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A NOVEL APPROACH FOR THE NONLINEAR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS USING CLIQUE POLYNOMIALS OF GRAPH

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Abstract This study proposed an efficient numerical technique for nonlinear elliptic partial differential equations (EPDEs) using the functional matrix generated by Clique polynomials of Complete Graph. Recently, Graph theory has attracted many mathematicians' attention due to its wide applications. Here, Three nonlinear problems have been considered to examine the proposed scheme proficiency. Some theorems on convergence are discussed. Here, the nonlinear elliptic PDEs are rehabilitated into a nonlinear algebraic equation system using the operational matrix of Clique polynomials and collocation technique. Using the Newton-Raphson method, we numerically solved this system of algebraic equations to the desired results. The proposed scheme results are compared with the literature's analytical and other method solutions through tables and graphs. Tables and graphs are used to support the proposed technique's efficacy and accuracy. The obtained results reveal that the current approach is more accurate than other methods. The theorems are used to draw the convergent analysis for the suggested approach. From the obtained results, we can conclude that to find a numerical solution for these kinds of nonlinear EPDEs, the method is extremely effective, requires less computational effort, and is easy to implement.

Keywords Nonlinear elliptic partial differential equations, collocation technique, complete graph, Clique polynomials.

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1. Introduction

Most physical events and technical processes require nonlinearity, leading to nonlinear differential equations. Since no universal method works everywhere, these

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nonlinear equations are typically challenging to solve. Because of this, each equation must be studied separately. Nonlinear EPDEs explain many diverse physical aspects of chemical theory, applied electrical engineering, static dynamics, mathematical biology, etc. The nonlinear PDE nature is highly unpredictable; determining the closed-form solution for such equations is highly complicated. With slow convergence, solutions to these issues typically demand more CPU time. There is no general scheme that works on such equations; that is, each different type of problem has to be considered as a separate problem. We considered three different types of PDEs, including parabolic, hyperbolic, and elliptic, while categorizing PDEs. Among these, the hyperbolic type is the one that attracts the most interest from researchers due to its enormous application. Many equations in the fields of applied genetics, genealogy, and economics are modeled with nonlinear elliptic PDEs. With additional applications in various sectors, such as optical devices, EPDEs are created as mathematical models that explain wave processes. Numerical techniques act as tools to study those equations and legalize the developed numerical scheme through PDEs analytic solution. Now, consider the elliptic PDEs as form:

$$\left[\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right]=f\left(x,y,\theta,\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y}\right). \tag{1.1}$$

With the following physical condition,

$$\theta(x,0)=g_1(x),\ \forall(x,y)\in[0,1)\times[0,1),$$
$$\theta(0,y)=g_2(y),\ \theta(b,y)=g_3(y).$$

Where, $f\left(x,y,\theta,\frac{\partial\theta}{\partial x},\frac{\partial\theta}{\partial y}\right)$ is the nonlinear term. b be the real constants, $g_1(x)$, $g_2(y)$, $g_3(y)$ are continuous real-valued functions. Many mathematicians contributed numerous techniques as follows: a New wavelet-based full-approximation scheme [27], regularized equations via wavelet technique [14], A finite difference method [25], Klein–Gordon Equation via the Operational-Matrix technique [15, 26], Numerical solution of elliptic PDEs on polygonal domains [10], The method of particular approximation [2], Chebyshev polynomials method [8], Method of oscillating coefficients [3], generalized conjugate gradient method [4], Schwarz alternating methods [21], linear moto note iteration and Schwarz methods [22], generalized finite difference method [7], polynomial wavelet bases through wavelet series collocation method for nonlinear lane-Emden type equations [28], Laguerre wavelets collocation method for the numerical solution of the Benjamina Bona Mohany equations [29], CAS wavelets analytic solution and Genocchi polynomials numerical solutions for the integral and integrodifferential equations [30], Hermite wavelets operational matrix of integration for the numerical solution of nonlinear singular initial value problems [31], The new operational matrix of integration for the numerical solution of integrodifferential equations via Hermite wavelet [18], Numerical investigation based on Laguerre wavelet for solving the hunter Saxton equation [33], Numerical solution for the fractional-order one-dimensional telegraph equation via wavelet technique [32], A new approach to the Benjamin-Bona-Mahony equation via ultraspherical wavelets collocation method [23] and Solving the generalized equal width wave equation via sextic B-spline collocation technique [24].

Let G be a graph that is free from multiple edges and loops. Clique polynomials and related work of Graphs are introduced by Hoede et al. [1, 9, 13]. The clique polynomial of a graph G, denoted by $h(G; y)$, is characterized by,

$$h(G; y) = \sum_{k=0}^{n} a_k y^k.$$

Where a_k represents the total distinct $k-$cliques in a Graph of size k, with $a_0 = 1$. For example, Here we considered a Complete Graph (every pair of vertices are connected) with four vertices as follows: By the definition of clique polynomial of

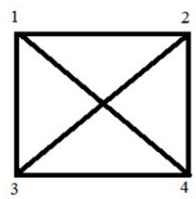

Figure 1. Complete graph with 4 vertices K_4

G concerning Figure 1, we get,

$$h(K_4; y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4.$$

a_1 represents the total number of different $1-$cliques in K_4 of size 1, therefore, $a_1 = 4$. $a_2 = 6$ represents the total number of different 2-cliques in K_4. $a_3 = 4$ represents the total number of different 3-cliques in K_4. $a_4 = 1$ represents the total number of different 4-cliques in K_4. Hence the required Clique polynomial for $K4$ is,

$$h(K_4; y) = (1 + y)^4.$$

In general, the clique polynomial of a complete graph K_n with n vertices is given by,

$$h(K_n; y) = (1 + y)^n.$$

We attempted and successfully obtained the numerical solutions of different mathematical problems through clique polynomial method such as the system of coupled ordinary differential equations [16], comparative study of Adomian decomposition method and clique polynomial method [17], a study on Homotopy analysis method and clique polynomial method [19], numerical solution of nonlinear Klein- Gordan equations [26], special types of boundary layer natural convection flow problems through the clique polynomial method [20], numerical solutions of time – fractional Klein-Gordan equations by clique polynomials [6], numerical simulations of fractional-order Brusselator chemical model using clique polynomials [12], new approaching method for linear neutral delay differential equations by using clique polynomials [34], some nonlinear nonlocal two-point BVPs using clique and QLM-clique matrix methods [11], numerical solutions of 2D stochastic time-fractional Sine-Gordan equation in the Caputo sense using clique polynomials [5]. As per our literature survey, we have not found any research article on elliptical PDEs through Complete Graph of Clique polynomials. This impetus us to propose the Clique polynomials method for the elliptic PDEs and proficiency of the current technique is revealed through tables and graph simulation.

2. Clique polynomials of the operational matrix of integration(OMI)

Let us choose the seven polynomials as follows [15, 26];

$$
\begin{aligned}
h(K_0;y) &= 1,\\
h(K_1;y) &= 1+y,\\
h(K_2;y) &= 1+2y+y^2,\\
h(K_3;y) &= 1+3y+3y^2+y^3,\\
h(K_4;y) &= 1+4y+6y^2+4y^3+y^4,\\
h(K_5;y) &= 1+5y+10y^2+10y^3+5y^4+y^5,\\
h(K_6;y) &= 1+6y+15y^2+20y^3+15y^4+6y^5+y^6.
\end{aligned}
$$

Where,

$$h_7(y) = \left[h(K_0;y), h(K_1;y), h(K_2;y), h(K_3;y), h(K_4;y), h(K_5;y), h(K_6;y)\right]^T.$$

Now, Integrate the given polynomials with respect to y limit from 0 to y and put it in matrix form; we get,

$$
\begin{aligned}
\int_0^y h\left(K_0;y\right)dy &= y\\
&= \left[-1,1,0,0,0,0,0\right]h_7(y),\\
\int_0^y h\left(K_1;y\right)dy &= y+\frac{y^2}{2}\\
&= \left[-\frac{1}{2},0,\frac{1}{2},0,0,0,0\right]h_7(y),\\
\int_0^y h\left(K_2;y\right)dy &= y+y^2+\frac{y^3}{3}\\
&= \left[-\frac{1}{3},0,0,\frac{1}{3},0,0,0\right]h_7(y),\\
\int_0^y h\left(K_3;y\right)dy &= y+\frac{3y^2}{2}+y^3+\frac{y^4}{4}\\
&= \left[-\frac{1}{4},0,0,0,\frac{1}{4},0,0\right]h_7(y),\\
\int_0^y h\left(K_4;y\right)dy &= y+2y^2+2y^3+y^4+\frac{y^5}{5}\\
&= \left[-\frac{1}{5},0,0,0,0,\frac{1}{5},0\right]h_7(y),\\
\int_0^y h\left(K_5;y\right)dy &= y+\frac{5y^2}{2}+\frac{10y^3}{3}+\frac{5y^4}{2}+y^5+\frac{y^6}{6}\\
&= \left[-\frac{1}{6},0,0,0,0,0,\frac{1}{6}\right]h_7(y),\\
\int_0^y h\left(K_6;y\right)dy &= y+3y^2+5y^3+5y^4+3y^5+y^6+\frac{y^7}{7}
\end{aligned}
$$

$$= \left[-\frac{1}{7}, 0, 0, 0, 0, 0, 0\right] h_7(y) + \frac{1}{7} h_7(y).$$

Thus,

$$\int_0^y h_7(y) dy = H_{7\times 7} h_7(y) + \overline{h_7(y)}.$$

Where,

$$H_{7\times 7} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{5} & 0 \\ -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ -\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad \overline{C_6(x)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{7} h\left(K_7; y\right) \end{bmatrix}.$$

Integrate the given polynomials twice with respect to y limit from 0 to y and put it in matrix form; we get

$$\begin{aligned}
\int_0^y \int_0^y h\left(K_0; y\right) dy\ dy &= \frac{y^2}{2} \\
&= \left[\frac{1}{2}, -1, \frac{1}{2}, 0, 0, 0, 0\right] h_7(y), \\
\int_0^y \int_0^y h\left(K_1; y\right) dy\ dy &= \frac{y^2}{2} + \frac{y^3}{6} \\
&= \left[\frac{1}{3}, -\frac{1}{2}, 0, \frac{1}{6}, 0, 0, 0\right] h_7(y), \\
\int_0^y \int_0^y Ch\left(K_2; y\right) dy\ dy &= \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{12} \\
&= \left[\frac{1}{4}, -\frac{1}{3}, 0, 0, \frac{1}{12}, 0, 0\right] h_7(y), \\
\int_0^y \int_0^y h\left(K_3; y\right) dy\ dy &= \frac{y^2}{2} + \frac{y^3}{2} + \frac{y^4}{4} + \frac{y^5}{20} \\
&= \left[\frac{1}{5}, -\frac{1}{4}, 0, 0, 0, \frac{1}{20}, 0\right] h_7(y), \\
\int_0^y \int_0^y h\left(K_4; y\right) dy\ dy &= \frac{y^2}{2} + \frac{2y^3}{3} + \frac{y^4}{2} + \frac{y^5}{5} + \frac{y^6}{30} \\
&= \left[\frac{1}{6}, -\frac{1}{5}, 0, 0, 0, 0, \frac{1}{30}\right] h_7(y), \\
\int_0^y \int_0^y h\left(K_5; y\right) dy\ dy &= \frac{y^2}{2} + \frac{5y^3}{6} + \frac{5y^4}{6} + \frac{y5}{2} + \frac{y^6}{6} + \frac{y^7}{42} \\
&= \left[\frac{1}{7}, -\frac{1}{6}, 0, 0, 0, 0, 0\right] h_7(y) + \frac{1}{42} h\left(K_7; y\right),
\end{aligned}$$

$$\int_0^y \int_0^y h\left(K_6;y\right)dy\ dy = \frac{y^2}{2}+y^3+\frac{5y^4}{4}+y^5+\frac{y^6}{2}+\frac{y^7}{7}+\frac{y^8}{56}$$
$$= \left[\frac{1}{8},-\frac{1}{7},0,0,0,0,0\right]h_7(y)+\frac{1}{56}h\left(K_8;y\right).$$

Thus,

$$\int_0^y \int_0^y h_7(y)dy\ dy = H'_{7\times 7}h_7(y)+\overline{h'_7(y)}.$$

Where,

$$H'_{7\times 7} = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{1}{3} & 0 & 0 & \frac{1}{12} & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{20} & 0 \\ \frac{1}{6} & -\frac{1}{5} & 0 & 0 & 0 & 0 & \frac{1}{30} \\ \frac{1}{7} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & -\frac{1}{7} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and } \overline{h'_7(y)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{42}h\left(K_7;y\right) \\ \frac{1}{56}h\left(K_8;y\right) \end{bmatrix}.$$

The generalized single integration of n-clique polynomials is denoted as:

$$\int_0^y h(y)dy = H_{n\times n}h(y)+\overline{h}(y).$$

Where,

$$H_{n\times n} = \begin{bmatrix} \frac{-1}{n-(n-1)} & \frac{1}{n-(n-1)} & 0 & 0 & \dots & 0 \\ \frac{-1}{n-(n-2)} & 0 & \frac{1}{n-(n-2)} & 0 & \dots & 0 \\ \frac{-1}{n-(n-3)} & 0 & 0 & \frac{1}{n-(n-3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-1}{(n-1)} & 0 & 0 & 0 & \dots & \frac{1}{(n-1)} \\ \frac{-1}{n} & 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

$$h(y) = \begin{bmatrix} h\left(K_0;y\right) \\ h\left(K_1;y\right) \\ h\left(K_2;y\right) \\ \vdots \\ h\left(K_{n-1};y\right) \end{bmatrix} \text{ and } \overline{h(y)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \frac{1}{n}h\left(K_n;y\right) \end{bmatrix}.$$

Generalized the operational matrix for double integration is as follows:

$$\int_0^y \int_0^y h(y)dy\ dy = H'_{n\times n} h(y) + \overline{h'}(y).$$

Where,

$$Z'_{n\times n} = \begin{bmatrix} \frac{1}{n-(n-2)} & \frac{-1}{n-(n-1)} & \frac{1}{(n-(n-2))(n-(n-1))} & 0 & \cdots & 0 \\ \frac{1}{n-(n-3)} & \frac{-1}{n-(n-2)} & 0 & \frac{1}{(n-(n-3))(n-(n-2))} & \cdots & 0 \\ \frac{1}{n-(n-4)} & \frac{-1}{n-(n-3)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{(n-1)} & \frac{-1}{(n-2)} & 0 & 0 & \cdots & \frac{1}{(n-1)(n-2)} \\ \frac{1}{n} & \frac{-1}{(n-1)} & 0 & 0 & \cdots & 0 \\ \frac{1}{(n+1)} & \frac{-1}{n} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$h(x) = \begin{bmatrix} h(K_0; y) \\ h(K_1; y) \\ h(K_2; y) \\ \vdots \\ h(K_n; y) \end{bmatrix} \text{ and } \overline{h_n}'(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \frac{1}{n(n-1)} h(K_n; y) \\ \frac{1}{(n+1)n} h(K_{n+1}; y) \end{bmatrix}.$$

3. Theoretical results on clique polynomials

Theorem 3.1. [26] *Let $\theta(x,y)$ be the bounded continuous function in $L^2(R\times R)$ defined on $[0,1]\times[0,1]$, then the clique polynomial expansion of $\theta(x,y)$ converges to it.*

Proof. Let $\theta(x,y)$ be the continuous function on $[0,1]\times[0,1]$ and $\theta(x,y)$ is bounded by some positive real numbers, say μ.

Consider,

$$\begin{aligned} \theta(x,y) &= S(y)^T \text{ M S}(x), \\ \theta(x,y) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}\ S(K_i; y)\, S(K_j; x). \end{aligned}$$

Where, $a_{ij} = \langle \theta(x,y),\ S(K_i; y)\, S(K_i; x)\rangle$ and $\langle\ .\ \rangle$ represents the inner product. The clique polynomials coefficients of $\theta(x,y)$ are defined as

$$\begin{aligned} \mathrm{a_{ij}} &= \int_0^1 \int_0^1 \theta(x,y) S(K_i, y)\ \ S(K_j, x)\, dx\, dy, \\ \mathrm{a_{ij}} &= \int_0^1 \int_0^1 \theta(x,y)\ S(K_j, x)\, dx\, S(K_i, y)\ \ dy. \end{aligned}$$

By generalized mean value theorem for integrals,

$$a_{ij} = \int_0^1 \theta(\eta, y) S\left(K_j, y\right) dy \quad \int_0^1 S\left(K_i; x\right) dx.$$

Where $\eta \in [0,1]$. Since $S(K_i, x)$ is continuous and integrable on $[0,1]$. Put $\int_0^1 S(K_i, x)dx = A$. Therefore

$$a_{ij} = A \int_0^1 \theta(\eta, y) S\left(K_j, y\right) dy.$$

By generalized mean value theorem for integrals,

$$a_{ij} = A\, \theta(\eta, \xi) \int_0^1 S\left(K_j, y\right) dy.$$

Where $\xi \in [0,1]$. Since $S(K_i, y)$ is continuous and integrable on $[0,1]$. Put $\int_0^1 S(K_i, y)dy = B$. Therefore

$$|a_{ij}| = |AB|\ |\theta(\eta, \xi)|.$$

Since $\theta(x, y)$ is bounded by μ. We have,

$$|a_{ij}| = |AB|\ \mu.$$

Therefore, $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij}$ is absolutely convergent. Hence, clique polynomial expansion of $\theta(x, y)$ converges to it. □

Theorem 3.2. [26] *Let $h(K_n, y)$ be the clique polynomials of the $n-1$ regular Graph. Then $h(K_n, y)$ are continuous uniformly on* [0,1].

Proof. Since clique polynomials are Lipschitz functions, then for every $\varepsilon > 0$, choose $\delta > \frac{\varepsilon}{k}$, such that $|S(K_n, x_1) - S(K_n, x_2)| < k\frac{\varepsilon}{k} = \varepsilon$, whenever $|x_1 - x_2| < \delta, \forall x_1, x_2 \in [0,1]$. □

Theorem 3.3. [26] *Let Clique polynomials $h(K_n, y)$ of the $n-1$ regular graph defined on $[0,1]$ are integrable continuous functions. Then integral of these polynomials is continuous on $[0,1]$, and they are bounded variation on* [0,1].

Proof. Let clique polynomials $h(K_n, y)$ of the complete graph with n vertices defined on [0,1] are non-negative functions. Then, given $\varepsilon > 0$ there is $\delta > 0$ such that for every $A \subseteq [0,1]$ with $m(A) < \delta$ and $\int_A^1 h(K_n, y) < \varepsilon$. Let $A = [x, x+h] \Rightarrow m(A) = h$. Choose $\int_0^x h(k_n, t)dt$.

Consider,

$$h(x+h) - h(x) = \int_0^{x+h} h(k_n, t)dt - \int_0^x h(k_n, t)dt = \int_0^{x+h} h(k_n, t)dt, \text{ exists.}$$

Therefore,

$$|h(x+h) - h(x)| \leq \int_x^{x+h} |h(k_n, t)|dt = \int_A^1 h(k_n, t) < \varepsilon.$$

Hence, $h(x) = \int_0^x h(k_n, t)dt$ is continuous.

Again,

$$\sum_{1}^{n} |h(x_k) - h(x_{k-1})| = \sum_{1}^{n} |\int_{x_{k-1}}^{x_k} h(k_n, t)dt|$$

$$\leq \sum_{1}^{n} |h(k_n, t)|dt,$$

$$\sum_{1}^{n} |h(x_k) - h(x_{k-1})| \leq \int_{0}^{1} |h(k_n, t)|dt \text{ is finite.}$$

Hence, $h(x) = \int_0^x h(k_n, t)dt$ bounded variation on $[0, 1]$. □

4. Novel Clique polynomial method

This division is dedicated to a novel method called the clique polynomials collocation technique to solve elliptic PDEs defined in Eqn. (1.1) with different physical conditions.

Consider,

$$\frac{\partial^3 \theta}{\partial y \partial x^2} = h(x)^T K\ h(y). \tag{4.1}$$

Integrate (4.1) concerning y from 0 to y,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial x^2}(x, 0) + h(x)^T K \left[H_{n\times n} h(y) + \overline{h}(y)\right]. \tag{4.2}$$

Integrate (4. 2) regarding x from 0 to x,

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial x}(0, y) + \frac{\partial \theta}{\partial x}(x, 0) - \frac{\partial \theta}{\partial x}(0, 0) + \left[H_{n\times n} h(x) + \overline{h}(x)\right]^T K \left[H_{n\times n} h(y) + \overline{h}(y)\right]. \tag{4.3}$$

Integrate (4.3) regarding x from 0 to x,

$$\begin{aligned}\theta(x, y) =& \theta(0, y) + x\left[\frac{\partial \theta}{\partial x}(0, y) - \frac{\partial \theta}{\partial x}(0, 0)\right] + \theta(x, 0) - \theta(0, 0) \\ &+ \left[H'_{n\times n} h(x) + \overline{h'}(x)\right]^T K \left[H_{n\times n} h(y) + \overline{h}(y)\right].\end{aligned} \tag{4.4}$$

Put $x = b$,

$$\begin{aligned}\theta(b, y) =& \theta(0, y) + b\left[\frac{\partial \theta}{\partial x}(0, y) - \frac{\partial \theta}{\partial x}(0, 0)\right] + \theta(b, 0) - \theta(0, 0) \\ &+ \left[H'_{n\times n} h(x) + \overline{h'}(x)\right]^T K \left[H_{n\times n} h(y) + \overline{h}(y)\right] |_{x=b},\end{aligned}$$

$$\begin{aligned}\left[\frac{\partial \theta}{\partial x}(0, y) - \frac{\partial \theta}{\partial x}(0, 0)\right] =& \frac{1}{b}\Bigg[\theta(b, y) - \theta(0, y) - \theta(b, 0) + \theta(0, 0) \\ &- \left[H'_{n\times n} h(x) + \overline{h'}(x)\right]^T K \left[H_{n\times n} h(y) + \overline{h}(y)\right] |_{x=b}\Bigg].\end{aligned} \tag{4.5}$$

Substitute (4.5) in (4.4) and (4.3) with given physical conditions,

$$\begin{aligned}\frac{\partial \theta}{\partial x} =& g_1(x) + \frac{1}{b}\Bigg[\theta(b,y) - \theta(0,y) - \theta(b,0) + \theta(0,0) \\ & - \left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right]|_{x=b}\Bigg] \\ & + \left[H_{n\times n}h(x) + \overline{h}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right],\end{aligned} \tag{4.6}$$

$$\begin{aligned}\theta(x,y) =& g_2(y) + g_1(x) - g_1(0) + \frac{x}{b}\Bigg[g_3(y) - g_2(y) - g_1(b) + g_1(0) \\ & - \left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right]|_{x=b}\Bigg] \\ & + \left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right].\end{aligned} \tag{4.7}$$

Differentiate (4.7) with respect y twice we get,

$$\begin{aligned}\frac{\partial \theta}{\partial y} =& g_2'(y) + \frac{x}{b}\Bigg[g_3'(y) - g_2'(y) \\ & - \frac{d}{dy}\left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right]|_{x=b}\Bigg] \\ & + \frac{d}{dy}\left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right],\end{aligned} \tag{4.8}$$

$$\begin{aligned}\frac{\partial^2 \theta}{\partial y^2} =& g_2''(y) + \frac{x}{b}\Bigg[g_3''(y) - g_2''(y) \\ & - \frac{d^2}{dy^2}\left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right]|_{x=b}\Bigg] \\ & + \frac{d^2}{dy^2}\left[H'_{n\times n}h(x) + \overline{h'}(x)\right]^T K\left[H_{n\times n}h(y) + \overline{h}(y)\right].\end{aligned} \tag{4.9}$$

Now substitute (4.2), (4.6), (4.7), (4.8), and (4.9) in equation (1.1) and collocate by using the grid points as,

$$x_i = y_i = \frac{2i-1}{2n^2}, \quad i = 1, 2, 3, \ldots, n^2.$$

Then, solve the obtained nonlinear system of algebraic equations by Newton's Raphson method. We get unknown coefficient values. Fit these values in equation (4.7) yields a Clique polynomial solution for the given nonlinear elliptic PDEs.

5. Applications

Application 5.1. Consider a nonlinear elliptic PDE of the form,

$$\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) - ye^y + 2\left[(x - x^2) + (y - y^2)\right] + (x - x^2)(y - y^2)e^{(x-x^2)(y-y^2)} = 0$$

With the following physical condition,

$$\theta(x,0) = x, \forall(x,y) \in (0,1)\times(0,1),$$
$$\theta(0,y) = 0 = \theta(1,y).$$

We found the results through the method explained in Section 4. We compared the obtained results with the other techniques, such as the New wavelet full approximation scheme (NWFAS), the Multigrid approximation scheme (FAS), the Wavelet full approximation scheme (WFAS), and the exact solution through tables and Figures. It is easy to see that the errors obtained using the proposed method are lesser than those obtained using other existing techniques. Table 1 reveals that the proposed technique is better than other methods. Figure 2. represents a geometrical analysis of the solution, for example, 1, at different values of N (size of the matrix). Given problem has an exact solution as $\theta(x,y) = (x - x^2)(y - y^2)$. Figure 3. represents the Graph at different values of fixed y.

Table 1. Assessment of Clique polynomial method (CPM) solution with the exact solution and other methods for the problem example 1.

x	y	Exact	CPM	NWFAS	FAS	WFAS	FDM
0.1111	0.1111	0.0098	0.0086	0.007	0.0066	0.0068	0.0063
0.2222	0.1111	0.0171	0.0122	0.0123	0.0114	0.0119	0.0109
0.3333	0.1111	0.0219	0.0201	0.0156	0.0149	0.0147	0.0140
0.4444	0.1111	0.0244	0.0232	0.0176	0.0161	0.0166	0.0157
0.5556	0.1111	0.0244	0.0228	0.0178	0.0164	0.0168	0.0160
0.6667	0.1111	0.0219	0.0212	0.0166	0.0146	0.0149	0.0149
0.7778	0.1111	0.0171	0.0123	0.0133	0.0121	0.0117	0.0121
0.8889	0.1111	0.0098	0.0086	0.008	0.0073	0.0073	0.0073
0.1111	0.2222	0.0171	0.0124	0.0122	0.0114	0.0119	0.0109
0.2222	0.2222	0.0299	0.0284	0.0215	0.0196	0.0213	0.0189
0.3333	0.2222	0.0384	0.0342	0.0274	0.0257	0.0258	0.0242
0.4444	0.2222	0.0427	0.0403	0.031	0.0279	0.0291	0.0272
0.5556	0.2222	0.0427	0.0428	0.0313	0.0285	0.0298	0.0278
0.6667	0.2222	0.0384	0.0343	0.0291	0.0255	0.0259	0.0259
0.7778	0.2222	0.0299	0.0287	0.0234	0.021	0.0199	0.0210
0.8889	0.2222	0.0171	0.0123	0.0142	0.0126	0.0126	0.0126

Application 5.2. Here, we considered the Poisson equation [27].

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \left(\frac{\partial\theta(x,y)}{\partial y}\right)^2 = 2y - x^4.$$

With the following physical condition,

$$\theta(x,0) = 0, \forall(x,y) \in (0,1)\times(0.1),$$
$$\theta(0,y) = 0, \theta(1,y) = y.$$

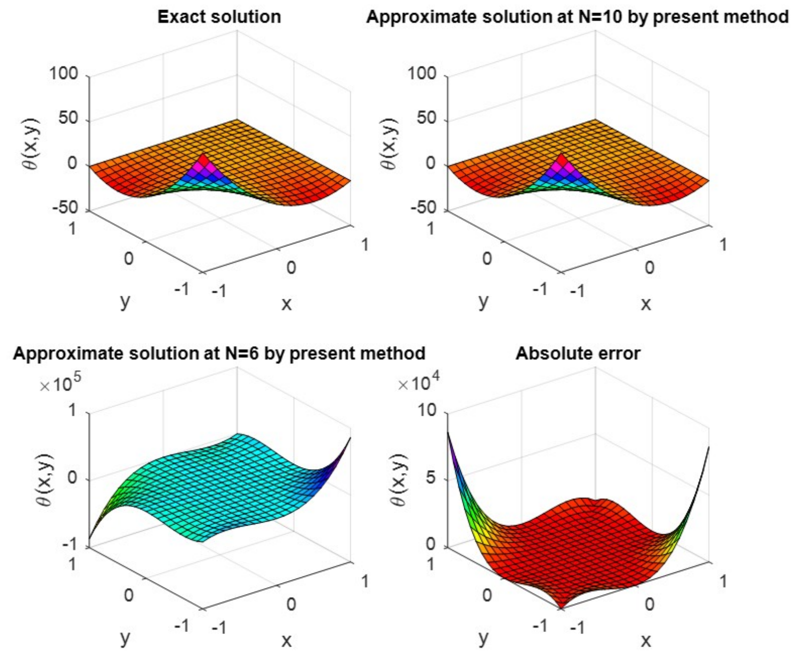

Figure 2. Assessment of Clique polynomial method with the exact solution for the example 1.

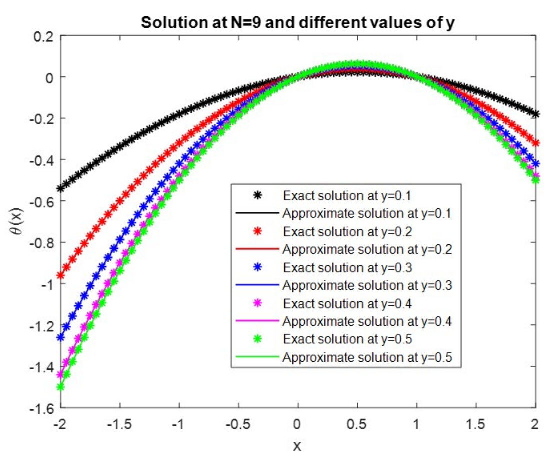

Figure 3. Comparison of Clique polynomial method with the exact solution for the example 1 at distinct values of y.

This problem has an exact solution as $\theta(x, y) = yx^2$. We obtained the numerical results of this model by the proposed procedure and compared them with the other schemes such as NWFAS, FAS, WFAS, and exact solutions through tables and Figures. It is easy to see that the errors obtained using the proposed method are lesser than those obtained using other existing techniques. Table 2 reveals that the proposed technique is better than other methods. Figure 4. reflects a graphical analysis of the solution, for example, two at different N (size of the matrix) values. Figure 5 and 6 illustrate the Graph to varying values of fixed y and x, respectively.

Table 2. Comparison of Clique polynomial method (CPM) solution with the exact solution and other techniques for example 2.

x	y	Exact	CPM	NWFAS	WFAS	FAS	FDM
0.1111	0.1111	0.0014	0.0014	0.0014	0.0014	0.0014	0.0014
0.2222	0.1111	0.0027	0.0027	0.0028	0.0029	0.0029	0.0029
0.3333	0.1111	0.0041	0.0042	0.0043	0.0045	0.0044	0.0045
0.4444	0.1111	0.0055	0.0053	0.0059	0.0062	0.0062	0.0062
0.5556	0.1111	0.0069	0.0069	0.0078	0.0082	0.0082	0.0082
0.6667	0.1111	0.0082	0.0082	0.0107	0.0113	0.0113	0.0113
0.7778	0.1111	0.0096	0.0096	0.0163	0.0174	0.0174	0.0174
0.8889	0.1111	0.011	0.0189	0.032	0.0334	0.0334	0.0334
0.1111	0.2222	0.0055	0.0054	0.0055	0.0056	0.0056	0.0056
0.2222	0.2222	0.011	0.0112	0.0111	0.0113	0.0112	0.0113
0.3333	0.2222	0.0165	0.0165	0.0167	0.017	0.017	0.017
0.4444	0.2222	0.022	0.0221	0.0225	0.0229	0.0229	0.023
0.5556	0.2222	0.0274	0.0277	0.0284	0.0292	0.0292	0.0292
0.6667	0.2222	0.0329	0.0321	0.035	0.0361	0.0361	0.0361
0.7778	0.2222	0.0384	0.0375	0.0418	0.0439	0.0439	0.0439
0.8889	0.2222	0.0439	0.0432	0.0487	0.0517	0.0517	0.0517

Application 5.3. Consider one more nonlinear equation [31].

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\sin(\theta)=\sin(x)$$

with the following physical conditions,

$$\theta(x,0)=x, \forall(x,y)\in(0,1)\times(0,1),$$
$$\theta(0,y)=0, \theta(1,y)=1.$$

This problem has an exact solution as $\theta(x,y)=x$. We obtained the numerical results of this model using the present technique as the exact solution. Table 3 reveals that the proposed approach is efficient. Figure 7. reflects a graphical analysis of the solution, for example, 3 at $N=2$ (size of the matrix).

Numerical implementation for Application 5.3: Consider,

$$\frac{\partial^3\theta}{\partial y\partial x^2}=h(x)^T K\,h(y). \tag{5.1}$$

Where,

$$h(x)^T=[1,1+x],\qquad K=\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix},\qquad h(y)=\begin{bmatrix}1\\ 1+y\end{bmatrix}.$$

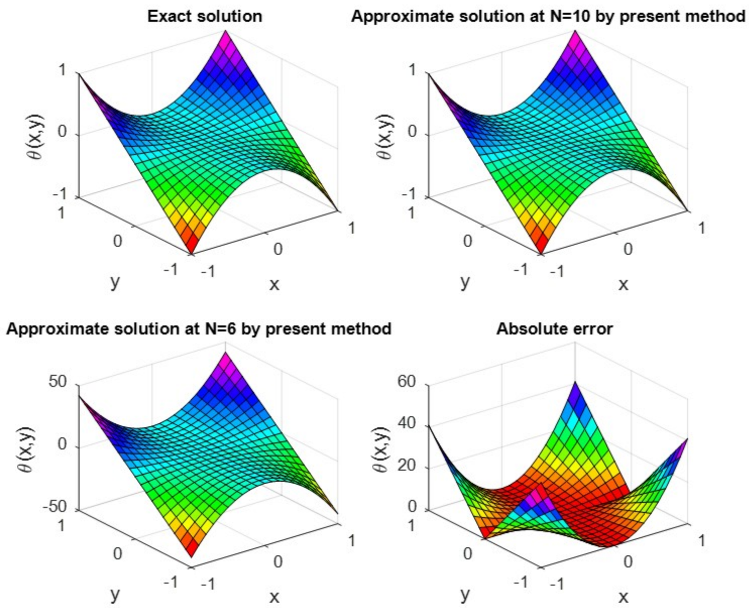

Figure 4. Assessment of Clique polynomial method with the exact solution for the example 2.

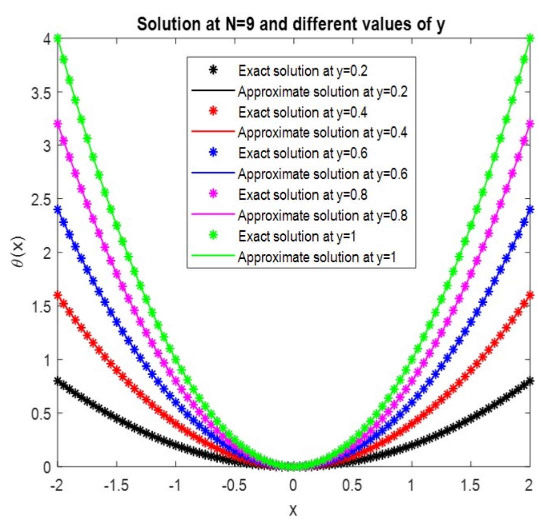

Figure 5. Assessment of Clique polynomial method with the exact solution, for example 2 at different values of y.

Integrate (5.1) relating to y from 0 to y,

$$\frac{\partial^2\theta}{\partial x^2} = h(x)^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]. \tag{5.2}$$

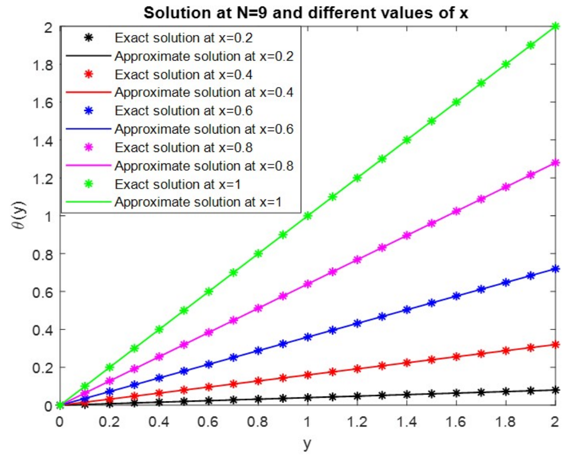

Figure 6. Assessment of Clique polynomial method with the exact solution for example 2 at different values of x.

Integrate (5.2) relating to x from 0 to x,

$$\frac{\partial\theta}{\partial x} = 1+\left[\frac{\partial\theta}{\partial x}(0,y) - \frac{\partial\theta}{\partial x}(0,0)\right]+\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]. \quad (5.3)$$

Integrate (5.3) relating to x from 0 to x,

$$\theta(x,y) = x+x\left[\frac{\partial\theta}{\partial x}(0,y) - \frac{\partial\theta}{\partial x}(0,0)\right]+\left[H^{'}_{2\times 2}h(x) + \overline{h^{'}}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]. \quad (5.4)$$

Put $x = 1$,

$$\begin{aligned}\theta(1,y) &= 1+\left[\frac{\partial\theta}{\partial x}(0,y) - \frac{\partial\theta}{\partial x}(0,0)\right]\\ &\quad+\left[H^{'}_{2\times 2}h(x) + \overline{h^{'}}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]|_{x=1},\\ \left[\frac{\partial\theta}{\partial x}(0,y) - \frac{\partial\theta}{\partial x}(0,0)\right] &= -\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]|_{x=1}. \quad (5.5)\end{aligned}$$

Substitute (5.5) in (5.4) and (5.3) with given physical conditions,

$$\begin{aligned}\frac{\partial\theta}{\partial x} &= 1-\left[\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]\right]|_{x=1}\\ &\quad+\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right], \quad (5.6)\\ \theta(x,y) &= x - x\left[\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]\right]|_{x=1}\\ &\quad+\left[H^{'}_{2\times 2}h(x) + \overline{h^{'}}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]. \quad (5.7)\end{aligned}$$

Now differentiate equation (5.7) with respect y twice we get,

$$\frac{\partial\theta}{\partial y} = -x\frac{d}{dy}\left[\left[H_{2\times 2}h(x) + \overline{h}(x)\right]^T K\left[H_{2\times 2}h(y) + \overline{h}(y)\right]\right]|_{x=1}$$

$$+\frac{d}{dy}\left[H'_{2\times 2}h(x)+\overline{h'}(x)\right]^T K\left[H_{2\times 2}h(y)+\overline{h}(y)\right], \tag{5.8}$$

$$\frac{\partial^2\theta}{\partial y^2} = -x\frac{d^2}{dy^2}\left[\left[H_{2\times 2}h(x)+\overline{h}(x)\right]^T K\left[H_{2\times 2}h(y)+\overline{h}(y)\right]\right]|_{x=1}$$
$$+\frac{d^2}{dy^2}\left[H'_{2\times 2}h(x)+\overline{h'}(x)\right]^T K\left[H_{2\times 2}h(y)+\overline{h}(y)\right]. \tag{5.9}$$

Now substitute (5.2), (5.6), (5.7), (5.8), and (5.9) in example 3 and collocate using the following grid points,

$$x_i = y_i = \frac{2i-1}{2n^2}, \quad i = 1,2,3,\ldots,n^2, \text{ here } n = 2.$$

Then, we get the following system of equations;

$$\frac{a_1}{8}+\frac{21a_2}{64}+\frac{9a_3}{64}+\frac{19a_4}{256}+\sin\left[\frac{1}{8}\right]$$
$$+\sin\left[\frac{7a_1}{1024}+\frac{119a_2}{16384}+\frac{77a_3}{8192}+\frac{1309a_4}{131072}-\frac{1}{8}\right]=0,$$
$$\frac{3a_1}{8}+\frac{21a_2}{64}+\frac{33a_3}{64}+\frac{113a_4}{256}+\sin\left[\frac{3}{8}\right]$$
$$+\sin\left[\frac{45a_1}{1024}+\frac{855a_2}{16384}+\frac{525a_3}{8192}+\frac{9975a_4}{131072}-\frac{3}{8}\right]=0,$$
$$\frac{5a_1}{8}+\frac{45a_2}{64}+\frac{65a_3}{64}+\frac{295a_4}{256}+\sin\left[\frac{5}{8}\right]$$
$$+\sin\left[\frac{75a_1}{1024}+\frac{1575a_2}{16384}+\frac{925a_3}{8192}+\frac{19425a_4}{131072}-\frac{5}{8}\right]=0,$$
$$\frac{7a_1}{8}+\frac{77a_2}{64}+\frac{105a_3}{64}+\frac{581a_4}{256}+\sin\left[\frac{7}{8}\right]$$
$$+\sin\left[\frac{49a_1}{1024}+\frac{1127a_2}{16384}+\frac{637a_3}{8192}+\frac{14651a_4}{131072}-\frac{7}{8}\right]=0.$$

Solve the obtained system of equations by Newton's Raphson method. We get values of unknown coefficients as;

$$\begin{bmatrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \end{bmatrix}.$$

Substituting these values in equation (5.7) yields a Clique polynomial solution for the given nonlinear elliptic PDEs that is the same as an exact solution $\theta(x, y) = x$.

6. Conclusion

We generated a new technique called the clique polynomial method to solve nonlinear elliptic PDEs in the present study. The main interest is to link two different

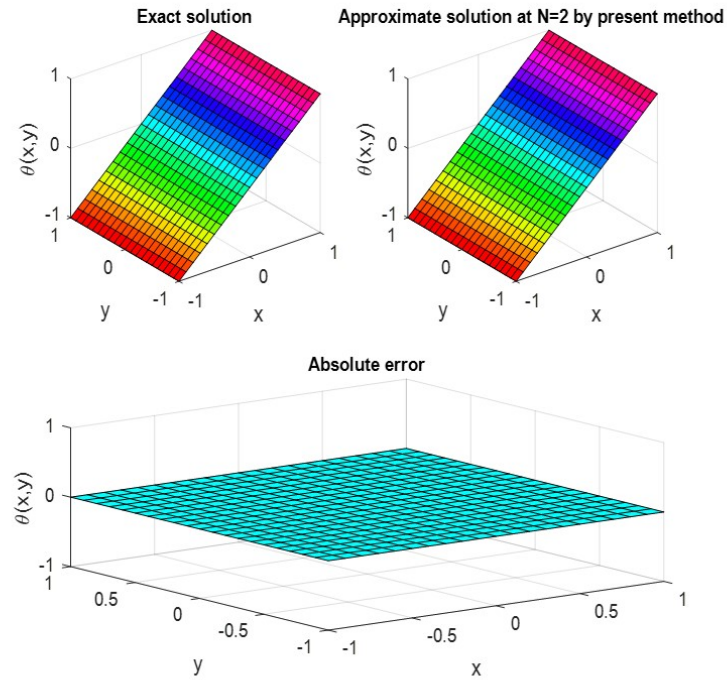

Figure 7. Assessment of Clique polynomial method with the exact solution for the example 3.

streams of subject, such as Graph theory and Numerical analysis. We constructed an operational matrix using clique polynomials [15] applied on nonlinear PDEs. Tables 1 and 2 revealed the efficiency of our proposed method and compared it with other literature techniques. The obtained computational output shows that our projected algorithm is useful and precise in comparison with other methods. Example 3 shows that the projected approach yields exact solutions for the problems whose solutions are in the finite degree polynomials. The findings in the tables and figures show that the suggested method is more accurate than the currently used numerical methods. Also, numerical illustrations support the claim that only a few terms are sufficient to attain suitable outcomes. The method yielded a very excellent result while being simple to use. It underlined our conviction that the method is a convenient method for handling EPDEs that are highly nonlinear. The technique that is being discussed is simple, uncomplicated to use, and requires less computation. As a result, we concluded that the method under consideration is a powerful tool compared to other techniques to obtain the numerical solutions of the nonlinear EPDEs.

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