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A NOVEL ANALYTICAL METHOD FOR TIME FRACTIONAL CONVECTION-DIFFUSION EQUATION THROUGH CLIQUE POLYNOMIALS OF THE COCKTAIL PARTY GRAPH

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Abstract This paper is devoted to providing a new approach to solve time fractional convection-diffusion equation (TFCDE) by utilizing Clique polynomials of the Cocktail party graph and collocation points. The main advantage of this method is converting the TFCDE into a system of ordinary fractional differential and algebraic equations. At this stage, Residual power series method (RPSM) is used to determine the unknown functions of the obtained system. Convergence analysis is given to substantiate the importance of the suggested method. Two numerical examples are presented to illustrate the implementation and effectiveness of the proposed method.

Keywords Fractional convection-diffusion equation, collocation points, Clique polynomials, residual power series method.

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1. Introduction

Last couple of decades, modelling scientific processes by fractional differential equations gains influential attention in various areas of science such as nonlinear waves, nuclear physics, thermodynamics, image and signal processing, visco-elasticity, acoustics, optics, aerodynamics, etc. [23]. As a result, fractional calculus becomes an essential branch of mathematics, physics and engineering. The fractional calculus contains arbitrary non-integer order of differentiation and integration. It provides various numerous a substantial features to be used in the analysis of miscellanous real-world phenomena. For instance, their non-local property plays a leading role in the modelling of memory-dependent phenomena such as porous media and anomalous diffusion [4, 13, 25, 26]. The mathematical models with fractional differential equations reflect the hereditary and memory of the phenomena [2, 18, 21, 28] which makes them more valuable compare to ordinary differential equations. A variety of fractional derivatives such as Grünwald-Letnikov, Riemann–Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu Caputo type, Atangana-Baleanu Rie-

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mann–Liouville type, etc. [7, 24] have been defined and used in the modelling of scientific processes by fractional differential equations based on their properties.

In large number of areas in science and engineering such as transport of mass and energy, weather prediction, dispersion of chemicals in reactors, the convection-diffusion equations [6, 20, 29] is an important tool to model scientific processes. Special polynomials such as Bernoulli polynomials, Legendre polynomials, Hermite polynomials, Chebyshev polynomials etc. [1, 8, 17, 27] play a substantial role to establish the solutions of fractional differential equations. They also form a basis for a special function spaces in which the solutions of the differential equations are constructed in series form. Therefore, utilizations of these polynomials arise in numerous fields of science to develop new methods for solving any kind of fractional differential equations. Some polynomials having orthogonality property attracts the attention of many researchers since the computation is easier with them.

Graphs are crucial tools to model various processes in real-world. Even though graphs provide single dimensional objects, it can be used in higher dimensional spaces in diverse fields. Graph theory is a combination of diverse branches of mathematics such as numerical analysis, matrix theory, topology, group theory, set theory, probability and combinatorics. In the development of numerical methods for attaining the solution of fractional differential equations a good many graph polynomials such as Clique polynomial, Characteristic polynomial, matching polynomials, Tutte polynomials, etc. [19, 22] have been used.

In the present work, we use the clique polynomial of the cocktail party graph instead of the clique polynomial of the complete graph to obtain the solution of following TFCDE [10]:

$$D_t^\alpha u(x,t) + b(x)u_x(x,t) + c(x)u_{xx}(x,t) = f(x,t),\ 0 \le x \le 1,\ 0 \le t \le T \tag{1.1}$$

with the initial and the boundary conditions

$$u(x,0) = \phi(x),\ 0 \le x \le 1, \tag{1.2}$$

$$u(0,t) = \mu_1(t),\ u(1,t) = \mu_2(t),\ 0 \le t \le T, \tag{1.3}$$

where $f(x,t)$ represents the source function and, $D_t^\alpha u(x,t)$ is Caputo's derivative of order $m-1 < \alpha \le m, m \in N$.

2. Preliminaries

In this section, fundamental definitions and notions are presented.

Definition 2.1. The Riemann-Liouville integral for α is [3, 9, 12, 16]:

$$J^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(x), & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. The α^{th} order fractional derivative in Caputo sense is given by [3, 9, 12, 16]

$$D^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & m-1 < \alpha < m,\ m \in N, \\ \dfrac{d^{(m)}}{dx^{(m)}} f(x), & \alpha = m. \end{cases} \tag{2.2}$$

Definition 2.3. A power series expansion of the form

$$\sum_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ..., \quad 0 \le m-1 < \alpha \le m,\ t \ge t_0 \tag{2.3}$$

is called fractional power series about $t = t_0$ [12].

3. Clique polynomial of cocktail party graph (CCPG)

In a complete subgraph, the number of cliques plays a vital role. The maximal clique G is defined as the highest clique in a graph G. A clique of size m is defined as the maximal set containing nodes at a distance not more than m. A maximal clique have the greatest possible number of vertices. In other words a maximal clique can not be extended to a larger clique by adding new vertex.

In a connected graph G, the clique polynomial is given in the following form:

$$C(G;x) = a_0(x) + \sum_{\theta=1}^{\rho(G)} a_\theta x^\theta \tag{3.1}$$

where a_θ represents total θ cliques in G, the constant $a_0(x)$ denotes the total zero cliques in G. Moreover, $\rho(G)$ denotes the maximal clique. The Clique polynomial of the m^{th}- order Cocktail party graph is obtained by substituting $\rho(G) = m$ in (3.1)

$$C(K_{m(2)};x) = (1+2x)^m \tag{3.2}$$

where $K_{m(2)}$ is the notation of complete cocktail party graph with m-partite. Notice that placing the values of a_θ in (3.1) leads to Eq. (3.2) [14]. A Cocktail graph have paired nodes on two rows and unpaired nodes are connected with straight lines. Therefore the distance among nodes are transitive and regular. Moreover they have antipodal feature. They are regarded as dual graph of the hypercube or complement of the ladder rung graph. Clique polynomials are not orthogonal but the clique polynomials of the cocktail party graph are orthogonal and the solution can be written in the series form in terms of clique polynomials of the cocktail party graph [5, 11, 15]. In other words, the exact solution can be constructed in terms of clique polynomials of the cocktail party graph unlike the clique polynomials.

4. Convergence analysis

Theorem 4.1. *Let R^n be the polynomial space of degree $n+1$ over the field R. The solution $F(x,t) : [a,b] \times [0,T] \to R^n$ of TFCDE is given as follows:*

$$F(x,t) = \sum_{m=1}^{\infty} a_m(t) C(K_{m(2)};x) \tag{4.1}$$

Proof. Let R^n is the polynomial space of degree $n+1$ over the field R, and $F(x,t) : [a,b] \to R^n$ is a solution of TFCDE of degree at most n. Then there is a basis $B = C(K_{1(2)};x), C(K_{2(2)};x), \dots, C(K_{n(2)};x), C(K_{n+1(2)};x)$, containing orthogonal polynomials of clique cocktail party graph (CCPG) polynomials, where $C(K_{1(2)};x), C(K_{2(2)};x), \dots, C(K_{n(2)};x), C(K_{n+1(2)};x)$ are CCPG polynomials of degree $0, 1, 2, \dots, n$ respectively. For fixed n,

$$F(x,t) = \sum_{m=1}^{n+1} a_m(t) C(K_{m(2)};x) \tag{4.2}$$

is a linear combination of elements of B. By equating the coefficients of the same degree of x on both sides, we get the values of $a_m(t)$. Hence $F(x,t)$ is approximated precisely as a linear combination of CCPG polynomials. □

Theorem 4.2. *Let $F(x,t)$ be the solution of TFCDE, which is a smooth real-valued bounded function on $[a,b] \times [0,T]$. $L_2[a,b]$ is the space generated by B, then the orthogonal CCPG polynomials expansion of $F(x,t)$ converges to it.*

Proof. Let us assume

$$F(x,t) = \sum_{m=1}^{\infty} a_m(t) C(K_{m(2)};x) \tag{4.3}$$

truncation of it leads to the following equation, we get,

$$F(x,t) = \sum_{m=1}^{n+1} a_m(t) C(K_{m(2)};x) \tag{4.4}$$

where, $a_m(t) =< F(x,t), C(K_{m(2)};x) >$, here $< . >$ denote inner product operator. Then

$$a_m(t) = \int_a^b F(x,t) C(K_{m(2)};x) dx. \tag{4.5}$$

Then,

$$\int_a^b \inf_t F(x,t) C(K_{m(2)};x) dx \leqslant a_m(t) \leqslant \int_a^b \sup_t F(x,t) C(K_{m(2)};x) dx. \tag{4.6}$$

By generalized mean value theorem, the following inequalities are obtained

$$\inf_t F(x_0,t) \int_a^b C(K_{m(2)};x) dx \leqslant a_m(t) \leqslant \sup_t F(x_1,t) \int_a^b C(K_{m(2)};x) dx, \tag{4.7}$$

for some x_0, x_1. Choose, $\int_a^b C(K_{m(2)};x)dx = \mu$ and F is bounded by some real constant K, then we get, $|a_m(t)| \leqslant |\mu K|$. Therefore $\sum a_i(t)$ converges absolutely. Hence a linear combination of $F(x,t)$, through the basis element of B, converges to it. □

5. Implementation of the presented method

In order to construct the approximate solution $u(x,t)$ for the problem (1.1)-(1.3) by the sets of special polynomials as

$$\sum_{i=0}^{\infty} a_i(t)C(K_{i(2)};x) \tag{5.1}$$

we follow the steps below:

Step 1. Plugging the m^{th} degree approximation of Eq.(5.1) into the Eq.(1.1) leads to the following equation:

$$\begin{aligned}&\sum_{i=0}^{m} D_t^{\alpha} a_i(t)C(K_{i(2)};x) + b(x)\sum_{i=0}^{m} a_i(t)C'(K_{i(2)};x) + c(x)L\sum_{i=0}^{m} a_i(t)C''(K_{i(2)};x)\\ =&f(x,t),\ n-1<\alpha\leqslant n.\end{aligned} \tag{5.2}$$

Step 2. Collocating Eq.(5.2) at the nodes $x_k = \frac{1}{2} + \frac{1}{2}cos(\frac{k\pi}{m}), k = 0,1,...m-1$, we have a system of fractional ordinary differential equations:

$$\begin{aligned}&\sum_{i=0}^{m} D_t^{\alpha} a_i(t)C(K_{i(2)};x_k) + b(x_k)\sum_{i=0}^{m} a_i(t)C'(K_{i(2)};x_k)\\ &+ c(x_k)L\sum_{i=0}^{m} a_i(t)C''(K_{i(2)};x_k)\\ =&f(x_k,t),\ n-1<\alpha\leqslant n.\end{aligned} \tag{5.3}$$

Step 3. Plugging the m^{th} degree approximation of Eq.(5.1) into in the initial and boundary conditions Eq.(1.2)-(1.3) leads to the following a system of algebraic equations, we can obtain $([\alpha]+1)$ equations as follows:

$$\begin{aligned}&\sum_{i=0}^{m} a_i(0)C(K_{i(2)};x) = \phi(x_k),\\ &\sum_{i=0}^{m} a_i(t)C(K_{i(2)};0) = \mu_1(t),\\ &\sum_{i=0}^{m} a_i(t)C(K_{i(2)};1) = \mu_2(t).\end{aligned} \tag{5.4}$$

Step 4. As a result, we have a system including fractional ordinary differential and algebraic equations. Solving this system by RPSM yields unknown functions $a_i(t), i = 0,1,2...m$ which are taken into account to form the approximate solution $u_m(x,t)$.

6. Special elucidative examples

The primary aim of this section is to illustrate the implementation of the method by presented examples and check their accuracy.

Table 1. The absolute error at $T = 0.1$ and $\alpha = 0.7, 0.9, 0.95$, respectively for Example 6.1.

	$\alpha = 0.7$		$\alpha = 0.9$		$\alpha = 0.95$	
x	$m = 6$ [10]	Present method	$m = 6$ [10]	Present method	$m = 6$ [10]	Present method
0.1	3.0250e-03	6.9389e-17	2.4473e-03	2.7756e-17	2.3521e-03	4.1633e-17
0.2	5.8222e-03	2.7756e-17	4.7146e-03	2.7756e-17	4.5138e-03	5.5511e-17
0.3	8.1614e-03	2.7756e-16	6.6114e-03	2.2204e-16	6.3227e-03	1.6653e-16
0.4	9.8394e-03	0	7.9728e-03	5.5511e-17	7.6213e-03	1.1102e-16
0.5	1.0675e-02	1.1102e-16	8.6566e-03	2.2204e-16	8.2740e-03	0
0.6	1.0492e-02	1.6653e-16	8.5537e-03	5.5511e-17	8.1765e-03	1.1102e-16
0.7	9.3727e-03	2.7756e-16	7.5997e-03	3.8858e-16	7.2674e-03	1.1102e-16
0.8	7.1396e-03	0	5.7900e-03	1.1102e-16	5.5422e-03	3.8858e-16
0.9	3.9436e-03	1.1102e-16	3.1971e-03	1.6653e-16	3.0699e-03	4.1633e-16

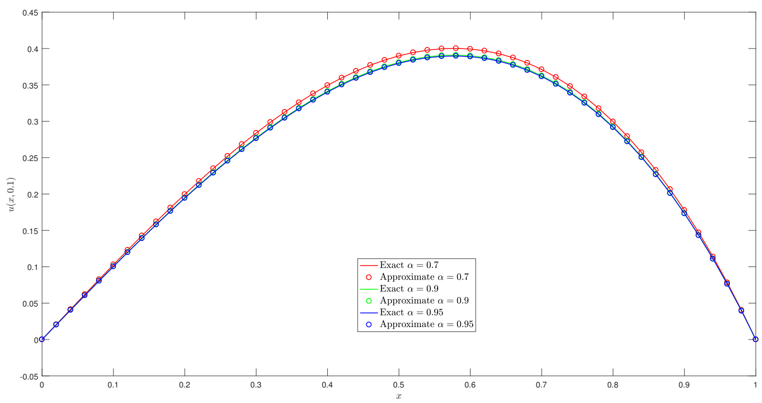

Figure 1. The graph of exact and numerical solution for various α values, ($m = 3$ and $T = 0.1$) for Example 6.1.

Example 6.1. Consider the following time fractional convection-diffusion equation:

$$D_t^{\alpha} u(x,t) + x u_x - u_{xx}(x,t) = f(x,t), 0 < \alpha \leqslant 1, x \in (0,1) \times (0,1] \tag{6.1}$$

with initial and boundary conditions

$$u(x,0) = x - x^3, \tag{6.2}$$
$$u(0,t) = u(1,t) = 0, \tag{6.3}$$

where $f(x,t) = \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha} t^{\alpha}(x - x^3) + (1 + t^{\alpha})(7x - 3x^3)$.

The exact solution of Example 6.1 is $u(x,t) = (1 + t^{2\alpha})(x - x^3)$. The absolute errors at $T = 0.1$ obtained by proposed method are given in Table 1 for $\alpha = 0.7, 0.9, 0.95$, respectively. In Figure 1, the graphs of exact and numerical solutions are presented for various values of α at $T = 0.1$ with $m = 3$. It is clear from Figure 1 that numerical results are in good agreement with exact solution.

Example 6.2. Consider the following time fractional convection-diffusion equation in the following form:

$$D_t^{\alpha} u(x,t) + x u_x(x,t) + u_{xx}(x,t) = f(x,t), 0 < \alpha \leq 1, x \in (0,1) \times (0,1] \tag{6.4}$$

Table 2. The absolute error at $T = 0.5$ and $\alpha = 0.5$ for Example 6.2.

	$\alpha = 0.5$	
x	$m = 5$ [10]	Present method
0.1	7.964e-06	0
0.2	3.912e-06	0
0.3	6.162e-06	0
0.4	5.953e-06	0
0.5	2.103e-06	0
0.6	7.639e-06	0
0.7	1.967e-06	0
0.8	8.103e-06	0
0.9	6.019e-06	0

with initial and boundary conditions

$$u(x,0) = x^2, \tag{6.5}$$

$$u(0,t) = 2\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha}t^{2\alpha}, \tag{6.6}$$

$$u(1,t) = 1 + 2\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha}t^{2\alpha}, \tag{6.7}$$

where $f(x,t) = 2t^{\alpha} + 2x^2 + 2$.

The exact solution of Example 6.2 is $u(x,t) = x^2 + 2\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha}t^{2\alpha}$. The absolute errors at $T = 0.5$ obtained by proposed method are given in Table 1 for $\alpha = 0.5$, respectively. In Figure 2, the graphs of exact and numerical solutions are presented for various values of α at $T = 0.5$ with $m = 2$. It is clear from Figure 2 that numerical results are in great agreement with exact solution.

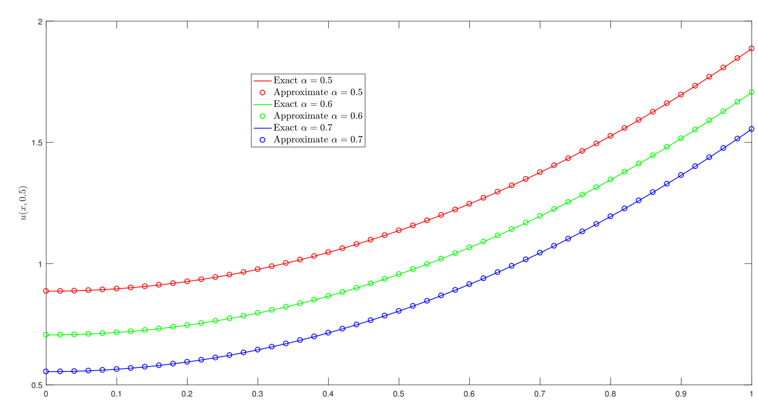

Figure 2. The graph of numerical and exact solution for $\alpha = 0.5$ at $T = 0.5$ for Example 6.2.

7. Conclusions

In this research, a new approach is developed by means of Clique polynomials and collocation points to establish the solution of TFCDE. First, TFCDE is reduced into a system of ordinary fractional differential and algebraic equations which allows us to acquire the solution without any difficulty. Later, utilization of RPSM let us to obtain the solution of the system. Convergence analysis is also presented to demonstrate significance of the proposed approach. Implementation of this approach is demonstrated by presenting two numerical examples which shows the effectiveness and accuracy of the suggested method.

In the future work, cocktail party graph with various polynomials will be used together to solve diverse nonlinear fractional problems. Moreover, RPSM will be changed by another numerical or approximate method to construct the solution of the problem.

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