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MULTIPLE SOLUTIONS FOR P-LAPLACIAN KIRCHHOFF-TYPE FRACTIONAL DIFFERENTIAL EQUATIONS WITH INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSES*

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Abstract In this paper, we consider a class of p-Laplacian Kirchhoff-type fractional differential equations with instantaneous and non-instantaneous impulses. The existence of at least two distinct weak solutions and infinitely many weak solutions is obtained based on variational methods.

Keywords Fractional differential equations, Kirchhoff-type, variational methods, instantaneous impulse, non-instantaneous impulse.

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1. Introduction

In recent years, there has been an increasing interest around the impulsive differential equations because of their numerous applications in various fields such as medicine, physics, biology and control theory. From the perspective of the duration of action, the impulses are divided into instantaneous and non-instantaneous impulses, which were first proposed by Milman-Myshkis [24] and Hernández-O'Regan [18], respectively. More details on these two types are available in [3]. To date, many methods have been used to investigate the differential equations with impulses, such as fixed point theory, theory of analytic semi-group, upper and lower solutions method, topological degree theory, and variational approach [4, 9, 11, 12, 14–18, 29].

Recently, the study of the fractional differential equations (FDEs for short) with instantaneous and non-instantaneous impulses using variational methods and critical point theory has attracted much attention. In [30], Zhang-Liu first considered a class of FDEs with instantaneous and non-instantaneous impulses and used

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variational approach to obtain at least one classical solution. Based on [30], Zhou-Deng-Wang [34] considered a class of FDEs involving the p-Laplacian operator with instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^\alpha \Phi_p({}_0^C D_t^\alpha y(t)) + g(t)|y(t)|^{p-2}y(t) = f_j(t, y(t)), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, ..., m, \\ \Delta\left({}_tD_T^{\alpha-1}\Phi_p({}_0^C D_t^\alpha y)\right)(t_j) = I_j(y(t_j)), \quad j = 1, 2, ..., m, \\ {}_tD_T^{\alpha-1}\Phi_p({}_0^C D_t^\alpha y)(t) = {}_tD_T^{\alpha-1}\Phi_p({}_0^C D_t^\alpha y)(t_j^+), \quad t \in (t_j, s_j],\ j = 1, 2, ..., m, \\ {}_tD_T^{\alpha-1}\Phi_p({}_0^C D_t^\alpha y)(s_j^+) = {}_tD_T^{\alpha-1}\Phi_p({}_0^C D_t^\alpha y)(s_j^-), \quad j = 1, 2, ..., m, \\ y(0) = y(T) = 0, \end{cases} \tag{1.1}$$

where $p \in [2, +\infty)$, $\alpha \in \left(\frac{1}{p}, 1\right]$, $0 = s_0 < t_1 < s_1 < t_2 < s_2 < ... < t_m < s_m < t_{m+1} = T$, ${}_0^C D_t^\alpha$ and ${}_tD_T^\alpha$ denote the left Caputo and the right Riemann-Liouville fractional derivative of order α, respectively, $f_j : (s_j, t_{j+1}] \times \mathbb{R} \to \mathbb{R}$ are continuous, $I_j : \mathbb{R} \to \mathbb{R}$ are continuous, there exists $j \in \{1, 2, ..., m\}$ such that $I_j(y(t_j)) \neq 0$, $g \in L^\infty([0, T])$. Authors obtained the problem (1.1) admits at least one classical solution via the critical point theory. Since then, there are many works that study the FDEs with instantaneous and non-instantaneous impulses by applying variational methods. We refer the readers to [22, 25, 31, 32].

On the other hand, Kirchhoff-type equation is an extension of the classical D'Alembert's wave equation. It was first presented by Kirchhoff [21] in 1883. Various problems of Kirchhoff-type are usually called non-local problems and have been extensively investigated up to now. However, there are relatively few studies on Kirchhoff-type impulsive differential equations in recent ten years. More precisely, in [2, 8, 13], Heidarkhani-Afrouzi-Moradi, Caristi-Heidarkhani-Salari, and Afrouzi-Heidarkhani-Moradi all considered second order Kirchhoff-type differential equations with instantaneous impulses on the half-line. Authors obtained at least one, two, three and infinitely many weak solutions by the virtue of variational methods. More recently, Wang-Tian [28] considered a class of Kirchhoff-type FDEs involving the (p, q)-Laplacian with instantaneous impulses:

$$\begin{cases} \quad M_\alpha\left(\|y\|_\alpha^p\right)\left({}_tD_T^\alpha(\rho(t)\Phi_p({}_0^C D_t^\alpha y(t))) + \kappa(t)|y(t)|^{p-2}y(t)\right) \\ = F_y(t, y(t), z(t)) + \lambda G_y(t, y(t), z(t)),\ t \neq t_j,\ a.e.\ t \in [0, T], \\ \quad M_\beta\left(\|z\|_\beta^q\right)\left({}_tD_T^\beta(\nu(t)\Phi_q({}_0^C D_t^\alpha z(t))) + \varpi(t)|z(t)|^{q-2}z(t)\right) \\ = F_z(t, y(t), z(t)) + \lambda G_z(t, y(t), z(t)),\ t \neq t_i',\ a.e.\ t \in [0, T], \\ \Delta\left(M_\alpha\left(\|y(t_j)\|_\alpha^p\right){}_tD_T^{\alpha-1}(\rho(t_j)\Phi_p({}_0^C D_t^\alpha y(t_j)))\right) = D_j(y(t_j)), \quad j = 1, 2, ..., m, \\ \Delta\left(M_\beta\left(\|z(t_i')\|_\beta^q\right){}_tD_T^{\beta-1}(\nu(t_i')\Phi_q({}_0^C D_t^\beta z(t_i')))\right) = L_i(z(t_i')), \quad i = 1, 2, ..., n, \\ y(0) = y(T) = z(0) = z(T) = 0, \end{cases} \tag{1.2}$$

where $p,\ q,\ \vartheta \in (1, +\infty)$, $\alpha \in \left(\frac{1}{p}, 1\right]$, $\beta \in \left(\frac{1}{q}, 1\right]$, $\Phi_\vartheta(s) = |s|^{\vartheta-2}s$ $(s \neq 0)$ is a ϑ-Laplacian operator, $\lambda \in (0, +\infty)$, $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $0 = t_0' < t_1' < ... < t_n' < t_{n+1}' = T$, ${}_0^C D_t^\alpha$, ${}_0^C D_t^\beta$ and ${}_tD_T^\alpha$, ${}_tD_T^\beta$ denote left Caputo and right Riemann-Liouville fractional derivatives, respectively. $F(t, y, z)$, $G(t, y, z)$:

$[0,T]\times\mathbb{R}^2\to\mathbb{R}$ are C^1 functions, F_s, G_s are the partial derivatives of F, G with respect to s, $D_j,\ L_i:\mathbb{R}\to\mathbb{R}$ are continuous, $M_\alpha,\ M_\beta:\mathbb{R}_0^+\to\mathbb{R}^+$ are continuous. Authors proved that the problem (1.2) admits at least two non-trivial solutions and infinitely many non-trivial solutions by mean of variational methods.

To our best knowledge, there are no published papers concerning the Kirchhoff-type FDEs with p-Laplacian operator and instantaneous and non-instantaneous impulses. To this end, our work aims to fill this gap. We shall apply variational methods to study the multiplicity of solutions for the following Kirchhoff-type fractional Dirichlet boundary value problem:

$$\begin{cases} \quad M\left(\|y\|_\alpha^p\right)\left({}_tD_T^\alpha(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))+a(t)\Phi_p(y(t))\right) \\ =\lambda f_j(t,y(t)),\ t\in(s_j,t_{j+1}],\ j=0,1,...,m, \\ \Delta\left(M\left(\|y(t_j)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j)\Phi_p({}_0^CD_t^\alpha y(t_j)))\right)=\mu I_j(y(t_j)),\ j=1,2,...,m, \\ \quad M\left(\|y\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))) \\ =M\left(\|y(t_j^+)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j^+)\Phi_p({}_0^CD_t^\alpha y(t_j^+))),\ t\in(t_j,s_j],\ j=1,2,...,m, \\ \quad M\left(\|y(s_j^+)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(s_j^+)\Phi_p({}_0^CD_t^\alpha y(s_j^+))) \\ =M\left(\|y(s_j^-)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(s_j^-)\Phi_p({}_0^CD_t^\alpha y(s_j^-))),\ j=1,2,...,m, \\ y(0)=y(T)=0, \end{cases} \tag{1.3}$$

where $p\in(1,+\infty)$, $\alpha\in\left(\frac{1}{p},1\right]$, $\Phi_p(y)=|y|^{p-2}y$, $0=s_0<t_1<s_1<...<s_m<t_{m+1}=T$, $\lambda,\ \mu$ are two positive parameters, ${}_0^CD_t^\alpha$ and ${}_tD_T^\alpha$ denote the left Caputo and the right Riemann-Liouville fractional derivative of order α, respectively, $f_j:(s_j,t_{j+1}]\times\mathbb{R}\to\mathbb{R}$ are continuous, $I_j:\mathbb{R}\to\mathbb{R}$ are continuous. $M:[0,+\infty)\to\mathbb{R}$ is a continuous function satisfying $m_0\le M(s)\le m_1$ for all $s\ge 0$, where m_0 and m_1 are positive constants. $h(t)\in L^\infty([0,T])$ with $h_0=\operatorname{ess\,inf}_{t\in[0,T]}h(t)>0$, $a(t)\in C([0,T])$ with $0<a_0=\min\limits_{t\in[0,T]}a(t)\le a(t)\le a^0=\max\limits_{t\in[0,T]}a(t)$. The norm $\|y\|_\alpha$ is specified later. The instantaneous impulses suddenly start to jump at the points t_j and the non-instantaneous impulses continue in the finite intervals $(t_j,s_j]$.

$$\begin{aligned} &\Delta\left(M\left(\|y(t_j)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j)\Phi_p({}_0^CD_t^\alpha y(t_j)))\right) \\ =&M\left(\|y(t_j^+)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j^+)\Phi_p({}_0^CD_t^\alpha y(t_j^+))) \\ &-M\left(\|y(t_j^-)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j^-)\Phi_p({}_0^CD_t^\alpha y(t_j^-))), \\ &M\left(\|y(t_j^\pm)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t_j^\pm)\Phi_p({}_0^CD_t^\alpha y(t_j^\pm))) \\ =&\lim_{t\to t_j^\pm}M\left(\|y\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))), \\ &M\left(\|y(s_j^\pm)\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(s_j^\pm)\Phi_p({}_0^CD_t^\alpha y(s_j^\pm))) \\ =&\lim_{t\to s_j^\pm}M\left(\|y\|_\alpha^p\right){}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))). \end{aligned}$$

The new contributions that we give are as follows. Firstly, a new class of Kirchhoff-type FDEs is presented and some new results on the multiple solutions are established depending on two real parameters μ and λ. Secondly, some results from the existing literature are extended. In fact, if $M=1$, the problem (1.3) becomes the usual FDEs of p-Laplacian with instantaneous and non-instantaneous impulses, such as [22, 25, 34]. It is obvious that the problem (1.3) is much more complicated than the problems studied in [22,25,34] because of the appearance of non-local term M. Furthermore, if $M = 1$, $p = 2$ and $t_j = s_j$, $j = 1, 2, ..., m$, the non-instantaneous impulses become the instantaneous impulses, and the problem (1.3) becomes the usual FDEs with instantaneous impulses, such as [1, 7, 10, 26, 33]. Based on the above assumptions, if $\alpha = 1$, the problem (1.3) will become the usual integer order differential equations with impulses. In brief, our main results generalize and supplement some previous results.

2. Preliminaries

In this part, we first recall some necessary definitions, lemmas and theorem which will be used later.

Definition 2.1 ([20]). Let y be a function defined on $[b, d]$. Then the left and right Riemann-Liouville fractional derivatives of order $\alpha \in [0, 1)$ are defined by

$$ {}_bD_t^{\alpha} y(t) = \frac{d}{dt}\, {}_bD_t^{\alpha-1} y(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_b^t (t-s)^{-\alpha} y(s) ds \right), \quad t \in [b, d], $$

and

$$ {}_tD_d^{\alpha} y(t) = -\frac{d}{dt}\, {}_tD_d^{\alpha-1} y(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_t^d (s-t)^{-\alpha} y(s) ds \right), \quad t \in [b, d]. $$

Definition 2.2 ([20]). Let $\alpha \in (0, 1)$ and $y \in AC([b, d], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order α for the function y, denoted by ${}_b^CD_t^{\alpha} y(t)$ and ${}_t^CD_d^{\alpha} y(t)$, are respectively defined by

$$ \begin{aligned} {}_b^CD_t^{\alpha} y(t) &= {}_bD_t^{\alpha-1} y'(t) = \frac{1}{\Gamma(1-\alpha)} \int_b^t (t-s)^{-\alpha} y'(s) ds, \quad t \in [b, d], \\ {}_t^CD_d^{\alpha} y(t) &= -{}_tD_d^{\alpha-1} y'(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^d (s-t)^{-\alpha} y'(s) ds, \quad t \in [b, d]. \end{aligned} $$

Remark 2.1. If the Caputo fractional derivatives ${}_b^CD_t^{\alpha} y(t)$ and ${}_t^CD_d^{\alpha} y(t)$ and the Riemann-Liouville fractional derivatives ${}_bD_t^{\alpha} y(t)$ and ${}_tD_d^{\alpha} y(t)$ all exist, the following relationships hold (see [20]):

$$ \begin{aligned} {}_b^CD_t^{\alpha} y(t) &= {}_bD_t^{\alpha} y(t) - \frac{y(b)}{\Gamma(1-\alpha)} (t-b)^{-\alpha}, \quad t \in [b, d], \\ {}_t^CD_d^{\alpha} y(t) &= {}_tD_d^{\alpha} y(t) - \frac{y(d)}{\Gamma(1-\alpha)} (d-t)^{-\alpha}, \quad t \in [b, d]. \end{aligned} $$

In particular, if $\alpha = 0$ or 1, then ${}_b^CD_t^0 y(t) = {}_t^CD_d^0 y(t) = y(t)$, ${}_b^CD_t^1 y(t) = y'(t)$ and ${}_t^CD_d^1 y(t) = -y'(t)$. If $y(b) = y(d) = 0$, then ${}_t^CD_d^{\alpha} y(t) = {}_tD_d^{\alpha} y(t)$ and

$$ {}_b^CD_t^{\alpha} y(t) = {}_bD_t^{\alpha} y(t). \tag{2.1} $$

Definition 2.3 ([19]). Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$\|y\| = \left(\int_0^T |{}_0^C D_t^\alpha y(t)|^p dt + \int_0^T |y(t)|^p dt\right)^{\frac{1}{p}}.$$

Since $h(t) \in L^\infty([0,T])$ with $h_0 = \operatorname{ess\,inf}_{t\in[0,T]} h(t) > 0$ and $a(t) \in C([0,T])$ with $0 < a_0 \le a(t) \le a^0$, we can obtain that $\|y\|$ is equivalent to the following norm:

$$\|y\|_{\alpha,p} = \left(\int_0^T h(t)|{}_0^C D_t^\alpha y(t)|^p dt + \int_0^T a(t)|y(t)|^p dt\right)^{\frac{1}{p}}.$$

Definition 2.4 ([23, Palais-Smale condition]). Let X be a real reflexive Banach space. For any sequence $\{y_n\} \subset X$, if $\{I_\lambda(y_n)\}$ is bounded and $I'_\lambda(y_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say that I_λ satisfies the Palais-Smale condition.

Lemma 2.1 ([20]). *If $\alpha \in (0,1]$ and $y \in AC([b,d],\mathbb{R}^N)$ or $y \in C^1([b,d],\mathbb{R}^N)$, then*

$${}_bD_t^{-\alpha}({}_b^C D_t^\alpha y(t)) = y(t) - y(b) \quad \text{and} \quad {}_tD_d^{-\alpha}({}_t^C D_d^\alpha y(t)) = y(t) - y(d).$$

Lemma 2.2 ([19]). *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2.3 ([19]). *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. For all $y \in E_0^{\alpha,p}$, if $1-\alpha \ge \frac{1}{p}$ or $\alpha > \frac{1}{p}$, we have $\|y\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha y\|_{L^p}$, where $\|y\|_{L^p} = \left(\int_0^T |y(t)|^p dt\right)^{\frac{1}{p}}$.*

Lemma 2.4 ([20]). *Let $\alpha \in (0,+\infty)$, p, $q \in [1,+\infty)$, $\frac{1}{p}+\frac{1}{q} \le 1+\alpha$ or $p \ne 1$, $q \ne 1$, $\frac{1}{p}+\frac{1}{q} = 1+\alpha$. If $y \in L^p([b,d],\mathbb{R}^N)$, $w \in L^q([b,d],\mathbb{R}^N)$, then*

$$\int_b^d ({}_bD_t^{-\alpha} y(t)) w(t) dt = \int_b^d ({}_tD_d^{-\alpha} w(t)) y(t) dt.$$

Lemma 2.5 ([19]). *Let $\alpha \in (0,1]$ and $p \in (1,+\infty)$. Assume that $\alpha \in \left(\frac{1}{p},+\infty\right)$ and the sequence $\{y_n\}$ converges weakly to y in $E_0^{\alpha,p}$, i.e., $y_n \rightharpoonup y$. Then $y_n \to y$ in $C([0,T],\mathbb{R}^N)$, i.e., $\|y_n - y\|_\infty \to 0$ as $n \to \infty$.*

Lemma 2.6 ([6, Theorem 2.1]). *Let X be a reflexive real Banach space, let ϕ, $\psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that ϕ is sequentially weakly lower semi-continuous, strongly continuous and coercive, and ψ is sequentially weakly upper semi-continuous. For every $r > \inf_X \phi$, let*

$$\varphi(r) := \inf_{y\in\phi^{-1}(-\infty,r)} \frac{\left(\sup\limits_{y\in\phi^{-1}(-\infty,r)} \psi(y)\right) - \psi(y)}{r - \phi(y)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r\to(\inf_X \phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *If* $\gamma < +\infty$, *then for each* $\lambda \in \left(0, \frac{1}{\gamma}\right)$, *the following alternative holds: either*

(a_1) $I_\lambda := \phi - \lambda\psi$ *possesses a global minimum, or*

(a_2) *there is a sequence* $\{y_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty} \phi(y_n) = +\infty.$$

(b) *If* $\delta < +\infty$, *then for each* $\lambda \in \left(0, \frac{1}{\delta}\right)$, *the following alternative holds: either*

(b_1) *there is a global minimum of* ϕ *which is a local minimum of* I_λ, *or*

(b_2) *there is a sequence* $\{y_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* ϕ, *with* $\lim_{n\to+\infty} \phi(y_n) = \inf_X \phi$.

Theorem 2.1 ([5, Theorem 3.2])**.** *Let* X *be a real Banach space and let* $\widetilde{\phi}$, $\widetilde{\psi} : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\widetilde{\phi}$ *is bounded from below and* $\widetilde{\phi}(0) = \widetilde{\psi}(0) = 0$. *Fix* $r > 0$ *such that* $\sup_{y\in\widetilde{\phi}^{-1}(-\infty,r)} \widetilde{\psi}(y) < +\infty$ *and assume that, for each* $\lambda \in \left(0, \frac{r}{\sup_{y\in\widetilde{\phi}^{-1}(-\infty,r)} \widetilde{\psi}(y)}\right)$, *the functional* $I_\lambda = \widetilde{\phi} - \lambda\widetilde{\psi}$ *satisfies the Palais-Smale condition and it is unbounded from below. Then, for each* $\lambda \in \left(0, \frac{r}{\sup_{y\in\tilde{\phi}^{-1}(-\infty,r)} \widetilde{\psi}(y)}\right)$, *the functional* I_λ *admits two distinct critical points.*

Let the space $E_0^{\alpha,p}$ equipped with the norm

$$\|y\|_\alpha = \left(\int_0^T h(t)|{}_0^C D_t^\alpha y(t)|^p dt + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} a(t)|y(t)|^p dt\right)^{\frac{1}{p}}.$$

Lemma 2.7. *For* $y \in E_0^{\alpha,p}$, *the norm* $\|y\|_{\alpha,p}$ *and the norm* $\|y\|_\alpha$ *are equivalent, that is, there exist constants* $m_3 > m_2 > 0$ *such that*

$$m_2\|y\|_{\alpha,p} \le \|y\|_\alpha \le m_3\|y\|_{\alpha,p}.$$

Proof. It is clear that $\|y\|_\alpha \le m_3\|y\|_{\alpha,p}$ for $m_3 = 1$. On the other hand, by Lemma 2.3 and (2.1), we can derive

$$\begin{aligned}
\|y\|_{\alpha,p}^p &= \int_0^T h(t)|{}_0^C D_t^\alpha y(t)|^p dt + \int_0^T a(t)|y(t)|^p dt \\
&\le \int_0^T h(t)|{}_0^C D_t^\alpha y(t)|^p dt + \frac{a^0}{h_0}\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p \int_0^T h(t)|{}_0^C D_t^\alpha y(t)|^p dt \\
&\le \left(1 + \frac{a^0}{h_0}\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p\right)\|y\|_\alpha^p.
\end{aligned}$$

Take $m_2 = \left(1 + \frac{a^0}{h_0}\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p\right)^{-\frac{1}{p}}$, we obtain $m_2\|y\|_{\alpha,p} \le \|y\|_\alpha$. □

Lemma 2.8. *For* $y \in E_0^{\alpha,p}$, $p \in (1, +\infty)$, $\alpha \in \left(\frac{1}{p}, +\infty\right)$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *there exists a constant* $K > 0$ *such that* $\|y\|_\infty \le K\|y\|_\alpha$, *where* $\|y\|_\infty = \max_{t\in[0,T]} |y(t)|$.

Proof. For any $y \in E_0^{\alpha,p}$, by Lemma 2.1 and the Hölder's inequality, we have

$$
\begin{aligned}
|y(t)| =& |{}_0D_t^{-\alpha}({}_0^CD_t^\alpha y(t))| \\
=& \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1}\ {}_0^CD_t^\alpha y(s)ds \right| \\
\le& \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)q} ds \right)^{\frac{1}{q}} \left(\int_0^T |{}_0^CD_t^\alpha y(s)|^p ds \right)^{\frac{1}{p}} \\
\le& \frac{T^{\alpha-\frac{1}{p}} h_0^{-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \left(\int_0^T h(t) |{}_0^CD_t^\alpha y(s)|^p ds \right)^{\frac{1}{p}} \\
\le& \frac{T^{\alpha-\frac{1}{p}} h_0^{-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|y\|_\alpha.
\end{aligned}
$$

Thus, we can choose $K := \frac{T^{\alpha-\frac{1}{p}} h_0^{-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}$ such that $\|y\|_\infty \le K\|y\|_\alpha$. □

Lemma 2.9. *We say that $y \in E_0^{\alpha,p}$ is a weak solution of the problem* (1.3), *if the following identity holds:*

$$
\begin{aligned}
&M(\|y\|_\alpha^p) \left(\int_0^T h(t)\Phi_p({}_0^CD_t^\alpha y(t)) {}_0^CD_t^\alpha w(t)dt + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} a(t)\Phi_p(y(t))w(t)dt \right) \\
&+ \mu \sum_{j=1}^m I_j(y(t_j))w(t_j) \\
=& \lambda \sum_{j=0}^m \int_{s_j}^{t_{j+1}} f_j(t, y(t))w(t)dt, \quad \forall w \in E_0^{\alpha,p}.
\end{aligned} \tag{2.2}
$$

Proof. For any $w \in E_0^{\alpha,p}$, from Lemma 2.4, we have

$$
\begin{aligned}
&\int_0^T M(\|y\|_\alpha^p) {}_tD_T^\alpha (h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w(t)dt \\
=& -\int_0^T M(\|y\|_\alpha^p) \frac{d}{dt} ({}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))))w(t)dt \\
=& -\sum_{j=0}^m \int_{s_j}^{t_{j+1}} M(\|y\|_\alpha^p) \frac{d}{dt} ({}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))))w(t)dt \\
&- \sum_{j=1}^m \int_{t_j}^{s_j} M(\|y\|_\alpha^p) \frac{d}{dt} ({}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))))w(t)dt \\
=& -\sum_{j=0}^m M(\|y\|_\alpha^p) {}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w(t)\Big|_{s_j^+}^{t_{j+1}^-} \\
&+ \sum_{j=0}^m \int_{s_j}^{t_{j+1}} M(\|y\|_\alpha^p) {}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w'(t)dt
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j=1}^{m} M(\|y\|_\alpha^p)\,{}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w(t)\Big|_{t_j^+}^{s_j^-}\\
&+\sum_{j=1}^{m}\int_{t_j}^{s_j} M(\|y\|_\alpha^p)\,{}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w'(t)dt\\
=&\mu\sum_{j=1}^{m} I_j(y(t_j))w(t_j)+\sum_{j=0}^{m} M(\|y\|_\alpha^p)\int_{s_j}^{t_{j+1}} h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0D_t^{\alpha-1}w'(t)dt\\
&+\sum_{j=1}^{m} M(\|y\|_\alpha^p)\int_{t_j}^{s_j} h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0D_t^{\alpha-1}w'(t)dt\\
=&\mu\sum_{j=1}^{m} I_j(y(t_j))w(t_j)+\sum_{j=0}^{m} M(\|y\|_\alpha^p)\int_{s_j}^{t_{j+1}} h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0^CD_t^\alpha w(t)dt\\
&+\sum_{j=1}^{m} M(\|y\|_\alpha^p)\int_{t_j}^{s_j} h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0^CD_t^\alpha w(t)dt\\
=&\mu\sum_{j=1}^{m} I_j(y(t_j))w(t_j)+M(\|y\|_\alpha^p)\int_0^T h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0^CD_t^\alpha w(t)dt. \qquad (2.3)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_0^T M(\|y\|_\alpha^p)\,{}_tD_T^\alpha(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w(t)dt\\
=&\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} M(\|y\|_\alpha^p)\,{}_tD_T^\alpha(h(t)\Phi_p({}_0^CD_t^\alpha y(t)))w(t)dt\\
&-\sum_{j=1}^{m}\int_{t_j}^{s_j} M(\|y\|_\alpha^p)\frac{d}{dt}({}_tD_T^{\alpha-1}(h(t)\Phi_p({}_0^CD_t^\alpha y(t))))w(t)dt\\
=&-\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} M(\|y\|_\alpha^p)a(t)\Phi_p(y(t))w(t)dt+\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y(t))w(t)dt. \qquad (2.4)
\end{aligned}
$$

Thus, combining (2.3) and (2.4), we can obtain (2.2) holds. □

Define the functional $I_\lambda : E_0^{\alpha,p}\to\mathbb{R}$ as follows:

$$
I_\lambda(y):=\frac{1}{p}\mathcal{M}(\|y\|_\alpha^p)-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,y(t))dt+\mu\sum_{j=1}^{m} J_j(y(t_j)),
$$

where $\mathcal{M}(y)=\int_0^y M(s)ds$, $F_j(t,y)=\int_0^y f_j(t,s)ds$ and $J_j(y)=\int_0^y I_j(s)ds$. Due to the continuity of M, f_j and I_j, we can easily obtain that I_λ is Gâteaux differentiable at any point $y\in E_0^{\alpha,p}$ and

$$
\begin{aligned}
\langle I_\lambda'(y),w\rangle =&M(\|y\|_\alpha^p)\left(\int_0^T h(t)\Phi_p({}_0^CD_t^\alpha y(t))\,{}_0^CD_t^\alpha w(t)dt\right.\\
&\left.+\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} a(t)\Phi_p(y(t))w(t)dt\right)-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y(t))w(t)dt \qquad (2.5)
\end{aligned}
$$

$$+\mu\sum_{j=1}^{m}I_j(y(t_j))w(t_j).$$

Obviously, the weak solutions of the problem (1.3) are the critical points of I_λ.

3. Main results

Our main results are obtained by using Lemma 2.6 and Theorem 2.1 in this section. Put

$$\zeta := \frac{\|h\|_\infty}{p\Gamma^p(1-\alpha)}\left(\frac{t_1^{1-p\alpha}+t_1^{-p\alpha}(s_m-t_1)}{(1-\alpha)^p}\right.$$
$$\left.+\frac{(T-s_m)\left(\max\{t_1^{-\alpha},(T-s_m)^{-\alpha}-t_1^{-\alpha}\}\right)^p}{(1-\alpha)^p}\right)+\frac{a^0}{p}\sum_{j=0}^{m}(t_{j+1}-s_j),$$
$$A_\infty := \frac{1}{m_0}\liminf_{x\to+\infty}\frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\max\limits_{|y|\le x}F_j(t,y)dt}{x^p},$$
$$B^\infty := \frac{1}{m_0}\limsup_{x\to+\infty}\frac{\sum_{j=1}^{m-1}\int_{s_j}^{t_{j+1}}F_j(t,x)dt}{x^p}.$$

Theorem 3.1. *Assume that*

$(H1)$ $F_j(t,y)\ge 0$ *for all* $(t,y)\in([0,t_1]\cup[s_m,T])\times\mathbb{R}^+$*;*
$(H2)$ $A_\infty<\frac{m_0}{pm_1\zeta K^p}B^\infty$.

Then, for every $\lambda\in\Lambda:=\left(\frac{m_1\zeta}{m_0B^\infty},\frac{1}{pK^pA_\infty}\right)$ *and for each continuous function* I_j, $j=1,2,...,m$ *such that*

$$-J_j(y)=-\int_0^y I_j(s)ds\ge 0,\quad \forall y\ge 0 \tag{3.1}$$

and

$$J^\infty := \frac{1}{m_0}\limsup_{x\to+\infty}\frac{\sum_{j=1}^{m}\max\limits_{|y|\le x}(-J_j(y))}{x^p}<+\infty, \tag{3.2}$$

if we put

$$\mu_{J,\lambda}:=\frac{1}{pK^pJ^\infty}(1-pK^p\lambda A_\infty),$$

where $\mu_{J,\lambda}=+\infty$ *when* $J^\infty=0$*, the problem* (1.3) *has an unbounded sequence of weak solutions for each* $\mu\in(0,\mu_{J,\lambda})$ *in* $E_0^{\alpha,p}$.

Proof. Define the functionals $\phi,\ \psi: E_0^{\alpha,p}\to\mathbb{R}$ as follows:

$$\phi(y)=\frac{1}{p}\mathcal{M}(\|y\|_\alpha^p),\quad \psi(y)=\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}F_j(t,y(t))dt-\frac{\mu}{\lambda}\sum_{j=1}^{m}J_j(y(t_j)),$$

then $I_\lambda(y)=\phi(y)-\lambda\psi(y)$.

In order to prove the theorem, we use Lemma 2.6(a). By standard arguments, ϕ is sequentially weakly lower semi-continuous, strongly continuous and coercive. Moreover, we can also get that ψ is sequentially weakly upper semi-continuous.

Pick $\lambda \in \Lambda$. Since $\lambda < \frac{1}{pK^p A_\infty}$, we have

$$\mu_{J,\lambda} = \frac{1}{pK^p J^\infty}(1 - pK^p\lambda A_\infty) > 0.$$

First, we prove that $\lambda < \frac{1}{\gamma}$. Let $\{x_n\}$ be a real sequence such that $\lim\limits_{n\to+\infty} x_n = +\infty$ and

$$\frac{1}{m_0}\lim_{n\to+\infty}\frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\max\limits_{|y|\le x_n} F_j(t,y)dt}{x_n^p} = A_\infty.$$

Put $r_n = \frac{m_0 x_n^p}{pK^p}$ for every $n \in \mathbb{N}$. For any $w \in E_0^{\alpha,p}$ with $\phi(w) < r_n$, by Lemma 2.8, we have $\phi(w) \ge \frac{m_0}{p}\|w\|_\alpha^p \ge \frac{m_0}{pK^p}\|w\|_\infty^p$, so that

$$\begin{aligned}\phi^{-1}(-\infty, r_n) =&\{w \in E_0^{\alpha,p} : \phi(w) \le r_n\}\\ \subseteq&\left\{w \in E_0^{\alpha,p} : \frac{m_0}{pK^p}\|w\|_\infty^p \le \frac{m_0 x_n^p}{pK^p}\right\}\\ =&\left\{w \in E_0^{\alpha,p} : \|w\|_\infty \le x_n\right\}.\end{aligned}$$

Since $0 \in \phi^{-1}(-\infty, r_n)$ and $\phi(0) = \psi(0) = 0$, we get

$$\begin{aligned}\varphi(r_n) =& \inf_{y\in\phi^{-1}(-\infty,r_n)}\frac{\left(\sup\limits_{w\in\phi^{-1}(-\infty,r_n)}\psi(w)\right) - \psi(y)}{r_n - \phi(y)}\\ \le& \frac{\sup\limits_{w\in\phi^{-1}(-\infty,r_n)}\psi(w)}{r_n}\\ \le& pK^p\left(\frac{\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\max\limits_{|y|\le x_n} F_j(t,y)dt}{m_0 x_n^p} + \frac{\mu}{\lambda}\frac{\sum_{j=1}^{m}\max\limits_{|y|\le x_n}(-J_j(y))}{m_0 x_n^p}\right).\end{aligned}$$

Therefore, from (H2) and (3.2), one has

$$\gamma \le \liminf_{n\to+\infty}\varphi(r_n) \le pK^p(A_\infty + \frac{\mu}{\lambda}J^\infty) < +\infty. \tag{3.3}$$

Taking into account $\mu \in (0, \mu_{J,\lambda})$, we have

$$\gamma \le pK^p(A_\infty + \frac{\mu}{\lambda}J^\infty) < pK^p A_\infty + \frac{1 - pK^p\lambda A_\infty}{\lambda}.$$

Hence,

$$\lambda = \frac{1}{pK^p A_\infty + \frac{1-pK^p\lambda A_\infty}{\lambda}} < \frac{1}{\gamma}. \tag{3.4}$$

According to (H2), (3.3) and (3.4), we can obtain

$$\Lambda \subseteq \left(0, \frac{1}{\gamma}\right).$$

Next, we verify that I_λ is unbounded from below for $\lambda \in \Lambda$. Since $\frac{1}{\lambda} < \frac{m_0 B^\infty}{m_1 \zeta}$, there exist a real sequence $\{\eta_n\}$ and $\tau > 0$ such that $\lim\limits_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{m_0}{m_1\zeta}\frac{1}{m_0}\frac{\sum_{j=1}^{m-1}\int_{s_j}^{t_{j+1}} F_j(t,\eta_n)dt}{\eta_n^p} = \frac{1}{m_1\zeta\eta_n^p}\sum_{j=1}^{m-1}\int_{s_j}^{t_{j+1}} F_j(t,\eta_n)dt \tag{3.5}$$

for each $n \in \mathbb{N}$ large enough. Let $\{\varsigma_n\} : [0,T] \to \mathbb{R}$ be a sequence in $E_0^{\alpha,p}$ given by

$$\varsigma_n(t) := \begin{cases} \dfrac{\eta_n}{t_1} t, & t \in [0,t_1], \\ \eta_n, & t \in [t_1, s_m], \\ \dfrac{\eta_n}{T-s_m}(T-t), & t \in [s_m, T]. \end{cases} \tag{3.6}$$

Clearly, one has

$$\varsigma_n'(t) := \begin{cases} \dfrac{\eta_n}{t_1}, & t \in (0,t_1), \\ 0, & t \in (t_1, s_m), \\ -\dfrac{\eta_n}{T-s_m}, & t \in (s_m, T), \end{cases}$$

and

$$\begin{aligned} {}_0^C D_t^\alpha \varsigma_n(t) =& \frac{1}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha}\varsigma_n'(s)ds\right) \\ =& \frac{1}{\Gamma(1-\alpha)}\begin{cases} \dfrac{\eta_n}{(1-\alpha)t_1} t^{1-\alpha}, & t \in [0,t_1], \\ \dfrac{\eta_n}{1-\alpha} t_1^{-\alpha}, & t \in [t_1, s_m], \\ \dfrac{\eta_n}{1-\alpha}\left(t_1^{-\alpha} - \dfrac{(t-s_m)^{1-\alpha}}{T-s_m}\right), & t \in [s_m, T], \end{cases} \end{aligned}$$

so that

$$\begin{aligned} &\phi(\varsigma_n) \\ \le& \frac{m_1}{p}\|\varsigma_n\|_\alpha^p \\ =& \frac{m_1}{p}\left(\int_0^T h(t)|{}_0^C D_t^\alpha \varsigma_n(t)|^p dt + \sum_{j=0}^m \int_{s_j}^{t_{j+1}} a(t)|\varsigma_n(t)|^p dt\right) \\ \le& \frac{m_1\|h\|_\infty \eta_n^p}{p\Gamma^p(1-\alpha)}\left(\frac{t_1^{1-p\alpha} + t_1^{-p\alpha}(s_m - t_1)}{(1-\alpha)^p}\right. \\ &\left. + \frac{(T-s_m)\left(\max\{t_1^{-\alpha}, (T-s_m)^{-\alpha} - t_1^{-\alpha}\}\right)^p}{(1-\alpha)^p}\right) + \frac{m_1 a^0}{p}\sum_{j=0}^m (t_{j+1} - s_j)\eta_n^p \\ =& m_1\zeta\eta_n^p. \end{aligned} \tag{3.7}$$

On the other hand, by (H1) and (3.1), we deduced that

$$\psi(\varsigma_n) = \sum_{j=0}^m \int_{s_j}^{t_{j+1}} F_j(t,\varsigma_n(t))dt + \frac{\mu}{\lambda}\sum_{j=1}^m (-J_j(\varsigma_n(t_j))) \ge \sum_{j=1}^{m-1}\int_{s_j}^{t_{j+1}} F_j(t,\eta_n)dt. \tag{3.8}$$

It follows from (3.5), (3.7) and (3.8) that

$$I_\lambda(\varsigma_n) \leq m_1\zeta\eta_n^p - \lambda\sum_{j=1}^{m-1}\int_{s_j}^{t_{j+1}} F_j(t,\eta_n)dt < m_1\zeta\eta_n^p(1-\lambda\tau)$$

for each $n\in\mathbb{N}$ large enough. In view of $\lambda\tau>1$, we have

$$\lim_{n\to+\infty} I_\lambda(\varsigma_n) = -\infty,$$

which implies that I_λ does not possess a global minimum. Hence, applying Lemma 2.6(a), I_λ admits a sequence $\{y_n\}$ of critical points such that $\lim\limits_{n\to+\infty}\|y_n\|_\alpha = +\infty$. □

Remark 3.1. Assume that $A_\infty = 0$ and $B^\infty = +\infty$. According to Theorem 3.1, the problem (1.3) has an unbounded sequence of weak solutions in $E_0^{\alpha,p}$ for every $\lambda>0$ and $\mu\in\left(0,\frac{1}{pK^pJ^\infty}\right)$. Furthermore, if $J^\infty=0$, the conclusion is still valid for every $\lambda>0$ and $\mu>0$.

Remark 3.2. Assume that f_j, $j=0,1,...,m$ are non-negative continuous functions. Then, condition (H1) holds, and (H2) becomes

$$(H2)' \;\; A'_\infty := \frac{1}{m_0}\liminf_{x\to+\infty}\frac{\sum_{j=0}^m\int_{s_j}^{t_{j+1}}F_j(t,x)dt}{x^p} < \frac{m_0}{pm_1\zeta K^p}B^\infty.$$

In this case, the condition (H2)$'$ ensures that the problem (1.3) possesses a sequence of weak solutions which is unbounded for every $\lambda\in\left(\frac{m_1\zeta}{m_0B^\infty},\frac{1}{pK^pA'_\infty}\right)$ and $\mu\in\left(0,\frac{1}{pK^pJ^\infty}(1-pK^p\lambda A'_\infty)\right)$ in $E_0^{\alpha,p}$.

Corollary 3.1. *Suppose that f_j, $j=0,1,...,m$ are non-negative continuous functions such that*

$$\liminf_{x\to+\infty}\frac{F_j(x)}{x^p}=0 \quad \textit{and} \quad 0<\widehat{B}^\infty := \limsup_{x\to+\infty}\frac{F_j(x)}{x^p}\leq+\infty,$$

where $F_j(x)=\int_0^x f_j(s)ds$ for $x\in\mathbb{R}$.

Then, for every $\lambda>\frac{m_1\zeta}{\sum_{j=1}^{m-1}(t_{j+1}-s_j)\widehat{B}^\infty}$, for every non-positive continuous function I_j, $j=1,2,...,m$ such that

$$\widehat{J}^\infty := \frac{1}{m_0}\limsup_{x\to+\infty}\frac{-\sum_{j=1}^m J_j(x)}{x^p}<+\infty,$$

and for each $\mu\in\left(0,\frac{1}{pK^p\widehat{J}^\infty}\right)$, the problem (1.3) possesses an unbounded sequence of weak solutions in $E_0^{\alpha,p}$.

Next, we present a special case of Theorem 3.1 with $\lambda=1$.

Corollary 3.2. *Assume that (H1) is fulfilled and*

$$A_\infty < \frac{1}{pK^p} \quad \textit{and} \quad B^\infty > \frac{m_1\zeta}{m_0}.$$

Then, for each continuous function I_j, $j = 1, 2, ..., m$ such that (3.1) *and* (3.2) *hold, and for each $\mu \in (0, \mu_J)$ where*

$$\mu_J := \frac{1}{pK^p J^\infty}(1 - pK^p A_\infty),$$

the problem (1.3) *possesses an unbounded sequence of weak solutions in $E_0^{\alpha,p}$.*

Furthermore, by utilizing Lemma 2.6(b) and arguing as in the proof of Theorem 3.1, put

$$A_0 := \frac{1}{m_0} \liminf_{x \to 0^+} \frac{\sum_{j=0}^m \int_{s_j}^{t_{j+1}} \max_{|y| \le x} F_j(t, y) dt}{x^p},$$
$$B^0 := \frac{1}{m_0} \limsup_{x \to 0^+} \frac{\sum_{j=1}^{m-1} \int_{s_j}^{t_{j+1}} F_j(t, x) dt}{x^p},$$

the following result will be obtained.

Theorem 3.2. *Suppose that (H1) holds and*

$(H3)$ $A_0 < \frac{m_0}{pm_1\zeta K^p} B^0$.

Then, for every $\lambda \in \widetilde{\Lambda} := \left(\frac{m_1\zeta}{m_0 B^0}, \frac{1}{pK^p A_0}\right)$, and for each continuous function I_j, $j = 1, 2, ..., m$ such that (3.1) *holds and*

$$J^0 := \frac{1}{m_0} \limsup_{x \to 0^+} \frac{\sum_{j=1}^m \max_{|y| \le x}(-J_j(y))}{x^p} < +\infty,$$

if we put

$$\widetilde{\mu}_{J,\lambda} := \frac{1}{pK^p J^0}(1 - pK^p \lambda A_0),$$

where $\widetilde{\mu}_{J,\lambda} = +\infty$ when $J^0 = 0$, for each $\mu \in (0, \widetilde{\mu}_{J,\lambda})$, the problem (1.3) *possesses a sequence of pairwise distinct weak solutions, which strongly converges to* 0 *in $E_0^{\alpha,p}$.*

Proof. Analogous to the proof of Theorem 3.1, we can obtain $\lambda < \frac{1}{\delta}$ and $\widetilde{\Lambda} \subseteq \left(0, \frac{1}{\delta}\right)$. In view of $\frac{1}{\lambda} < \frac{m_0 B^0}{m_1 \zeta}$, there exist a real sequence $\{\varsigma_n\}$ with η_n defined in (3.6), and $\hat{\tau} > 0$ such that $\lim_{n \to +\infty} \eta_n = 0^+$ and

$$\frac{1}{\lambda} < \hat{\tau} < \frac{1}{m_1 \zeta \eta_n^p} \sum_{j=1}^{m-1} \int_{s_j}^{t_{j+1}} F_j(t, \eta_n) dt$$

for each $n \in \mathbb{N}$ large enough. Obviously, the sequence $\{\varsigma_n\}$ strongly converges to 0 in $E_0^{\alpha,p}$. Similarly to Theorem 3.1, we also can obtain $I_\lambda(\varsigma_n) < 0$ for each n large enough. Taking $I_\lambda(0) = 0$ into account, we get that I_λ does not possess a local minimum at 0. Therefore, by Lemma 2.6(b), there is a sequence $\{y_n\}$ in $E_0^{\alpha,p}$ of critical points of I_λ such that $\lim_{n \to +\infty} \|y_n\|_\alpha = 0$. $\square$

Remark 3.3. Using Theorem 3.2, we also can obtain analogous results to Corollaries 3.1 and 3.2. The discussions are omitted here.

Theorem 3.3. *Assume that*

(H4) *There exist constants $C_1 \geq 0$, $\ell_j > p$ and $p < \iota_j < \ell$, such that*

$$0 < \ell_j F_j(t,y) \leq y f_j(t,y), \quad \text{for } t \in (s_j, t_{j+1}],\ |y| \geq C_1,\ j = 0,1,...,m,$$

and

$$0 < y I_j(y) \leq \iota_j J_j(y), \quad \text{for } y \in \mathbb{R}\backslash\{0\},\ j = 1,2,...,m,$$

where $\ell = \min_{0\leq j\leq m}\{\ell_j\}$ satisfies $\ell m_0 - p m_1 > 0$.

Then, for $\lambda \in \left(0, \frac{m_0 \varrho^p}{pK^p \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max_{|y|\leq\varrho} F_j(t,y)dt}\right)$, the problem (1.3) with $\mu = 1$ has at least two distinct weak solutions.

Proof. Define the functionals $\widetilde{\phi}$, $\widetilde{\psi} : E_0^{\alpha,p} \to \mathbb{R}$ as follows:

$$\widetilde{\phi}(y) = \frac{1}{p}\mathcal{M}(\|y\|_\alpha^p) + \sum_{j=1}^{m} J_j(y(t_j)), \quad \widetilde{\psi}(y) = \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,y(t))dt.$$

Clearly $I_\lambda(y) = \widetilde{\phi}(y) - \lambda\widetilde{\psi}(y)$.

Because $0 < J_j(y)$, $y \in \mathbb{R}\backslash\{0\}$, one has

$$\widetilde{\phi}(y) = \frac{1}{p}\mathcal{M}(\|y\|_\alpha^p) + \sum_{j=1}^{m} J_j(y(t_j)) \geq \frac{1}{p}\mathcal{M}(\|y\|_\alpha^p) \geq \frac{m_0}{p}\|y\|_\alpha^p, \tag{3.9}$$

which implies that $\widetilde{\phi}$ is bounded from below.

Now, we show that I_λ satisfies the Palais-Smale condition. Let $\{y_n\} \subset E_0^{\alpha,p}$ such that $\{I_\lambda(y_n)\}$ is a bounded sequence and $I'_\lambda(y_n) \to 0$. Taking into account (H4), one has

$$\begin{aligned}
&\ell I_\lambda(y_n) - \langle I'_\lambda(y_n), y_n\rangle \\
=&\frac{\ell}{p}\mathcal{M}(\|y_n\|_\alpha^p) - \ell\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} F_j(t,y_n(t))dt + \ell\sum_{j=1}^{m} J_j(y_n(t_j)) \\
&- M(\|y_n\|_\alpha^p)\|y_n\|_\alpha^p + \lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} f_j(t,y_n(t))y_n(t)dt - \sum_{j=1}^{m} I_j(y_n(t_j))y_n(t_j) \\
\geq&(\frac{\ell m_0}{p} - m_1)\|y_n\|_\alpha^p - \lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \max_{y_n\in[-C_1,C_1]} |\ell F_j(t,y_n(t)) - f_j(t,y_n(t))y_n(t)|\,dt,
\end{aligned}$$

which implies that $\{y_n\}$ is bounded in $E_0^{\alpha,p}$. By (2.5), we have

$$\begin{aligned}
&\langle I'_\lambda(y_n) - I'_\lambda(y), y_n - y\rangle \\
=&M(\|y_n\|_\alpha^p)\int_0^T h(t)(\Phi_p({}_0^C D_t^\alpha y_n(t)) - \Phi_p({}_0^C D_t^\alpha y(t)))_0^C D_t^\alpha(y_n(t) - y(t))dt \\
&+ (M(\|y_n\|_\alpha^p) - M(\|y\|_\alpha^p))\int_0^T h(t)\Phi_p({}_0^C D_t^\alpha y(t))_0^C D_t^\alpha(y_n(t) - y(t))dt \\
&+ M(\|y_n\|_\alpha^p)\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} a(t)(\Phi_p(y_n(t)) - \Phi_p(y(t)))(y_n(t) - y(t))dt
\end{aligned}$$

$$+ (M(\|y_n\|_\alpha^p) - M(\|y\|_\alpha^p)) \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} a(t)\Phi_p(y(t))(y_n(t) - y(t))dt$$
$$- \lambda \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} (f_j(t, y_n(t)) - f_j(t, y(t)))(y_n(t) - y(t))dt$$
$$+ \sum_{j=1}^{m} (I_j(y_n(t_j)) - I_j(y(t_j)))(y_n(t_j) - y(t_j)). \tag{3.10}$$

According to Lemma 2.5 and the boundedness of $M(\|y_n\|_\alpha^p) - M(\|y\|_\alpha^p)$, we have

$$(M(\|y_n\|_\alpha^p) - M(\|y\|_\alpha^p)) \int_0^T h(t)\Phi_p({}_0^C D_t^\alpha y(t)) {}_0^C D_t^\alpha (y_n(t) - y(t))dt \to 0, \tag{3.11}$$

$$(M(\|y_n\|_\alpha^p) - M(\|y\|_\alpha^p)) \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} a(t)\Phi_p(y(t))(y_n(t) - y(t))dt \to 0, \tag{3.12}$$

$$\sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} (f_j(t, y_n(t)) - f_j(t, y(t)))(y_n(t) - y(t))dt \to 0, \tag{3.13}$$

$$\sum_{j=1}^{m} (I_j(y_n(t_j)) - I_j(y(t_j)))(y_n(t_j) - y(t_j)) \to 0. \tag{3.14}$$

Since $y_n \rightharpoonup y$ and $I'_\lambda(y_n) \to 0$, one has

$$\langle I'_\lambda(y_n) - I'_\lambda(y), y_n - y\rangle \to 0. \tag{3.15}$$

By [27, Eq (2.2)], there exist constants $c_p, d_p > 0$, such that

$$M(\|y_n\|_\alpha^p) \int_0^T h(t)(\Phi_p({}_0^C D_t^\alpha y_n(t)) - \Phi_p({}_0^C D_t^\alpha y(t))) {}_0^C D_t^\alpha (y_n(t) - y(t))dt$$
$$+ M(\|y_n\|_\alpha^p) \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} a(t)(\Phi_p(y_n(t)) - \Phi_p(y(t)))(y_n(t) - y(t))dt$$
$$\geq \begin{cases} c_p M(\|y_n\|_\alpha^p)\Big(\int_0^T h(t)|{}_0^C D_t^\alpha y_n(t) - {}_0^C D_t^\alpha y(t)|^p dt \\ + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} a(t)|y_n(t) - y(t)|^p dt\Big), & p \geq 2, \\ d_p M(\|y_n\|_\alpha^p)\Big(\int_0^T \frac{h(t)|{}_0^C D_t^\alpha y_n(t) - {}_0^C D_t^\alpha y(t)|^2}{(|{}_0^C D_t^\alpha y_n(t)| + |{}_0^C D_t^\alpha y(t)|)^{2-p}} dt \\ + \sum_{j=0}^{m} \int_{s_j}^{t_{j+1}} \frac{a(t)|y_n(t) - y(t)|^2}{(|y_n(t)| + |y(t)|)^{2-p}} dt\Big), & 1 < p < 2. \end{cases} \tag{3.16}$$

If $p \geq 2$, it follows from (3.10)-(3.16) that $\|y_n - y\|_\alpha \to 0$ in $E_0^{\alpha,p}$.
If $1 < p < 2$, based on the proof of [31, Lemma 3.4], we can obtain that

$$\int_0^T h(t)|{}_0^C D_t^\alpha y_n(t) - {}_0^C D_t^\alpha y(t)|^p dt$$
$$\leq 2^{\frac{(p-1)(2-p)}{2}} \left(\int_0^T \frac{h(t)|{}_0^C D_t^\alpha y_n(t) - {}_0^C D_t^\alpha y(t)|^2}{(|{}_0^C D_t^\alpha y_n(t)| + |{}_0^C D_t^\alpha y(t)|)^{2-p}} dt \right)^{\frac{p}{2}} (\|y_n\|_\alpha + \|y\|_\alpha)^{\frac{(2-p)p}{2}}, \tag{3.17}$$

and

$$\begin{aligned}&\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} a(t)|y_n(t)-y(t)|^p dt\\ \le& 2^{\frac{(p-1)(2-p)}{2}}\left(\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} \frac{a(t)|y_n(t)-y(t)|^2}{(|y_n(t)|+|y(t)|)^{2-p}}dt\right)^{\frac{p}{2}}(\|y_n\|_\alpha+\|y\|_\alpha)^{\frac{(2-p)p}{2}}.\end{aligned} \tag{3.18}$$

It follows from (3.16), (3.17) and (3.18) that

$$\begin{aligned}&M(\|y_n\|_\alpha^p)\left(\int_0^T h(t)(\Phi_p({}_0^C D_t^\alpha y_n(t))-\Phi_p({}_0^C D_t^\alpha y(t)))_0^C D_t^\alpha(y_n(t)-y(t))dt\right.\\ &\left.+\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} a(t)(\Phi_p(y_n(t))-\Phi_p(y(t)))(y_n(t)-y(t))dt\right)\\ \ge&\frac{d_pM(\|y_n\|_\alpha^p)}{2^{\frac{(p-1)(2-p)}{p}}(\|y_n\|_\alpha+\|y\|_\alpha)^{2-p}}\left(\left(\int_0^T h(t)|_0^C D_t^\alpha y_n(t)-_0^C D_t^\alpha y(t)|^p dt\right)^{\frac{2}{p}}\right.\\ &\left.+\left(\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}} a(t)|y_n(t)-y(t)|^p dt\right)^{\frac{2}{p}}\right)\\ \ge&\frac{d_pM(\|y_n\|_\alpha^p)}{2^{\frac{(p-1)(2-p)}{p}}\max\{2^{\frac{2}{p}-1},1\}}\frac{\|y_n-y\|_\alpha^2}{(\|y_n\|_\alpha+\|y\|_\alpha)^{2-p}}.\end{aligned} \tag{3.19}$$

In view of (3.10)-(3.15) and (3.19), we obtain that $\|y_n-y\|_\alpha\to 0$ in $E_0^{\alpha,p}$, i.e., $\{y_n\}$ strongly converges to y in $E_0^{\alpha,p}$.

On the other hand, from (H4), there exist ν_j, A_j, χ_j, $B_j>0$, such that

$$F_j(t,y(t))\ge \nu_j|y|^{\ell_j}-A_j \quad\text{and}\quad J_j(y)\le \chi_j|y|^{\iota_j}+B_j.$$

Let $\|y\|_\alpha=1$, it follows that

$$\begin{aligned}I_\lambda(\varkappa y)\le&\frac{1}{p}\mathcal{M}(\|\varkappa y\|_\alpha^p)-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}(\nu_j|\varkappa y|^{\ell_j}-A_j)dt+\sum_{j=1}^{m}(\chi_j|\varkappa y|^{\iota_j}+B_j)\\ \le&\frac{m_1}{p}\|\varkappa y\|_\alpha^p-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\nu_j|\varkappa y|^{\ell_j}dt+\sum_{j=1}^{m}\chi_j|\varkappa y|^{\iota_j}\\ &+\lambda\sum_{j=0}^{m}A_j(t_{j+1}-s_j)+\sum_{j=1}^{m}B_j\\ \le&\frac{m_1}{p}\|\varkappa y\|_\alpha^p-\lambda\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\nu_j|\varkappa y|^{\ell_j}dt+\sum_{j=1}^{m}\chi_jK^{\iota_j}\|\varkappa y\|_\alpha^{\iota_j}\\ &+\lambda\sum_{j=0}^{m}A_j(t_{j+1}-s_j)+\sum_{j=1}^{m}B_j\\ \le&\frac{m_1}{p}\varkappa^p-\lambda\sum_{j=0}^{m}\varkappa^\ell\int_{s_j}^{t_{j+1}}\nu_j|y|^{\ell_j}dt+\sum_{j=1}^{m}\chi_j(K\varkappa)^{\iota_j}\end{aligned}$$

$$+\lambda\sum_{j=0}^{m}A_j(t_{j+1}-s_j)+\sum_{j=1}^{m}B_j.$$

Since $\ell_j > p$, $p < \iota_j < \ell$ and $\int_{s_j}^{t_{j+1}} \nu_j|y|^{\ell_j}dt > 0$, we get $I_\lambda(\varkappa y) \to -\infty$ as $\varkappa \to +\infty$. Thus, I_λ is unbounded from below.

Put $r = \frac{m_0\varrho^p}{pK^p}$. By Lemma 2.8 and (3.9), we have $\|y\|_\infty \leq \varrho$. So

$$\sup_{y\in\widetilde{\phi}^{-1}(-\infty,r)}\widetilde{\psi}(y) \leq \sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\max_{|y|\leq\varrho}F_j(t,y)dt < +\infty.$$

Therefore, by Theorem 2.1, for every $\lambda \in \left(0, \frac{m_0\varrho^p}{pK^p\sum_{j=0}^{m}\int_{s_j}^{t_{j+1}}\max_{|y|\leq\varrho}F_j(t,y)dt}\right)$, I_λ admits two distinct critical points, that is, the problem (1.3) with $\mu = 1$ possesses at least two distinct weak solutions. □

4. Examples

Example 4.1. Let $\alpha = \frac{1}{2}$, $h(t) = a(t) = T = 1$, $m = 2$, $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{1}{2} < t_2 = \frac{7}{12} < s_2 = \frac{2}{3} < t_3 = 1$, $p = 3$, $M(y) = \frac{3}{2} + \frac{\sin y}{2}$, $I_1(y) = -\frac{1}{5}y^2$ and $I_2(y) = -e^{-y}$. Then $m_0 = 1$, $m_1 = 2$ and

$$\widehat{J}^\infty = \limsup_{x\to+\infty}\frac{-J_1(x)-J_2(x)}{x^3} = \lim_{x\to+\infty}\frac{\frac{1}{15}x^3+1-e^{-x}}{x^3} = \frac{1}{15}.$$

Put

$$\hat{a}_n := \frac{2n!(n+2)!-1}{4(n+1)!}, \quad \hat{b}_n := \frac{2n!(n+2)!+1}{4(n+1)!}, \quad \forall n\in\mathbb{N},$$

and consider the non-negative continuous functions $f_j : \mathbb{R}\to\mathbb{R}$, $j = 1, 2$,

$$f_j(y) = \begin{cases} \dfrac{32(n+1)!^2\left((n+1)!^3-n!^3\right)}{\pi}\sqrt{\dfrac{1}{16(n+1)!^2}-\left(y-\dfrac{n!(n+2)}{2}\right)^2}, \\ y\in[\hat{a}_n,\hat{b}_n], \\ 0,\ y\neq\cup_{n\in\mathbb{N}}[\hat{a}_n,\hat{b}_n]. \end{cases}$$

One has

$$\int_{\hat{a}_n}^{\hat{b}_n} f_j(y)dy = (n+1)!^3 - n!^3, \quad \forall n\in\mathbb{N}.$$

Then

$$\lim_{n\to+\infty}\frac{F_j(\hat{a}_n)}{\hat{a}_n^3} = 0, \quad \lim_{n\to+\infty}\frac{F_j(\hat{b}_n)}{\hat{b}_n^3} = 8.$$

So

$$\liminf_{x\to+\infty}\frac{F_j(x)}{x^3} = 0, \quad \widehat{B}^\infty = \limsup_{x\to+\infty}\frac{F_j(x)}{x^3} = 8.$$

Through direct calculation, we obtain that $\zeta \approx 2.7384$, $K \approx 1.4217$. Hence, from Corollary 3.1, for every $\lambda > 8.2153$ and $\mu \in (0, 1.7401)$, the problem (1.3) possesses an unbounded sequence of weak solutions in $E_0^{\alpha,p}$.

Example 4.2. Let $\alpha = 0.6$, $h(t) = a(t) = T = m = 1$, $p = 3$. Consider the following problem:

$$\begin{cases} \quad M(\|y\|_{0.6}^3)({}_tD_1^{0.6}\left(\Phi_3({}_0^CD_t^{0.6}y(t))\right) + \Phi_3(y(t))) \\ = \lambda f_j(t, y(t)), \quad t \in (s_j, t_{j+1}],\ j = 0, 1, \\ \Delta\left(M(\|y(t_1)\|_{0.6}^3){}_tD_1^{-0.4}\left(\Phi_3({}_0^CD_t^{0.6}y(t_1))\right)\right) = I_1(y(t_1)), \\ \quad M(\|y\|_{0.6}^3){}_tD_1^{-0.4}\left(\Phi_3({}_0^CD_t^{0.6}y(t))\right) \\ = M(\|y(t_1^+)\|_{0.6}^3){}_tD_1^{-0.4}\left(\Phi_3({}_0^CD_t^{0.6}y(t_1^+))\right), \quad t \in (t_1, s_1], \\ \quad M(\|y(s_1^+)\|_{0.6}^3){}_tD_1^{-0.4}\left(\Phi_3({}_0^CD_t^{0.6}y(s_1^+))\right) \\ = M(\|y(s_1^-)\|_{0.6}^3){}_tD_1^{-0.4}\left(\Phi_3({}_0^CD_t^{0.6}y(s_1^-))\right), \\ y(0) = y(1) = 0, \end{cases} \tag{4.1}$$

where $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = 1$, $M(y) = 5 + \frac{y}{1+y}$ for all $y \in \mathbb{R}^+$, $f_j(t, y) = y^5$, $I_1(y) = y^3$. Obviously, $m_0 = 5$. If $\ell_j = 5$ and $\iota_1 = \frac{9}{2}$, we can obtain

$$\begin{aligned} &0 < \frac{5}{6}y^6 = \ell_j F_j(t, y) \leq y f_j(t, y) = y^6, \quad j = 0, 1, \\ &0 < y^4 = y I_1(y) \leq \iota_1 J_1(y) = \frac{9}{8}y^4. \end{aligned}$$

Thus, (H4) holds. Let $\varrho = 1$. By direct calculation, $\frac{m_0 \varrho^p}{pK^p \sum_{j=0}^m \int_{s_j}^{t_{j+1}} \max_{|y| \leq \varrho} F_j(t,y)dt}$ ≈ 7.9261. Applying Theorem 3.3, for each $\lambda \in (0, 7.9261)$, the problem (4.1) possesses at least two distinct weak solutions.

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References

[1] G. A. Afrouzi and A. Hadjian, *A variational approach for boundary value problems for impulsive fractional differential equations*, Fract. Calc. Appl. Anal., 2018, 21(6), 1565–1584.

[2] G. A. Afrouzi, S. Heidarkhani and S. Moradi, *Multiple solutions for a Kirchhoff-type second-order impulsive differential equation on the half-line*, Quaest. Math., 2022, 45(1), 109–141.

[3] R. Agarwal, D. O'Regan and S. Hristova, *Stability by Lyapunov like functions of nonlinear differential equations with non-instantaneous impulses*, J. Appl. Math. Comput., 2017, 53(1), 147–168.

[4] C. Z. Bai, *Existence result for boundary value problem of nonlinear impulsive fractional differential equation at resonance*, J. Appl. Math. Comput., 2012, 39, 421–443.

[5] G. Bonanno, *Relations between the mountain pass theorem and local minima*, Adv. Nonlinear Anal., 2012, 1(3), 205–220.

[6] G. Bonanno and G. Molica Bisci, *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities*, Bound. Value Probl., 2009, 2009, 1–20.

[7] G. Bonanno, R. Rodríguez-López and S. Tersian, *Existence of solutions to boundary value problem for impulsive fractional differential equations*, Fract. Calc. Appl. Anal., 2014, 17, 717–744.

[8] G. Caristi, S. Heidarkhani and A. Salari, *Variational approaches to Kirchhoff-type second-order impulsive differential equations on the half-line*, Results Math., 2018, 73, 1–31.

[9] V. Colao, L. Muglia and H. Xu, *Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay*, Ann. Mat. Pura Appl., 2016, 195(3), 697–716.

[10] X. L. Fan, T. T. Xue and Y. S. Jiang, *Dirichlet problems of fractional p-Laplacian equation with impulsive effects*, Math. Biosci. Eng., 2023, 20(3), 5094–5116.

[11] J. R. Graef, S. Heidarkhani, L. J. Kong and S. Moradi, *Existence results for impulsive fractional differential equations with p-Laplacian via variational methods*, Math. Bohem., 2022, 147(1), 95–112.

[12] J. R. Graef, S. Heidarkhani, L. J. Kong and S. Moradi, *Three solutions for impulsive fractional boundary value problems with p-Laplacian*, Bull. Iran Math. Soc., 2022, 48(4), 1413–1433.

[13] S. Heidarkhani, G. A. Afrouzi and S. Moradi, *Existence results for a Kirchhoff-type second-order differential equation on the half-line with impulses*, Asymptotic Anal., 2017, 105(3–4), 137–158.

[14] S. Heidarkhani, A. Cabada, G. A. Afrouzi, et al., *A variational approach to perturbed impulsive fractional differential equations*, J. Comput. Appl. Math., 2018, 341, 42–60.

[15] S. Heidarkhani and A. Salari, *Nontrivial solutions for impulsive fractional differential systems through variational methods*, Math. Meth. Appl. Sci., 2020, 43(10), 6529–6541.

[16] S. Heidarkhani, A. Salari and G. Caristi, *Infinitely many solutions for impulsive nonlinear fractional boundary value problems*, Adv. Differ. Equ., 2016, 2016, 196.

[17] S. Heidarkhani, Y. L. Zhao, G. Caristi, et al., *Infinitely many solutions for perturbed impulsive fractional differential systems*, Appl. Anal., 2017, 96(8), 1401–1424.

[18] E. Hernández and D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc., 2013, 141(5), 1641–1649.

[19] F. Jiao and Y. Zhou, *Existence results for fractional boundary value problem via critical point theory*, Int. J. Bifurcation Chaos, 2012, 22(4), 1250086.

[20] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, New York, 2006.

[21] G. Kirchhoff, *Vorlesungen über Mathematische Physik: Mechanik*, Teubner, Leipzig, 1883.

[22] D. P. Li, F. Q. Chen, Y. H. Wu and Y. K. An, *Multiple solutions for a class of p-Laplacian type fractional boundary value problems with instantaneous and non-instantaneous impulses*, Appl. Math. Lett., 2020, 106, 106352.

[23] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989.

[24] V. D. Milman and A. D. Myshkis, *On the stability of motion in the presence of impulses*, Sibirian Math. J., 1960, 1(2), 233–237.

[25] Y. Qiao, F. Q. Chen and Y. K. An, *Variational method for p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses*, Math. Meth. Appl. Sci., 2021, 44(11), 8543–8553.

[26] R. Rodríguez-López and S. Tersian, *Multiple solutions to boundary value problem for impulsive fractional differential equations*, Fract. Calc. Appl. Anal., 2014, 17(4), 1016–1038.

[27] J. Simon, *Régularité de la solution d'une équation non linéaire dans* $\mathbb{R}^N$, Lect. Notes Math., 1978, 665, 205–227.

[28] Y. Wang and L. X. Tian, *Existence and multiplicity of solutions for* (p, q)*-Laplacian Kirchhoff-type fractional differential equations with impulses*, Math. Meth. Appl. Sci., 2023, 46(13), 14177–14199.

[29] L. H. Zhang, J. J. Nieto and G. T. Wang, *Extremal solutions for a nonlinear impulsive differential equations with multi-orders fractional derivatives*, J. Appl. Anal. Comput., 2017, 7(3), 814–823.

[30] W. Zhang and W. B. Liu, *Variational approach to fractional Dirichlet problem with instantaneous and non-instantaneous impulses*, Appl. Math. Lett., 2020, 99, 105993.

[31] W. Zhang and J. B. Ni, *Study on a new p-Laplacian fractional differential model generated by instantaneous and non-instantaneous impulsive effects*, Chaos Solitons Fractals, 2023, 168, 113143.

[32] W. Zhang, Z. Y. Wang and J. B. Ni, *Variational method to the fractional impulsive equation with Neumann boundary conditions*, J. Appl. Anal. Comput., 2024, 14(5), 2890–2902.

[33] Y. L. Zhao, H. B. Chen and C. J. Xu, *Nontrivial solutions for impulsive fractional differential equations via Morse theory*, Appl. Math. Comput., 2017, 307, 170–179.

[34] J. W. Zhou, Y. M. Deng and Y. N. Wang, *Variational approach to p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses*, Appl. Math. Lett., 2020, 104, 106251.