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GENERAL CONFORMABLE FRACTIONAL DOUBLE LAPLACE-SUMUDU TRANSFORM AND ITS APPLICATION

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Abstract A new deformation of the Laplace-Sumudu transform that called general fractional conformable double Laplace-Sumudu transform (FCDLST) has been introduced. Its excellent properties are proved, then, fractional partial differential equations is solved by using the proposed transform. Besides, illustrative examples are provided to demonstrate the validity and applicability of the presented method.

Keywords Fractional conformable derivatives, conformable Laplace transform, general fractional conformable double Laplace-Sumudu transform, fractional partial differential equations.

MSC(2010) 35A20, 35A22.

1. Introduction

In the last few decades, fractional partial differential equations (FPDEs) have been modeled many applications in sciences and engineering, such as mechanics, applied mathematical, physics, etc. [1,3,4,8,10,15], in the meantime, all sorts of definitions of fractional derivatives have been reported, such as Riemann-Liouville, Caputo, Hadamard and so on. These types of fractional derivatives do not obey chain rule, product, and quotient rule of two functions, these disadvantages complicate scientific applications or calculations. In 2014, Khalil et. al [15] proposed a new type of derivative called the conformable fractional derivative (CFD) which has all properties of excellent classical derivatives.

Recently, various analytic methods are proposed to solve fractional partial differential equations(FPDEs), such as conformable fractional Sumudu transform method [8], double integral transform (Laplace-Sumudu transform) method [3–5], homotopy perturbation sumudu transform method [19], fractional natural adomian decomposition method [7], Exponential rational function method [14], conformable Laplace transform method (CLT) [13], conformable double Laplace transform methods(CDLTM) [6, 10], etc. These integrals transform dealt with some components of them, definitions, and theorem, besides, some researchers addressed these transforms combining them with other method such as variational iteration method, differential transform approach, Adomian decomposition method and Homotopy per-

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turbation technique [2,12,17,18,20] to solve fractional partial differential equations. In [11], Fractional partial differential equations were determined by novel fractional double Laplace-Sumudu integral transform, which is the first paper studied the fractional partial differential equations by using fractional double Laplace-Sumudu integral transform method.

In this paper, we proposed a new coupling method which called general conformable fractional double Laplace-Sumudu transform, it combines conformable fractional Laplace transform with conformable fractional Sumudu transform to solve the fractional partial differential equations(FPDEs) with arbitrary order derivative, double Laplace-Sumudu transform has been studied in [3–5], in which reported integer order Laplace-Sumudu transform method to solve integer order partial differential equations, however, we extend them to the fractional derivative with arbitrary order fractional derivative, besides, a list of new excellent properties of this extension are given, Lastly, two examples for conformable fractional Laplace-Sumudu transform are presented to demonstrate the validity and applicability of the presented method.

2. Conformable fractional double Laplace-Sumudu transform

Conformable fractional derivatives were proposed in [1, 15]. in the following, we recall the definition of conformable fractional derivatives, and then generalize the conformable fractional Laplace transform [8,10] and conformable fractional Sumudu transform [8] to higher order, lastly, we modify and generalize the conformable double Laplace-Sumudu transform studied in [3–5], these definitions will be used later.

Definition 2.1. Let $f : (0, \infty) \to R$, the conformable fractional derivative of f order $\alpha > 0$ by Khalil et el. [15] is defined as:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f^{\lceil \alpha \rceil - 1}\left(x + \varepsilon x^{\lceil \alpha \rceil + \alpha}\right) - f^{\lceil \alpha \rceil - 1}(x)}{\varepsilon}, n-1 < \alpha \leq n, x > 0, \tag{2.1}$$

where $\lceil \alpha \rceil$ is the smallest integer number greater than or equal α and $n \in N$.

As a special case, if $0 < \alpha \leq 1$, then we have:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x) - f(x)}{\varepsilon}, x > 0.$$

Definition 2.2. Given a real-valued function $f(x,t)$ with two real variables $(x,t) \in R^{+} \times R^{+}$. Then, we have the following conformable partial fractional derivative (CPFD) of higher orders $\alpha, \beta \in (n, n+1]$ as follows:

$$D_x^{\alpha} f(x,t) = \lim_{\varepsilon \to 0} \frac{f^{\lceil \alpha \rceil - 1}\left(x + \varepsilon x^{\lceil \alpha \rceil - \alpha}, t\right) - f^{\lceil \alpha \rceil - 1}(x,t)}{\varepsilon}, n-1 < \alpha \leq n, x, t > 0, \tag{2.2}$$

$$D_t^{\beta} f(x,t) = \lim_{\varepsilon \to 0} \frac{f^{\lceil \beta \rceil - 1}\left(x, t + \varepsilon t^{\lceil \beta \rceil - \beta}\right) - f^{\lceil \alpha \rceil - 1}(x,t)}{\varepsilon}, n-1 < \beta \leq n, x, t > 0.$$

As a special case, if $0 < \alpha, \beta \leq 1$, equation (2.2) reduces to [18]:

$$D_x^\alpha f(x,t) = \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}, t\right) - f(x,t)}{\varepsilon}, 0 < \alpha \leq 1, x, t > 0,$$

$$D_t^\beta f(x,t) = \lim_{\varepsilon \to 0} \frac{f\left(x, t + \varepsilon t^{1-\beta}\right) - f(x,t)}{\varepsilon}, 0 < \beta \leq 1, x, t > 0.$$

Definition 2.3. Let $f(x,t), x \geq 0$ be a real value function, the conformable Laplace transform of $f(x,t)$ with respect to x is defined by:

$$\begin{aligned} L_x^{\lceil \alpha \rceil - \alpha} f(x,t) =& F_{\lceil \alpha \rceil - \alpha}(s,t) \\ =& \int_0^\infty e^{-s \frac{x^{\lceil \alpha \rceil - \alpha}}{\lceil \alpha \rceil - \alpha}} f(x,t) t^{\alpha - \lceil \alpha \rceil} dx, n-1 < \alpha \leq n, x > 0. \end{aligned} \tag{2.3}$$

As a special case, if $0 < \alpha \leq 1$, then we have:

$$L_x^\alpha (f(x,t)) = F_\alpha(s,t) = \int_0^\infty e^{-s \frac{x^\alpha}{\alpha}} f(x,t) x^{\alpha - 1} dx, x > 0.$$

Definition 2.4. Over the following set of functions:

$$A_\beta = \left\{ f(x,t) : \exists M, \tau_1, \tau_2 > 0, |f(x,t)| < M e^{-\frac{\left| f^{\lceil \beta \rceil - \beta} \right|}{u(\lceil \beta \rceil - \beta)}}, \right. \\ \left. \text{if } t^{\lceil \beta \rceil - \beta} \in (-1)^j \times [0, \infty), j = 1, 2 \right\},$$

then the conformable fractional Sumudu transform of $f(x,t)$ with respect to t can be generalized by:

$$S_\beta^t [f(x,t)] = F_\beta(x,u) = \frac{1}{u} \int_0^\infty e^{-\frac{t^{\lceil \beta \rceil - \beta}}{u(\lceil \beta \rceil - \beta)}} f(x,t) t^{\beta - \lceil \beta \rceil} dt, n-1 < \beta \leq n, t > 0,$$

then the conformable fractional Sumudu transform of $f(x,t)$ with respect to can be generalized by:

$$S_\beta^t [f(x,t)] = F_\beta(x,u) = \frac{1}{u} \int_0^\infty e^{-\frac{t^{\lceil \beta \rceil - \beta}}{u(\lceil \beta \rceil - \beta)}} f(x,t) t^{\beta - \lceil \beta \rceil} dt, n-1 < \beta \leq n, t > 0. \tag{2.4}$$

Provided the integral exists.

As a special case, if $0 < \beta \leq 1$, then we have:

$$S_\beta^t [f(x,t)] = F_\beta(x,u) = \frac{1}{u} \int_0^\infty e^{-\frac{t^\beta}{u\beta}} f(x,t) t^{\beta - 1} dt, t > 0.$$

Provided the integral exists.

Definition 2.5. The conformable fractional double Laplace-Sumudu transform of the function $f(x,t)$ of two variable $x > 0$ and $t > 0$ is denoted by:

$$\begin{aligned} & L_x^{\lceil \alpha \rceil - \alpha} S_t^{\lceil \beta \rceil - \beta} f(x,t) = U(s,u) \\ =& \frac{1}{u} \int_0^\infty \int_0^\infty e^{-s \frac{x^{\lceil \alpha \rceil - \alpha}}{\lceil \alpha \rceil - \alpha} - \frac{t^{\lceil \beta \rceil - \beta}}{(\lceil \beta \rceil - \beta) u}} f(x,t) x^{\alpha - \lceil \alpha \rceil} t^{\beta - \lceil \beta \rceil} dx dt, \\ & n-1 < \alpha, \beta \leq n, x, t > 0. \end{aligned} \tag{2.5}$$

As a special case, if $0 < \alpha, \beta \leq 1$, then we have:

$$
\begin{aligned}
L_x^\alpha S_t^\beta f(x,t) =& U(s,u) \\
=& \frac{1}{u}\int_0^\infty \int_0^\infty e^{-s\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta u}} f(x,t) x^{\alpha-1} t^{\beta-1} dxdt, x,t > 0.
\end{aligned}
$$

Definition 2.6. The conformable fractional inverse double Laplace-Sumudu transform denoted by:

$$
\begin{aligned}
L_x^{-1} S_t^{-1}[U(s,u)] =& f(x,t) \\
=& \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{s\lceil\alpha\rceil T - \alpha}{\lceil\alpha\rceil - \alpha}} ds \\
& \times \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{u} e^{\frac{t^{\lceil\beta\rceil - \beta}}{(\lceil\beta\rceil - \beta)u}} U(s,u) du, n-1 < \alpha, \beta \leq n.
\end{aligned} \tag{2.6}
$$

As a special case, if $0 < \alpha, \beta \leq 1$, then we have:

$$
\begin{aligned}
L_x^{-1} S_t^{-1}[U(s,u)] &= f(x,t) \\
&= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\frac{x^\alpha}{\alpha}} ds \frac{1}{2\pi i}\int_{\omega - i\infty}^{\omega + i\infty} \frac{1}{u} e^{\frac{t^\beta}{\beta u}} U(s,u) du.
\end{aligned} \tag{2.7}
$$

Theorem 2.1. *Let $f(x,t)$ be function that $f(x,t), L_x^\alpha f(x,t), \alpha \in (n-1, n]$ are continuous with respect to x, then, we have* [16]*:*

$$
\begin{aligned}
L_x^\alpha \left(D^{\lceil\alpha\rceil} f(x,t)\right) =& s^{\lceil\alpha\rceil} F_\alpha(s,t) - s^{\lceil\alpha\rceil - 1} f(0,t) - s^{\lceil\alpha\rceil - 2} F_\alpha(0,t) \\
& - \cdots - s^{(\lceil\alpha\rceil - 2)} F_\alpha(0,t) - s^{(\lceil\alpha\rceil - 1)} F_\alpha(0,t),
\end{aligned} \tag{2.8}
$$

Theorem 2.2. *Let $f(x,t)$ be a n times differentiable real value function with respect to t, then we have:*

$$
\begin{aligned}
S_t^\beta \left[D^{n\beta} f(x,t)\right] =& \frac{S_t^\beta [f(x,t)]}{u^n} - \frac{f(x,0)}{u^n} \\
=& \frac{L_t^\beta \left[f\left(x, t^\beta\right)^{\frac{1}{\beta}}\right]_{s=\frac{1}{u}} - \frac{f(x,0)}{u^n}}{u^{n+1}}, 0 < \beta \leq 1, n \in N.
\end{aligned} \tag{2.9}
$$

Where s, u are Laplace transform and Sumudu transform variables respectively.

Theorem 2.3. *Let $f(x,t)$ be a given real value function, $0 < \beta \leq 1$, then we have:*

$$
S_t^\beta [f(x,t)] = \frac{1}{u} L_t^\beta \left[f\left(x, t^\beta\right)^{\frac{1}{\beta}}\right]_{s=\frac{1}{u}}. \tag{2.10}
$$

Theorem 2.4. *Let $f(x,t)$ be a given real value function, $0 < \beta \leq 1$, then we have:*

$$
S_t^\beta [f(x,t)] = \frac{1}{u} F_\beta \left(x, \frac{1}{u}\right). \tag{2.11}
$$

Theorem 2.5. *Let $f(x,t)$ be a given real value function, $0 < \alpha \leq 1$, then we have* [8]*:*

$$
L_x^\alpha (f(x,t)) = F_\alpha(s,t) = L_x^\alpha \left(f\left((\alpha x)^{\frac{1}{\alpha}}, t\right)\right)(s).
$$

3. Some results and theorems of the general conformable double Laplace-Sumudu transform

Theorem 3.1. *If $0 < \alpha, \beta \leq 1$, then conformable Double Laplace-Sumudu transform for some certain functions are given by:*

(a) $L_x^\alpha S_t^\beta(c) = \dfrac{c}{s}, c$ *is a real constant.*

(b) $L_x^\alpha S_t^\beta\left[\left(\dfrac{x^\alpha}{\alpha}\right)^m\left(\dfrac{t^\beta}{\beta}\right)^n\right] = \dfrac{m!n!}{s^{m+1}}u^n, m, n \in N.$

(c) $L_x^\alpha S_t^\beta\left[e^{-\frac{x^\alpha}{\alpha}+\tau\frac{t^\beta}{\beta}}\right] = \dfrac{1}{(s-\lambda)(1-\tau u)}, s > \lambda, u > \dfrac{1}{\tau}.$

(d) $L_x^\alpha S_t^\beta\left[\sin\left(\dfrac{ax^\alpha}{\alpha}\right)\cos\left(\dfrac{bt^\beta}{\beta}\right)\right] = \dfrac{a}{(s^2+a^2)(1+b^2u^2)}, s > 0, u > \dfrac{1}{|b|}.$

(e) $L_x^\alpha S_t^\beta\left[\sinh\left(\dfrac{ax^\alpha}{\alpha}\right)\cosh\left(\dfrac{bt^\beta}{\beta}\right)\right] = \dfrac{a}{(s^2-a^2)(1-b^2u^2)}, s > |a|, u > \dfrac{1}{|b|}.$

(f) $L_x^\alpha S_t^\beta\left[x^p t^q\right] = \alpha^{\frac{p}{\alpha}}\beta^{\frac{q}{\beta}}\dfrac{u^{\frac{q}{\beta}}}{s^{1+\frac{p}{\alpha}}}\Gamma\left(1+\dfrac{p}{\alpha}\right)\Gamma\left(1+\dfrac{q}{\beta}\right), \dfrac{p}{\alpha}, \dfrac{q}{\beta} > -1.$

Proof. (a) Applying the Proposition 1 in [9] and Theorem 2.3 in [8], we get: $L_x^\alpha S_t^\beta(c) = L_x^\alpha(1)S_t^\beta(c) = \frac{1}{s}c = \frac{c}{s}$.

(b) Applying the Proposition 1 in [9] and Theorem 2.3 in [8], we get:

$$\begin{aligned} L_x^\alpha S_t^\beta\left[\left(\frac{x^\alpha}{\alpha}\right)^m\left(\frac{t^\beta}{\beta}\right)^n\right] &= L_x^\alpha\left(\frac{x^\alpha}{\alpha}\right)^m S_t^\beta\left(\frac{t^\beta}{\beta}\right)^n \\ &= \frac{m!}{s^{m+1}}n!u^n. \end{aligned}$$

Similarly, we can prove (c-f) easily. □

Theorem 3.2. *If function $f(x,t)$ is continuous in every finite internal $(0, X)$ and $(0, T)$ of exponential order $e^{c\frac{x^\alpha}{\alpha}+d\frac{t^\beta}{\beta}}$, then the conformable double Laplace-Sumudu transform of $f(x,t)$ exists for all s and $\frac{1}{u}$ provided $s > c, \frac{1}{u} > d, 0 < \alpha, \beta \leq 1$.*

Proof. From the definition 2.5, we have:

$$\begin{aligned} &\left|L_x^{\lceil\alpha\rceil-\alpha}S_t^{\lceil\beta\rceil-\beta}f(x,t)\right| \\ =&|U(s,u)| \\ =&\left|\frac{1}{u}\int_0^\infty\int_0^\infty e^{-s\frac{x^{\lceil\alpha\rceil-\alpha}}{\lceil\alpha\rceil-\alpha}-\frac{t^{\lceil\beta\rceil-\beta}}{(\lceil\beta\rceil-\beta)u}}f(x,t)x^{\alpha-\lceil\alpha\rceil}t^{\beta-\lceil\beta\rceil}dxdt\right| \\ \leq&M\int_0^\infty e^{-\frac{(s-c)x^{\lceil\alpha\rceil-\alpha}}{\lceil\alpha\rceil-\alpha}}x^{\alpha-\lceil\alpha\rceil}dx\int_0^\infty\frac{1}{u}e^{-\frac{\left(\frac{1}{u}-d\right))^{\lceil\beta\rceil-\beta}}{(\lceil\beta\rceil-\beta)}}t^{\beta-\lceil\beta\rceil}dt \\ \leq&\frac{M}{(s-c)(1-du)} \\ \to&0, s > c, \frac{1}{u} > d, x, t \to \infty. \end{aligned}$$

Thus, the proof is completed. □

Theorem 3.3. *Some Derivative Properties of the conformable double Laplace-Sumudu Transform.*

Let $f_1(x,t), f_2(x,t)$ be two functions that have the conformable double Laplace-Sumudu Transform. then, we have:

(a) $L_x^\alpha S_t^\beta\left(c_1 f_1(x,t)+c_2 f_2(x,t)\right) = c_1 L_x^\alpha S_t^\beta\left(f_1(x,t)\right)+c_2 L_x^\alpha S_t^\beta\left(f_2(x,t)\right), c_1, c_2$ *are real constants.*

(b) $L_x^\alpha S_t^\beta\left[e^{-a\frac{x^\alpha}{\alpha}-b\frac{t^\beta}{\beta}} f(x,t)\right] = \frac{1}{u}U\left(s+a, \frac{1}{u}+b\right), a, b$ *are real constants.*

(c) $L_x^\alpha S_t^\beta[f(\gamma x, \sigma t)] = \frac{1}{r}U\left(\frac{s}{\gamma^\alpha}, \frac{u}{\sigma^\beta}\right), r=\gamma^\alpha\sigma^\beta.$

(d) $(-1)^{m+n} L_x^\alpha S_t^\beta\left[\frac{x^{m\alpha}}{\alpha^m}\frac{t^{n\beta}}{\beta^n} f(x,t)\right] = \frac{1}{u}\frac{\partial^m F_\alpha(s)}{\partial s^m}\left[\frac{\partial^n}{\partial s^n}F_\beta(s)\right]_{s=\frac{1}{u}}.$

Proof. (a) By applying the definition 2.5 of conformable double Laplace-Sumudu transform, (a) can be proved easily.

(b) By using the Theorems 2.4 and 2.5, we get:

$$
\begin{aligned}
& L_x^\alpha S_t^\beta\left[e^{--\frac{x^\alpha}{\alpha}-b\frac{t^\beta}{\beta}} f(x,t)\right] \\
= & \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}-a\frac{x^\alpha}{\alpha}} x^{\alpha-1}\left(\frac{1}{u}\int_0^\infty e^{-\frac{t^\beta}{u\beta}-b\frac{t^\beta}{\beta}} t^{\beta-1} f(x,t)dt\right)dx \\
= & \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}-a^{x^\alpha}\frac{\alpha}{\alpha}} x^{\alpha-1}\frac{1}{u}\left(L_t^\beta\left(e^{-b\frac{t^\beta}{\beta}} f(x,t)\bigg|_{s=\frac{1}{u}}\right)\right)dx \\
= & \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}-a\frac{x^\alpha}{\alpha}} x^{\alpha-1}\frac{1}{u}L_t^\beta\left(e^{-bt} f\left(x,(\beta t)^{\frac{1}{\beta}}\right)\bigg|_{s=\frac{1}{u}}\right)dx \\
= & \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}-a\frac{x^\alpha}{\alpha}} x^{\alpha-1}\frac{1}{u}F_\beta\left(\frac{1}{u}+b\right)dx \\
= & \frac{1}{u}U\left(s+a, \frac{1}{u}+b\right).
\end{aligned}
$$

(c)

$$
\begin{aligned}
& L_x^\alpha S_t^\beta[f(\gamma x, \sigma t)] \\
= & \int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}\left(\frac{1}{u}\int_0^\infty e^{-\frac{t^\beta}{\beta u}} f(\gamma x, \sigma t)t^{\beta-1}dt\right)x^{\alpha-1}dx \\
\overset{\chi=\sigma t}{=} & \frac{1}{\sigma^\beta}\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}\left(\frac{1}{u}\int_0^\infty e^{-\frac{t^\beta}{\beta u\sigma^\beta}} f(\gamma x, \chi)\chi^{\beta-1}d\chi\right)x^{\alpha-1}dx \\
= & \frac{1}{\sigma^\beta}\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}\left(U\left(\gamma x, \frac{u}{\sigma^\beta}\right)\right)x^{\alpha-1}dx \\
\overset{\tau=\gamma x}{=} & \\
= & \frac{1}{\gamma^\alpha\sigma^\beta}\int_0^\infty e^{-s\frac{\tau^\alpha}{\alpha\gamma^\alpha}}\left(U\left(\tau, \frac{u}{\sigma^\beta}\right)\right)\tau^{\alpha-1}d\tau \\
= & \frac{1}{r}U\left(\frac{s}{\gamma^\alpha}, \frac{u}{\sigma^\beta}\right), \ r=\gamma^\alpha\sigma^\beta.
\end{aligned}
$$

(d) due to the order of differentiation and integration can be changed (convergence of improper integral), we get:

$$\begin{aligned}
&(-1)^{m+n} L_x^\alpha S_t^\beta \left[\frac{x^{m\alpha}}{\alpha^m} \frac{t^{n\beta}}{\beta^n} f(x,t) \right] \\
=& \frac{1}{u} \frac{\partial^m F_\alpha(s)}{\partial s^m} \left[\frac{\partial^n}{\partial s^n} F_\beta(s) \right]_{s=\frac{1}{u}} \\
=& \int_0^\infty \frac{\partial^m}{\partial s^m} e^{-s\frac{x^\alpha}{\alpha}} \left[\frac{1}{u} \frac{\partial^n}{\partial s^n} F_\beta(s) \right]_{s=\frac{1}{u}} x^{\alpha-1} dx.
\end{aligned}$$

Then differentiating with respect to two times and using Theorems 2.4 and 2.5,we get the desired results. □

Theorem 3.4. *(The Convolution Theorem for Conformable fractional Double Laplace-SumuduTransform). If* $L_x^\alpha S_t^\beta [f_1(x,t)] = U_1(s,u)$, $L_x^\alpha S_t^\beta [f_2(x,t)] = U_2(s,u)$ *exist, then, we have:*

$$L_x^\alpha S_t^\beta [f_1(x,t) * f_2(x,t)] = uU_1(s,u)U_2(s,u).$$

Where $f_1(x,t) * f_2(x,t)$ *is the convolution of the functions* $f_1(x,t), f_2(x,t)$.

Proof. By using the definition 2.5 and Theorem 6 in [4] and using Heaviside unit function, we have,

$$\begin{aligned}
&L_x^\alpha S_t^\beta [f_1(x,t) ** f_2(x,t)] \\
=& \frac{1}{u} \int_0^\infty \int_0^\infty e^{-s\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta u}} (f_1(x,t) ** f_2(x,t))\, x^{\alpha-1} t^{\beta-1} dxdt \\
=& \frac{1}{u} \int_0^\infty \int_0^\infty e^{-s\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta u}} \left[\int_0^x \int_0^t f_1(x-\delta, t-\varepsilon) f_2(\delta,\varepsilon) \right] x^{\alpha-1} t^{\beta-1} dxdt \\
=& \int_0^\infty \int_0^\infty f_2(\delta,\varepsilon) d\delta d\varepsilon \\
&\times \left[\frac{1}{u} \int_0^\infty \int_0^\infty e^{-s\frac{x^\alpha}{\alpha} - \frac{t^\beta}{\beta u}} f_1(x-\delta, t-\varepsilon) H(x-\delta, t-\varepsilon) x^{\alpha-1} t^{\beta-1} dxdt \right] \\
=& \int_0^\infty \int_0^\infty f_2(\delta,\varepsilon) d\delta d\varepsilon \left[\frac{1}{u} \int_0^\infty \int_0^\infty e^{-s\delta - \frac{\varepsilon}{u}} U_1(s,u) \right] \\
=& U_1(s,u) \frac{1}{u} \int_0^\infty \int_0^\infty e^{-s\delta - \frac{\varepsilon}{u}} f_2(\delta,\varepsilon) d\delta d\varepsilon \\
=& uU_1(s,u)U_2(s,u).
\end{aligned}$$

□

Lemma 3.1. *If* $0 < \alpha, \beta \le 1$, *then the conformable fractional Double Laplace-Sumudu transform of* $\frac{\partial^\alpha}{\partial x^\alpha} f(x,t)$, $\frac{\partial^\beta}{\partial t^\beta} f(x,t)$ *are given below:*

$$(a)\ L_x^\alpha S_t^\beta \left[\frac{\partial^\alpha}{\partial x^\alpha} f(x,t) \right] = sU(s,u) - S_t^\beta [f(0,t)],$$

$$(b)\ L_x^\alpha S_t^\beta \left[\frac{\partial^\beta}{\partial t^\beta} f(x,t) \right] = \frac{1}{u} U(s,u) - \frac{1}{u} L_x^\alpha (f(x,0)).$$

Proof. (a) Applying the definition of Conformable fractional Double Laplace-Sumudu transform,we have:

$$\begin{aligned}
&L_x^\alpha S_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}f(x,t)\right]\\
=&\frac{1}{u}\int_0^\infty\int_0^\infty e^{-\frac{x^\alpha}{\alpha}}\frac{\partial}{\partial x}f(x,t)x^{1-\alpha}x^{\alpha-1}e^{-\frac{t^\beta}{\beta u}}t^{\beta-1}dxdt\\
=&\frac{1}{u}\int_0^\infty\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}\frac{\partial}{\partial x}f(x,t)e^{\frac{t^\beta}{\beta u}}t^{\beta-1}dxdt.
\end{aligned}$$

Taking integration by part, yields:

$$\begin{aligned}
&=\frac{1}{u}\int_0^\infty\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}\frac{\partial}{\partial x}f(x,t)e^{\frac{t^\beta}{\beta u}}t^{\beta-1}dxdt\\
&=-\frac{1}{u}\int_0^\infty f(0,t)e^{-\frac{x^\beta}{\beta u}}t^{\beta-1}dt+\frac{s}{u}\int_0^\infty\int_0^\infty e^{-s\frac{x^\alpha}{\alpha}}e^{-\frac{t^\beta}{\beta u}}f(x,t)x^{\alpha-1}t^{\beta-1}dt\\
&=sU(s,u)-S_t^\beta[f(0,t)].
\end{aligned}$$

(b) can be proved similarly. □

Theorem 3.5. *If* $0<\alpha,\beta\leq 1$*, then the Conformable fractional Double Laplace-Sumudur transform of* $\frac{\partial^{2\beta}}{\partial t^{2\beta}}f(x,t)$*,* $\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(x,t)$*, are given below:*

$$\begin{aligned}
L_x^\alpha S_t^\beta\left[\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}f(x,t)\right]&=s^2U(s,u)-sS_t^\beta[f(0,t)]-S_t^\beta\left[\frac{\partial^\alpha}{\partial x^\alpha}f(0,t)\right],\\
L_x^\alpha S_t^\beta\left[\frac{\partial^{2\beta}}{\partial t^{2\beta}}f(x,t)\right]&=\frac{1}{u^2}U(s,u)-\frac{1}{u^2}L_x^\alpha[f(x,0)]-\frac{1}{u}L_x^\alpha\left[\frac{\partial^\beta}{\partial t^\beta}f(x,0)\right].
\end{aligned}$$

Proof. Follows by similar process as Lemma 3.1. □

Theorem 3.6. *If* $0<\alpha,\beta\leq 1$*, then the Conformable fractional Double Laplace-Sumudu transform of* $\frac{\partial^{m\alpha}}{\partial x^{m\alpha}}f(x,t)$*,* $\frac{\partial^{n\beta}}{\partial t^{n\beta}}f(x,t)$ *are given below:*

$$\begin{aligned}
L_x^\alpha S_t^\beta\left[\frac{\partial^{m\alpha}}{\partial x^{m\alpha}}f(x,t)\right]&=s^mU(s,u)-\sum_{i=0}^{m-1}s^{m-1-i}S_t^\beta\left[\frac{\partial^{i\alpha}}{\partial x^{i\alpha}}f(0,t)\right],\\
L_x^\alpha S_t^\beta\left[\frac{\partial^{n\beta}}{\partial t^{n\beta}}f(x,t)\right]&=\frac{1}{u^n}U(s,u)-\sum_{j=0}^{n-1}\frac{1}{u^{n-j}}L_x^\alpha\left[\frac{\partial^{j\beta}}{\partial t^{j\beta}}f(x,0)\right].
\end{aligned}$$

Proof. Follows by using the induction process on n and Lemma 3.1 and Theorems 3.5. □

4. Principle of the FCDLST method

In this section, we adopt a new technique called FCDLST method for solving FPDEs. The main idea of the proposed approach is to apply CDLST on the given FPDE with conformable derivatives to obtain the equation in a new space. Finally, we apply the inverse FCDLST to obtain the solution of the following nonhomogeneous linear fractional partial differential equations with conformable derivatives in the original space.

$$A\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u(x,t)+B\frac{\partial^{2\beta}}{\partial t^{2\beta}}u(x,t)+C\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(x,t)+D\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t)+Eu(x,t)=g(x,t). \tag{4.1}$$

Subjecting to the following initial and boundary conditions:

$$u(x,0)=h_1(x),\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,0)=h_2(x), \tag{4.2}$$

$$u(0,t)=h_3(t),\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(0,t)=h_4(t), \tag{4.3}$$

where A,B,C,D,E are real constants and $g(x,t)$ is the nonhomogeneous source term.

Then by applying the property of partial derivative of the conformable double Laplace-Sumudu transform to Eq. (4.1), single conformable Laplace transform to Eq. (4.2) and single conformable Sumudu transform to Eq. (4.3), lastly, yields the simplified Eq. (4.4):

$$\begin{aligned} &U(x,t)\\ =&\frac{1}{As^2+Cs+\frac{B}{u^2}+\frac{D}{u}+E}\left\{\begin{array}{l}Ash_3(u)+Ah_4(u)+\frac{B}{u^2}h_1(s)\\ +\frac{B}{u}h_2(s)+Ch_3(u)+\frac{D}{u}h_1(s)+G(s,u)\end{array}\right\}. \end{aligned} \tag{4.4}$$

Performing the inverse of conformable double Laplace-Sumudu transform to Eq. (4.4), yields the analytic solution of Eq. (4.1) as following:

$$\begin{aligned} U(x,t)=\ & L_x^{-1}S_t^{-1}\left\{\frac{1}{As^2+Cs+\frac{B}{u^2}+\frac{D}{u}+E}\right.\\ &\left.\times\left\{\begin{array}{l}Ash_3(u)+Ah_4(u)+\frac{B}{u^2}h_1(s)\\ +\frac{B}{u}h_2(s)+Ch_3(u)+\frac{D}{u}h_1(s)+G(s,u)\end{array}\right\}\right\}. \end{aligned} \tag{4.5}$$

5. Illustrating examples

Example 5.1. Consider the following conformable fractional homogeneous wave equation:

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}u(x,t)-\frac{\partial^{2\beta}}{\partial t^{2\beta}}u(x,t)=0, \tag{5.1}$$

the initial and boundary conditions are as below:

$$\begin{aligned} &u(x,0)=\sin\left(\frac{x^{\alpha}}{\alpha}\right),\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,0)=2, \qquad (5.2)\\ &u(0,t)=2\left(\frac{t^{\beta}}{\beta}\right),\frac{\partial^{\alpha}}{\partial x^{\alpha}}u(0,t)=\cos\left(\frac{t^{\beta}}{\beta}\right). \end{aligned}$$

Substituting $h_1(s)=\frac{1}{s^2+1},h_2(s)=\frac{2}{s},h_3(u)=2u,h_4(u)=\frac{1}{u^2+1},G(s,u)=0$ into Eq. (4.5), yields the solution of Eq. (5.1):

$$U(x,t)=L_x^{-1}S_t^{-1}\left[\frac{2u}{s}+\frac{1}{(s^2+1)(u^2+1)}\right]=2\frac{t^{\beta}}{\beta}+\sin\left(\frac{x^{\alpha}}{\alpha}\right)\cos\left(\frac{t^{\beta}}{\beta}\right). \tag{5.3}$$

This result is exactly the same as the solution in [3] when $\alpha = \beta = 1$.

Example 5.2. Consider the following conformable fractional nonhomogeneous heat equation:

$$\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} u(x,t) - \frac{\partial^{\beta}}{\partial t^{\beta}} u(x,t) - 3u(x,t) = -3. \tag{5.4}$$

Subjecting to the conditions:

$$\begin{aligned} &u(x,0) = \sin\left(\frac{x^\alpha}{\alpha}\right) + 1, \frac{\partial^\beta}{\partial t^\beta} u(x,0) = 0, \\ &u(0,t) = 1, \frac{\partial^\alpha}{\partial x^\alpha} u(0,t) = e^{-4\left(\frac{t^\beta}{\beta}\right)}. \end{aligned} \tag{5.5}$$

Substituting $h_1(s) = \frac{1}{s^2+1} + \frac{1}{s}, h_2(s) = 0, h_3(u) = 1, h_4(u) = \frac{1}{4u+1}, G(s,u) = \frac{3}{s}$ into Eq. (4.5), yields the solution of Eq. (5.4):

$$U(x,t) = L_x^{-1} S_t^{-1}\left[\frac{1}{s} + \frac{1}{(s^2+1)(4u+1)}\right] = 1 + \sin\left(\frac{x^\alpha}{\alpha}\right) e^{-4\frac{t^\beta}{\beta}}. \tag{5.6}$$

This result is exactly the same as the solution in [3] when $\alpha = \beta = 1$.

6. Conclusion

In this manuscript, the fractional conformable double Laplace-Sumudu transform (FCDLST)method for solving fractional conformable partial differential equations is proposed. We presented the related theorems and some properties of the new fractional transform and some examples are given. Examples shows that the fractional conformable double Laplace-Sumudu transform was a effective approach to solve these equations, besides, we can conclude the following conclusions:

I. The advantages of the proposed approach over other methods are: (i) its simplicity and ease of operation of the technique aimed to determine exact solutions to a large class of nonhomogeneous fractional partial differential equations; (ii) The (FCDLST) can solve the conformable fractional partial differential equations easily by turning these equations into algebraic ones; (iii) The (FCDLST) has a rapid convergence of the exact solution without any restrictive assumption of the solution compared to other techniques [21].

II. However, it should be noted that the solutions obtained by using this method are valid only when the inverse of this double Laplace-Sumudu transform exists.

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