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FINITE SPECTRUM OF STURM-LIOUVILLE PROBLEMS WITH N TRANSMISSION CONDITIONS AND SPECTRAL PARAMETERS IN THE BOUNDARY CONDITIONS*

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Abstract In this paper, we mainly study the finite spectrum of Sturm-Liouville problems with n transmission conditions and spectral parameters in the boundary conditions. For any positive integer n and a set of positive integers $m_i, i = 0, 1, \cdots, n$, it has at most $m_0+m_1+\cdots+m_n+2n+1$ eigenvalues. And further we show that these $m_0 + m_1 + \cdots + m_n + 2n + 1$ eigenvalues can be distributed arbitrarily throughout the complex plane in the non-self-adjoint case and anywhere along the real line in the self-adjoint case. The key to this analysis is an iterative construction of the characteristic function, the main tool used in this paper is Rouche's theorem and iterative construction of initial value.

Keywords Transmission conditions, spectral parameters, Sturm-Liouville problems, characteristic function.

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1. Introduction

Sturm-Liouville problems (SLPs for short) [14, 15, 17] with transmission conditions and spectral parameters in the boundary conditions have always been an important research topic in mathematical physics. Such a problem connected with many assortment of physics problems, such as heat conduction and the chord vibration of the boundary on the slider.

As is well-known, the classic Sturm-Liouville theory [22] states that the spectrum of a regular or singular, self-adjoint SLP is unbounded and therefore infinite. This result is generally established under the assumption that the leading cofficient p and the weight function w are both positive. Atkinson in his book [8] studied that if the cofficients of Sturm-Liouville equation satisfy some conditions, it may have finite

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eigenvalues, but he did not elaborate with an example. In 2001, Kong et al. [14] constructed a class of SLP with finite eigenvalues. In 2011, Ao et al. [6] obtained that the following regular SLP with a transmission condition

$$\begin{cases} -(py')' + qy = \lambda wy, \quad t \in J, \\ AY(a) + BY(b) = 0, \\ CY(c-) + DY(c+) = 0 \end{cases}$$

has exactly n eigenvalues, where n is positive integer and n is connected with the partition of the interval $J = (a, c) \cup (c, b)$, A, B, C and D are all matrices, and the coefficients satisfy the minimal conditions $r = \frac{1}{p}$, q, $w \in L(J, \mathbb{C})$. Their technique was a combination of the iterative construction of characteristic function and the fundamental theorem of Algebra. In 2013, by applying the iteration of the characteristic function and the fundamental theorem of Algebra, Ao et al. [7] obtained that the following regular SLP with a transmission condition and spectral parameters in the boundary conditions

$$\begin{cases} -(py')' + qy = \lambda wy, \quad t \in J, \\ A_\lambda Y(a) + B_\lambda Y(b) = 0, \\ CY(c-) + DY(c+) = 0 \end{cases}$$

has at most $m + n + 4$ eigenvalues, where m and n are positive integer, and m, n are connected with the partition of the interval $J = (a, c) \cup (c, b)$. It is divided into $a = a_0 < a_1 < a_2 < \cdots < a_{2m} < a_{2m+1} = c$, $c = b_0 < b_1 < b_2 < \cdots < b_{2n} < b_{2n+1} = b$. Recently, Xu et al. [21] researched that the following SLP with n transmission conditions

$$\begin{cases} -(py')' + qy = \lambda wy, \quad t \in J, \\ AY(a) + BY(b) = 0, \\ C_iY(c_i-) + D_iY(c_i+) = 0 \end{cases}$$

has exactly $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues for any positive integer n and a set of positive integers m_i, $i = 1, 2, \cdots, n+1$, where m_i and n are connected with the partition of the interval $J = (a, c_1) \cup (c_1, c_2) \cup \cdots \cup (c_n, b)$. They used similar tools to [7]. These results indicate the existence of finite spectrum of SLP. It also should be pointed out that although many excellent achievements have been made in researches on the finite spectrum of SLP, such as literature [7, 14, 15, 18–20, 22] and its references, but the conditions involved are relatively simple. It is worth mentioning that some scholars have done outstanding work on boundary value problems of differential equations with finite spectrum [1, 4, 5, 9–11, 13, 16, 22, 23]. In addition, for other articles on whether boundary value problems of differential equations have finite spectrum, please refer to Ao and Sun's articles [2, 3]. Motivated and inspired by the above-mentioned works, in this paper, we consider the following SLP

$$-(py')' + qy = \lambda wy, \tag{1.1}$$

$$A_\lambda Y(a) + B_\lambda Y(b) = 0, \tag{1.2}$$

$$C_iY(c_i-) + D_iY(c_i+) = 0, \tag{1.3}$$

where $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$, $y = y(t)$, $t \in J = (a, c_1) \cup (c_1, c_2) \cup ... \cup (c_n, b)$, $-\infty < a < b < +\infty$, $c_i \in (a, b)$, $C_i, D_i \in M_2(\mathbb{R}), det(C_i) = \rho_i > 0, det(D_i) = \theta_i > 0, i = 1, 2, \cdots, n$. $A_\lambda = \begin{pmatrix} \lambda\alpha_1' - \alpha_1 & -\lambda\alpha_2' + \alpha_2 \\ 0 & 0 \end{pmatrix}$, $B_\lambda = \begin{pmatrix} 0 & 0 \\ \lambda\beta_1' + \beta_1 & -\lambda\beta_2' - \beta_2 \end{pmatrix}$, $\alpha_j, \alpha_j', \beta_j, \beta_j' \in \mathbb{R}, j = 1, 2$, and satisfies $det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_1' & \alpha_2' \end{pmatrix} \neq 0$, $det \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_1' & \beta_2' \end{pmatrix} \neq 0$. λ is the spectral parameter. The coefficients satisfy the minimal conditions

$$r = \frac{1}{p},\ q,\ w \in L(J,\ \mathbb{C}), \tag{1.4}$$

where $L(J,\ \mathbb{C})$ denotes the complex-valued functions which are Lebesgue integrable on J. Condition (1.4) is minimal in the sense that it is necessary and sufficient for all initial value problems of to have unique solutions on $[a, b]$; see [12]. In this paper, we assume that condition (1.4) holds and we will prove that SLP (1.1)∼(1.3) still has finite spectrum.

2. Notation and preliminaries

In this section, we let $u = y, v = py'$. Then (1.1) can be transferred into the following first order system

$$u' = rv, v' = (q - \lambda w)u. \tag{2.1}$$

This can be written in the following matrix form

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & r \\ q - \lambda w & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Definition 2.1. By a trivial solution of equation (1.1) on some interval we mean a solution y which is identically zero and whose quasi-derivative $v = py'$ is also identically zero on this interval.

In this part, we give some related concepts to introduce Lemma 2.1.

Let $u_1(t, \lambda), v_1(t, \lambda)$ be two linearly independent solutions of equation (1.1) on (a, c_1) satisfying the following initial conditions

$$u_1(a, \lambda) = 1, (pu_1')(a, \lambda) = 0, v_1(a, \lambda) = 0, (pv_1')(a, \lambda) = 1.$$

Now we can define the solutions $u_{i+1}(t, \lambda), v_{i+1}(t, \lambda)(i = 1, 2, ..., n)$ of equation (1.1) on $(c_i, c_{i+1})(c_{n+1} = b)$ satisfying the following initial conditions

$$\begin{aligned} &u_{i+1}(c_i+, \lambda) = g_{i,11}u_i(c_i-, \lambda) + g_{i,12}(pu_i')(c_i-, \lambda), \\ &(pu_{i+1}')(c_i+, \lambda) = g_{i,21}u_i(c_i-, \lambda) + g_{i,22}(pu_i')(c_i-, \lambda), \\ &v_{i+1}(c_i+, \lambda) = g_{i,11}v_i(c_i-, \lambda) + g_{i,12}(pv_i')(c_i-, \lambda), \end{aligned}$$

$$(pv'_{i+1})(c_i+,\lambda)=g_{i,21}v_i(c_i-,\lambda)+g_{i,22}(pv'_i)(c_i-,\lambda).$$

For convenience, we let

$$\phi_{11}(t,\lambda)=\begin{cases}u_1(t,\lambda), & t\in(a,c_1),\\ u_{i+1}(t,\lambda), & t\in(c_i,c_{i+1})(i=1,2,...,n,c_{n+1}=b),\end{cases}$$

$$\phi_{12}(t,\lambda)=\begin{cases}v_1(t,\lambda), & t\in(a,c_1),\\ v_{i+1}(t,\lambda), & t\in(c_i,c_{i+1})(i=1,2,...,n,c_{n+1}=b),\end{cases}$$

$$\phi_{21}(t,\lambda)=\begin{cases}(pu'_1)(t,\lambda), & t\in(a,c_1),\\ (pu'_{i+1})(t,\lambda), & t\in(c_i,c_{i+1})(i=1,2,...,n,c_{n+1}=b),\end{cases}$$

$$\phi_{22}(t,\lambda)=\begin{cases}(pv'_1)(t,\lambda), & t\in(a,c_1),\\ (pv'_{i+1})(t,\lambda), & t\in(c_i,c_{i+1})(i=1,2,...,n,c_{n+1}=b).\end{cases}$$

Then

$$\Phi(t,\lambda)=\begin{pmatrix}\phi_{11}(t,\lambda) & \phi_{12}(t,\lambda)\\ \phi_{21}(t,\lambda) & \phi_{22}(t,\lambda)\end{pmatrix},\ t\in J.$$

So $\Phi(t,\lambda)=[\phi_{ef}(t,\lambda)](e,f=1,2,t\in J)$ denotes the fundamental matrix of the system (2.1) determined by the initial condition $\Phi(a,\lambda)=I$.

Lemma 2.1. *The complex number λ is an eigenvalue of the SLP* (1.1)∼(1.3) *if and only if*

$$\triangle(\lambda)=det[A_\lambda+B_\lambda\Phi(b,\lambda)]=0. \tag{2.2}$$

Particularly, $\triangle(\lambda)$ can be written as

$$\triangle(\lambda)=h_{11}(\lambda)\phi_{11}(b,\lambda)+h_{12}(\lambda)\phi_{12}(b,\lambda)+h_{21}(\lambda)\phi_{21}(b,\lambda)+h_{22}(\lambda)\phi_{22}(b,\lambda), \tag{2.3}$$

where

$$H(\lambda)=\begin{pmatrix}h_{11}(\lambda) & h_{12}(\lambda)\\ h_{21}(\lambda) & h_{22}(\lambda)\end{pmatrix}$$

$$:=\begin{pmatrix}(\lambda\alpha'_2-\alpha_2)(\lambda\beta'_1+\beta_1) & (\lambda\alpha'_1-\alpha_1)(\lambda\beta'_1+\beta_1)\\ -(\lambda\alpha'_2-\alpha_2)(\lambda\beta'_2+\beta_2) & -(\lambda\alpha'_1-\alpha_1)(\lambda\beta'_2+\beta_2)\end{pmatrix}.$$

Proof. If λ is an eigenvalue of the SLP (1.1)∼(1.3), then there exists a non-trivial solution

$$y(t,\lambda)=\begin{cases}k_1u_1+l_1v_1, & t\in(a,c_1),\\ k_2u_2+l_2v_2, & t\in(c_1,c_2),\\ ...\\ k_{n+1}u_{n+1}+l_{n+1}v_{n+1}, & t\in(c_n,b)\end{cases} \tag{2.4}$$

of equation (1.1), where $k_i,l_i(i=1,2,...,n+1)$ are not all zero. Since $y(t,\lambda)$ satisfies (1.2), we have

$$A_\lambda\Phi(a,\lambda)\begin{pmatrix}k_1\\ l_1\end{pmatrix}+B_\lambda\Phi(b,\lambda)\begin{pmatrix}k_{n+1}\\ l_{n+1}\end{pmatrix}=0. \tag{2.5}$$

From (1.3), we get

$$D_i\Phi(c_i+,\lambda) = -C_i\Phi(c_i-,\lambda). \tag{2.6}$$

When $i=1$, we can obtain

$$C_1\Phi(c_1-,\lambda)\begin{pmatrix} k_1 \\ l_1 \end{pmatrix} + D_1\Phi(c_1+,\lambda)\begin{pmatrix} k_2 \\ l_2 \end{pmatrix} = 0,$$

so

$$C_1\Phi(c_1-,\lambda)\begin{pmatrix} k_1-k_2 \\ l_1-l_2 \end{pmatrix} = 0.$$

It means that $k_1=k_2, l_1=l_2$. Using the same method we can get the following results

$$k_1=k_2=\ldots=k_{n+1}, l_1=l_2=\ldots=l_{n+1},$$

so we have

$$A_\lambda\Phi(a,\lambda)\begin{pmatrix} k_1 \\ l_1 \end{pmatrix} + B_\lambda\Phi(b,\lambda)\begin{pmatrix} k_1 \\ l_1 \end{pmatrix} = 0. \tag{2.7}$$

Since k_1 and l_1 are not all zero, then $\triangle(\lambda) = det[A_\lambda + B_\lambda\Phi(b,\lambda)] = 0$.

Let $\triangle(\lambda)=0$. Then (2.7) has non-trivial solution. Now we consider the next initial value problem

$$\begin{cases} -(py')'+qy=\lambda wy, & t\in J, \\ y(a,\lambda)=\lambda\alpha_2'-\alpha_2, \\ (py')(a,\lambda)=\lambda\alpha_1'-\alpha_1, \end{cases}$$

we have

$$y(t,\lambda) = (\lambda\alpha_2'-\alpha_2)\phi_{11}(t,\lambda) + (\lambda\alpha_1'-\alpha_1)\phi_{12}(t,\lambda), \quad t\in J.$$

Substituting $y(t,\lambda)$ into (1.2), we have

$$\begin{aligned} &(\lambda\alpha_1'-\alpha_1)y(a,\lambda) + (\lambda\alpha_1'-\alpha_1)(py')(a,\lambda) \\ =&(\lambda\alpha_1'-\alpha_1)(\lambda\alpha_2'-\alpha_2) - (\lambda\alpha_2'-\alpha_2)(\lambda\alpha_1'-\alpha_1) \\ =&0. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} &(\lambda\beta_1'+\beta_1)y(b,\lambda) - (\lambda\beta_2'+\beta_2)(py')(b,\lambda) \\ =&(\lambda\beta_1'+\beta_1)(\lambda\beta_2'+\beta_2) - (\lambda\beta_2'+\beta_2)(\lambda\beta_1'+\beta_1) \\ =&0. \end{aligned}$$

So $y(t,\lambda)$ satisfies (1.2). Recalling that the solution $y(t,\lambda)$ satisfies (1.3), it's means that $y(t,\lambda)$ is an eigenfunction of the SLP (1.1)∼(1.3) corresponding to eigenvalue λ. And (2.3) comes from a straightforward computation. □

Definition 2.2. The SLP (1.1)∼(1.3), or equivalently $(2.1),(1.2),(1.3)$ is said to be degenerate if in (2.2) either $\triangle(\lambda)\equiv 0$ for all $\lambda\in\mathbb{C}$ or $\triangle(\lambda)\neq 0$ for any $\lambda\in\mathbb{C}$.

In the derivation of our main results an important role is played by the "Continuity Principle" established in Kong et al. See [13].

3. Statement of the problem

In this section, we assume that there exists a partition of the interval J

$$\begin{cases} a = a_0 < a_1 < a_2 < \cdots < a_{2m_0} < a_{2m_0+1} = c_1-, \\ c_1+ = c_{1,0} < c_{1,1} < c_{1,2} < \cdots < c_{1,2m_1} < c_{1,2m_1+1} = c_2-, \\ \cdots, \\ c_{n-1}+ = c_{n-1,0} < c_{n-1,1} < c_{n-1,2} < \cdots < c_{n-1,2m_{n-1}} < c_{n-1,2m_{n-1}+1} = c_n-, \\ c_n+ = c_{n,0} < c_{n,1} < c_{n,2} < \cdots < c_{n,2m_n} < c_{n,2m_n+1} = b, \end{cases} \tag{3.1}$$

for some positive integers $m_0, m_1, \cdots, m_n$, when $r(t) = \frac{1}{p(t)} = 0$, such that

$$\begin{cases} \displaystyle\int_{a_{2k}}^{a_{2k+1}} w(t)dt \neq 0, k = 0, 1, \cdots, m_0, t \in (a_{2k}, a_{2k+1}), \\ \displaystyle\int_{c_{1,2i}}^{c_{1,2i+1}} w(t)dt \neq 0, i = 0, 1, \cdots, m_1, t \in (c_{1,2i}, c_{1,2i+1}), \\ \cdots, \\ \displaystyle\int_{c_{n,2z}}^{c_{n,2z+1}} w(t)dt \neq 0, z = 0, 1, \cdots, m_n, t \in (c_{n,2z}, c_{n,2z+1}), \end{cases} \tag{3.2}$$

and when $q(t) = w(t) = 0$, we have

$$\begin{cases} \displaystyle\int_{a_{2k+1}}^{a_{2k+2}} r(t)dt \neq 0, k = 0, 1, \cdots, m_0 - 1, t \in (a_{2k+1}, a_{2k+2}), \\ \displaystyle\int_{c_{1,2i+1}}^{c_{1,2i+2}} r(t)dt \neq 0, i = 0, 1, \cdots, m_1 - 1, t \in (c_{1,2i+1}, c_{1,2i+2}), \\ \cdots, \\ \displaystyle\int_{c_{n,2z+1}}^{c_{n,2z+2}} r(t)dt \neq 0, z = 0, 1, \cdots, m_n, t \in (c_{n,2z+1}, c_{n,2z+2}). \end{cases} \tag{3.3}$$

Let

$$\begin{cases} q_k = \displaystyle\int_{a_{2k}}^{a_{2k+1}} q(t)dt, k = 0, 1, \cdots, m_0, \\ w_k = \displaystyle\int_{a_{2k}}^{a_{2k+1}} w(t)dt, k = 0, 1, \cdots, m_0, \\ r_k = \displaystyle\int_{a_{2k+1}}^{a_{2k+2}} r(t)dt, k = 0, 1, \cdots, m_0 - 1, \end{cases}$$

$$\begin{cases} q_{1,i} = \displaystyle\int_{c_{1,2i}}^{c_{1,2i+1}} q(t)dt, i = 0, 1, \cdots, m_1, \\ w_{1,i} = \displaystyle\int_{c_{1,2i}}^{c_{1,2i+1}} w(t)dt, i = 0, 1, \cdots, m_1, \\ r_{1,i} = \displaystyle\int_{c_{1,2i+1}}^{c_{1,2i+2}} r(t)dt, i = 0, 1, \cdots, m_1 - 1, \end{cases}$$

$\cdots,$

$$\begin{cases} q_{n,z} = \int_{c_{n,2z}}^{c_{n,2z+1}} q(t)dt, z = 0, 1, \cdots, m_n, \\ w_{n,z} = \int_{c_{n,2z}}^{c_{n,2z+1}} w(t)dt, z = 0, 1, \cdots, m_n, \\ r_{n,z} = \int_{c_{n,2z+1}}^{c_{n,2z+2}} r(t)dt, z = 0, 1, \cdots, m_n - 1. \end{cases}$$

In the following Lemma and Theorem, we let (3.1)∼(3.3) always hold.

Lemma 3.1. *For each $\lambda \in \mathbb{C}$,*

$\Phi(t, \lambda) = [\phi_{ef}(t, \lambda)](t \in (a, c_1))$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Phi(a, \lambda) = I$*;*

$\Psi_i(t, \lambda) = [\psi_{i,ef}(t, \lambda)](t \in (c_i, c_{i+1}), c_{n+1} = b = c_{n,2m_n+1}, i = 1, 2, ..., n)$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Psi_i(c_i+, \lambda) = I$ *(here* $\Psi_i(c_i+, \lambda) = \Psi_i(c_{i,0}, \lambda) = \Phi(c_i+, \lambda)$*).*

So we have

(1)

$$\Phi(a_1, \lambda) = \begin{pmatrix} 1 & 0 \\ q_0 - \lambda w_0 & 1 \end{pmatrix}, \tag{3.4}$$

$$\Phi(a_3, \lambda) = \begin{pmatrix} 1 + (q_0 - \lambda w_0)r_0 & r_0 \\ \phi_{21}(a_3, \lambda) & 1 + (q_1 - \lambda w_1)r_0 \end{pmatrix}, \tag{3.5}$$

where

$$\phi_{21}(a_3, \lambda) = (q_0 - \lambda w_0) + (q_1 - \lambda w_1) + (q_0 - \lambda w_0)(q_1 - \lambda w_1)r_0.$$

In general, for $1 \le k \le m_0$*,*

$$\Phi(a_{2k+1}, \lambda) = \begin{pmatrix} 1 & r_{k-1} \\ q_k - \lambda w_k & 1 + (q_k - \lambda w_k)r_{k-1} \end{pmatrix} \Phi(a_{2k-1}, \lambda). \tag{3.6}$$

(2)

$$\Psi_i(c_{i,1}, \lambda) = \begin{pmatrix} 1 & 0 \\ q_{i,0} - \lambda w_{i,0} & 1 \end{pmatrix}, \tag{3.7}$$

$$\Psi_i(c_{i,3}, \lambda) = \begin{pmatrix} 1 + (q_{i,0} - \lambda w_{i,0})r_{i,0} & r_{i,0} \\ \psi_{i,21}(c_{i,3}, \lambda) & 1 + (q_{i,1} - \lambda w_{i,1})r_{i,0} \end{pmatrix}, \tag{3.8}$$

where

$$\psi_{i,21}(c_{i,3}, \lambda) = (q_{i,0} - \lambda w_{i,0}) + (q_{i,1} - \lambda w_{i,1}) + (q_{i,0} - \lambda w_{i,0})(q_{i,1} - \lambda w_{i,1})r_{i,0}.$$

In general, for $1 \le \kappa \le m_i(\kappa = i, j, ..., z)$*,*

$$\Psi_i(c_{i,2\kappa+1}, \lambda) = \begin{pmatrix} 1 & r_{i,\kappa-1} \\ q_{i,\kappa} - \lambda w_{i,\kappa} & 1 + (q_{i,\kappa} - \lambda w_{i,\kappa})r_{i,\kappa-1} \end{pmatrix} \Psi_i(c_{i,2\kappa-1}, \lambda). \tag{3.9}$$

Proof. We can see from the system (2.1) that u is constant on each subinterval where r identically zero and v is constant on each subinterval where both q and w are identically zero. The result follows from repeated applications of system (2.1). $\square$

Lemma 3.2. *For each* $\lambda \in \mathbb{C}$,

$\Phi(t,\lambda) = [\phi_{ef}(t,\lambda)](t \in (a, c_1))$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Phi(a,\lambda) = I$;

$\Psi_i(t,\lambda) = [\psi_{i,ef}(t,\lambda)](t \in (c_i, c_{i+1}), c_{n+1} = b, i = 1, 2, ..., n)$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Psi_i(c_i+,\lambda) = I$.

So we have

$$\Phi(b,\lambda) = \Psi_n(b,\lambda)G_n\Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda),$$

where

$$G_i = [g_{i,ef}]_{2\times 2}(i = 1, 2, ..., n; e, f = 1, 2).$$

Proof. From (1.3), we know that

$$C_i\Phi(c_i-,\lambda) + D_i\Phi(c_i+,\lambda) = 0,$$

so

$$\Phi(c_i+,\lambda) = -D_i^{-1}C_i\Phi(c_i-,\lambda) = G_i\Phi(c_i-,\lambda),$$

where

$$G_i = [g_{i,ef}]_{2\times 2}(i = 1, 2, ..., n; e, f = 1, 2).$$

When $i = 1$, $\Psi_1(c_1+,\lambda) = I$, combining Lemma 3.1

$$\Psi_1(t,\lambda) = \Phi(t,\lambda)[G_1\Phi(c_1-,\lambda)]^{-1}, \quad c_1+ \le t \le c_2-,$$

let $t = c_2-$, then

$$\begin{aligned}&\Psi_1(c_2-,\lambda) = \Phi(c_2-,\lambda)[G_1\Phi(c_1-,\lambda)]^{-1},\\&\Phi(c_2-,\lambda) = \Psi_1(c_2-,\lambda)G_1\Phi(c_1-,\lambda).\end{aligned}$$

When $i = 2$, $\Psi_2(c_2+,\lambda) = I$, we find that condition $\Phi(c_i+,\lambda) = -D_i^{-1}C_i\Phi(c_i-,\lambda)$ $= G_i\Phi(c_i-,\lambda)$ always holds, so

$$\Psi_2(t,\lambda) = \Phi(t,\lambda)[G_2\Phi(c_2-,\lambda)]^{-1}, \quad c_2+ \le t \le c_3-,$$

let $t = c_3-$, then

$$\begin{aligned}&\Psi_2(c_3-,\lambda) = \Phi(c_3-,\lambda)[G_2\Phi(c_2-,\lambda)]^{-1},\\&\Phi(c_3-,\lambda) = \Psi_2(c_3-,\lambda)G_2\Phi(c_2-,\lambda),\\&\cdots.\end{aligned}$$

By repeated application of the above process, we have

$$\Phi(b,\lambda) = \Psi_n(b,\lambda)G_n\Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda).$$

$\square$

Lemma 3.3. *For each* $\lambda \in \mathbb{C}$,

$\Phi(t,\lambda) = [\phi_{ef}(t,\lambda)](t \in (a,c_1))$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Phi(a,\lambda) = I$;

$\Psi_i(t,\lambda) = [\psi_{i,ef}(t,\lambda)](t \in (c_i,c_{i+1}), c_{n+1} = b, i = 1,2,...,n)$ *denotes the fundamental matrix of the system* (2.1) *determined by* $\Psi_i(c_i+,\lambda) = I$.

For $\Phi(b,\lambda)$, *we have the following result*

$$\begin{aligned}
&\phi_{11}(b,\lambda)\\
=&R\prod_{i=1}^{n} R_i G^* G^{**} \prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j})]\\
&+\phi'_{11}(b,\lambda),\\
&\phi_{12}(b,\lambda)\\
=&R\prod_{i=1}^{n} R_i G^* G^{**} \prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j})]\\
&+\phi'_{12}(b,\lambda),\\
&\phi_{21}(b,\lambda)\\
=&R\prod_{i=1}^{n} R_i G^* G^{**} \prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j})]\\
&+\phi'_{21}(b,\lambda),\\
&\phi_{22}(b,\lambda)\\
=&R\prod_{i=1}^{n} R_i G^* G^{**} \prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j})]\\
&+\phi'_{22}(b,\lambda),
\end{aligned}$$

where

$$\begin{aligned}
G^* =& g_{1,12}(q_{m_0}-\lambda w_{m_0})(q_{1,0}-\lambda w_{1,0}) + g_{1,11}(q_{1,0}-\lambda w_{1,0})\\
&+ g_{1,22}(q_{m_0}-\lambda w_{m_0}) + g_{1,21},\\
G^{**} =& \prod_{i=2}^{n}\{[g_{i,11}+g_{i,12}(q_{i-1,m_{i-1}}-\lambda w_{i-1,m_{i-1}})](q_{i,0}-\lambda w_{i,0})\\
&+[g_{i,21}+g_{i,22}(q_{i-1,m_{i-1}}-\lambda w_{i-1,m_{i-1}})]\},\\
R =& \prod_{k=0}^{m_0-1} r_k, R_i = \prod_{j=0}^{m_i-1} r_{i,j}, \phi'_{ef}(b,\lambda) = o(R\prod_{i=1}^{n} R_i).
\end{aligned}$$

Proof. From Lemma 3.1 we know that

$$\begin{aligned}
\Phi(c_1-,\lambda) =&\Phi(a_{2m_0+1},\lambda)\\
=&\begin{pmatrix} 1 & r_{m_0-1} \\ q_{m_0}-\lambda w_{m_0} & 1+(q_{m_0}-\lambda w_{m_0})r_{m_0-1}\end{pmatrix}\Phi(a_{2m_0-1},\lambda)
\end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & r_{m_0-1} \\ q_{m_0} - \lambda w_{m_0} & 1 + (q_{m_0} - \lambda w_{m_0}) r_{m_0-1} \end{pmatrix} \cdot \Phi(a_{2m_0-3}, \lambda) \\
&\quad \times \begin{pmatrix} 1 & r_{m_0-2} \\ q_{m_0-1} - \lambda w_{m_0-1} & 1 + (q_{m_0-1} - \lambda w_{m_0-1}) r_{m_0-2} \end{pmatrix} \\
&= \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \Phi(a_{2m_0-3}, \lambda),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_{11} =& 1 + r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1}) \\
=& r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1}) + o(r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})), \\
\theta_{12} =& r_{m_0-2} + r_{m_0-1} + r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1}) \\
=& r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1}) + o(r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})), \\
\theta_{21} =& (q_{m_0-1} - \lambda w_{m_0-1}) + (q_{m_0} - \lambda w_{m_0}) + r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})(q_{m_0} - \lambda w_{m_0}) \\
=& r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})(q_{m_0} - \lambda w_{m_0}) \\
&+ o(r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})(q_{m_0} - \lambda w_{m_0})), \\
\theta_{22} =& r_{m_0-2}(q_{m_0} - \lambda w_{m_0}) + 1 + r_{m_0-2}(q_{m_0-1} - \lambda w_{m_0-1}) + r_{m_0-1}(q_{m_0} - \lambda w_{m_0}) \\
&+ r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})(q_{m_0} - \lambda w_{m_0}) \\
=& r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1}) \\
&\times (q_{m_0} - \lambda w_{m_0}) + o(r_{m_0-2} r_{m_0-1}(q_{m_0-1} - \lambda w_{m_0-1})(q_{m_0} - \lambda w_{m_0})),
\end{aligned}
$$

and

$$
\Phi(a_{2m_0-3}, \lambda) = \begin{pmatrix} 1 & r_{m_0-3} \\ q_{m_0-2} - \lambda w_{m_0-2} & 1 + (q_{m_0-2} - \lambda w_{m_0-2}) r_{m_0-3} \end{pmatrix} \Phi(a_{2m_0-5}, \lambda),
$$

so we have

$$
\begin{aligned}
\Phi(c_1-, \lambda) &= \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \Phi(a_{2m_0-3}, \lambda) \\
&= \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} 1 & r_{m_0-3} \\ q_{m_0-2} - \lambda w_{m_0-2} & 1 + (q_{m_0-2} - \lambda w_{m_0-2}) r_{m_0-3} \end{pmatrix} \\
&\quad \times \Phi(a_{2m_0-5}, \lambda) \\
&= \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \Phi(a_{2m_0-5}, \lambda),
\end{aligned}
$$

where

$$
\eta_{11} = \theta_{11} + (q_{m_0-2} - \lambda w_{m_0-2}) \theta_{12}
$$

$$
\begin{aligned}
&= r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1}) \\
&\quad + o(r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})), \\
\eta_{12} &= r_{m_0-3}\theta_{11} + (1+(q_{m_0-2}-\lambda w_{m_0-2}) r_{m_0-3})\theta_{12} \\
&= r_{m_0-3} r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1}) \\
&\quad + o(r_{m_0-3} r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})), \\
\eta_{21} &= \theta_{21} + (q_{m_0-2}-\lambda w_{m_0-2})\theta_{22} \\
&= r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})(q_{m_0}-\lambda w_{m_0}) \\
&\quad + o(r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})(q_{m_0}-\lambda w_{m_0})), \\
\eta_{22} &= r_{m_0-3}\theta_{21} + (1+(q_{m_0-2}-\lambda w_{m_0-2}) r_{m_0-3})\theta_{22} \\
&= r_{m_0-3} r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})(q_{m_0}-\lambda w_{m_0}) \\
&\quad + o(r_{m_0-3} r_{m_0-2} r_{m_0-1}(q_{m_0-2}-\lambda w_{m_0-2})(q_{m_0-1}-\lambda w_{m_0-1})(q_{m_0}-\lambda w_{m_0})), \\
&\cdots .
\end{aligned}
$$

By repeated application of the above method, finally we can get

$$
\Phi(c_1-,\lambda) = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \Phi(a_1,\lambda),
$$

where

$$
\begin{aligned}
\xi_{11} &= \prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0-1} (q_k-\lambda w_k) + o(\prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0-1} (q_k-\lambda w_k)), \\
\xi_{12} &= \prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0-1} (q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0-1} (q_k-\lambda w_k)), \\
\xi_{21} &= \prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0} (q_k-\lambda w_k) + o(\prod_{k=1}^{m_0-1} r_k \prod_{k=1}^{m_0} (q_k-\lambda w_k)), \\
\xi_{22} &= \prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0} (q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0} (q_k-\lambda w_k)).
\end{aligned}
$$

And

$$
\Phi(a_1,\lambda) = \begin{pmatrix} 1 & 0 \\ q_0-\lambda w_0 & 1 \end{pmatrix},
$$

so

$$
\begin{aligned}
\Phi(c_1-,\lambda) &= \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \Phi(a_1,\lambda) \\
&= \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q_0-\lambda w_0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \xi_{11}+\xi_{12}(q_0-\lambda w_0) & \xi_{12} \\ \xi_{21}+\xi_{22}(q_0-\lambda w_0) & \xi_{22} \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} \phi_{11}(c_1-,\lambda) & \phi_{12}(c_1-,\lambda) \\ \phi_{21}(c_1-,\lambda) & \phi_{22}(c_1-,\lambda) \end{pmatrix}.$$

It means that

$$\begin{aligned} \phi_{11}(c_1-,\lambda) &= \prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0-1}(q_k-\lambda w_k)), \\ \phi_{12}(c_1-,\lambda) &= \prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0-1}(q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0-1}(q_k-\lambda w_k)), \\ \phi_{21}(c_1-,\lambda) &= \prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0}(q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=0}^{m_0}(q_k-\lambda w_k)), \\ \phi_{22}(c_1-,\lambda) &= \prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0}(q_k-\lambda w_k) + o(\prod_{k=0}^{m_0-1} r_k \prod_{k=1}^{m_0}(q_k-\lambda w_k)), \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \psi_{1,11}(c_2-,\lambda) &= \prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=0}^{m_1-1}(q_{1,i}-\lambda w_{1,i}) + o(\prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=0}^{m_1-1}(q_{1,i}-\lambda w_{1,i})), \\ \psi_{1,12}(c_2-,\lambda) &= \prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i}) + o(\prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})), \\ \psi_{1,21}(c_2-,\lambda) &= \prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=0}^{m_1}(q_{1,i}-\lambda w_{1,i}) + o(\prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=0}^{m_1}(q_{1,i}-\lambda w_{1,i})), \\ \psi_{1,22}(c_2-,\lambda) &= \prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=1}^{m_1}(q_{1,i}-\lambda w_{1,i}) + o(\prod_{i=0}^{m_1-1} r_{1,i} \prod_{i=1}^{m_1}(q_{1,i}-\lambda w_{1,i})). \end{aligned} \tag{3.11}$$

By repeated application of the above method, then

$$\begin{aligned} \psi_{i,11}(c_{i+1}-,\lambda) &= \prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=0}^{m_i-1}(q_{i,j}-\lambda w_{i,j}) + o(\prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=0}^{m_i-1}(q_{i,j}-\lambda w_{i,j})), \\ \psi_{i,12}(c_{i+1}-,\lambda) &= \prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j}) + o(\prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j})), \\ \psi_{i,21}(c_{i+1}-,\lambda) &= \prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=0}^{m_i}(q_{i,j}-\lambda w_{i,j}) + o(\prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=0}^{m_i}(q_{i,j}-\lambda w_{i,j})), \\ \psi_{i,22}(c_{i+1}-,\lambda) &= \prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j}) + o(\prod_{j=0}^{m_i-1} r_{i,j} \prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j})), \\ & i = 2,3,\cdots,n. \end{aligned} \tag{3.12}$$

From Lemma 3.2, we have

$$\Phi(c_n-,\lambda) = \Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda).$$

In combination with (3.10)∼(3.12), and

$$\Phi(c_2-,\lambda)=\Psi_1(c_2-,\lambda)G_1\Phi(c_1-,\lambda),$$

we can obtain

$$\phi_{11}(c_2-,\lambda)=RR_1G^*\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})+\phi'_{11}(c_2-,\lambda),$$
$$\phi_{12}(c_2-,\lambda)=RR_1G^*\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})+\phi'_{12}(c_2-,\lambda),$$
$$\phi_{21}(c_2-,\lambda)=RR_1G^*\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1}(q_{1,i}-\lambda w_{1,i})+\phi'_{21}(c_2-,\lambda),$$
$$\phi_{22}(c_2-,\lambda)=RR_1G^*\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1}(q_{1,i}-\lambda w_{1,i})+\phi'_{22}(c_2-,\lambda),$$

where

$$\begin{aligned}G^*=&\,g_{1,12}\,(q_{m_0}-\lambda w_{m_0})(q_{1,0}-\lambda w_{1,0})+g_{1,11}\,(q_{1,0}-\lambda w_{1,0})\\&+g_{1,22}\,(q_{m_0}-\lambda w_{m_0})+g_{1,21}\,,\\R=&\prod_{k=0}^{m_0-1}r_k,R_1=\prod_{i=0}^{m_1-1}r_{1,i}\,,\phi'_{ef}(c_2-,\lambda)=o(RR_1).\end{aligned}$$

Similarly, we know that

$$\Phi(c_3-,\lambda)=\Psi_2(c_3-,\lambda)G_2\Phi(c_2-,\lambda),$$

so

$$\begin{aligned}&\phi_{11}(c_3-,\lambda)\\=&RR_1R_2G^*G^{2*}\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{j=1}^{m_2-1}(q_{2,j}-\lambda w_{2,j})\\&+\phi'_{11}(c_3-,\lambda),\\&\phi_{12}(c_3-,\lambda)\\=&RR_1R_2G^*G^{2*}\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{j=1}^{m_2-1}(q_{2,j}-\lambda w_{2,j})\\&+\phi'_{12}(c_3-,\lambda),\\&\phi_{21}(c_3-,\lambda)\\=&RR_1R_2G^*G^{2*}\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{j=1}^{m_2}(q_{2,j}-\lambda w_{2,j})\\&+\phi'_{21}(c_3-,\lambda),\\&\phi_{22}(c_3-,\lambda)\end{aligned}$$

$$=RR_1R_2G^*G^{2*}\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{j=1}^{m_2}(q_{2,j}-\lambda w_{2,j})$$
$$+\phi'_{22}(c_3-,\lambda),$$

where

$$\begin{aligned}
G^* &= g_{1,12}(q_{m_0}-\lambda w_{m_0})(q_{1,0}-\lambda w_{1,0})+g_{1,11}(q_{1,0}-\lambda w_{1,0})\\
&\quad+g_{1,22}(q_{m_0}-\lambda w_{m_0})+g_{1,21},\\
G^{2*} &= [g_{2,11}+g_{2,12}(q_{1,m_1}-\lambda w_{1,m_1})](q_{2,0}-\lambda w_{2,0})\\
&\quad+[g_{2,21}+g_{2,22}(q_{1,m_1}-\lambda w_{1,m_1})],\\
R &= \prod_{k=0}^{m_0-1}r_k,\ R_1=\prod_{i=0}^{m_1-1}r_{1,i},\ R_2=\prod_{j=0}^{m_2-1}r_{2,j},\ \phi'_{ef}(c_3-,\lambda)=o(RR_1R_2),\\
&\cdots.
\end{aligned}$$

Similarly, because

$$\Phi(b,\lambda)=\Psi_n(b,\lambda)G_n\Psi_{n-1}(c_n-,\lambda)G_{n-1}\Psi_{n-2}(c_{n-1}-,\lambda)\cdots G_1\Phi(c_1-,\lambda),$$

we have

$$\begin{aligned}
\phi_{11}(b,\lambda) &= R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-i}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j})]\\
&\quad+\phi'_{11}(b,\lambda),\\
\phi_{12}(b,\lambda) &= R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-i}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i-1}(q_{i,j}-\lambda w_{i,j})]\\
&\quad+\phi'_{12}(b,\lambda),\\
\phi_{21}(b,\lambda) &= R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=0}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-i}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j})]\\
&\quad+\phi'_{21}(b,\lambda),\\
\phi_{22}(b,\lambda) &= R\prod_{i=1}^{n}R_iG^*G^{**}\prod_{k=1}^{m_0-1}(q_k-\lambda w_k)\prod_{i=1}^{m_1-1}(q_{1,i}-\lambda w_{1,i})\prod_{i=2}^{n}[\prod_{j=1}^{m_i}(q_{i,j}-\lambda w_{i,j})]\\
&\quad+\phi'_{22}(b,\lambda),
\end{aligned}$$

where

$$\begin{aligned}
G^* &= g_{1,12}(q_{m_0}-\lambda w_{m_0})(q_{1,0}-\lambda w_{1,0})+g_{1,11}(q_{1,0}-\lambda w_{1,0})\\
&\quad+g_{1,22}(q_{m_0}-\lambda w_{m_0})+g_{1,21},\\
G^{**} &= \prod_{i=2}^{n}\{[g_{i,11}+g_{i,12}(q_{i-1,m_{i-1}}-\lambda w_{i-1,m_{i-1}})](q_{i,0}-\lambda w_{i,0})\\
&\quad+[g_{i,21}+g_{i,22}(q_{i-1,m_{i-1}}-\lambda w_{i-1,m_{i-1}})]\},
\end{aligned}$$

$$R=\prod_{k=0}^{m_0-1} r_k, R_i=\prod_{j=0}^{m_i-1} r_{i,j}\,, \phi'_{ef}(b,\lambda)=o(R\prod_{i=1}^{n} R_i).$$

Therefore, the conclusion is proved. □

Theorem 3.1. *Let* $m_i\in\mathbb{N}(i=0,1,\cdots,n)$, $g_{1,12}\,g_{i,12}\neq 0, i=2,3,\cdots,n,$ *and* $H(\lambda)=(h_{ij}(\lambda))_{2\times 2}$ *be defined as in Lemma* 2.1. *Then*

(1) *If* $h_{21}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+2n+1)$ *eigenvalues.*

(2) *If* $h_{21}(\lambda)=0, h_{11}(\lambda)w_0+\prod_{i=2}^{n} h_{22}(\lambda)w_{i,m_i}\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+2n)$ *eigenvalues.*

(3) *If* $h_{21}(\lambda)=h_{11}(\lambda)=h_{22}(\lambda)=0, h_{12}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+n+1)$ *eigenvalues.*

(4) *If none of the above conditions holds, then the SLP* (1.1)∼(1.3) *either has k eigenvalues,* $k\in\{1,2,\cdots,m_0+m_1+\cdots+m_n+n\}$ *or is degenerate.*

Proof. From Lemma 2.1 we know

$$\triangle(\lambda)=h_{11}(\lambda)\phi_{11}(b,\lambda)+h_{12}(\lambda)\phi_{12}(b,\lambda)+h_{21}(\lambda)\phi_{21}(b,\lambda)+h_{22}(\lambda)\phi_{22}(b,\lambda),$$

and observe that from the Lemma 3.3 the degree of λ of $\phi_{11}(b,\lambda),\phi_{12}(b,\lambda),\phi_{21}(b,\lambda)$, $\phi_{22}(b,\lambda)$ in $\triangle(\lambda)$ are $m_0+m_1+\cdots+m_n+n, m_0+m_1+\cdots+m_n+n-1, m_0+m_1+\cdots+m_n+2n-1, m_0+m_1+\cdots+m_n+2n-2$, respectively. Thus when $h_{21}(\lambda)\neq 0$, we can deduce from (2.2) that the characteristic function $\triangle(\lambda)$ is also a polynomial function of λ and with the degree is $m_0+m_1+\cdots+m_n+2n+1$. Hence from Fundamental Theorem of Algebra, we know that $\triangle(\lambda)$ has exactly $m_0+m_1+\cdots+m_n+2n+1$ roots, i.e. SLP (1.1)∼(1.3) has exactly $m_0+m_1+\cdots+m_n+2n+1$ eigenvalues. Then we complete the proof of case (1), and the other cases can be proved in the same way. □

Theorem 3.2. *Let* $m_i\in\mathbb{N}(i=0,1,\cdots,n)$, $g_{1,12}\,g_{i,12}=0, i=2,3,\cdots,n,$ *but* $g_{1,12}\prod_{i=2}^{n}(g_{i,11}\,w_{i,0}+g_{i,22}\,w_{i-1,m_{i-1}})\neq 0$. *Then*

(1) *If* $h_{21}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+\cdots+m_n+n+2)$ *eigenvalues.*

(2) *If* $h_{21}(\lambda)=0, h_{11}(\lambda)w_0+\prod_{i=2}^{n} h_{22}(\lambda)w_{i,m_i}\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+1)$ *eigenvalues.*

(3) *If* $h_{21}(\lambda)=h_{11}(\lambda)=h_{22}(\lambda)=0, h_{12}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+2)$ *eigenvalues.*

(4) *If none of the above conditions holds, then the SLP* (1.1)∼(1.3) *either has k eigenvalues,* $k\in\{1,2,\cdots,m_0+m_1+\cdots+m_n+1\}$ *or is degenerate.*

Proof. The proof is similar to Theorem 3.1. □

Theorem 3.3. *Let* $m_i\in\mathbb{N}(i=0,1,\cdots,n)$, $g_{1,12}=0$, *but* $(g_{1,11}\,w_{1,0}+g_{1,22}\,w_{m_0})\times g_{i,12}\neq 0, i=2,3,\cdots,n$. *Then*

(1) *If* $h_{21}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+\cdots+m_n+2n)$ *eigenvalues.*

(2) *If* $h_{21}(\lambda)=0, h_{11}(\lambda)w_0+\prod_{i=2}^{n} h_{22}(\lambda)w_{i,m_i}\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+2n-1)$ *eigenvalues.*

(3) *If* $h_{21}(\lambda)=h_{11}(\lambda)=h_{22}(\lambda)=0, h_{12}(\lambda)\neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0+m_1+m_2+\cdots+m_n+n)$ *eigenvalues.*

(4) *If none of the above conditions holds, then the SLP* (1.1)∼(1.3) *either has k eigenvalues,* $k\in\{1,2,\cdots,m_0+m_1+\cdots+m_n+n-1\}$ *or is degenerate.*

Proof. The proof is similar to Theorem 3.1. □

Theorem 3.4. *Let* $m_i \in \mathbb{N}(i = 0, 1, 2, \cdots, n)$, $g_{1,12} = (g_{1,11}\, w_{1,0} + g_{1,22}\, w_{m_0})g_{i,12} = 0$, *but* $(g_{1,11}\, w_{1,0} + g_{1,22}\, w_{m_0})(g_{i,11}\, w_{i,0} + g_{i,22}\, w_{i-1,m_{i-1}}) \neq 0, i = 2, 3, \cdots, n$. *Then*

(1) *If* $h_{21}(\lambda) \neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0 + m_1 + \cdots + m_n + n + 1)$ *eigenvalues.*

(2) *If* $h_{21}(\lambda) = 0, h_{11}w_0 + \prod_{i=2}^{n} h_{22}(\lambda)w_{i,m_i} \neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0 + m_1 + m_2 + \cdots + m_n + n)$ *eigenvalues.*

(3) *If* $h_{21}(\lambda) = h_{11}(\lambda) = h_{22}(\lambda) = 0, h_{12}(\lambda) \neq 0$, *then the SLP* (1.1)∼(1.3) *has exactly* $(m_0 + m_1 + m_2 + \cdots + m_n + 1)$ *eigenvalues.*

(4) *If none of the above conditions holds, then the SLP* (1.1)∼(1.3) *either has* k *eigenvalues, for* $k \in \{1, 2, \cdots, m_0 + m_1 + \cdots + m_n\}$ *or is degenerate.*

Proof. The proof is similar to Theorem 3.1. □

4. Main result

Theorem 4.1. *Given any* γ *disjoint open sets* $N_l, N_l \in \mathbb{C}$ *and any* γ *integers* $n_l (l = 1, 2, \ldots, \gamma)$, *there exists an SLP* (1.1)∼(1.3) *with exactly* $n_l + 2$ *eigenvalues in* N_l.

Proof. By constructing the SLP (1.1)∼(1.3), we assume that (1.4) and (3.1)∼(3.3) hold, $g_{1,12}\, g_{i,12} \neq 0$, $a_{21} = a_{22} = b_{11} = b_{12} = 0$, and $a_{11} = \lambda\alpha_1' - \alpha_1, a_{12} = -\lambda\alpha_2' + \alpha_2, b_{21} = \lambda\beta_1' + \beta_1, b_{22} = -\lambda\beta_2' - \beta_2$. Let $m_0 + m_1 + \cdots + m_n + n = \sum_{l=0}^{\gamma} n_l$. Then by Lemma 3.3 the characteristic function defined by equation (2.3), $\triangle(\lambda) = h_{11}(\lambda)\phi_{11}(b, \lambda) + h_{12}(\lambda)\phi_{12}(b, \lambda) + h_{21}(\lambda)\phi_{21}(b, \lambda) + h_{22}(\lambda)\phi_{22}(b, \lambda)$. Because the calculation of $\triangle(\lambda)$ is rather tedious, it is omitted here. Then it follows from Rouche's theorem that the $\triangle(\lambda)$ has exactly $n_l + 2$ roots in N_l. □

5. A case study

In order to demonstrate the analysis results we have obtained, we consider the following SLP with three transmission conditions and spectral parameters in the boundary conditions:

$$
\begin{cases}
-(py')' + qy = \lambda wy, \quad t \in J = (-6, -3) \cup (-3, 0) \cup (0, 3) \cup (3, 9), \\
\lambda y(-6) + (py')(-6) = 0, \\
3y(9) + (\lambda - 1)(py')(9) = 0, \\
-2(py')(-3-) + y(-3+) = 0, \\
y(-3-) + (py')(-3+) = 0, \\
-(py')(0-) + y(0+) = 0, \\
2y(0-) + (py')(0+) = 0, \\
2(py')(3-) + y(3+) = 0, \\
-y(3-) + (py')(3+) = 0.
\end{cases}
\tag{5.1}
$$

Let $n = 2$, we choose $m_0 = 1, m_1 = 1, m_2 = 2, m_3 = 2$ and suppose p, q, w are piecewise polynomial functions defined as follows:

$$p(t)=\begin{cases}\infty, & t\in(-6,-5),\\ 1, & t\in(-5,-4),\\ \infty, & t\in(-4,-3),\\ \infty, & t\in(-3,-2),\\ \frac{1}{2}, & t\in(-2,-1),\\ \infty, & t\in(-1,0),\\ \infty, & t\in(0,1),\\ 1, & t\in(1,2),\\ \infty, & t\in(2,3),\\ 1, & t\in(3,4),\\ \infty, & t\in(4,5),\\ \infty, & t\in(5,6),\\ \frac{1}{2}, & t\in(6,7),\\ \infty, & t\in(7,8),\\ 1, & t\in(8,9);\end{cases}\quad q(t)=\begin{cases}1, & t\in(-6,-5),\\ 0, & t\in(-5,-4),\\ 1, & t\in(-4,-3),\\ 1, & t\in(-3,-2),\\ 2, & t\in(-2,-1),\\ 3, & t\in(-1,0),\\ 1, & t\in(0,1),\\ 1, & t\in(1,2),\\ 1, & t\in(2,3),\\ 0, & t\in(3,4),\\ 1, & t\in(4,5),\\ 2, & t\in(5,6),\\ 0, & t\in(6,7),\\ 1, & t\in(7,8),\\ 0, & t\in(8,9);\end{cases}\quad w(t)=\begin{cases}0, & t\in(-6,-5),\\ 0, & t\in(-5,-4),\\ 1, & t\in(-4,-3),\\ 3, & t\in(-3,-2),\\ 0, & t\in(-2,-1),\\ 1, & t\in(-1,0),\\ 1, & t\in(0,1),\\ \frac{1}{2}, & t\in(1,2),\\ 1, & t\in(2,3),\\ 0, & t\in(3,4),\\ 1, & t\in(4,5),\\ 1, & t\in(5,6),\\ 0, & t\in(6,7),\\ 2, & t\in(7,8),\\ 0, & t\in(8,9).\end{cases}\tag{5.2}$$

From the SLP (5.1), we have

$$A_\lambda=\begin{pmatrix}1 & -\lambda\\ 0 & 0\end{pmatrix},\ B_\lambda=\begin{pmatrix}0 & 0\\ 3 & \lambda-1\end{pmatrix},$$

$$C_1=\begin{pmatrix}0 & -2\\ 1 & 0\end{pmatrix},\ D_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ C_2=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix},$$

$$D_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ C_3=\begin{pmatrix}0 & 2\\ -1 & 0\end{pmatrix},\ D_3=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and

$det(C_1)=det(C_2)=det(C_3)=2>0,\ det(D_1)=det(D_2)=det(D_3)=1>0,$

$$G_1=-D_1^{-1}C_1=\begin{pmatrix}0 & 2\\ -1 & 0\end{pmatrix},\ G_2=-D_2^{-1}C_2=\begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix},\ G_3=-D_3^{-1}C_3=\begin{pmatrix}0 & -2\\ 1 & 0\end{pmatrix},$$

$g_{1,12}=2\neq 0,\quad g_{2,12}=1\neq 0.$

We can deduce that the the characteristic function

$$\triangle(\lambda)=-3\lambda^8-22\lambda^7+43\lambda^6-15\lambda^5-126\lambda^4+138\lambda^3-63\lambda^2+13\lambda-1,$$

so the SLP (5.1) has exactly $m_0+m_1+m_2+m_3+n=8$ eigenvalues

$$\lambda_1=-8.9338,\ \lambda_2=-1.6971,\ \lambda_3=0.2107-0.0438i,$$

$\lambda_4 = 0.2107 + 0.0438i,\ \lambda_5 = 0.3401 - 0.2543i,\ \lambda_6 = 0.3401 + 0.2543i,$
$\lambda_7 = 1.0979 - 1.1932i,\ \lambda_8 = 1.0979 + 1.1932i.$

6. Conclusion

By using the construction method of discontinuous function solution, it is concluded that the the finite spectrum of SLP with n transmission conditions and spectral parameters in the boundary conditions has at most $m_0 + m_1 + \cdots + m_n + 2n + 1$ eigenvalues. In addition, we show that these $m_0+m_1+\cdots+m_n+2n+1$ eigenvalues can be distributed arbitrarily throughout the complex plane in the non-self-adjoint case and anywhere along the real line in the self-adjoint case. Finally, we give a specific example to verify the accuracy of this conclusion.

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