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LIE SYMMETRY REDUCTION FOR (2+1)-DIMENSIONAL FRACTIONAL SCHRÖDINGER EQUATION*

Panpan Wang[1], Xiufang Feng[1,†] and Shangqin He[2]

Abstract This study investigates Lie symmetry analysis, exact solutions, and conservation laws for a (2+1)-dimensional fractional Schrödinger equation. The original equations have been reduced to fractional ODEs employing the obtained vector field. For the considered equation, exact solutions are also established. Furthermore, the resulting exact solutions are demonstrated for convergence. Conservation laws for this equation have been investigated employing the Ibragimov theorem.

Keywords Lie symmetry analysis, fractional Schrödinger equation, exact solutions, conservation laws.

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1. Introduction

A classical equation that describes signal propagation in optical fibers, the nonlinear Schrödinger equation (NLSE) [4, 6, 15] is widely used in fiber optic communication systems. The description of the dispersion effects, which characterize compression or pulse broadening processes, and the nonlinear effects, which describe the self-interaction of light wave packets in optical fibers, have been extensively utilized [21]. Therefore, establishing exact solutions to NLSE has become essential in nonlinear science.

In this article, we consider fractional Schrödinger equation [13] in the following form:

$$iD_t^\alpha q + aq_{xx} - bq_{yy} + \gamma q|q|^2 = 0, \quad t > 0, \quad 0 < \alpha \leq 1, \tag{1.1}$$

in which a, b, and γ are constants, γ is the coefficient for the self-phase modulation term, $q = q(x, y, t)$ typifies the complex function, the terms aq_{xx} and bq_{yy} denote the group velocity dispersion effect, and the fourth term represents the self-phase modulation effect. Eq. (1.1) represents a model for the electromagnetic wave equation

[†]The corresponding author.
[1]School of Mathematics and Statistics, Ningxia University, Ningxia 750021, China
[2]School of Mathematics and Information Sciences, North Minzu University, Ningxia 750021, China
[*]The authors were supported by Scientific research project of Ningxia Education Department (Grant No. NYG2022062), National Science Foundation of Ningxia Province (Grant No. 2023AAC03257), National Natural Science Foundation of China (Grant No. 11961054).
Email: 12022140038@stu.nxu.edu.cn(P. Wang), xf_feng@nxu.edu.cn(X. Feng), hsq101@163.com(S. He)

in two-dimensional weakly guided structures, with applications in electromagnetic wave propagation, underwater acoustics, quantum physics, and optoelectronic device design.

Numerous researchers have studied exact solutions to Eq. (1.1). The novel approach [5], the extended experimental equation approach [12], (G'/G)-expansion approach [13], and the F-expansion approach [19] have all been applied in searching for exact solutions to Eq. (1.1).

One of the current effective methods for obtaining exact solutions to PDEs is the Lie symmetry analysis approach [14]. In 1998, Buckwar and Luchko [7] employed scale transformation groups to establish group-invariant solutions for fractional diffusion wave equations. This marked the beginning of the development of the Lie symmetry analysis approach for fractional PDEs. Gazizov and colleagues [8] have recently utilized the Lie symmetry analysis approach for some fractional differential equations. Many nonlinear models [1–3, 16–18] of fractional order with physical backgrounds were then considered. The time fractional order of complicated functions is a rather unexplored area.

This work investigates similarity reduction, exact solutions, convergence analysis, and conservation laws of Eq. (1.1). The conserved vectors for Eq. (1.1) have been derived by applying a new conservation theorem [10] and formal Lagrangian operators.

We have the following framework. The method of Lie symmetry analysis has been presented in Sect. 2. The Lie symmetry admitted by Eq. (1.1) is obtained, and similarity reduction is made on the basis of the fractional integral operator in Sect. 3. Sect. 4 focuses on constructing power series solutions as exact solutions to Eq. (1.1) and demonstrating that the resulting exact solutions are convergent. Sect. 5 presents conservation laws of Eq. (1.1), which depends on the new conservation theorem.

2. Lie symmetry analysis of PDEs with fractional order

We first give a quick overview of the definition and practical applications of the results. The Riemann-Liouville fractional derivative is given by:

$$
\begin{aligned}
&D_t^{\alpha} v(t,x)\\
&= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\partial^m}{\partial t^m} \displaystyle\int_0^t (t-w)^{m-1-\alpha} v(w,x)dw, \ 0 \le m-1 < \alpha < m, \ m \in \mathbf{N}, \\ \dfrac{\partial^m v(t,x)}{\partial t^m}, \quad \alpha = m \in \mathbf{N}, \end{cases}
\end{aligned}
\tag{2.1}
$$

the Euler gamma function, denoted by $\Gamma(z)$, is defined as $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$.

Definition 2.1. The Erdélyi-Kober fractional differential operator is defined as:

$$
(\mathcal{P}_{\Omega}^{\tau,\alpha} \mathcal{F})(z) := \prod_{i=0}^{m-1} (\tau + i - \frac{1}{\Omega} z \frac{d}{dz})(\mathcal{K}_{\Omega}^{\tau+\alpha, m-\alpha} \mathcal{F})(z), \tag{2.2}
$$

$$m = \begin{cases} \alpha, & \alpha \in \mathbf{N}, \\ [\alpha]+1, & \alpha \notin \mathbf{N}, \end{cases}$$

where

$$(\mathcal{K}_{\Omega}^{\tau,\alpha}\mathcal{F})(z) := \begin{cases} \mathcal{F}(z), & \alpha = 0, \\ \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^{\infty}(u-1)^{\alpha-1}u^{-(\tau+\alpha)}\mathcal{F}(zu^{\frac{1}{\Omega}})du, & \alpha > 0, \end{cases} \tag{2.3}$$

denotes Erdélyi-Kober fractional integral operator.

Here, we introduce the basic concepts of Lie symmetry theory for fractional PDEs. The fractional PDEs take the subsequent form:

$$\begin{cases} \Omega_1 = R_1(t,x,y,v,u,D_t^{\alpha}u,v_x,\cdots), \\ \Omega_2 = R_2(t,x,y,v,u,D_t^{\alpha}v,v_x,\cdots), \end{cases} \quad 0<\alpha\leq 1. \tag{2.4}$$

Assuming that fractional PDEs (2.4) are invariant under one-parameter Lie group of point transformations,

$$\begin{aligned}
&\widehat{t} = t + \varepsilon\tau(t,x,y,v,u) + o(\varepsilon^2), \\
&\widehat{x} = x + \varepsilon\xi(t,x,y,v,u) + o(\varepsilon^2), \\
&\widehat{y} = y + \varepsilon\rho(t,x,y,v,u) + o(\varepsilon^2), \\
&\widehat{u} = u + \varepsilon\eta(t,x,y,v,u) + o(\varepsilon^2), \\
&\widehat{v} = v + \varepsilon\phi(t,x,y,v,u) + o(\varepsilon^2), \\
&D_{\widehat{t}}^{\alpha}\widehat{u} = D_t^{\alpha}u + \varepsilon\eta^{\alpha,t}(t,x,y,v,u) + o(\varepsilon^2), \\
&D_{\widehat{t}}^{\alpha}\widehat{v} = D_t^{\alpha}v + \varepsilon\phi^{\alpha,t}(t,x,y,v,u) + o(\varepsilon^2), \\
&\frac{\partial\widehat{u}}{\partial\widehat{x}} = \frac{\partial u}{\partial x} + \epsilon\eta^{x}(t,x,y,v,u) + o(\epsilon^2), \\
&\frac{\partial\widehat{v}}{\partial\widehat{x}} = \frac{\partial v}{\partial x} + \epsilon\phi^{x}(t,x,y,v,u) + o(\epsilon^2), \\
&\frac{\partial^2\widehat{u}}{\partial\widehat{x}^2} = \frac{\partial^2 u}{\partial x^2} + \epsilon\eta^{xx}(t,x,y,v,u) + o(\epsilon^2), \\
&\frac{\partial^2\widehat{v}}{\partial\widehat{x}^2} = \frac{\partial^2 v}{\partial x^2} + \epsilon\phi^{xx}(t,x,y,v,u) + o(\epsilon^2).
\end{aligned} \tag{2.5}$$

In (2.5), τ, ξ, ρ, η, and ϕ are the infinitesimals of the forms for the dependent and independent variables, respectively, and $\varepsilon \ll 1$ is the Lie group parameter.

Let us now consider the explicit expressions of η^x, ϕ^x, η^{xx}, and ϕ^{xx}, which are

$$\begin{aligned}
\eta^{x} &= D_x(\eta) - u_tD_x(\tau) - u_xD_x(\xi) - u_yD_x(\rho), \\
\phi^{x} &= D_x(\phi) - v_tD_x(\tau) - v_xD_x(\xi) - v_yD_x(\rho), \\
\eta^{xx} &= D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi) - u_{xy}D_x(\rho), \\
\phi^{xx} &= D_x(\phi^x) - v_{xt}D_x(\tau) - v_{xx}D_x(\xi) - v_{xy}D_x(\rho),
\end{aligned} \tag{2.6}$$

wherein, D_x represents the total differential operator concerning x, defined as follows:

$$D_x = \frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v} + u_x\frac{\partial}{\partial u}. \tag{2.7}$$

In the corresponding Lie algebra of symmetries, there is a set of vector fields,

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v}. \tag{2.8}$$

The expression below represents the prolongation for vector field V,

$$Pr^{(m)}V(\Omega_1)|_{\Omega_1=0} = 0, \quad Pr^{(m)}V(\Omega_2)|_{\Omega_2=0} = 0, \quad m = 1, 2, \cdots, \tag{2.9}$$

where Pr represents the prolongation operator. Since the Riemann-Liouville fractional differential operator (2.1) has a determined lower limit of the integral, it is invariant under the transformations (2.5) and satisfies the following invariant condition:

$$\tau(t, x, y, v, u)|_{t=0} = 0, \tag{2.10}$$

and the α-th prolonged infinitesimals [9] involving Riemann-Liouville fractional derivative with Eq. (2.10) are

$$\begin{cases} \eta^{\alpha,t} = D_t^\alpha\eta + (\eta_u - \alpha D_t\tau)D_t^\alpha u - uD_t^\alpha\eta_u - vD_t^\alpha\phi_v + \eta_v D_t^\alpha\eta_v \\ \qquad + \mu_2 + \sum_{m=1}^{\infty}\left[\binom{\alpha}{m}\frac{\partial^m\eta_u}{\partial t^m} - \binom{\alpha}{m+1}D_t^{m+1}(\tau)\right]D_t^{\alpha-m}(u) \\ \qquad - \sum_{m=1}^{\infty}\binom{\alpha}{m}D_t^m(\xi)D_t^{\alpha-m}(u_x) + \sum_{m=1}^{\infty}\binom{\alpha}{m}\frac{\partial^m\eta_v}{\partial t^n}D_t^{\alpha-m}(v), \\ \phi^{\alpha,t} = D_t^\alpha\phi + (\phi_v - \alpha D_t\tau)D_t^\alpha v - uD_t^\alpha\phi_u - vD_t^\alpha\phi_v + \phi_u D_t^\alpha\eta_u \\ \qquad + \mu_1 + \sum_{m=1}^{\infty}\left[\binom{\alpha}{m}\frac{\partial^m\phi_v}{\partial t^m} - \binom{\alpha}{m+1}D_t^{m+1}(\tau)\right]D_t^{\alpha-m}(v) \\ \qquad + \sum_{m=1}^{\infty}\binom{\alpha}{m}\frac{\partial^m\phi_u}{\partial t^m}D_t^{\alpha-m}(v) - \sum_{m=1}^{\infty}\binom{\alpha}{m}D_t^m(\xi)D_t^{\alpha-m}(v_x), \end{cases} \tag{2.11}$$

where

$$\begin{cases} \mu_1 = \sum_{m=2}^{\infty}\sum_{m_1+m_2=2}^{m}\sum_{s_1=0}^{m_1}\sum_{r_1=0}^{s_1}\sum_{s_2=0}^{m_2}\sum_{r_2=0}^{s_2}\binom{\alpha}{m}\binom{m}{m_1}\binom{m-m_1}{m_2}\binom{s_1}{r_1}\binom{s_2}{r_2}\frac{1}{s_1!s_2!} \\ \qquad \times\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}[-u]^{r_1}[-v]^{r_2}\frac{\partial^{m_1}u^{s_1-r_1}}{\partial t^{m_1}}\frac{\partial^{m_2}v^{s_2-r_2}}{\partial t^{m_2}}\frac{\partial^{n-m_1-m_2+s_1+s_2}\phi}{\partial t^{n-m_1-m_2}\partial u^{k_1}\partial v^{k_2}}, \\ \mu_2 = \sum_{m=2}^{\infty}\sum_{m_1+m_2=2}^{m}\sum_{s_1=0}^{m_1}\sum_{r_1=0}^{s_1}\sum_{s_2=0}^{m_2}\sum_{r_2=0}^{s_2}\binom{\alpha}{m}\binom{m}{m_1}\binom{m-m_1}{m_2}\binom{s_1}{r_1}\binom{s_2}{r_2}\frac{1}{s_1!s_2!} \\ \qquad \times\frac{t^{m-\alpha}}{\Gamma(m+1-\alpha)}[-u]^{r_1}[-v]^{r_2}\frac{\partial^{m_1}u^{s_1-r_1}}{\partial t^{m_1}}\frac{\partial^{m_2}v^{s_2-r_2}}{\partial t^{m_2}}\frac{\partial^{n-m_1-m_2+s_1+s_2}\eta}{\partial t^{n-m_1-m_2}\partial u^{s_1}\partial v^{s_2}}. \end{cases} \tag{2.12}$$

In the next theorem, the functions $u = \Theta_1(x, y, t)$ and $v = \Theta_2(x, y, t)$ are called invariant under certain conditions.

Theorem 2.1. [22] *Invariant solutions of Eq. (2.4) are* $u = \Theta_1(x, y, t)$ *and* $v = \Theta_2(x, y, t)$ *if and only if* *(i)* $u = \Theta_1(x, y, t)$ *and* $v = \Theta_2(x, y, t)$ *are invariant surfaces, that is to say,*

$$\begin{cases} V\Theta_1 = 0 \Leftrightarrow (\xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial v} + \eta\frac{\partial}{\partial u})\Theta_1(x, y, t) = 0, \\ V\Theta_2 = 0 \Leftrightarrow (\xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial v} + \eta\frac{\partial}{\partial u})\Theta_2(x, y, t) = 0. \end{cases} \tag{2.13}$$

(ii) $u = \Theta_1(x, y, t)$ *and* $v = \Theta_2(x, y, t)$ *are solutions for Eq. (2.4).*

3. Lie symmetry analysis and reduction

This section investigates and develops the characteristic formulas of vector fields, which are going to be utilized to obtain the reduction equations. Here, we apply Erdélyi-Kober fractional differential operator to reduce Eq. (1.1) to a nonlinear fractional PDE and then solve it thoroughly.

The present research investigates Lie symmetry analysis of Eq. (1.1) from the standpoint of Eq. (2.1). Initially, presuming that

$$q(t, x, y) = iu(t, x, y) + v(t, x, y), \tag{3.1}$$

after substituting Eq. (3.1), the following equations are obtained by separating the real and imaginary parts of Eq. (1.1),

$$\begin{cases} D_t^\alpha u - av_{xx} + bv_{yy} - \gamma(v^3 + vu^2) = 0, \\ D_t^\alpha v + au_{xx} - bu_{yy} + \gamma(uv^2 + u^3) = 0. \end{cases} \tag{3.2}$$

Assume that (3.2) is invariant under one-parameter transformations (2.5). This leads to the following transformed equation,

$$\begin{cases} D_t^\alpha \widehat{u} - a\widehat{v}_{xx} + b\widehat{v}_{yy} - \gamma(\widehat{v}^3 + \widehat{v}\widehat{u}^2) = 0, \\ D_t^\alpha \widehat{v} + a\widehat{u}_{xx} - b\widehat{u}_{yy} + \gamma(\widehat{u}\widehat{v}^2 + \widehat{u}^3) = 0. \end{cases} \tag{3.3}$$

The invariance conditions are given in this manner via the second prolongation for Eq. (3.3),

$$\begin{cases} \eta^{\alpha,t} - a\phi^{xx} + b\phi^{yy} - \gamma(3v^2\phi + u^2\phi + 2uv\eta) = 0, \\ \phi^{\alpha,t} + a\eta^{xx} - b\eta^{yy} + \gamma(v^2\eta + 2uv\phi + 3u^2\eta) = 0. \end{cases} \tag{3.4}$$

By substituting (2.6) and (2.11) into (3.4), we obtain the following determining equations,

$$\begin{cases} 2\xi_x - \alpha\tau_t = 0, \quad \tau_x = \tau_u = 0 = \xi_u = 0 = \rho_u = \rho_v = \phi_u = 0, \\ -\alpha\tau_t - \phi_v + \eta_u + 2\xi_x = 0, \quad \alpha\tau_t - 2\rho_y - \eta_u + \phi_v = 0, \\ -\alpha\tau_t v - 3\phi + \eta_u v = 0, \quad \eta_u uv - \phi u - \alpha\tau_t uv - 2\eta v = 0, \\ \eta_u - \phi_v - 2\xi_x + \alpha\tau_t = 0, \quad -\eta_u + \phi_v + 2\rho_y - \alpha\tau_t = 0, \\ \alpha\tau_t u - \phi_v u + 3\eta = 0, \quad 2\phi u - \phi_v uv + \alpha\tau_t uv + \eta v = 0, \\ \binom{\alpha}{m}\frac{\partial^m \eta_u}{\partial t^m} - \binom{\alpha}{m+1}D_t^{m+1}(\tau) = 0, \qquad m = 1, 2, \cdots, \\ \binom{\alpha}{m}\frac{\partial^m \phi_v}{\partial t^m} - \binom{\alpha}{m+1}D_t^{m+1}(\tau) = 0, \qquad m = 1, 2, \cdots. \end{cases} \tag{3.5}$$

Here are some solutions we found for the previously mentioned system after tedious and lengthy calculations:

$$\xi = C_1\alpha x + C_2, \quad \rho = C_1\alpha y + C_3, \quad \tau = 2C_1 t, \quad \eta = -C_1\alpha u, \quad \phi = -C_1\alpha v, \tag{3.6}$$

here C_1, C_2, C_3 are arbitrary parameters. Thus, the infinitesimal generator of Eq. (3.2) is given by

$$V = 2C_1 t\frac{\partial}{\partial t} + (C_1\alpha x + C_2)\frac{\partial}{\partial x} + (C_1\alpha y + C_3)\frac{\partial}{\partial y} - C_1\alpha u\frac{\partial}{\partial u} - C_1\alpha v\frac{\partial}{\partial v}. \tag{3.7}$$

This means that the Lie algebra of symmetry group of Eq. (3.2) consists of three vector fields,

$$V_1 = \frac{\partial}{\partial y}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u} - \alpha v\frac{\partial}{\partial v}. \tag{3.8}$$

Since vector fields V_1 and V_2 do not give physically significant results, we are considering the case for V_3 only. Solving the given characteristic equation yields the similarity variables and transformations for V_3. Given vector field V_3, its characteristic equation is

$$\frac{dx}{\alpha x} = \frac{dy}{\alpha y} = \frac{dt}{2t} = \frac{du}{-\alpha u} = \frac{dv}{-\alpha v}, \tag{3.9}$$

and the corresponding group invariant solutions are

$$z = (x-y)t^{-\frac{\alpha}{2}}, \quad u = f(z)t^{-\frac{\alpha}{2}}, \quad v = g(z)t^{-\frac{\alpha}{2}}. \tag{3.10}$$

By employing the similarity variables and group invariant solutions, Eq.(3.2) yields the following results.

Theorem 3.1. *The transformation (3.10) reduces the (2+1)-dimensional fractional Schrödinger equation, namely Eq. (3.2) to FODEs:*

$$\begin{cases} (\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3}{2}\alpha,\alpha} f)(z) - \gamma(f^2(z)g(z) + g^3(z)) + (b-a)g''(z) = 0, \\ (\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3}{2}\alpha,\alpha} g)(z) + \gamma(f^3(z) + g^2(z)f(z)) + (a-b)f''(z) = 0, \end{cases} \tag{3.11}$$

with Erdélyi-Kober fractional differential operator, as outlined in Section 2.

Proof. For $n-1 < \alpha < n$, where $n = 1, 2, 3, \cdots$, there is

$$D_t^\alpha u = \frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-p)^{n-1-\alpha}p^{-\frac{\alpha}{2}}f((x-y)p^{-\frac{\alpha}{2}})dp\right]. \tag{3.12}$$

Letting $h = \frac{t}{p}$, we have $dp = -\frac{t}{h^2}dh$, Eq. (3.12) becomes:

$$\begin{aligned} &D_t^\alpha u \\ =&\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t p^{n-\alpha-1}(\frac{t}{p}-1)^{n-\alpha-1}p^{-\frac{\alpha}{2}}f((x-y)\frac{t^{\frac{\alpha}{2}}}{p^{\frac{\alpha}{2}}}t^{-\frac{\alpha}{2}})dp\right] \\ =&\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{p^{n-\alpha-1-\frac{\alpha}{2}}}{t^{n-\alpha-1-\frac{\alpha}{2}}}t^{n-\alpha-1-\frac{\alpha}{2}}(\frac{t}{p}-1)^{n-\alpha-1}f((x-y)t^{-\frac{\alpha}{2}}(\frac{t}{p})^{\frac{\alpha}{2}})dp\right] \end{aligned}$$

$$
\begin{aligned}
=&\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha-1-\frac{\alpha}{2}}}{\Gamma(n-\alpha)}\int_0^t (\frac{p}{t})^{n-\alpha-1-\frac{\alpha}{2}}(\frac{t}{p}-1)^{n-\alpha-1} f((x-y)t^{-\frac{\alpha}{2}}(\frac{t}{p})^{\frac{\alpha}{2}})dp\right]\\
=&\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha-1-\frac{\alpha}{2}}}{\Gamma(n-\alpha)}\int_{+\infty}^1 (\frac{1}{h})^{n-\alpha-1-\frac{\alpha}{2}}(h-1)^{n-\alpha-1} f(zh^{\frac{\alpha}{2}})(-\frac{t}{h^2})dh\right]\\
=&\frac{\partial^n}{\partial t^n}\left[\frac{t^{n-\alpha-\frac{\alpha}{2}}}{\Gamma(n-\alpha)}\int_1^\infty h^{-(n+1-\alpha-\frac{\alpha}{2})}(h-1)^{n-1-\alpha} f(zh^{\frac{\alpha}{2}})dh\right]\\
=&\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2}}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z)\right].
\end{aligned}
\tag{3.13}
$$

We simplify the right-hand side of (3.13) by utilizing the connection $z=(x-y)t^{-\frac{\alpha}{2}}$ with $\Phi \in C^1(0,\infty)$. The expression becomes

$$
t\frac{\partial}{\partial t}\Phi(z) = t\frac{\partial z}{\partial t}\frac{\partial \Phi(z)}{\partial z} = -\frac{\alpha}{2}z\frac{\partial \Phi(z)}{\partial z}. \tag{3.14}
$$

By Eq. (3.14), we acquire

$$
\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2}}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z)\right]\\
=&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}(t^{n-\alpha-\frac{\alpha}{2}}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z))\right]\\
=&\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\frac{3}{2}\alpha-1}(n-\frac{3}{2}\alpha-\frac{\alpha}{2}z\frac{\partial}{\partial z})(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z)\right].
\end{aligned}
\tag{3.15}
$$

By applying the above steps for $n-1$ times, we have the following results

$$
\begin{aligned}
&\frac{\partial^n}{\partial t^n}\left[t^{n-\frac{3}{2}\alpha}(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z)\right]\\
=&\cdots\\
=&t^{-\frac{3}{2}\alpha}\prod_{j=1}^{n}\left(1-\frac{3}{2}\alpha+j-\frac{\alpha}{2}z\frac{\partial}{\partial z}\right)(\mathcal{K}_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2},n-\alpha} f)(z).
\end{aligned}
\tag{3.16}
$$

According to the definition of Erdélyi-Kober fractional differential operator, the result in (3.16) becomes

$$
D_t^\alpha u = t^{-\frac{3}{2}\alpha}(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha} f)(z). \tag{3.17}
$$

Utilizing the same procedure as before, the second equation in Eq. (3.2) becomes

$$
D_t^\alpha v = t^{-\frac{3}{2}\alpha}(\mathcal{P}_{\frac{2}{\alpha}}^{1-\frac{3\alpha}{2},\alpha} g)(z). \tag{3.18}
$$

By substituting Eqs. (3.10), (3.17), and (3.18) into Eq. (3.2), we validate Eq. (3.11). After a great deal of work, we have reached the main point of this theorem: to transform the governing equation into nonlinear FDEs in order to make it easier to solve, as described in the following section. □

4. Exact solutions

We have completed exact solutions of Eq. (3.2) through power series approach [11,20] and symbolic calculations in the present section. Assume that the solutions of Eq. (3.11) are expressed in terms of power series, as given by

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \quad g(z) = \sum_{m=0}^{\infty} b_m z^m. \tag{4.1}$$

Eq. (4.1) shows that a_m and b_m are some undetermined constants, and there are

$$\begin{aligned} f''(z) &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}z^m, \\ g''(z) &= \sum_{m=0}^{\infty} (m+2)(m+1)b_{m+2}z^m. \end{aligned} \tag{4.2}$$

When Eq. (4.1) and Eq. (4.2) are substituted into Eq.(3.11), we acquire

$$\begin{cases} \sum_{m=0}^{\infty} \frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} a_m z^m + (b-a)\sum_{m=0}^{\infty}(m+2)(m+1)b_{m+2}z^m \\ -\gamma\left[(\sum_{m=0}^{\infty} a_m z^m)^2(\sum_{m=0}^{\infty} b_m z^m) + (\sum_{m=0}^{\infty} b_m z^m)^3\right] = 0, \\ \sum_{m=0}^{\infty} \frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} b_m z^m + (a-b)\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}z^m \\ +\gamma\left[(\sum_{m=0}^{\infty} a_m z^m)(\sum_{m=0}^{\infty} b_m z^m)^2 + (\sum_{m=0}^{\infty} a_m z^m)^3\right] = 0. \end{cases} \tag{4.3}$$

Comparing coefficients in Eq. (4.3) with $m=0$ yields

$$\begin{cases} a_2 = \frac{1}{2(b-a)} \times \left[\frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2})} b_0 + \gamma(b_0^2 a_0 + a_0^3)\right], \\ b_2 = \frac{1}{2(a-b)} \times \left[\frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2})} a_0 - \gamma(b_0^3 + b_0 a_0^2)\right]. \end{cases} \tag{4.4}$$

But when $m \geq 1$, we obtain

$$\begin{cases} a_{m+2} = \frac{1}{(b-a)(m+1)(m+2)} \left[\frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} b_m \right. \\ \qquad \left. + \gamma(\sum_{k=0}^{m}\sum_{j=0}^{k} a_j b_{k-j} b_{m-k} + \sum_{k=0}^{m}\sum_{j=0}^{k} a_j a_{k-j} a_{m-k})\right], \\ b_{m+2} = \frac{1}{(a-b)(m+1)(m+2)} \left[\frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} a_m \right. \\ \qquad \left. - \gamma(\sum_{k=0}^{m}\sum_{j=0}^{k} b_j b_{k-j} b_{m-k} + \sum_{k=0}^{m}\sum_{j=0}^{k} b_j a_{k-j} a_{m-k})\right]. \end{cases} \tag{4.5}$$

For this reason, given a_0, a_1, b_0, and b_1, any coefficients a_m and $b_m (m \geq 2)$ in Eq. (4.1) can be derived. This indicates that there are exact solutions to Eq. (3.11), while Eqs. (4.4) and (4.5) provide the coefficients of these solutions. Eq. (3.11) is represented in the following forms:

$$\begin{aligned} &f(z) \\ =&a_0 + a_1 z + a_2 z^2 + \sum_{m=1}^{\infty} a_{m+2} z^{m+2} \\ =&a_0 + a_1 z + \frac{1}{2(b-a)}[\frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2})} b_0 + \gamma(b_0^2 a_0 + a_0^3)] z^2 \\ &+ \sum_{m=1}^{\infty} \frac{1}{(b-a)(m+2)(m+1)} \\ &\times [\frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} b_m + \gamma(\sum_{k=0}^{m}\sum_{j=0}^{k} a_j a_{k-j} a_{m-k}) + \sum_{k=0}^{m}\sum_{j=0}^{k} a_j b_{k-j} b_{m-k}] z^{m+2}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &g(z) \\ =&b_0 + b_1 z + b_2 z^2 + \sum_{m=1}^{\infty} b_{m+2} z^{m+2} \\ =&b_0 + b_1 z + \frac{1}{2(a-b)}[\frac{\Gamma(2-\frac{\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2})} a_0 - \gamma(b_0^3 + b_0 a_0^2)] z^2 \\ &+ \sum_{m=1}^{\infty} \frac{1}{(a-b)(m+2)(m+1)} \\ &\times [\frac{\Gamma(2-\frac{\alpha}{2}+\frac{m\alpha}{2})}{\Gamma(2-\frac{3\alpha}{2}+\frac{m\alpha}{2})} a_m - \gamma(\sum_{k=0}^{m}\sum_{j=0}^{k} b_j b_{k-j} b_{m-k} + \sum_{k=0}^{m}\sum_{j=0}^{k} b_j a_{k-j} a_{m-k})] z^{m+2}. \end{aligned} \tag{4.7}$$

Exact solutions of Eq. (1.1) have been found by applying Eqs. (3.10), (4.6), and (4.7):

$$q(t,x,y) = i\sum_{m=0}^{\infty} a_m (x-y)^m t^{-\frac{(m+1)\alpha}{2}} + \sum_{m=0}^{\infty} b_m (x-y)^m t^{-\frac{(m+1)\alpha}{2}}, \tag{4.8}$$

among these, Eqs. (4.4) and (4.5) define a_m and b_m, where the initial conditions are $f(0) = a_0$, $f'(0) = a_1$, $g(0) = b_0$, $g'(0) = b_1$.

Figs. 1-2 of power series solutions (4.8) are plotted with appropriate parameters to aid our analysis of the power series solutions' attributes. This approach provides a more precise understanding of the features of exact solutions and wave transmission laws. These derived exact solutions are important for elucidating a range of physical phenomena within applied mathematics.

To further contextualize this study, the convergence of explicit power series solutions (4.8) will be investigated in depth. For expressions (4.5), it can be easily seen that

$$|a_{m+2}| \leq H[|b_m| + \sum_{k=0}^{m}\sum_{j=0}^{k}(|a_j||b_{k-j}||b_{m-k}| + |a_j||a_{k-j}||a_{m-k}|)],$$

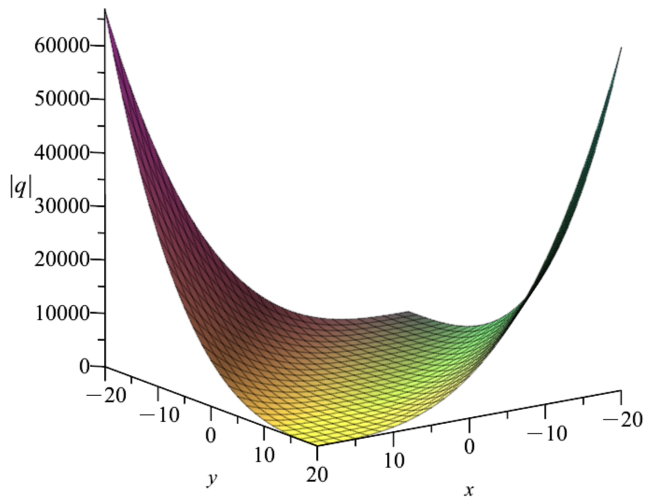

Figure 1. Numerical simulation of power series solutions $|q|$ at $a_0 = b_0 = a_1 = b_1 = 0.1$, $a - b = -1$, $\gamma = 0.1$, and $\alpha = 0.8$.

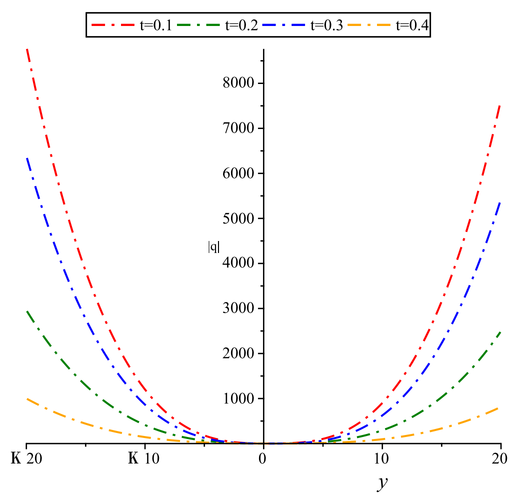

Figure 2. Wave disseminate pattern along y-axis at $a_0 = b_0 = a_1 = b_1 = 0.1$, $a - b = -1$, $\gamma = 0.1$, $\alpha = 0.8$, and $x = 0.2$.

$$|b_{m+2}| \leq H[|a_m| + \sum_{k=0}^{m}\sum_{j=0}^{k}(|b_j||b_{k-j}||b_{m-k}| + |b_j||a_{k-j}||a_{m-k}|)]. \tag{4.9}$$

$H = \max\left\{|\frac{1}{b-a}|, |\frac{\gamma}{b-a}|\right\}$ and $m = 0, 1, 2, \cdots$.

To proceed, we introduce two power series

$$C(z) = \sum_{m=0}^{\infty} c_m z^m, \quad D(z) = \sum_{m=0}^{\infty} d_m z^m. \tag{4.10}$$

Letting $c_j = |a_j|$ and $d_j = |b_j|$ $(j = 0, 1, 2, \cdots)$, there are

$$\begin{aligned} c_{m+2} &= H[d_m + \sum_{k=0}^{m}\sum_{j=0}^{k}(c_j d_{k-j} d_{m-k} + c_j c_{k-j} c_{m-k})], \\ d_{m+2} &= H[c_m + \sum_{k=0}^{m}\sum_{j=0}^{k}(d_j d_{k-j} d_{m-k} + d_j c_{k-j} c_{m-k})], \end{aligned} \tag{4.11}$$

here $m = 0, 1, 2, \cdots$. It's obvious that $|a_m| \leq c_m$ and $|b_m| \leq d_m$ $(m = 0, 1, 2, \cdots)$. That is to say, the majorant series of Eq. (4.1) are Eq. (4.10). Through the calculations, we found

$$\begin{aligned} C(z) =& c_0 + c_1 z + \sum_{m=0}^{\infty} c_{m+2} z^{m+2} \\ =& c_0 + c_1 z + H\sum_{m=0}^{\infty} d_m z^{m+2} + H\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{k}(c_j d_{k-j} d_{m-k} + c_j c_{k-j} c_{m-k}) z^{m+2} \\ =& c_0 + c_1 z + H(D + CD^2 + C^3) z^2, \\ D(z) =& d_0 + d_1 z + \sum_{m=0}^{\infty} d_{m+2} z^{m+2} \\ =& d_0 + d_1 z + H(\sum_{m=0}^{\infty} c_m z^{m+2} + \sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{k}(d_j d_{k-j} d_{m-k} + d_j c_{k-j} c_{m-k}) z^{m+2}) \\ =& d_0 + d_1 z + H(C + D^3 + C^2 D) z^2. \end{aligned} \tag{4.12}$$

Next, we shall show that $C(z)$ and $D(z)$ have a positive radius of convergence. Regarding the implicit function equations of the independent variable z, we will demonstrate

$$\begin{aligned} F(z, C, D) &= C - c_0 - c_1 z - H(D + CD^2 + C^3) z^2, \\ G(z, C, D) &= D - d_0 - d_1 z - H(C + D^3 + C^2 D) z^2. \end{aligned} \tag{4.13}$$

Because $F(z, C, D)$ and $G(z, C, D)$ are analytic in the neighborhood of $(0, c_0, d_0)$, where $F(0, c_0, d_0) = G(0, c_0, d_0) = 0$. In addition, the jacobian

$$\frac{\partial(F, G)}{\partial(C, D)}|_{(0, c_0, d_0)} = 1 \neq 0. \tag{4.14}$$

Subsequently, we arrive at convergence via the implicit function theorem.

5. Conservation laws of Eq. (1.1)

In time fractional PDEs, conservation laws play a crucial role, particularly in proving the existence and uniqueness of solutions. In this section, we construct conservation laws for Eq. (1.1) via the Ibragimov theorem [10]. Consider a vector $T = (T^x, T^y, T^t)$ that satisfies the conservation equation,

$$[D_x(T^x) + D_t(T^t) + D_y(T^y)]|_{(3.2)} = 0. \tag{5.1}$$

Here, $T^x = T^x(x,t,u,v,...)$, $T^y = T^y(x,t,u,v,...)$, and $T^t = T^t(x,t,u,v,...)$ are called conserved vectors of Eq. (3.2). According to the new conservation theorem by Ibragimov, the formal Lagrangian of (3.2) is expressed as

$$\begin{aligned} L = \mathcal{A}(t,x,y)(D_t^\alpha u - av_{xx} + bv_{yy} - \gamma(v^3 + vu^2)) \\ + \mathcal{B}(t,x,y)(D_t^\alpha v + au_{xx} - bu_{yy} + \gamma(v^2 u + u^3)), \end{aligned} \tag{5.2}$$

where $\mathcal{A}(t,x,y)$ and $\mathcal{B}(t,x,y)$ are sufficiently smooth functions.

We have an action integral based on Eq. (5.2), that is defined as:

$$\int_0^T \int_{\Delta_x} \int_{\Delta_y} L(t,x,y,u,v,\mathcal{B},\mathcal{A},D_t^\alpha u, u_{xx}, u_{yy}, D_t^\alpha v, v_{xx}, v_{yy}) dxdydt. \tag{5.3}$$

The adjoint Euler-Lagrange equations for Eq. (3.2) are

$$\frac{\delta L}{\delta u} = 0, \qquad \frac{\delta L}{\delta v} = 0, \tag{5.4}$$

which defines the Euler-Lagrange operators as

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} \frac{\partial}{\partial u_{i_1,\cdots,i_k}}, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha v} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} \frac{\partial}{\partial v_{i_1,\cdots,i_k}}. \end{aligned} \tag{5.5}$$

$(D_t^\alpha)^*$ is the adjoint operator to D_t^α, which is represented as

$$(D_t^\alpha)^* = (-1)^n I_c^{n-\alpha}(D_t^n) =_t^C D_c^\alpha. \tag{5.6}$$

Here, $I_c^{n-\alpha}$ is the right-sided operator of fractional integration of order $n-\alpha$, which is expressed as

$$I_c^{n-\alpha} R(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_t^C \frac{R(\tau,x)}{(\tau-t)^{1+\alpha-n}} d\tau. \tag{5.7}$$

For the case of two variables $u(t,x,y)$ and $v(t,x,y)$, we have

$$\widehat{V} + D_x(\xi)I + D_y(\rho)I + D_t(\tau)I = W_1 \frac{\delta}{\delta u} + W_2 \frac{\delta}{\delta v} + D_t T^t + D_x T^x + D_y T^y, \tag{5.8}$$

here, I is the identity operator, and $\widehat{V}$ is defined as

$$\begin{aligned} \widehat{V} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v} + \eta^{\alpha,t}\frac{\partial}{\partial D_t^\alpha u} + \phi^{\alpha,t}\frac{\partial}{\partial D_t^\alpha v} + \eta^x\frac{\partial}{\partial u_x} \\ + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \phi^x\frac{\partial}{\partial v_x} + \phi^{xx}\frac{\partial}{\partial v_{xx}} + \phi^{yy}\frac{\partial}{\partial v_{yy}}. \end{aligned}$$

The Lie characteristic functions are presented as

$$\begin{aligned} W_1 &= \eta - \xi u_x - \rho u_y - \tau u_t, \\ W_2 &= \phi - \xi v_x - \rho v_y - \tau v_t. \end{aligned} \tag{5.9}$$

Eq. (3.2), which employs Riemann-Liouville fractional derivative, allows us to find the component of a conserved vector V_i as

$$\begin{aligned} T^t = & \sum_{k=0}^{m-1}(-1)^k_0 D_t^{\alpha-1-k}(W_1)D_t^k\frac{\partial L}{\partial_0 D_t^\alpha u} - (-1)^m J(W_1, D_t^m\frac{\partial L}{\partial D_t^\alpha u}) \\ & + \sum_{k=0}^{m-1}(-1)^k_0 D_t^{\alpha-1-k}(W_2)D_t^k\frac{\partial L}{\partial_0 D_t^\alpha v} - (-1)^m J(W_2, D_t^m\frac{\partial L}{\partial D_t^\alpha v}), \end{aligned} \tag{5.10}$$

where $J(.)$ is expressed as

$$J(f, f_1) = \frac{1}{\Gamma(m-\alpha)}\int_0^t\int_t^s \frac{f(\zeta, x, y)f_1(\mu, x, y)}{(\mu-\zeta)^{\alpha+1-m}}d\mu d\zeta. \tag{5.11}$$

Furthermore, the x and y components of conserved vectors T^x and T^y are calculated below,

$$T^i = W_1[\frac{\partial L}{\partial u_i} - D_j\frac{\partial L}{\partial u_{ij}}] + W_2[\frac{\partial L}{\partial v_i} - D_j\frac{\partial L}{\partial v_{ij}}] + D_j(W_1)\frac{\partial L}{\partial u_{ij}} + D_j(W_2)\frac{\partial L}{\partial v_{ij}}. \tag{5.12}$$

Now, we apply the fundamental definitions provided in Eqs. (5.10), (5.11), and (5.12) to find the conservation laws in Eq. (3.2). By doing so, we have derived the following components of the conservation laws for Eq. (3.2). Therefore, we obtain conserved vectors of vector field V_3,

$$\begin{aligned} T^t = & D_t^{\alpha-1}(-\alpha u - \alpha x u_x - \alpha y u_y - 2tu_t)\mathcal{A} + J(-\alpha u - \alpha x u_x - \alpha y u_y - 2tu_t, \mathcal{A}_t) \\ & + D_t^{\alpha-1}(-\alpha v - \alpha x v_x - \alpha y v_y - 2tv_t)\mathcal{B} + J(-\alpha v - \alpha x v_x - \alpha y v_y - 2tv_t, \mathcal{B}_t), \\ T^x = & a\mathcal{B}_x(\alpha u + \alpha x u_x + \alpha y u_y + 2tu_t) - a\mathcal{A}_x(\alpha v + \alpha x v_x + \alpha y v_y + 2tv_t) \\ & - a\mathcal{B}(2\alpha u_x + \alpha x u_{xx} + \alpha y u_{xy} + 2tu_{xt}) + a\mathcal{A}(2\alpha v_x + \alpha x v_{xx} + \alpha y v_{xy} + 2tv_{xt}), \\ T^y = & -b\mathcal{B}_y(\alpha u + \alpha x u_x + \alpha y u_y + 2tu_t) + b\mathcal{A}_y(\alpha v + \alpha x v_x + \alpha y v_y + 2tv_t) \\ & + b\mathcal{B}(2\alpha u_y + \alpha x u_{xy} + \alpha y u_{yy} + 2tu_{yt}) - b\mathcal{A}(2\alpha v_y + \alpha x v_{xy} + \alpha y v_{yy} + 2tv_{yt}). \end{aligned} \tag{5.13}$$

There are conserved vectors for vector field V_1

$$\begin{aligned} T^t &= -\mathcal{A}D_t^{\alpha-1}(u_x) + J(-u_x, \mathcal{A}_t) - \mathcal{B}D_t^{\alpha-1}(v_x) + J(-v_x, \mathcal{B}_t), \\ T^x &= a\mathcal{B}_x u_x - a\mathcal{A}_x v_x - a\mathcal{B}u_{xx} + a\mathcal{A}v_{xx}, \\ T^y &= -b\mathcal{B}_y u_x + b\mathcal{A}_y v_x + b\mathcal{B}u_{xy} - b\mathcal{A}v_{xy}. \end{aligned} \tag{5.14}$$

Through calculation, the expressions of conserved vectors of vector field V_2 are obtained,

$$\begin{aligned} T^t &= -\mathcal{A}D_t^{\alpha-1}(u_y) + J(-u_y, \mathcal{A}_t) - \mathcal{B}D_t^{\alpha-1}(v_y) + J(-v_y, \mathcal{B}_t), \\ T^x &= a\mathcal{B}_x u_y - a\mathcal{A}_x v_y - a\mathcal{B}u_{xy} + a\mathcal{A}v_{xy}, \\ T^y &= -b\mathcal{B}_y u_y + b\mathcal{A}_y v_y + b\mathcal{B}u_{yy} - b\mathcal{A}v_{yy}. \end{aligned} \tag{5.15}$$

6. Conclusions

The invariance properties in the sense of Lie point symmetry of the time fractional Schrödinger equation are presented in this paper. The Lie symmetries of the equations are acquired and the related fractional PDEs are reduced to FODEs. Besides, we derive the power series solutions for the reduced systems. Furthermore, for time fractional Schrödinger equation, the conservation analysis is performed using the new conservation theorem, which is applied here for the identification of conserved vectors.

This article solely focuses on the Lie symmetry analysis approach for time fractional PDEs with constant coefficients. In future research endeavors, we aim to address time fractional PDEs with variable coefficients via the Lie symmetry analysis method.

Author's contribution

All authors participated equally to reach the final version of the paper. All authors read and approved the final version of the manuscript.

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