha\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}\right|\|I-TT^+\|_{\infty}\Bigg(\Big|\int_0^1(1-s)^{\alpha-1}Nu(s)ds\Big|\\
&\quad+\Big|A\int_0^1\int_0^t(t-s)^{\alpha-1}Nu(s)dsdh(t)\Big|\Bigg)\\
&\le\left|\frac{\alpha\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}\right|\|I-TT^+\|_{\infty}\\
&\quad\times\left(\int_0^1|Nu(s)|ds+\|A\|_{\infty}\int_0^1\int_0^1|Nu(s)|dsdh(t)\right)\\
&\le\left|\frac{\alpha\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}\right|\|I-TT^+\|_{\infty}\left(1+\parallel A\parallel_{\infty}\Big|\int_0^1 dh(t)\Big|\right)\|m_R\|_1.
\end{aligned}
$$

Thus, $QN(\bar{\Omega})$ is bounded.

For $u\in\bar{\Omega}$,

$$
\begin{aligned}
&\phi(QNu)-\phi(Nu)\\
=&I^{\alpha}_{0+}QNu(t)|_{t=1}-A\int_0^1 I^{\alpha}_{0+}QNu(t)dh(t)-\phi(Nu)\\
=&\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}QNu(s)ds-A\int_0^1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}QNu(s)dsdh(t)-\phi(Nu)\\
=&\frac{G(I-TT^+)\phi(Nu)}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}ds\\
&-\frac{AG(I-TT^+)\phi(Nu)}{\Gamma(\alpha)}\int_0^1\int_0^t(t-s)^{\alpha-1}dsdh(t)-\phi(Nu)\\
=&\frac{G(I-TT^+)}{\Gamma(\alpha+1)}\phi(Nu)-\frac{AG(I-TT^+)}{\Gamma(\alpha+1)}\phi(Nu)\int_0^1 t^{\alpha}dh(t)-\phi(Nu)\\
=&\frac{\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}\Big(I-A\int_0^1 t^{\alpha}dh(t)\Big)(I-TT^+)\phi(Nu)-\phi(Nu)\\
=&D\phi(Nu),
\end{aligned}
$$

where $D=\frac{\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^{\alpha})dh(t)}\Big(I-A\int_0^1 t^{\alpha}dh(t)\Big)(I-TT^+)-I$.

For every $u\in\Omega$, we have

$$
\begin{aligned}
K_p(I-Q)Nu(t)=&I^{\alpha}_{0+}Nu(t)-t^{\gamma-1}T^+\phi Nu(t)-I^{\alpha}_{0+}QNu(t)+t^{\gamma-1}T^+\phi(QNu(t))\\
=&I^{\alpha}_{0+}Nu(t)-I^{\alpha}_{0+}QNu(t)+t^{\gamma-1}T^+D\phi(Nu) \qquad (3.5)
\end{aligned}
$$

and

$$
D^{\alpha-1}_{0+}K_p(I-Q)Nu(t)=I^1_{0+}Nu(t)-I^1_{0+}QNu(t)+I^{\beta(n-\alpha)}_{0+}\Gamma(\gamma)T^+D\phi(Nu). \qquad (3.6)
$$

Combining (3.4) and (3.5) we have

$$
\begin{aligned}
&\Big|K_p(I-Q)Nu(t)\Big|\\
=&\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Nu(s)ds-\frac{G(I-TT^+)}{\Gamma(\alpha+1)}\phi(Nu)t^\alpha+t^{\gamma-1}T^+D\phi(Nu)\right|\\
\leq&\frac{1}{\Gamma(\alpha)}\int_0^1 \mid Nu(s)\mid ds+\left|\frac{G(I-TT^+)}{\Gamma(\alpha+1)}\phi(Nu)\right|+|T^+D\phi(Nu)|\\
\leq&\frac{1}{\Gamma(\alpha)}\|m_R\|_1+\left(\left|\frac{G}{\Gamma(\alpha+1)}\right|\|I-TT^+\|_\infty+\|T^+D\|_\infty\right)|\phi(Nu)|\\
=&\frac{\|m_R\|_1}{\Gamma(\alpha)}\left(\left|\frac{\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\gamma-1}-t^\alpha)dh(t)}\right|\|I-TT^+\|_\infty+\|T^+D\|_\infty\right)\left(1+\|A\|_\infty\Big|\int_0^1 dh(t)\Big|\right)\\
&+\frac{\|m_R\|_1}{\Gamma(\alpha)}
\end{aligned}
$$

and

$$
\begin{aligned}
&|D_{0+}^{\alpha-1}K_p(I-Q)Nu(t)|\\
=&|I_{0+}^1Nu(t)-I_{0+}^1QNu(t)+I_{0+}^{\beta(n-\alpha)}\Gamma(\gamma)T^+D\phi(Nu)|\\
\leq&\int_0^1|Nu(s)|ds+\int_0^1|QNu(s)|ds+|I_{0+}^{\beta(n-\alpha)}\Gamma(\gamma)T^+D\phi(Nu)|\\
\leq&\|m_R\|_1+|G(I-TT^+)\phi(Nu)|+\frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)}\|T^+D\|_\infty|\phi(Nu)|\\
=&\left(\frac{|\alpha\int_0^1 t^{\gamma-1}dh(t)|\|I-TT^+\|_\infty}{|\int_0^1(t^{\gamma-1}-t^\alpha)dh(t)|}+\frac{\Gamma(\gamma)\|T^+D\|_\infty}{\Gamma(\alpha)\Gamma(\beta(n-\alpha)+1)}\right)\\
&\times\left(1+\|A\|_\infty\Big|\int_0^1 dh(t)\Big|\right)\|m_R\|_1+\|m_R\|_1.
\end{aligned}
$$

That is, $K_p(I-Q)N(\bar{\Omega})$ is uniformly bounded in X.

For $0\leq t_1<t_2\leq 1, u\in\bar{\Omega}$, we have

$$
\begin{aligned}
&\Big|K_p(I-Q)Nu(t_2)-K_p(I-Q)Nu(t_1)\Big|\\
\leq&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}Nu(s)ds-\frac{G(I-TT^+)\phi(Nu)}{\Gamma(\alpha+1)}t_2^\alpha+t_2^{\gamma-1}T^+D\phi(Nu)\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}Nu(s)ds-\frac{G(I-TT^+)\phi(Nu)}{\Gamma(\alpha+1)}t_1^\alpha+t_1^{\gamma-1}T^+D\phi(Nu)\bigg|\\
=&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}Nu(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}Nu(s)ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}Nu(s)ds\\
&+\frac{G(I-TT^+)\phi(Nu)}{\Gamma(\alpha+1)}(t_1^\alpha-t_2^\alpha)+(t_2^{\gamma-1}-t_1^{\gamma-1})T^+D\phi(Nu)\bigg|
\end{aligned}
$$

$$\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]m_R(s)ds+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}m_R(s)ds$$
$$+\left|\frac{G(I-TT^+)\phi(Nu)}{\Gamma(\alpha+1)}\right||t_1^\alpha-t_2^\alpha|+|t_2^{\gamma-1}-t_1^{\gamma-1}|\|T^+D\|_\infty|\phi(Nu)|.$$

According to the uniform continuity of binary functions, for any $\varepsilon>0$, there is always a positive integer δ that only depends on ε, so that for all points $(t_1,t_2)\in[0,1]\times[0,1]$, as long as $|t_1-t_2|<\delta$, there is $|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}|<\varepsilon$.

$$\left|D_{0+}^{\alpha-1,\beta}K_p(I-Q)Nu(t_2)-D_{0+}^{\alpha-1,\beta}K_p(I-Q)Nu(t_1)\right|$$
$$=\left|\int_0^{t_2}Nu(s)ds-G(I-TT^+)\phi(Nu)t_2+\frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)}T^+D\phi(Nu)\right.$$
$$\left.-\int_0^{t_1}Nu(s)ds+G(I-TT^+)\phi(Nu)t_1-\frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)}T^+D\phi(Nu)\right|$$
$$=\left|\int_{t_1}^{t_2}Nu(s)ds+G(I-TT^+)\phi(Nu)(t_1-t_2)\right|$$
$$\le\left|\frac{\alpha\int_0^1 t^{\gamma-1}dh(t)}{\int_0^1(t^{\alpha-1}-t^\alpha)dh(t)}\right|\|I-TT^+\|_\infty\Big(1+\|A\|_\infty\Big|\int_0^1 dh(t)\Big|\Big)|t_1-t_2|\|m_R\|_1$$
$$+\int_{t_1}^{t_2}m_R(s)ds.$$

Thus, $K_p(I-Q)N(\bar\Omega)$ is equicontinuous. By the Ascoli-Arzela theorem, we can conclude that the operator N is L-compact in $\bar\Omega$. □

Theorem 3.1. *Suppose* $(H_1),(H_2)$ *and the following conditions hold:*
(H_3) *There exists a constant* $\delta_1>0$ *such that for* $u\in domL$, *if* $|t^{1-\gamma}u(t)|>\delta_1$ *for all* $t\in[0,1]$, *then*

$$\phi f(t,u(t),D_{0+}^{\alpha-1,\beta}u(t))\notin ImT.$$

(H_4) *There exist three nonnegative functions* $a,b,c\in C^1[0,1]$ *such that*

$$|f(t,u,v)|\le a(t)|u|+b(t)|v|+c(t),\quad \textit{for all } t\in[0,1],u,v\in\mathbb{R}^m,$$

where $B\Gamma(\alpha+1)>2(\|a\|B+\|b\|D)$, $B=\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1-\|b\|)-\|b\|\Gamma(\gamma)$ *and* $D=[\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]$.
(H_5) *There exists a constant* $\delta_2>0$ *such that for any* $\beta\in\mathbb{R}^m$, *satisfying* $\beta=\int_0^1 t^{\gamma-1}dh(t)A\beta$ *and* $|\beta|>\delta_2$, *either*

$$\langle\beta,QN\beta\rangle\ge 0\quad \textit{or}\quad \langle\beta,QN\beta\rangle\le 0,$$

where $\langle\cdot,\cdot\rangle$ *is the scalar product in* $\mathbb{R}^m$.
Then the problem (1.1) *has at least one solution in* X.

Before we prove theorem 3.1, we show three Lemmas.

Lemma 3.4. *Let* $\Omega_1=\Big\{u|u\in domL\backslash KerL, Lu=\lambda Nu,\lambda\in(0,1)\Big\}$. *Assume* $(H_1)-(H_4)$ *hold. Then* Ω_1 *is bounded in* X.

Proof. Let $u \in \Omega_1$, we have $Lu = \lambda Nu$, $Nu \in ImL$, we get $\phi(Nu) \in ImT$. It follows from (H_3) that there exists a constant $t_0 \in [0,1]$ such that $|t_0^{1-\gamma}u(t_0)| \leq \delta_1$.

By $Lu = \lambda Nu(t)$ and boundary condition, we have

$$u(t) = \lambda I_{0+}^{\alpha} Nu(t) + \xi t^{\gamma-1}. \tag{3.7}$$

Taking $t = t_0$ into equation (3.6), we have $u(t_0) = \lambda I_{0+}^{\alpha} Nu(t_0) + \xi t_0^{\gamma-1}$. That means

$$\begin{aligned}|\xi| &\leq |t_0^{1-\gamma}u(t_0)| + |\lambda t_0^{1-\gamma} I_{0+}^{\alpha} Nu(t_0)| \\ &\leq \delta_1 + t_0^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_0^{t_0}(t_0-s)^{\alpha-1}|Nu(s)|ds \\ &\leq \delta_1 + t_0^{1-\gamma}t_0^{\alpha}\frac{1}{\Gamma(\alpha+1)}(\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}u\|_\infty + \|c\|) \\ &\leq \delta_1 + \frac{1}{\Gamma(\alpha+1)}(\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}u\|_\infty + \|c\|).\end{aligned}$$

Based on

$$\begin{aligned}&|D_{0+}^{\alpha-1,\beta}u(t)| = |\lambda I_{0+}^{1}Nu(t) + I_{0+}^{\beta(n-\alpha)}\xi\Gamma(\gamma)| \\ \leq&\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}u\|_\infty + \|c\| \\ &+\left(\delta_1 + \frac{\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}u\|_\infty + \|c\|}{\Gamma(\alpha+1)}\right)\frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)} \\ \leq&\frac{\|a\|\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)}\|u\|_\infty + \frac{\|b\|\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)}\|D_{0+}^{\alpha-1,\beta}u\|_\infty \\ &+\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}u\|_\infty + \|c\| + \big(\delta_1 + \frac{\|c\|}{\Gamma(\alpha+1)}\big)\frac{\Gamma(\gamma)}{\Gamma(\beta(n-\alpha)+1)},\end{aligned}$$

we obtain

$$\begin{aligned}\|D_{0+}^{\alpha-1,\beta}u\|_\infty \leq&\frac{[\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]\|a\|}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1-\|b\|)-\|b\|\Gamma(\gamma)}\|u\|_\infty \\ &+\frac{[\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]\|c\| + \delta_1\Gamma(\alpha+1)\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1-\|b\|)-\|b\|\Gamma(\gamma)}.\end{aligned}$$

Therefor, we get

$$\begin{aligned}|\xi| \leq&\delta_1 + \frac{\|b\|}{\Gamma(\alpha+1)}\left\{\frac{[\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]\|a\|}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1-\|b\|)-\|b\|\Gamma(\gamma)}\|u\|_\infty\right. \\ &\left.+\frac{[\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]\|c\| + \delta_1\Gamma(\alpha+1)\Gamma(\gamma)}{\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1-\|b\|)-\|b\|\Gamma(\gamma)}\right\} + \frac{\|c\|}{\Gamma(\alpha+1)} \\ &+\frac{\|a\|}{\Gamma(\alpha+1)}\|u\|_\infty \\ \leq&\delta_1 + \frac{\|a\|(B+\|b\|D)}{\Gamma(\alpha+1)B}\|u\|_\infty + \frac{\|b\|E+\|c\|B}{\Gamma(\alpha+1)B},\end{aligned}$$

where $E = [\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)+\Gamma(\gamma)]\|c\| + \delta_1\Gamma(\alpha+1)\Gamma(\gamma)$.

By simple calculation, we can get

$$\|u\| \leq\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|Nu(s)|ds + |\xi|$$

$$\le \frac{1}{\Gamma(\alpha+1)}(\|a\|\|u\|_\infty + \|b\|\|D_{0+}^{\alpha-1,\beta}\|_\infty + \|c\|) + \delta_1 + \frac{\|a\|(B+\|b\|D)}{\Gamma(\alpha+1)B}\|u\|_\infty$$
$$+ \frac{\|b\|E+\|c\|B}{\Gamma(\alpha+1)B}$$
$$\le \delta_1 + 2\frac{\|a\|(B+\|b\|D)}{\Gamma(\alpha+1)B}\|u\|_\infty + 2\frac{\|b\|E+\|c\|B}{\Gamma(\alpha+1)B}.$$

Therefore,

$$\|u\|_\infty \le \frac{\delta_1 B\Gamma(\alpha+1) + 2(\|b\|E+\|c\|B)}{B\Gamma(\alpha+1) - 2(\|a\|B+\|b\|D)}.$$

We can conclude that Ω_1 is bounded in X. □

Lemma 3.5. *Let* $\Omega_2 = \Big\{u | u \in KerL, Nu \in ImL\Big\}$. *Suppose* $(H_1)-(H_3)$ *hold. Then* Ω_2 *is bounded in* X.

Proof. Let $u \in \Omega_2$, for any $t \in [0,1]$, we have $u(t) = \xi t^{\gamma-1}$, $\xi \in KerT$. Since $Nu \in ImL$, then $\phi(Nu) \in ImT$. According to (H_3), there exists $t_0 \in [0,1]$ such that $|t_0^{1-\gamma}u(t_0)| \le \delta_1$. Thus, we get that $|\xi| = |t_0^{1-\gamma}u(t_0)| \le \delta_1$. Therefore, Ω_2 is bounded in X. □

Lemma 3.6. *Let* $\Omega_3 = \Big\{u \in KerL | \rho\lambda u + (1-\lambda)QNu = \theta, \lambda \in [0,1]\Big\}$ *and*

$$\rho = \begin{cases} 1, & \text{if } \langle\beta, QN\beta\rangle \ge 0 \text{ holds}, \\ -1, & \text{if } \langle\beta, QN\beta\rangle \le 0 \text{ holds}. \end{cases}$$

Then Ω_3 *is bounded in* X.

Proof. Let $u \in \Omega_3$, we know that $u(t) = \beta t^{\gamma-1}$ with $\beta \in KerT$ and $(1-\lambda)QNu = -\rho\lambda u$.
If $\lambda = 0$, then $QNu = \theta$, $Nu \in KerL = ImL$. Thus, we have $u \in \Omega_2$, so $\|u\| \le \delta_1$.
If $\lambda \in (0,1]$ and $\rho = 1$, suppose $|\beta| > \delta_2$. Then, from (H_5), we get a contradiction

$$0 > -\lambda|\beta|^2 = -\lambda\langle\beta,\beta\rangle = (1-\lambda)\langle\beta, QN\beta\rangle \ge 0.$$

Thus, we have $\|u\| \le \delta_2$.
If $\lambda \in (0,1]$ and $\rho = -1$, suppose $|\beta| > \delta_2$. Similarly,

$$0 < \lambda|\beta|^2 = \lambda\langle\beta,\beta\rangle = (1-\lambda)\langle\beta, QN\beta\rangle \le 0.$$

Thus, $\|u\| \le \delta_2$. In conclusion, Ω_3 is bounded in X. □

The following is the proof of Theorem 3.4.

Proof. Let $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3} \cup \{\theta\}$ be a bounded open subset of X. It follows from Lemma 3.3 that N is L-compact on $\bar{\Omega}$. By Lemmas 3.4 and 3.5, we have

(1) $Lu \ne \lambda Nu$, for every $(u,\lambda) \in [(doml \backslash KerL) \cap \partial\Omega] \times (0,1)$;

(2) $Nu \notin ImL$ for every $u \in KerL \cap \partial\Omega$.

We need only to prove $deg(JQN|_{KerL}, \Omega \cap KerL, \theta) \ne 0$. Take $H(u,\lambda) = \lambda Ju + \rho(1-\lambda)QNu$. According to Lemma 3.6, we know that $H(u,\lambda) \ne \theta$ for $u \in \partial\Omega \cap KerL$. Therefore, via the homotopy property of degree, we obtain

$$deg(JQN|_{KerL}, \Omega \cap KerL, \theta) = deg(\rho H(\cdot, 0), \Omega \cap KerL, \theta)$$

$$
\begin{aligned}
&= deg(\rho H(\cdot,1), \Omega \cap KerL, \theta) \\
&= deg(\pm\rho I, \Omega \cap KerL, \theta) \\
&= \pm 1 \\
&\neq 0.
\end{aligned}
$$

Applying Lemma 2.1, we conclude that the problem (1.1) has at least one solution in X. The proof is completed. □

4. Conclusions

This paper mainly studied a class of Hilfer fractional differential boundary value problem systems at resonance that state variable $u \in \mathbb{R}^m$ and gave a new theorem on the existence of solutions in kernel spaces by using the Mawhin coincidence degree theorem. We provided an example to illustrate the obtained results. Our results also provide some methods for φ-Hilfer and Hadamard fractional differential. These contributions will advance research in other fields.

5. Example

In this section, we present an example to verify our main results. Let's consider the following boundary value problem at resonance:

$$
\begin{cases}
D_{0+}^{\frac{7}{2},\frac{1}{2}} x(t) = f_1(t, x(t), y(t), D_{0+}^{\frac{5}{2},\frac{1}{2}} x(t), D_{0+}^{\frac{5}{2},\frac{1}{2}} y(t)), \quad t \in [0,1], \\
D_{0+}^{\frac{7}{2},\frac{1}{2}} y(t) = f_2(t, x(t), y(t), D_{0+}^{\frac{5}{2},\frac{1}{2}} x(t), D_{0+}^{\frac{5}{2},\frac{1}{2}} y(t)), \quad t \in [0,1], \\
x(0) = y(0) = 0, \\
D_{0+}^{\frac{7}{4}} x(0) = D_{0+}^{\frac{7}{4}} y(0) = 0, \\
x(1) = \dfrac{357}{52} \displaystyle\int_0^1 y(t) d(t^2 - t), \\
y(1) = \dfrac{357}{52} \displaystyle\int_0^1 y(t) d(t^2 - t).
\end{cases} \tag{5.1}
$$

The problem (5.1) has a solution if and only if the problem (1.1) has a solution, where $\alpha = \frac{7}{2}$, $\beta = \frac{1}{2}$, $h(t) = t^2 - t$ and $A = \begin{bmatrix} 0 & \frac{285}{44} \\ 0 & \frac{285}{44} \end{bmatrix}$. Define the function $f_i : [0,1] \times \mathbb{R}^4 \to \mathbb{R}^2$, $i = 1, 2$ by

$$
\begin{aligned}
f(t,u,v) &= (f_1(t, x_1, y_1, x_2, y_2), f_2(t, x_1, y_1, x_2, y_2))^T \\
&= (\frac{t^2}{16}(x_1 + y_1) + \frac{t}{3}(|x_2| + |y_2|), \frac{t^2}{16}(x_1 + y_1) + \frac{t}{3}(\sin^2 x_2 + \sin^2 y_2))^T
\end{aligned} \tag{5.2}
$$

for all $t \in [0,1]$ and $u = (x_1, y_1)$, $v = (x_2, y_2) \in \mathbb{R}^2$. Let $T = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $T^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 \end{bmatrix}$, we can get $I - TT^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then the problem (5.1) has one solution if and only if the problem (1.1) with A and f defined as above has one solution. So we only need prove that the conditions of Theorem 3.1 are satisfied.

First of all, we prove the first condition of theorem 3.1. Let $a(t) = \frac{t^2}{8}$, $b(t) = \frac{2}{3}t$, $c(t) = 1$. It follows from (5.2) that $|f(t,u,v)| \leq a(t)|u| + b(t)|v| + c(t)$ for all $t \in [0,1]$ and $u, v \in \mathbb{R}^2$. By simple calculation we have $B = \Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1)(1 - \|b\|) - \|b\|\Gamma(\gamma) \approx 5.5533$, $D = [\Gamma(\alpha+1)\Gamma(\beta(n-\alpha)+1) + \Gamma(\gamma)] \approx 14.9644$ and $B\Gamma(\alpha+1) - 2(\|a\|B + \|b\|D) \approx 58.9026 > 0$. Hence, (H3) is satisfied.

Next, let's check (H_4), we note that

$$|f_2(t, x_1, y_1, x_2, y_2)| > 0$$

for all $u = (x_1, y_1)$, $v = (x_2, y_2) \in \mathbb{R}^2$. We calculated that

$$|\phi(f_2)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f_2(s)ds - \frac{285}{44} \int_0^1 \int_0^1 f_2(s)dsd(t^2 - t) \right| \neq 0.$$

Therefore,

$$\phi f(t, x_1, y_1, x_2, y_2) = \big(\phi f_1(t, x_1, y_1, x_2, y_2), \phi f_2(t, x_1, y_1, x_2, y_2)\big) \notin ImT$$

due to $ImT = \{(\eta, 0)^\top; \eta \in \mathbb{R}\}$. Hence, (H_4) is satisfied.

Finally, let's prove the condition (H_5). For any $\beta \in \mathbb{R}^2$, satisfying $\beta = \frac{285}{44}A\beta$ and $|\beta| > 0$. β can be written as $\beta = (\beta_0, \beta_0)^\top$ for $\beta_0 \in \mathbb{R}$. By (3.1) and (4.2) we have

$$N\beta = \big(f_1(t,\beta,0), f_2(t,\beta,0)\big)^\top = \Big(\frac{t^2}{8}\beta_0, \frac{t^2}{8}\beta_0\Big)^\top,$$

and we get $G = 138.4249$, $\phi(N\beta) = \Big(0.0376\beta_0, 0.0376\beta_0\Big)^\top$. It follows from (3.3) that

$$Q(N\beta) = 138.4249\big(0, 0.0376\beta_0\big)^\top$$

and

$$\langle \beta, Q(N\beta) \rangle = 0.1957\beta_0^2 > 0.$$

Then, the condition (H_5) holds. Therefore, by an application of Theorem 3.1, we obtain that the problem (4.1) has at least one solution.

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