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NOETHERIAN SOLVABILITY FOR CONVOLUTION SINGULAR INTEGRAL EQUATIONS WITH FINITE TRANSLATIONS IN THE CASE OF NORMAL TYPE

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Abstract In this paper, we mainly study the solvability for some classes of convolution singular integral equations with finite translations in the case of normal type. Via using Fourier transforms, we transform these equations into Riemann boundary value problems with nodes. By means of the classical theory of Riemann-Hilbert problems and the principle of analytic continuation, we discuss the general solutions and conditions of solvability in the normal type. Due to the coefficients of Riemann boundary value problems contain discontinuous points, thus we discuss the solvable conditions and the properties for the equation near the nodes. Unlike the general convolution equations, the unknown function in the questions has finite translations on the real axis, so it is a further generalization of the classical theory of singular integral equations.

Keywords Singular integral equations, Riemann boundary value problems, Noetherian solvability, finite translations, normal type.

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1. Introduction

Singular integral equations is a class of important equations involving the intersection and fusion of many branches of mathematics. Its development can be traced back to the work of some mathematicians in the late nineteenth and early twentieth centuries, and it has broad application value in many fields. For example, in electromagnetism, singular integral equations can be used to solve the distribution and boundary problems of electromagnetic fields. In fluid mechanics, it can be used to describe and calculate the velocity field and pressure of a fluid. Singular integral equations are also used to solve perturbation theory problems and to calculate scattering processes in quantum mechanics. This has been extensively studied by many scholars, see [6, 13, 22–24, 26, 27, 30, 32, 33] for more details. Singular integral equation with shifts generalize further the theory of the classical equations, and it is widely used in the theory of boundary value problems with the differential operators (see [4] and references). At present, many mathematicians have systematically in-

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vestigated the singular integral equations with shift and achieved a series of results. Chuan and Nguyen [8] discussed the solvability and explicit solutions for a class of singular integral equations on the unit circle with Carleman shift and degenerate kernels using the Riemann-Hilbert method. Amer and Dardery [1] studied a nonlinear singular integral equation with Carleman shift on a closed Lyapunov curve and obtained solution in the generalized Hölder space. Recently, Bliev and Tulenov [5] gave the conditions of Noether solvability and index formulas for singular integral equations with Cauchy kernel and Carleman shift in Besov space. Therefore, in addition to Carleman shift, singular integral equation with translations is also the focus of research.

In this paper, we mainly study the following four classes of singular integral equations with translations on the real axis.

(1) Singular integral equations with one convolution kernel and finite translations

$$af(t+\lambda)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{f(\tau+\lambda)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k(t-\tau)\Im f(\tau+\lambda)d\tau=g(t),\quad t\in\mathbb{R}.$$

(2) Singular integral equations with two convolution kernels and finite translations

$$\begin{aligned}&af(t+\lambda)+\frac{b}{\pi i}\int_{-\infty}^{+\infty}\frac{f(t+\lambda)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}k(t-\tau)\Im f(\tau+\lambda)d\tau\\&+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}h(t-\tau)\Im f(\tau+\lambda)d\tau=g(t),\quad t\in\mathbb{R}.\end{aligned}$$

(3) Singular integral equations of Wiener-Hopf type with convolution and finite translations

$$af(t+\lambda)+\frac{b}{\pi i}\int_{0}^{+\infty}\frac{f(\tau+\lambda)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty}k(t-\tau)\Im f(\tau+\lambda)d\tau=g(t),\quad t\in\mathbb{R}^{+}.$$

(4) Singular integral equations of dual type with convolution and finite translations

$$\begin{cases}\quad a_1f(t+\lambda)+\dfrac{b_1}{\pi i}\displaystyle\int_{-\infty}^{+\infty}\frac{f(\tau+\lambda)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_1(t-\tau)\Im f(\tau+\lambda)d\tau\\ =g(t),\ t\in\mathbb{R}^{+},\\ \quad a_2f(t+\lambda)+\dfrac{b_2}{\pi i}\displaystyle\int_{-\infty}^{+\infty}\frac{f(\tau+\lambda)}{\tau-t}d\tau+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k_2(t-\tau)\Im f(\tau+\lambda)d\tau\\ =g(t),\ t\in\mathbb{R}^{-}.\end{cases}$$

For the above equations (1)-(4), their notations can be found in Section 2.

In this paper, the classical theory of Fredholm integral equations is no longer applicable to solving singular integral equations (1)-(4). Thus, we use the Fourier transform to transform singular integral equations into Riemann boundary value problems with discontinuous coefficients, and we study the existence and asymptotic property of solutions. In the course of solution, since the coefficients of equations possess the nodes, to overcame the difficulties appearing at nodes, we separate the discontinuous point of equations. Our method is innovative and effective, and we further generalize the results presented in [2, 9, 29].

Our paper is constructed as follows. In Section 2, we introduce the necessary definitions and lemmas, especially the function classes H_1 and H_2. Furthermore, we prove the connection between the Fourier transform and a specific class of Cauchy type integrals. In Sections 3-6, we apply Fourier analysis theory, complex analysis theory, and harmonic analysis theory to solving equations (1)-(4) in the case of normal type, and we discuss in detail the behavior of solutions near the nodes.

2. Definitions and lemmas

In this section, we give some definitions and lemmas. We mainly introduce two important function classes H_1, H_2, and prove their properties.

Definition 2.1. Let $F(x)$ be a continuous function on $\mathbb{R}$, if there exists $L, B \in \mathbb{R}^+$, such that

$$|F(x_1) - F(x_2)| \leq L\,|x_1 - x_2|^{\omega}, \quad \forall x_1, x_2 \in [-B, B], \tag{2.1}$$

and

$$|F(x_1) - F(x_2)| \leq L\left|\frac{1}{x_1} - \frac{1}{x_2}\right|^{\omega}, \quad \forall x_1, x_2 \in \mathbb{R} \setminus [-B, B], \tag{2.2}$$

are fulfilled, we say that $F(x) \in \hat{H}$, where $0 < \omega \leq 1$. If the following inequality

$$\int_{-\infty}^{+\infty} |F(x)|^2\, dx < +\infty, \quad x \in \mathbb{R}, \tag{2.3}$$

is true, then we say that $F(x) \in L^2(\mathbb{R})$. Obviously, the set of functions

$$\mathrm{O}\left(|x|^{-\mu}\right) = \left\{F(x) : |x^{\mu} F(x)| \leq M \text{ with } \mu > \frac{1}{2} \text{ and } M \in \mathbb{R}^+\right\} \tag{2.4}$$

is a subspace of $L^2(\mathbb{R})$. we say that $H_1 = \hat{H} \cap L^2(\mathbb{R})$ and $H_2 = \hat{H} \cap \mathrm{O}(|x|^{-\mu})$.

It is obvious that $H_2 \subseteq H_1$.

Definition 2.2. The Fourier transform and inverse transform are defined as follows

$$\begin{aligned} F(x) = \mathcal{F}f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\exp(ixt)dt, \\ f(t) = \mathcal{F}^{-1}F(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(x)\exp(ixt)dx. \end{aligned} \tag{2.5}$$

When $F(x) \in H_j$, we say that $f(t) \in H_j^*$ $(j = 1, 2)$. As we know above $H_2^* \subseteq H_1^*$.

Definition 2.3. The convolution of two functions $k(t)$, $f(t)$ can be defined as follows

$$k(t) * f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} k(\tau - t) f(t) dt, \quad \tau \in \mathbb{R}. \tag{2.6}$$

Obviously, $k(t) * f(t) = f(t) * k(t)$ and $\mathcal{F}(k(t) * f(t)) = K(x)F(x)$, where $K(x) = \mathcal{F}k(t)$, $F(x) = \mathcal{F}f(t)$.

Definition 2.4. Define the Hilbert transform as follows

$$\mathcal{T}f(t) = \mathrm{P.V.}\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \mathbb{R}, \tag{2.7}$$

where P.V. stands for Cauchy principal value integral.

According to the Poincaré-Bertrand formula [15, 23, 26], we get $\mathcal{T}^2 = \mathcal{I}$.

Definition 2.5. The reflection and symbolic operators are defined as follows

$$\mathcal{L}f(t) = f(-t), \quad \mathcal{S}f(t) = f(t)\mathrm{sgn}(t). \tag{2.8}$$

It is easy to prove that

$$\mathcal{L}^2 = \mathcal{I}, \quad \mathcal{S}^2 = \mathcal{I}.$$

Definition 2.6. The positive and negative parts of a function $f(t)$ are defined as follows

$$f_{\pm}(t) = \frac{\mathcal{S}f(t) \pm \mathcal{I}f(t)}{2}. \tag{2.9}$$

Obviously, we have

$$f(t) = f_+(t) - f_-(t).$$

The relationship between Fourier transform and boundary values of analytic functions, as elucidated by Lemma 2.1, plays a pivotal role in this paper.

Lemma 2.1. *Assume that $f(t) \in L^2(\mathbb{R})$, then we have*

$$\mathcal{F}f_{\pm}(t) = (\mathcal{T}F(x))^{\pm}. \tag{2.10}$$

Proof. By Sokhotski-Plemelj formula and Poincaré-Bertrand formula [22, 23, 26], we have

$$(\mathcal{T}F(x))^{\pm} = \pm\frac{1}{2}F(x) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{F(\tau)}{\tau - x}d\tau,$$

then we get

$$\begin{aligned}(\mathcal{T}F(x))^{\pm} =& \frac{\pm 1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\exp(ixt)dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{\tau - x}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\exp(i\tau t)dt\right)d\tau \\ =& \frac{\pm 1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\exp(ixt)dt + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t)\mathrm{sgn}(t)\exp(ixt)dt \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\mathcal{S}f(t) \pm \mathcal{I}f(t)}{2}\exp(ixt)dt,\end{aligned} \tag{2.11}$$

thus the positive (or negative) Fourier transform of $f(t)$ is equal to the positive (or negative) boundary value of the Hilbert transform of $F(x)$, and therefore (2.10) is valid. $\square$

Lemma 2.2. *Assume that $\mathcal{F}$, $\mathcal{T}$, $\mathcal{L}$, $\mathcal{S}$ are as described above, we have*

$$(1)\ \mathcal{T}\mathcal{F} = \mathcal{F}\mathcal{S}; \quad (2)\ \mathcal{F}^2 = \mathcal{L}; \quad (3)\ \mathcal{F}^{-1} = \mathcal{L}\mathcal{F} = \mathcal{F}\mathcal{L}. \tag{2.12}$$

Proof. Without loss of generality, we only prove the case (1). By Sokhotski-Plemelj formula, we obtain

$$\mathcal{T}\mathcal{F}f(t) = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{F(\tau)}{\tau - x}d\tau = (\mathcal{T}F(x))^+ + (\mathcal{T}F(x))^-. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}\mathcal{F}\mathcal{S}f(t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)\mathrm{sgn}(t)\exp(ixt)dt\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f_{+}(t)\exp(ixt)dt+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f_{-}(t)\exp(ixt)dt.\end{aligned}\tag{2.14}$$

According to Lemma 2.1, we know that (2.13) and (2.14) are equal, thus (1) is valid.

The similar methodology can be applied to prove (2) and (3). □

Lemma 2.3. *Assume that* $f(t)\in H_1^*$, $\lambda\in\mathbb{R}$ *and* $|\lambda|<+\infty$, *we have*

$$\begin{aligned}&(1)\ \mathcal{F}f(t+\lambda)=F(x)\exp(-i\lambda x),\\&(2)\ \mathcal{F}\mathcal{T}f(t+\lambda)=-F(x)\mathrm{sgn}(x)\exp(-i\lambda x).\end{aligned}\tag{2.15}$$

Proof. (1) This can be proved by simply substituting $s=t+\lambda$ and performing a transformation of variables.

(2) According to Lemma 2.2, we have

$$\begin{aligned}\mathcal{F}\mathcal{T}f(t+\lambda)&=\mathcal{F}\mathcal{F}\mathcal{S}\mathcal{F}^{-1}f(t+\lambda)\\&=\mathcal{L}\mathcal{S}\mathcal{L}\mathcal{F}f(t+\lambda)\\&=\mathcal{L}\mathcal{S}F(-x)\exp(-i\lambda x)\\&=-F(x)\mathrm{sgn}(x)\exp(-i\lambda x).\end{aligned}\tag{2.16}$$

□

Lemma 2.4. *If* $f(t)\in H_j^*$ *and* $F(0)=0$, *then* $\mathcal{T}f(t)\in H_j^*$ $(j=1,2)$.

Proof. Since $f(t)\in H_j^*$ and $F(0)=0$, we have $F(x)\in H_j$ and

$$F(\infty)=F(0)=0,$$

thus $F(x)\mathrm{sgn}(x)\in H_j$. In Lemma 2.3, if we take $\lambda=0$, we have

$$\mathcal{F}\mathcal{T}f(t)=-F(x)\mathrm{sgn}(x)\in H_j,\quad j=1,2,\tag{2.17}$$

thus $\mathcal{T}f(t)\in H_j^*$ $(j=1,2)$. □

3. Singular integral equations with one convolution kernel and finite translations

First, we solve a basic class of convolution equations with finite translations by rewriting equation (1) in the following form

$$af(t+\lambda)+b\mathcal{T}f(t+\lambda)+k(t)*\mathcal{T}f(t+\lambda)=g(t),\quad t\in\mathbb{R},\tag{3.1}$$

where a, b, $\lambda\in\mathbb{R}$ with $ab\neq0$ and $|\lambda|<+\infty$, $k(t)$, $g(t)\in H_1^*$. We require that the unknown function $f(t)\in H_1^*$. Taking Fourier transforms on the both sides of (3.1), we get the following equation

$$[a-b\mathrm{sgn}(x)+K(x)\mathrm{sgn}(x)]\exp(-i\lambda x)F(x)=G(x),\quad x\in\mathbb{R},\tag{3.2}$$

where

$$K(x) = \mathcal{F}k(t), \quad G(x) = \mathcal{F}g(t), \quad F(x) = \mathcal{F}f(t). \tag{3.3}$$

Here, we only consider the case of normal type, that is

$$K(x) \neq \begin{cases} b-a, & x \in \mathbb{R}^+, \\ b+a, & x \in \mathbb{R}^-. \end{cases} \tag{3.4}$$

From (3.2), we have

$$F(x) = \frac{G(x)}{a - b\mathrm{sgn}(x) + K(x)\mathrm{sgn}(x)} \exp(i\lambda x). \tag{3.5}$$

On the other hand, since $G(x) \in H_1$, it follows from (3.2) that

$$G(0) = \mathcal{F}g(0) = 0, \tag{3.6}$$

which implies that $F(0) = 0$. Denote

$$G_0(x) = \frac{G(x)}{a - b\mathrm{sgn}(x)} \exp(i\lambda x), \quad P_0(x) = \frac{K(x)\mathrm{sgn}(x)}{a - b\mathrm{sgn}(x) + K(x)\mathrm{sgn}(x)} \exp(i\lambda x), \tag{3.7}$$

then, (3.5) can be rewritten as

$$F(x) = G_0(x) - G_0(x)P_0(x), \tag{3.8}$$

therefore, (3.1) has a unique solution

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}F(x) \\ &= \mathcal{F}^{-1}G_0(x) - \mathcal{F}^{-1}\left[G_0(x)P_0(x)\right] \\ &= g_0(t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g_0(t-\tau)p_0(\tau)d\tau, \end{aligned} \tag{3.9}$$

where

$$g_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{G(x)}{a - b\mathrm{sgn}(x)} \exp(ix(\lambda - t))dx, \tag{3.10}$$

and

$$p_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{K(x)\mathrm{sgn}(x)}{a - b\mathrm{sgn}(x) + K(x)\mathrm{sgn}(x)} \exp(ix(\lambda - t))dx. \tag{3.11}$$

Based on the above discussion, we obtain

Theorem 3.1. *If k, $g \in H_1^*$, and* (3.4) *is valid, then* (3.1) *is solvable in H_1 if and only if* (3.6) *holds. In this case,* (3.1) *has a unique solution*

$$f(t) = \mathcal{F}^{-1}F(x), \tag{3.12}$$

where $F(x)$ is given by (3.5) *or* (3.8).

4. Singular integral equations with two convolution kernels and finite translations

In this section, we consider the equation with two convolution kernels and finite translations, and equation (2) can be rewritten as

$$af(t+\lambda)+b\mathfrak{T}f(t+\tau)+k(t)*\mathfrak{T}f_+(t+\lambda)+h(t)*\mathfrak{T}f_-(t+\lambda)=g(t), \quad t\in\mathbb{R}, \tag{4.1}$$

where a, b, $\lambda \in \mathbb{R}$ with $ab \neq 0$, $|\lambda| < +\infty$, $k(t)$, $h(t)$, $g(t) \in H_1^*$, and the unknown function $f(t) \in H_1^*$. Similar to Section 3, taking the Fourier transforms on the both sides of (4.1), we get

$$\begin{aligned}&[a-b\mathrm{sgn}(x)]\,F(x)-K(x)\mathrm{sgn}(x)F^+(x)+H(x)\mathrm{sgn}(x)F^-(x)\\ =&G(x)\exp(i\lambda x), \quad x\in\mathbb{R},\end{aligned} \tag{4.2}$$

where

$$K(x)=\mathcal{F}k(t), \quad G(x)=\mathcal{F}g(t), \quad H(x)=\mathcal{F}h(t), \quad F(x)=\mathcal{F}f(t), \tag{4.3}$$

and

$$F^{\pm}(x)=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\mathrm{sgn}(t)\pm 1)f(t)\exp(ixt)dt. \tag{4.4}$$

By further simplifying (4.2), we obtain the following Riemann boundary value problem with discontinuous coefficients

$$F^+(x)=\Omega(x)F^-(x)+\Lambda(x), \quad x\in\mathbb{R}, \tag{4.5}$$

where

$$\Omega(x)=\frac{a-b\mathrm{sgn}(x)-H(x)\mathrm{sgn}(x)}{a-b\mathrm{sgn}(x)-K(x)\mathrm{sgn}(x)}, \tag{4.6}$$

and

$$\Lambda(x)=\frac{G(x)}{a-b\mathrm{sgn}(x)+K(x)\mathrm{sgn}(x)}\exp(i\lambda x). \tag{4.7}$$

Here, we still consider the normal types with restrictions on $K(x)$ and $H(x)$, that is

$$K(x)\neq\begin{cases}a-b, & x\in\mathbb{R}^+,\\ a+b, & x\in\mathbb{R}^-,\end{cases} \quad H(x)\neq\begin{cases}a-b, & x\in\mathbb{R}^+,\\ a+b, & x\in\mathbb{R}^-.\end{cases} \tag{4.8}$$

We can easily see that the coefficients $\Omega(x)$ (or $\Lambda(x)$) of Riemann boundary value problem (4.5) may exist the discontinuities points.

Next, we consider the continuity of $\Omega(x)$ at $x=0$. Since

$$\lim_{x\to+0}\Omega(x)=\frac{a-b-H(0)}{a-b-K(0)}, \quad \lim_{x\to-0}\Omega(x)=\frac{a+b+H(0)}{a+b+K(0)}, \tag{4.9}$$

and $ab\neq 0$, thus

$$\lim_{x\to+0}\Omega(x)\neq\lim_{x\to-0}\Omega(x),$$

then we know that $x=0$ is the node of (4.5). Furthermore, at $x=\infty$, one has

$$\lim_{x\to+\infty}\Omega(x)=\lim_{x\to-\infty}\Omega(x)=1, \tag{4.10}$$

thus $x=\infty$ is not the node of problem (4.5). It can be observed that (4.5) only contains one node $x=0$.

Denote

$$\gamma_0=\alpha_0+i\beta_0=\frac{1}{2\pi i}\{\log\Omega(-0)-\log\Omega(+0)\}, \tag{4.11}$$

we choose the integer $\kappa=[\alpha_0]$ such as

$$0\le\alpha_0-\kappa<1, \tag{4.12}$$

then we say that κ is the index of (4.5).

To solve (4.5), we need to consider the following function

$$Y(z)=\begin{cases}\exp(\Gamma(z)), & z\in\mathbb{Z}^+,\\ (\dfrac{z+i}{z-i})^{\kappa}(\exp\Gamma(z)), & z\in\mathbb{Z}^-,\end{cases} \tag{4.13}$$

where

$$\Gamma(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\log\Omega_0(x)}{x-z}dx, \tag{4.14}$$

and

$$\Omega_0(x)=(\frac{x+i}{x-i})^{\kappa}\Omega(x), \tag{4.15}$$

here $\Omega_0(x)$ is a holomorphic function, but we have taken the definite branch of

$$\log\Omega_0(x)=\kappa\log\frac{x+i}{x-i}+\log\Omega(x), \tag{4.16}$$

and chosen

$$\lim_{x\to\infty}\log\frac{x+i}{x-i}=0,\quad \lim_{x\to\pm 0}\log\frac{x+i}{x-i}=\pm i\pi. \tag{4.17}$$

The Sokhotski-Plemelj formula for (4.14) yields the following result

$$\Gamma^{\pm}(x)=\pm\frac{1}{2}\log\Omega_0(x)+\mathfrak{T}\Omega_0(x), \tag{4.18}$$

so we obtian

$$Y^+(x)=\sqrt{\Omega_0(x)}\exp(\mathfrak{T}\Omega_0(x)),\quad Y^-(x)=\left(\frac{x+i}{(x-i)\Omega_0(x)^{\frac{1}{2\kappa}}}\right)^{\kappa}\exp(\mathfrak{T}\Omega_0(x)). \tag{4.19}$$

According to the principle of analytic continuation and Liouville theorem [16, 18, 23], we have the following conclusion.

When $\kappa\ge 0$, the general solution of (4.5) is

$$F(z)=\frac{Y(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{\Lambda(x)}{Y^+(x)(x-z)}dx+\frac{Y(z)}{(z+i)^{\kappa}}P_{\kappa-1}(z), \tag{4.20}$$

where

$$P_{\kappa-1}(z)=C_0+C_1z+\ldots+C_{\kappa-1}z^{\kappa-1}, \tag{4.21}$$

is a polynomial of the order $\kappa-1$ and $P_{-1}(z)\equiv 0$.

When $\kappa \le -1$, (4.5) has a unique solution

$$F(z) = \frac{Y(z)}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Lambda(x)}{Y^+(x)(x-z)}dx. \tag{4.22}$$

Observing (4.13), it can be prove that $Y(z)$ has a pole of the order $-\kappa$ at $z=-i$. For which the following solvability conditions needs to be satisfied

$$\int_{-\infty}^{+\infty} \frac{\Lambda(x)}{Y^+(x)(x+i)^l}dx = 0, \quad l = 1, 2, \ldots, -\kappa, \tag{4.23}$$

thus, (4.5) has a unique solution (4.22) if and only if (4.23) holds. We also know that, when $\kappa \ge 0$, (4.5) has κ linearly independent solutions; when $\kappa \le -1$, (4.5) has $-\kappa$ solvability conditions. In summary, we say that (4.5) has $|\kappa|$ degrees of freedom. Applying the Sokhotski-Plemelj formula to (4.20), we can obtain

$$F^+(x) = \frac{\Lambda(x)}{2} + \frac{Y^+(x)}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Lambda(t)}{Y^+(t)(t-x)}dx + \frac{Y^+(x)}{(x+i)^\kappa}P_{\kappa-1}(x), \tag{4.24}$$

and

$$F^-(x) = -\frac{\Lambda(x)}{2\Omega(x)} + \frac{Y^-(x)}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Lambda(t)}{Y^+(t)(t-x)}dx + \frac{Y^-(x)}{(x+i)^\kappa}P_{\kappa-1}(x), \tag{4.25}$$

thus

$$\begin{aligned} F(x) =& F^+(x) - F^-(x) \\ =& \frac{\Lambda(x)}{2}\left(1 + \frac{1}{\Omega(x)}\right) + (Y^+(x) - Y^-(x))\Bigg(\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Lambda(t)}{Y^+(t)(t-x)}dx \\ &+ \frac{P_{\kappa-1}(x)}{(x+i)^\kappa}\Bigg). \end{aligned} \tag{4.26}$$

It is easy to verify that $F^\pm(x)$, $F(x) \in H_1$ in any closed interval except $x = 0$.

Next we discuss the solvability conditions for (4.5) near the node $x = 0$, and we denote

$$\alpha = \alpha_0 - \kappa$$

and

$$\gamma = \alpha + i\beta_0.$$

Now, we will discuss two cases, respectively.

(1) Suppose that $x = 0$ is an ordinary node, that is, $0 < \alpha < 1$, we have

$$Y^+(x) = x^\gamma\sqrt{\Omega_0(x)}\exp\Gamma_0(x), \tag{4.27}$$

where

$$\Gamma_0(x) = \Im\log\Omega_0(x) - \gamma\log x.$$

Denote

$$\Phi(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Lambda(x)}{x^\gamma(x-z)\sqrt{\Omega_0(x)}}\exp(-\Gamma_0(x))dx, \tag{4.28}$$

by [2, 14, 26, 31], when $x > 0$, we have

$$\begin{aligned}\Phi(x) =& \frac{\Lambda(+0)\exp(-\Gamma_0(x)+i\gamma\pi)}{4ix^\gamma \sin\gamma\pi\sqrt{\Omega_0(+0)}} \\ &+ \frac{\Lambda(+0)\sqrt{\Omega_0(-0)}-2\Lambda(-0)\sqrt{\Omega_0(+0)}}{4ix^\gamma \sin\gamma\pi\sqrt{\Omega_0(+0)\Omega_0(-0)}}\exp(-\Gamma_0(x)-i\gamma\pi)+\Phi^*(x),\end{aligned} \tag{4.29}$$

where

$$\Phi^*(x) = \frac{\Phi^{**}(x)}{|x|^{\alpha'}}, \quad 0<\alpha'<\alpha<1, \quad \Phi^{**}(x)\in\hat{H}. \tag{4.30}$$

In addition, we have

$$\lim_{x\to+0} x^\gamma\sqrt{\Omega_0(x)}\exp\Gamma_0(x)\left(\Phi^*(x)+\frac{P_{\kappa-1}(x)}{(x+i)^\kappa}\right)=0, \tag{4.31}$$

thus, we get

$$F^+(+0) = \frac{\Lambda(+0)}{2i\sin\gamma\pi}\exp i\gamma\pi - \frac{\Lambda(-0)}{2i\sin\gamma\pi}\exp(-2i\gamma\pi). \tag{4.32}$$

Similarly, when $x < 0$, we have

$$\begin{aligned}\Phi(x) =& \frac{2\Lambda(+0)\sqrt{\Omega_0(-0)}-\Lambda(-0)\sqrt{\Omega_0(+0)}}{4ix^\gamma \sin\gamma\pi\sqrt{\Omega_0(+0)\Omega_0(-0)}}\exp(-\Gamma_0(x)+i\gamma\pi) \\ &- \frac{\Lambda(-0)\exp(-\Gamma_0(x)-i\gamma\pi)}{4ix^\gamma \sin\gamma\pi\sqrt{\Omega_0(-0)}}+\Phi^*(x),\end{aligned} \tag{4.33}$$

and

$$F^+(-0) = \frac{\Lambda(+0)}{2i\sin\gamma\pi}\exp(2i\gamma\pi) - \frac{\Lambda(-0)}{2i\sin\gamma\pi}\exp(-2i\gamma\pi). \tag{4.34}$$

Since $F^+(x)$ is continuous at $x = 0$, we must have

$$\Lambda(+0)\exp 3i\gamma\pi = \Lambda(-0). \tag{4.35}$$

It can be shown that $F^+(\pm 0) = 0$ if and only if (4.35) holds. Moreover, in order to $F^-(\pm 0) = 0$, (3.6) is once again holds, so we have

$$F(0) = F^+(0) = F^-(0) = 0.$$

From the above discussion, we can obtian $F(x) \in H_1$.

(2) Suppose that $x = 0$ is a special node, that is, $\alpha = 0$.

(2a) If $\beta_0 \neq 0$, (4.31) is not necessarily holds. In this case, it is necessary to assume that (3.6) and (4.35) are also valid. Since

$$\Lambda(0) = G(0) = 0,$$

we have

$$\Phi(0) = \Phi^*(0).$$

Then $F(x)$ is continuous at $x = 0$ if and only if

$$\Phi(0) = \begin{cases} C_0 i^{-\kappa-2}, \ \kappa \geq 0, \\ 0, \qquad\quad \kappa \leq -1, \end{cases} \tag{4.36}$$

that is, when $\kappa \geq 0$, the constant term of $P_{\kappa-1}(z)$ satisfies the following equality

$$C_0 = \frac{i^{\kappa+1}}{2\pi} \int_{-\infty}^{+\infty} \frac{\Lambda(x)}{Y^+(x)x} dx, \tag{4.37}$$

when $\kappa \leq -1$, we have

$$\int_{-\infty}^{+\infty} \frac{\Lambda(x)}{Y^+(x)x} dx = 0. \tag{4.38}$$

(4.37) and (4.38) ensure that $F(x)$ is continuous at $x = 0$, that is, $F(0) = F^{\pm}(0) = 0$.

(2b) If $\beta_0 = 0$, $\gamma = 0$, then $\Omega(x)$ is continuous at $x = 0$. In this case, the coefficients of (4.5) does not contain node.

Based on the above discussion, we have

Theorem 4.1. *Under supposition, in the case that* (4.8) *is satisfied,* (4.1) *is solvable in* H_1 *provided that* (3.6) *holds. Consequently, we have*

(1) Let $x = 0$ *be an ordinary node,* (4.35) *is valid. When* $\kappa \geq 0$, (4.1) *always has the solution; when* $\kappa \leq -1$, *if* (4.23) *is true,* (4.1) *has a unique solution.*

(2) Let $x = 0$ *be a special node, and* (4.35) *is still valid. When* $\kappa \geq 0$, (4.1) *has a general solution if and only if* (4.37) *holds; when* $\kappa \leq -1$, *both* (4.23) *and* (4.38) *are holds,* (4.1) *has a unique solution.*

In short, if the above conditions are satisfied, the solution of (4.1) *is*

$$f(t) = \mathcal{F}^{-1} F(x), \tag{4.39}$$

where $F(x)$ *is given by* (4.26).

5. Singular integral equations of Wiener-Hopf type with convolution and finite translations

In this section, we investigate the Wiener-Hopf type equations on the positive real axis. Equation (3) can be reformulated as follows

$$af(t+\lambda) + b\mathfrak{T}f_+(t+\tau) + k(t) * \mathfrak{T}f_+(t+\lambda) = g(t), \quad t \in \mathbb{R}^+, \tag{5.1}$$

where a, b, $\lambda \in \mathbb{R}$ with $ab \neq 0$ and $|\lambda| < +\infty$, $k(t)$, $g(t) \in H_2^*$, $f(t)$ is required to be in H_2^*. Add an unknown function $f_-(t)$ in the right side of (5.1) to extend the equation to the whole real axis, we have

$$af_+(t+\lambda) + b\mathfrak{T}f_+(t+\tau) + k(t) * \mathfrak{T}f_+(t+\lambda) = g(t) + f_-(t), \quad t \in \mathbb{R}. \tag{5.2}$$

Taking Fourier transforms on the both sides of (5.2), we obtian

$$[a - b\mathrm{sgn}(x) - K(x)\mathrm{sgn}(x)]\, F^+(x)\exp(-i\lambda x) = G(x) + F^-(x), \quad x \in \mathbb{R}. \tag{5.3}$$

Further simplifying (5.3), we obtain the following Riemann boundary value problem

$$F^{+}(x)=\Delta(x)F^{-}(x)+\Theta(x),\quad x\in\mathbb{R}, \tag{5.4}$$

where

$$\Delta(x)=\frac{\exp(i\lambda x)}{a-b\mathrm{sgn}(x)-K(x)\mathrm{sgn}(x)},\quad \Theta(x)=G(x)\Delta(x), \tag{5.5}$$

and we also require that $K(x)$ satisfy the following conditions

$$K(x)\neq\begin{cases}a-b, & x\in\mathbb{R}^{+},\\ -b, & x=0,\\ -a-b, & x\in\mathbb{R}^{-}.\end{cases} \tag{5.6}$$

It is clear that $\Delta(x)$ is discontinuous at both $x=0$ and $x=\infty$, so (5.4) has two nodes. Denote

$$\sigma_{\infty}=\nu_{\infty}+i\iota_{\infty}=\frac{1}{2\pi i}(\log\Delta(-\infty)-\log\Delta(+\infty)), \tag{5.7}$$

form [17,20,21], we must have $0\leq\nu_{\infty}<1$ and $\sigma_{\infty}\neq 0$. Again take

$$\sigma_{0}=\nu_{0}+i\iota_{0}=\frac{1}{2\pi i}(\log\Delta(-0)-\log\Delta(+0)), \tag{5.8}$$

and $\chi=[\nu_0]$ as the index of (5.4). Note that $0\leq\nu=\nu_0-\chi<1$, we take $\sigma=\nu+\iota_0$ and $\sigma\neq 0$. Once more, using the principle of analytical continuation and Liouville theorem, we can obtain

When $\chi\geq -1$, (5.4) always has a general solution

$$F(z)=\frac{V(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{z+i}{x+i}\frac{\Theta(x)}{V^{+}(x)(x-z)}dx+\frac{V(z)}{(z+i)^{\chi}}P_{\chi}(z), \tag{5.9}$$

where

$$V(z)=\begin{cases}\exp\Xi(z), & z\in\mathbb{Z}^{+},\\ (\dfrac{z+i}{z-i})^{\chi}\exp\Xi(z), & z\in\mathbb{Z}^{-},\end{cases} \tag{5.10}$$

and

$$\Xi(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{z+i}{x+i}\frac{\log\Delta_0(x)}{x-z}dx,\quad \Delta_0(x)=(\frac{x+i}{x-i})^{\chi}\Delta(x), \tag{5.11}$$

$P_{\chi}(z)$ is a polynomial of order χ similar to the form (4.23); when $\chi\leq -2$, (5.4) has a unique solution

$$F(z)=\frac{V(z)}{2\pi i}\int_{-\infty}^{+\infty}\frac{z+i}{x+i}\frac{\Theta(x)}{V^{+}(x)(x-z)}dx, \tag{5.12}$$

if and only if the conditions

$$\int_{-\infty}^{+\infty}\frac{\Theta(x)}{V^{+}(x)(x+i)^{l}}dx=0,\quad l=2,3,\ldots,-\chi, \tag{5.13}$$

are satisfied. Thus, via using Sokhotski-Plemelj formula, we get

$$F^+(x) = \frac{\Theta(x)}{2} + \frac{V^+(x)}{2\pi i}\int_{-\infty}^{+\infty} \frac{x+i}{t+i}\frac{\Theta(t)}{V^+(t)(t-x)}dt + \frac{V^+(x)}{(x+i)^\chi}P_\chi(x). \tag{5.14}$$

Similarly to (4.24), we know that $F^+(x) \in H_2$ in any closed interval excluding $x = 0$ and $x = \infty$.

Next, we consider the behavior of (5.14) near $x = 0$ and $x = \infty$. Again denote

$$\Psi(x) = \frac{x+i}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Theta(t)}{V^+(t)(t-x)(t+i)}dt. \tag{5.15}$$

(1) Suppose that $x = \infty$ is an ordinary node. Note that $V(x) = x^{-\sigma_\infty}V^*(x)$ in the neighborhood of $x = \infty$, where $V^*(x) \in H_2$.

(1a) When $\frac{1}{2} < \nu_\infty < \mu < 1$, since $G(x) \in H_2$, we have

$$\Theta(x) = \mathrm{O}(|x|^{-\mu}), \tag{5.16}$$

and (5.15) is bounded, we get

$$V^+(x)\Psi(x) = \mathrm{O}(|x|^{-\nu_\infty}). \tag{5.17}$$

If $\chi \geq -1$, we know that

$$\frac{V^+(x)}{(x+i)^\chi}P_\chi(x) = \mathrm{O}(|x|^{-\nu_\infty}). \tag{5.18}$$

If $\chi \leq -2$, then $P_\chi(x) = 0$. From (5.16)–(5.18), we get

$$F^+(x) = \mathrm{O}(|x|^{-\mu}). \tag{5.19}$$

(1b) When $\frac{1}{2} < \mu \leq \nu_\infty < 1$, (5.16) and (5.18) still hold, but for (5.17) we can prove that there exists a sufficiently small positive real number $\varepsilon > 0$ such that $\nu_\infty > \frac{1+2\varepsilon}{2}$, then (5.17) is expressed in the following form

$$V^+(x)\Psi(x) = \mathrm{O}(|x|^{-\nu_\infty+\varepsilon}), \tag{5.20}$$

we have

$$F^+(x) = \mathrm{O}(|x|^{-\nu_\infty+\varepsilon}). \tag{5.21}$$

Denote $\xi = \min\{\mu, \nu_\infty - \varepsilon\}$, when $\nu_\infty > \frac{1}{2}$, we have

$$F^+(x) = \mathrm{O}(|x|^{-\xi}). \tag{5.22}$$

(1c) When $\nu_\infty \leq \frac{1}{2} < \mu < 1$, (5.15) still holds, from [18,23,26], (5.17) and (5.20) can transformed into the following form

$$V^+(x)\Psi(x) = \mathrm{O}(|x|^{-\mu}). \tag{5.23}$$

But in this case, (5.18) may not be valid.

If $\chi \geq 0$, the highest power term coefficient C_χ of $P_\chi(z)$ should be satisfied

$$C_\chi = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\Theta(x)}{V^+(x)(x+i)}dx. \tag{5.24}$$

If $\chi \leq -1$, then $P_\chi(z) = 0$, and we need to add the following solvability condition

$$\int_{-\infty}^{+\infty} \frac{\Theta(x)}{V^+(x)(x+i)} dx = 0. \tag{5.25}$$

Thus, (5.13) can be rewritten as

$$\int_{-\infty}^{+\infty} \frac{\Theta(x)}{V^+(x)(x+i)^l} dx = 0, \quad l = 1, 2, \ldots, -\chi. \tag{5.26}$$

It can be seen that when (5.24) and (5.26) holds, (5.22) still holds.

(2) Suppose that $x = \infty$ is a special node, then $\nu_\infty = 0$, $\sigma_\infty = i\iota_\infty \neq 0$. In this case, we have $F^+(x) \in H_2$, and the discussion is similar to case (1c) above.

Next we consider the case in the neighborhood of $x = 0$. The discussion is similar to Section 4, and we only state the differences from the previous text. According to

$$F^+(+0) = F^-(+0),$$

we get the following equality

$$G(0) \left[\Delta(-0) \exp 3i\sigma\pi - \Delta(+0)\right] = 0. \tag{5.27}$$

Note that (5.8), we have

$$\Delta(-0) = \Delta(+0) \exp 2i\sigma_0\pi,$$

thus (5.27) can be rewritten as

$$G(0)\Delta(+0)(1 - \exp(-i\sigma\pi)) = 0. \tag{5.28}$$

Thus, we can obtain that (3.6) holds, if and only if (5.27) is valid.

On the other hand, we also note that when $\nu_\infty > \frac{1}{2}$ and $\chi \geq 0$, the following equality is also fulfilled

$$P_\chi(0) = \frac{i^{\kappa-1}}{2\pi} \int_{-\infty}^{+\infty} \frac{\Theta(x)}{V^+(x)(x+i)x} dx. \tag{5.29}$$

And when $\chi \leq -1$, then (4.38) should be rewritten as the following form

$$\int_{-\infty}^{+\infty} \frac{\Theta(x)}{V^+(x)(x+i)x} dx = 0. \tag{5.30}$$

In all of the above cases, we know that $F^+(x) \in H_2$ undoubtedly.

Based on the above discussion, we can formulate the main results about solutions of the equation (5.1) in the following form

Theorem 5.1. *Under supposition,* (5.6) *holds, then* (5.1) *is solvable in* H_2 *if and only if* (3.6) *holds.*

(1) *Let* $x = \infty$ *be an ordinary node. When* $\nu_\infty > \frac{1}{2}$*, if* $\chi \geq -1$*,* (5.1) *always has the solution; if* $\chi \leq -2$*,* (5.1) *is solvable only if* (5.13) *is valid. When* $\nu_\infty \leq \frac{1}{2}$*, if* $\chi \geq 0$*, the necessary condition for* (5.1) *is* (5.24)*; if* $\chi \leq -1$*,* (5.1) *is solvable if and only if* (5.25) *holds.*

(2) *Let $x = \infty$ be a special node. When $\chi \geq 0$, the solvability condition for* (5.1) *is* (5.24)*; when $\chi \leq -1$, the solvability condition for* (5.1) *is* (5.25).

(3) *Let $x = 0$ be an ordinary node,* (5.27) *holds. When $\nu_\infty > \frac{1}{2}$, if $\chi \geq -1$,* (5.1) *always has the solution; if $\chi \leq -2$,* (5.1) *is solvable when* (5.13) *holds. When $\nu_\infty \leq \frac{1}{2}$, if $\chi \geq 0$,* (5.24) *is a solvable condition for* (5.1)*; if $\chi \leq -1$,* (5.1) *is solvable in H_2 if and only if* (5.26) *holds.*

(4) *Let $x = 0$ be a special node,* (5.27) *still holds. When $\nu_\infty > \frac{1}{2}$, if $\chi \geq 0$, (*(5.1) *has a general solution if and only if* (5.26) *holds; if $\chi \leq -1$,* (5.1) *has a unique solution if and only if both* (5.26) *and* (5.30) *holds. When $\nu_\infty \leq \frac{1}{2}$, if $\chi \geq 0$,* (5.1) *has a general solution when both* (5.24) *and* (5.29) *hold; if $\chi \leq -1$, both* (5.24) *and* (5.27) *hold, and* (5.1) *has a unique solution.*

If cases (1)-(4) are satisfied, (5.1) *is solvable in H_2, and the solution is*

$$f^+(t) = \mathcal{F}^{-1} F^+(x), \tag{5.31}$$

where $F^+(x)$ is given by (5.14).

6. Singular integral equations of dual type with convolution and finite translations

Finally, we address the dual equation through convolution and finite translations by reformulating equation (4) as

$$\begin{cases} a_1 f(t+\lambda) + b_1 \mathcal{T} f(t+\lambda) + k_1(t) * \mathcal{T} f(t+\lambda) = g(t),\, t \in \mathbb{R}^+, \\ a_2 f(t+\lambda) + b_2 \mathcal{T} f(t+\lambda) + k_2(t) * \mathcal{T} f(t+\lambda) = g(t),\, t \in \mathbb{R}^-, \end{cases} \tag{6.1}$$

where a_j, b_j, $\lambda \in \mathbb{R}$ with $a_j b_j \neq 0$ and $|\lambda| < +\infty$, $g(t)$, $k_j(t) \in H_2^*$ $(j = 1, 2)$. We requires that the unknown function $f(t) \in H_2^*$. In order to extend the equation to the whole real axis, we add the unknown function $\upsilon^\pm(t) \in H_2^*$ to the right hand side of (6.1), respectively, and we get

$$\begin{cases} a_1 f(t+\lambda) + b_1 \mathcal{T} f(t+\lambda) + k_1(t) * \mathcal{T} f(t+\lambda) = g(t) + \upsilon^-(t), \\ a_2 f(t+\lambda) + b_2 \mathcal{T} f(t+\lambda) + k_2(t) * \mathcal{T} f(t+\lambda) = g(t) + \upsilon^+(t), \end{cases} \quad t \in \mathbb{R}. \tag{6.2}$$

Taking the Fourier transforms on the both sides of (6.2), we get

$$\begin{cases} [a_1 - b_1 \mathrm{sgn}(x) - K_1(x)\mathrm{sgn}(x)]\, F(x) \exp(-i\lambda x) = G(x) + \Upsilon^-(x), \\ [a_2 - b_2 \mathrm{sgn}(x) - K_2(x)\mathrm{sgn}(x)]\, F(x) \exp(-i\lambda x) = G(x) + \Upsilon^+(x), \end{cases} \quad x \in \mathbb{R}. \tag{6.3}$$

We still consider the normal type, that is

$$K_j(x) \neq \begin{cases} a_j - b_j, & x \in \mathbb{R}^+, \\ -b_j, & x = 0, \\ -a_j - b_j, & x \in \mathbb{R}^-, \end{cases} \quad (j = 1, 2). \tag{6.4}$$

By further simplifying (6.3), we obtain

$$\frac{G(x)+\Upsilon^{-}(x)}{a_1-b_1\mathrm{sgn}(x)-K_1(x)\mathrm{sgn}(x)}=\frac{G(x)+\Upsilon^{+}(x)}{a_2-b_2\mathrm{sgn}(x)-K_2(x)\mathrm{sgn}(x)}, \tag{6.5}$$

thus we get the following Riemann boundary value problem

$$\Upsilon^{+}(x)=M(x)\Upsilon^{-}(x)+N(x), \quad x\in\mathbb{R}, \tag{6.6}$$

where

$$M(x)=\frac{a_2-b_2\mathrm{sgn}(x)-K_2(x)\mathrm{sgn}(x)}{a_1-b_1\mathrm{sgn}(x)-K_1(x)\mathrm{sgn}(x)}, \quad N(x)=G(x)(M(x)-1). \tag{6.7}$$

Note that (6.4) is continuous at $x=0$, we have

$$G(0)+\Upsilon^{\pm}(0)=0, \tag{6.8}$$

we get (3.6) holds again. Similar to Section 5, $x=0$ and $x=\infty$ are still nodes of (6.6). Therefore, all the results in Theorem 5.1 still holds and we get $f(t)=\mathcal{F}^{-1}F(x)$ where $F(x)$ is given by (6.1).

In addition, we can generalize equations (1)-(4) to singular integral equations on closed curves with Carleman shifts [7,11,28], as well as singular integral equations and boundary value problems in the function of several complex variables [3,10,12,19,25,34].

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