Journal of Applied Analysis and Computation
Volume 15, Number 2, April 2025, 1002–1019
Website:http://www.jaac-online.com
DOI:10.11948/20240243

# ADVANCING LOTKA-VOLTERRA SYSTEM SIMULATION WITH VARIABLE FRACTIONAL ORDER CAPUTO DERIVATIVE FOR ENHANCED DYNAMIC ANALYSIS

Parvaiz Ahmad Naik[1,†], S. Naveen[2], V. Parthiban[2], Sania Qureshi[3], Marwan Alquran[4] and Mehmet Senol[5]

**Abstract** This study investigates the application of the Caputo derivative with a variable fractional order to time-dependent models of Ordinary Differential Equations (ODEs), aiming to enhance the simulation accuracy of dynamic systems characterized by complex, nonlinear temporal behaviors. The proposed approach provides a more refined understanding and predictive capability for non-constant real-world phenomena, contributing to the development of advanced scientific and engineering solutions. The research centers on a variable-order Lotka-Volterra predator-prey model, employing the Arzelà-Ascoli and Schaefer fixed point theorems to establish the existence of solutions, and the Banach fixed point theorem to demonstrate their uniqueness. Numerical analyses are conducted to compare the proposed model with its integer-order, fractional-order, and variable-order counterparts, utilizing various time-varying and constant delay functions. The findings validate the efficacy of the proposed method in accurately modeling dynamic systems.

**Keywords** Caputo, chaotic behavior, existence and uniqueness, fixed point theory, Arzela-Ascoli and Schaefer, numerical analysis.

**MSC(2010)** 34A08, 34C28, 47H10, 65L05.

## 1. Introduction

Fractional differential equations (FDEs) have recently attracted significant interest because of their extensive applicability in mathematical modeling across various ar-

[†]The corresponding author.
[1]Department of Mathematics and Computer Science, Youjiang Medical University for Nationalities, Baise-533000, Guangxi, China
[2]School of Advanced Sciences, Vellore Institute of Technology, 600127 Chennai, India
[3]Department of Computer Science and Mathematics, Lebanese American University, 13-5053, Beirut, Lebanon
[4]Department of Mathematics and Statistics, Jordan University of Science & Technology Irbid 22110, Jordan
[5]Department of Mathematics, Nevşehir Hacı Bektaş Veli University, Nevşehir, Turkey
Email: naik.parvaiz@ymun.edu.cn(P. A. Naik),
naveen.2020a@vitstudent.ac.in(S. Naveen),
parthiban.v@vit.ac.in(V. Parthiban), sania.shahid@lau.edu.lb(S. Qureshi),
marwan04@just.edu.jo(M. Alquran), msenol@nevsehir.edu.tr(M. Senol)

eas, such as physics, engineering, biological systems, and viscoelastic materials. The equations we analyze belong to a specific category characterized by the presence of fractional derivatives. The field of applied mathematics has recently experienced a significant surge in interest in the study of differential equations. The fractional-order differential operator is currently recognized as a more comprehensive form of the conventional integer-order differential operator. The Riemann-Liouville (R-L), Caputo, and Grunwald-Letnikov definitions are the three most frequently employed definitions. Furthermore, fractional calculus regularly uses the Caputo derivative and R-L operators for a wide range of scientific and practical applications. Fractional derivatives provide various benefits, but in some cases, their use may be restricted. The R-L derivative presents certain challenges when modeling real-world systems with fractional differential equations. It is important to note that in the context of R-L differentiation, the derivative of a constant is non-zero. If we examine the fractional differentiation of Mittag-Leffler and exponential functions, we find that every function that remains constant at the origin also has a singularity. The extent to which the R-L fractional derivative can be used efficiently is limited by the prior identified difficulties. However, it should be emphasized that to determine whether a function is differentiable while calculating its fractional derivative, the Caputo derivative requires stricter regularity criteria. Only differentiable functions can be employed with these derivatives. However, R-L fractional derivatives of any order can be obtained for functions without a first-order derivative [20, 34].

In recent years, fractional differential equations have gained popularity across a wide range of domains. The controllability of damped dynamical systems represented by Hilfer fractional derivatives was studied by Naveen et al. [31]. In [30], they also offered a qualitative study of an RLC circuit through the use of the Hilfer derivative and numerical methods via the Lagrange polynomial approach. The qualitative analysis of variable-order fractional differential equations with constant delay is explored by Naveen et al. in [29]. The predator-prey model was originally examined in [38], and the shift from simple to complex dynamics in a predator-prey-parasite model was investigated by Bairagi et al. [6]. The association between incubation latency and infection rate is also examined in the article by Bairagi et al. [11] investigates how interactions between predators, prey, and subsidies on stepping-stone domains affect populations when there are delays in dispersal. In the meantime, a fractional-order prey-predator system with time delay and a Monod-haldane functional response was studied for stability by C. Rajivgandhi et al. [9]. Considering the distribution of prey, Alidousti et al. [5] investigated the stability and bifurcation of a time-delay fractional predator-prey system. Fractional-order delayed predator-prey systems with Holling type-II functional responses were studied by Rihan et al. [36]. A study on the modeling, analysis, and bifurcation control of a delayed fractional-order predator-prey model was carried out by Huang et al. [17]. A study on the hybrid control of Hopf bifurcation in a Lotka-Volterra predator-prey model with two delays was carried out by Peng et al. [33]. Finally, staged-structured Lotka-Volterra predator-prey models and their possible use in pest control were examined by Shi et al. [37]. The ergodic property of the Lotka-Volterra predator-prey model under regime switching was investigated by Zu et al. [41] using white noise higher-order perturbation.

The prevailing importance of fractional-order models can be gauged in several articles. The research work spans a diverse array of mathematical modeling techniques applied to various ecological and epidemiological systems. For instance, Izadi et al. [18] introduced a novel Touchard polynomial-based spectral matrix collocation method to solve the Lotka-Volterra competition system with diffusion, offering new insights into these classical biological systems. Boulaaras et al. [8] explore the co-dynamics of vector-borne infections using optimal control theory, contributing to the understanding of disease spread and control strategies. Naik et al. [26] and Naik et al. [25] delve into the complexities of predator-prey dynamics through bifurcation analysis and chaos theory, highlighting the intricate behavior of these ecological systems. Additionally, Danane et al. [10] model a three-species prey-predator system using stochastic methods to incorporate Lévy jumps, providing a more realistic depiction of ecological interactions. Naik and colleagues [12,23,27,28] also investigate the interplay between HIV and HCV co-infection using fractional-order models, as well as the stability and bifurcation in predator-prey systems with specific refuge and harvesting effects. These studies collectively advance the understanding of ecological and epidemiological systems through sophisticated mathematical techniques.

Recent research in fractional calculus has significantly advanced the modeling and analysis of complex nonlinear systems across various scientific disciplines. Higazy et al. in [16] applied fractional-order derivatives to predator-prey models, capturing memory effects in ecological interactions, while Abdul et al. in [1] provided novel solutions to fractional logistic equations, offering insights into population dynamics. Ganie et al. in [15] extended soliton theory by simulating fractional Hirota–Satsuma Korteweg–de Vries systems, and Abdul et al. [2] analyzed the Zakharov–Kuznetsov equations, enhancing stability analysis in plasma physics. Furthermore, Naik et al. in [24] explored bifurcations in discrete-time chemical models using fractional techniques, and Abdul et al. [3] examined the stability of solutions in coupled fractional differential equations. Together, these studies underscore the growing importance of fractional calculus in understanding complex systems.

In the Lotka-Volterra model, which is also called the predator-prey model, it is common to use a set of first-order, nonlinear ODEs to show how two species interact, with one acting as a predator and the other as a prey. This important paradigm in ecological and mathematical sciences was independently proposed by Alfred Lotka in 1925 and Vito Volterra in 1926. Two equations in the model describe the dynamic changes in the two species populations. Let's say $P$ represents the predator population and $N$ represents the prey population. Given is the Lotka-Volterra model:

$$\frac{dN}{dp} = rN - aNP, \tag{1.1}$$

where $\frac{dN}{dp}$ reflects the prey population change rate over time. The prey's growth term is $rN$, where $r$ is its natural growth rate without predators, $a$ is a constant denoting predator efficiency, and $aNP$ is the rate at which predators kill prey. The other equation involves:

$$\frac{dP}{dp} = -sP + bNP, \tag{1.2}$$

where the change in the predator population over time is denoted by $\frac{dP}{dp}$, $s$ is the predator's natural mortality rate when there is no prey around, and $-sP$ is the predator's death term. Predator population growth due to prey eating is denoted by the constant $bNP$, where $b$ represents the efficiency with which prey is converted into predatory progeny. The population sizes of both species tend to fluctuate with time, according to the Lotka-Volterra model. As a result of an increase in the number of potential meals, predator numbers will rise in tandem. If there are more predators than prey, there will be fewer prey, which will cause there to be fewer predators. This cycle continues, giving rise to the distinctive oscillations observed in many predator-prey systems in nature.

Fractional calculus is a topic of active study. Several scientists have used fractional-order derivatives to model dynamical systems. Some researchers use a fixed fractional order, while others opt for a more flexible one. Some research also integrates the time delay components into their dynamics, making the model more adaptable and realistic. For example, in [22], R.M. May discussed the time delay versus stability in population models with two and three trophic levels of the form given below:

$$\begin{aligned}\dot{x_1}(p) =& x_1(p)\left[k_1 - ax_1(p-\tau) - bx_2(p)\right],\\ \dot{x_2}(p) =& x_2(p)\left[-k_2 + cx_1(p) - dx_2(p)\right].\end{aligned}$$

Similarly, in [39], Yan et al investigated the Hopf bifurcation and global periodic solutions in a delayed predator-prey system as follows:

$$\begin{aligned}\dot{x_1}(p) =& x_1(p)\left[k_1 - ax_1(p-\tau) - bx_2(p)\right],\\ \dot{x_2}(p) =& x_2(p)\left[-k_2 + cx_1(p) - dx_2(p-\tau)\right].\end{aligned}$$

In [13], T. Faria examined the Hopf bifurcation and stability of a system with two discrete delays and instantaneous feedback control

$$\begin{aligned}\dot{x_1}(p) =& x_1(p)\left[k_1 - ax_1(p) - bx_2(p-\tau_2)\right],\\ \dot{x_2}(p) =& x_2(p)\left[-k_2 + cx_1(p-\tau_1) - dx_2(p)\right].\end{aligned}$$

In [40], Yan et al discussed a Hopf bifurcation in a delayed Lotka- Volterra predator-prey system as follows:

$$\begin{aligned}\dot{x_1}(p) =& x_1(p)\left[k_1 - ax_1(p-\tau) - bx_2(p-\tau)\right],\\ \dot{x_2}(p) =& x_2(p)\left[-k_2 + cx_1(p-\tau) - dx_2(p-\tau)\right].\end{aligned}$$

In [21], Li et al discussed the bifurcation for a fractional-order Lotka-Volterra predator-prey model with delay feedback control such as the one given below:

$$\begin{aligned}D^{\vartheta}x_1(p) =& x_1(p)\left[k_1 - ax_1(p-\tau) - bx_2(p-\tau)\right],\\ D^{\vartheta}x_2(p) =& x_2(p)\left[-k_2 + cx_1(p-\tau) - dx_2(p-\tau)\right].\end{aligned}$$

Numerous studies have delved into the intricate complexities of the Lotka-Volterra system within the framework of fractional calculus. In this new research, we investigate the role of variable-order fractional derivatives with time-varying delay in understanding the behavior of the model's solutions.

The novelty of this research lies in its exploration of the Lotka-Volterra predator-prey model by incorporating variable-order fractional derivatives with time-varying delay. While previous studies have extensively examined the model using fixed-order fractional derivatives and time delays, this research introduces the concept of variable-order fractional derivatives, which allows for a more flexible and accurate representation of the dynamic interactions between predator and prey populations. The inclusion of time-varying delay further enhances the model's realism, making it more applicable to real-world ecological systems where delays and interactions are not constant. This approach offers new insights into the complex behavior of predator-prey systems, potentially leading to a deeper understanding of ecological dynamics.

The article's structure is as follows: In Section 2, the model is formulated and its novelty is explained in detail. In Section 3, we explain the motivation of the variable-order derivative. Then, in Section 4, we look into whether solution(s) to the initial value problem for the variable-order time-varying delay differential equations exists and is unique. In Section 5, we discussed the integer-order proposed system of time-varying delay using the Adams-Bashforth-Moulton(ABM) predictor-corrector method. In Section 5.1, we look at the results and discuss the proposed system's different variable orders and time-varying delays. Finally, the findings and future remarks of the present study are outlined in Section 6.

## 2. Formulation of the model

This present research work is to study the existence and uniqueness of results for the Lotka-Volterra predator-prey system of variable order with time-varying delay components with initial conditions. Such a scenario can be represented by the following equations:

$$\mathbb{D}_{0,p}^{\vartheta(p)}x(p) = x(p)\left[k_1 - ax(p-\tau(p)) - by(p-\tau(p))\right], x(0) = x_0, \tag{2.1}$$

$$\mathbb{D}_{0,p}^{\vartheta(p)}y(p) = y(p)\left[-k_2 + cx(p-\tau(p)) - dy(p-\tau(p))\right], y(0) = y_0, \tag{2.2}$$

where $\mathbb{D}_{0,p}^{\vartheta(p)}$ is a Caputo differential operator with fractional variable order function $\vartheta(p)$, while $k_1, k_2,\ a, b, c, d$ are the model's parameters, and $\tau(p)$ is a time-varying delay component. Now the proposed model is converted into a general coupled system based upon Caputo fractional variable order derivative of the following form:

$$\mathbb{D}_{0,p}^{\vartheta(p)}x(p) = g_1(p, x(p), x(p-\tau(p)), y(p), y(p-\tau(p))),\quad x(0) = x_0, \tag{2.3}$$

$$\mathbb{D}_{0,p}^{\vartheta(p)}y(p) = g_2(p, x(p), x(p-\tau(p)), y(p), y(p-\tau(p))),\quad y(0) = y_0, \tag{2.4}$$

where $p \in [0, T]$, $g_1, g_2$ are in general nonlinear functions, and $\tau(p)$ is a time-varying delay component.

The primary contribution of the present research study can be outlined as follows:

1. In recent studies, the fractional order Lotka-Volterra system with constant delay is analyzed under the various distinct delays by Li et al in [21] including some references cited therein. In the present study, however, fractional variable order is used for the model under time-varying delay components.
2. By employing fixed point theory, the solution to the Caputo derivative of variable-order time-varying delay differential equations (VOTDDEs) with the initial condition has been attained. This approach establishes both the solutions' existence and uniqueness while examining the chaotic characteristics of the proposed systems featuring time-dependent delays.
3. We employ the innovative hypothesis to confirm the existence and uniqueness of solutions for VOTDDEs, specifically those described by (2.3)-(2.4) with

time-varying delays. Subsequently, we demonstrate chaotic behavior, thus validating the theoretical findings.

4. Explore the variable order Lotka-Volterra predator-prey system of time-varying delay differential equations (VOTDDEs) with an initial condition to analyze its chaotic behavior. Next, compare the effects of time-varying delay for both integer and variable orders to validate the obtained results. Finally, implement these findings effectively in real-world scenarios.

## 3. Motivation for fractional-order systems

This is a big step forward in the modeling of complex dynamic systems, especially where standard integer-order models fail. It uses ODE models with Caputo fractional variable order derivatives and time delays. The Caputo fractional derivative, a frequently used operator in fractional calculus, offers a more generalized and flexible approach compared to standard derivatives [4,7,14,19,32,35]. Its main characteristic is the use of non-integer-order differentiation, which improves the representation of systems' memories and genetic qualities. This is especially important when modeling phenomena whose current state is affected not just by their recent past but also by their longer-term background, as is the case in many biological, physical, and engineering processes.

The Caputo operator's variable fractional order increases its complexity even further. When modeling complex systems with potentially evolving dynamics, variable-order derivatives have the advantage of allowing the order of the derivative to alter across time or space. In materials science, for instance, this allows for a more precise depiction of the mechanical characteristics of viscoelastic materials as they evolve over time or are subjected to varying degrees of stress. In neuroscience, variable-order models are superior for capturing the neural network's plasticity across time. These models' accuracy is further improved by the incorporation of temporal delays. Systems such as population dynamics and control systems require time delays because of the time lag between cause and effect. Time delays improve future state prediction, which in turn improves planning and control across a wide range of scientific and technical applications. There has been a major stride forward in the modeling and understanding of complex, dynamic systems across many scientific disciplines since the introduction of the Caputo fractional variable order derivatives and time delays.

Additionally, the utilization of the Caputo fractional variable order derivative is crucial in disciplines such as geophysics and finance, as it enables the capture of non-local dynamics. This is particularly important in scenarios where long-range interactions and memory effects have a significant impact. In the field of geophysics, the intricate phenomena occurring within the Earth, such as the propagation of seismic waves over different geological strata, demonstrate characteristics that can be more accurately elucidated through the application of fractional calculus. This mathematical framework allows for a more comprehensive understanding of the medium's nuanced and history-dependent interactions. In financial markets, the utilization of variable-order derivatives offers an improved representation of non-linear dynamics that are dependent on memory, such as market volatility and swings in asset prices. This technique offers a more realistic and predictive framework compared to conventional models, facilitating enhanced understanding and more resilient forecasting.

The incorporation of time delay serves to enhance the effectiveness of these models by acknowledging the temporal gap between acts and their discernible outcomes, a prevalent occurrence in economic and ecological systems. The integration of variable order and time delay into these extensive modeling capabilities establishes the Caputo fractional derivative as a potent instrument in contemporary scientific study and its practical implementation.

# 4. Main theoretical results

This section focuses on the examination of the presence and unique nature of solutions utilizing fixed-point theory. Next, we analyze the contrast between the conventional Lotka-Volterra predator-prey model and the variable-order system with a time-dependent delay. Before establishing the theoretical analysis in this section, readers must familiarize themselves with the fundamental principles of fractional calculus. The ideas mentioned are the Rieman-Liouville integral of variable fractional order, the Caputo differential operator with variable-order derivative, the continuous operator, complete mapping, fixed points, and a uniformly bounded sequence. The primary principles can be located in the reference [29].

## 4.1. Existence and uniqueness results

First, let us prove the existence theorem by using Arzela-Ascoli and Schaefer's fixed point theorem and next, the uniqueness theorem by using Banach contraction principles for the problem (2.3)-(2.4). For this, we make the following assumptions.
$(A_1)$ Consider the function $g_i : \mathbb{J} \times \Omega \times \Omega \to \Omega$, where $g_i$ is continuous, and there exist positive constants $\beta_1$ and $\beta_2$ such that

$$|g_i(p, \delta_1, \eta_1) - g_i(p, \delta_2, \eta_2)| \leq \beta_1 |\delta_1 - \delta_2| + \beta_2 |\eta_1 - \eta_2|, \text{ for } i = 1, 2.$$

$(A_2)$ Assume the function is continuous and there exists a positive constant $\beta_3$ such that

$$|g_i(p, s, \delta_1) - g_i(p, s, \delta_2|) \leq \beta_3 |\delta_1 - \delta_2|.$$

$(A_3)$ Assuming the function $g_i$ is completely continuous, there exists $\beta_4(\cdot) \in \mathbb{L}^1(\mathbb{J}, \mathbb{R})$ such that

$$|g_i(p, \delta, \eta)| \leq \beta_4(p), \quad p \in \mathbb{J}, \quad \delta, \eta \in \Omega.$$

**Theorem 4.1.** *Assume that* $(A_3)$ *holds, then the system* (2.3)-(2.4) *has at least one solution on* $\mathbb{J}$.

**Proof.** Let us consider the operator $\mathcal{A}$ is continuous and completely continuous.

**Step 1.** To illustrate the continuity of $\mathcal{A}$, we examine a sequence $x_n$ converging to a point $x \in \mathbb{C}$

$$\begin{aligned} & |(\mathcal{A}x_n)(p) - (\mathcal{A}x)(p)| \\ =& \Big| \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x_n(s), x_n(s - \tau(p)))) ds \\ & - \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x(s), x(s - \tau(p))) ds \Big| \end{aligned}$$

$$\leq \left| \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x_n(s), x_n(s-\tau(p)))ds \right|,$$

$$\left| - \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x(s), x(s-\tau(p)))ds \right|$$

$$\leq \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} \Big| g_1(s, x_n(s), x_n(s-\tau(p))) - g_1(s, x(s), x(s-\tau(p))) \Big| ds$$

$$\leq \frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p)+1)} \Big\| g_1(s, x_n(s), x_n(s-\tau(p))) - g_1(s, x(s), x(s-\tau(p))) \Big\|_{\mathbb{C}}.$$

As function $f$ exhibits continuous behavior, we can conclude that

$$|(\mathcal{A}x_n)(p) - (\mathcal{A}x)(p)|$$
$$\leq \frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p)+1)} \Big\| g_1(s, x_n(s), x_n(s-\tau(p))) - g_1(s, x(s), x(s-\tau(p))) \Big\| \to 0 \text{ as } p \to \infty.$$

Similarly,

$$|(\mathcal{A}y_n)(p) - (\mathcal{A}y)(p)|$$
$$\leq \frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p)+1)} \Big\| g_2(s, y_n(s), y_n(s-\tau(p))) - g_2(s, y(s), y(s-\tau(p))) \Big\| \to 0 \text{ as } p \to \infty.$$

**Step 2.** The operator $\mathcal{A}$ is bounded within its own set, where $r > 0$ is a positive constant. For any $x$ belonging to $B_r = \{x \in C : |x| \leq r\}$, it holds that $|\mathcal{A}x| \leq l$, where $l$ is a positive constant

$$\begin{aligned}
|(\mathcal{A}x(p))| &= \left| \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x(s), x(s-\tau(p)))ds \right| \\
&\leq \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} \left| g_1(s, x(s), x(s-\tau(p))) \right| ds \\
&\leq \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} \left| \beta_4(s) \right| ds \\
&\leq \frac{1}{\Gamma(\vartheta(p))} \int_0^p (p-s)^{\vartheta(p)-1} \left\| \beta_4(s) \right\| ds, \\
|(\mathcal{A}x(p))| &\leq T^{\vartheta(p)} \frac{\|\beta_4(s)\|_{\mathbb{C}}}{\Gamma(\vartheta(p)+1)}.
\end{aligned}$$

Similarly,

$$|(\mathcal{A}y(p))| \leq T^{\vartheta(p)} \frac{\|\beta_4(s)\|_{\mathbb{C}}}{\Gamma(\vartheta(p)+1)}.$$

**Step 3.** The operator $\mathcal{A}$ maps bounded sets to equicontinuous sets in $\mathbb{C}$. Consider $0 \leq p_1, p_2 \leq T$, where $B_r$ is a bounded set in $\mathbb{C}$, and $x$ belongs to $B_r$

$$\begin{aligned}
&|(\mathcal{A}x)(p_2) - (\mathcal{A}x)(p_1)| \\
\leq & \frac{1}{\Gamma(\vartheta(p))} \int\limits_0^{p_2} (p_2 - s)^{\vartheta(p)-1} \left| g_1(s, x(s), x(s - \tau(p))) \right| ds \\
& - \frac{1}{\Gamma(\vartheta(p))} \int\limits_0^{p_1} (p_1 - s)^{\vartheta(p)-1} \left| g_1(s, x(s), x(s - \tau(p))) \right| ds \\
\leq & \frac{1}{\Gamma(\vartheta(p))} \int\limits_0^{p_2} (p_2 - s)^{\vartheta(p)-1} \left\| g_1(s, x(s), x(s - \tau(p))) \right\| ds \\
& - \frac{1}{\Gamma(\vartheta(p))} \int\limits_0^{p_1} (p_1 - s)^{\vartheta(p)-1} \left\| g_1(s, x(s), x(s - \tau(p))) \right\| ds \\
\leq & \frac{1}{\Gamma(\vartheta(p))} \int\limits_0^{p_2} \left[ (p_2 - s)^{\vartheta(p)-1} - (p_1 - s)^{\vartheta(p)-1} \right] \left\| g_1(s, x(s), x(s - \tau(p))) \right\| ds \\
& + \frac{1}{\Gamma(\vartheta(p))} \int\limits_{p_1}^{p_2} \left[ (p_2 - s)^{\vartheta(p)-1} \right] \left\| g_1(s, x(s), x(s - \tau(p)))) \right\| ds \\
\leq & \frac{\|\beta_4(s)\|_{\mathbb{C}}}{\Gamma(\vartheta(p))} \int\limits_0^{p_2} \left[ (p_2 - s)^{\vartheta(p)-1} - (p_1 - s)^{\vartheta(p)-1} \right] ds \\
& + \frac{\|\beta_4(s)\|_{\mathbb{C}}}{\Gamma(\vartheta(p))} \int\limits_{p_1}^{p_2} (p_2 - s)^{\vartheta(p)-1} ds.
\end{aligned}$$

Given $p_2 > p_1$, the right side of the inequality becomes 0. Utilizing the completely continuous definitions, it becomes apparent that the operator $\mathcal{A} : \mathbb{C} \to \mathbb{C}$ demonstrates complete continuity. By applying the Arzelà-Ascoli theorem, we establish its compactness. Moreover, by identifying $\mathcal{A}$ as a fixed point, it follows, by the principles outlined in Schaefer's fixed point theorem, that it constitutes a solution for (2.3)-(2.4). Consequently, the existence of at least one solution within the interval $\mathbb{J}$ for the system is affirmed. □

Next, let us prove the uniqueness theorem of the system (2.3)-(2.4) by using the Banach contraction principles.

**Theorem 4.2.** *Suppose that the assumptions* $(A_1)$, $(A_2)$ *and the following inequality holds*

$$(\beta_1 + \beta_2\beta_3) \frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p+1))} < 1, \tag{4.1}$$

*then the problem* (2.3)-(2.4) *has a unique solution of* $\mathbb{J}$.

**Proof.** We define the operator $\mathcal{A} : \mathbb{C} \to \mathbb{C}$ to map the system (2.3)-(2.4) into a fixed-point problem.

$$(\mathcal{A}x)(p) = x_0 + \frac{1}{\Gamma(\vartheta(p))}\int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x(s), x(s-\tau(p)))ds.$$

Now, let us consider the system has as another solution $x(p)$. Letting $x, z \in \mathbb{C}(\mathbb{J}, \mathbb{R})$ and $p \in \mathbb{J}$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(p) - (\mathcal{A}z)(p)| \\
=&|\frac{1}{\Gamma(\vartheta(p))}\int_0^p (p-s)^{\vartheta(p)-1} g_1(s, x(s), x(s-\tau(p)))ds \\
&- \frac{1}{\Gamma(\vartheta(p))}\int_0^p (p-s)^{\vartheta(p)-1} g_1(s, z(s), z(s-\tau(p)))ds| \\
\leq&\frac{1}{\Gamma(\vartheta(p))}\int_0^p (p-s)^{\vartheta(p)-1} |g_1(s, x(s), x(s-\tau(p))) - g_1(s, z(s), z(s-\tau(p)))|ds.
\end{aligned}$$

By using the assumptions $(A_1)$ and $(A_2)$, we get

$$\begin{aligned}
&\leq (\beta_1 + \beta_2\beta_3)\frac{|x(s) - z(s)|}{\Gamma(\vartheta(p))}\int_0^p (p-s)^{\vartheta(p)-1} ds \\
&\leq (\beta_1 + \beta_2\beta_3)\frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p)+1)}\|x - z\|_{\mathbb{C}}.
\end{aligned}$$

Similarly,

$$|(\mathcal{A}y)(p) - (\mathcal{A}z)(p)| \leq (\beta_1 + \beta_2\beta_3)\frac{T^{\vartheta(p)}}{\Gamma(\vartheta(p)+1)}\|y - z\|_{\mathbb{C}}.$$

The equation (4.1) establishes that the operator $\mathcal{A}$ is a contraction. Applying the Banach contraction principle confirms the existence of a fixed point for $\mathcal{A}$, thereby demonstrating the uniqueness of the problem (2.3)-(2.4). □

# 5. Numerical approach

Within this section, we utilize the ABM predictor-corrector technique to carry out the numerical solution of nonlinear VOTDDEs. Let's analyze the resulting variable-order fractional system outlined in equation (2.3)-(2.4). By applying the fractional integrator to both sides of the system (2.3)-(2.4), we assert the following:

$$x(p_{r+1}) = x_0 + \frac{1}{\Gamma(\vartheta(p_{r+1}))}\int_0^{p_{r+1}} (p_{r+1}-s)^{\vartheta(p_{r+1})-1} g_1(s, x(s), x(s-\tau(p)))ds. \quad (5.1)$$

Subsequently, we employ the product trapezoidal quadrature formula to compute the integral in equation (5.1). This process yields the ensuing corrector formula:

$$x(p_{r+1}) = x_0 + \frac{h^{\vartheta(p_{r+1})}}{\Gamma(\vartheta(p_{r+1})+2)} g_1(p_{r+1}, x(p_{r+1}), x(p_{r+1} - \tau(p)))$$

$$+ \frac{h^{\vartheta(p_{r+1})}}{\Gamma(\vartheta(p_{r+1})+2)} \sum_{i=0}^{r} b_{i,r+1}\ g_1(p_i, x(p_i), x(p_i - \tau(p))),$$

and

$$x^q\left(p_{r+1}\right) = x_0 + \frac{1}{\Gamma\left(\vartheta\left(p_{r+1}\right)\right)} \sum_{i=0}^{r} c_{i,r+1}\ g_1(p_i, x(p_i), x(p_i - \tau(p))), \tag{5.2}$$

where

$$\begin{aligned} & b_{i,r+1} \\ =& \begin{cases} r^{\vartheta(p_{r+1})+1} - \left(r - \vartheta\left(p_{r+1}\right)\right)(r+1)^{\vartheta(p_{r+1})}, & i=0, \\ (r-i+2)^{\vartheta(p_{r+1})+1} + (r-i)^{\vartheta(p_{r+1})+1} - 2(r-i+1)^{\vartheta(p_{r+1})+1}, & 1 \leq i \leq r, \\ 1, & i = r+1, \end{cases} \end{aligned} \tag{5.3}$$

$$\begin{aligned} & c_{i,r+1} \\ =& \frac{h^{\vartheta(p_{r+1})}}{\vartheta\left(p_{r+1}\right)} \left((r-i+1)^{\vartheta(p_{r+1})} - (r-i)^{\vartheta(p_{r+1})}\right). \end{aligned} \tag{5.4}$$

The above technique is a well-known approach to solving fractional dynamical systems. Therefore we employ it to discuss the numerical applications in the subsection that follows.

## 5.1. Numerical applications

Here, we use the ABM predictor-corrector approach as given above, an effective approximation scheme for the numerical solution of the Caputo fractional variable order derivative of the proposed model (2.3)-(2.4).

Consider the variable order Caputo fractional Lotka- Volterra predator-prey model with the time-varying delay of the form given in (2.1-2.2). Let us take the system's parameter values $k_1 = k_2 = 1$ and $a = b = 1$, $c = 2, d = 1$. While the model has its limitations and simplifications, it has been widely used in various fields to understand the interactions between species in ecosystems. In ecology and conservation biology, the model helps ecologists understand the population dynamics and interactions between predators and prey in ecosystems. It can be used to predict how changes in one population might affect the other and how disturbances (natural or human-induced) can impact the stability of ecosystems. In epidemiology, it has been adapted to study the dynamics of infectious diseases in populations. In this context, one species represents the infected individuals (prey), and the other species represents the predators (e.g., the immune system or medical interventions). This can help in understanding the spread and control of diseases.

The chaotic behavior shows the relationship between the prey populations and the predator population $x, y$; respectively. The time response of the state equation $x(p)$ represents the prey population as a function of time $p$. It shows how the prey population changes over time according to the Lotka-Volterra equations. The curve may exhibit oscillations or other patterns, reflecting the dynamic nature of the predator-prey interaction. The time response of the state equation $y(p)$ represents the predator population as a function of time $p$. It shows how the predator population changes over time, according to the model. The curves in various plots may also display oscillations or other patterns, illustrating the impact of the prey population on the predator population and vice versa.

Let us consider the system (2.1-2.2) to be an integer-order model, that is, when the order $\vartheta(p) = 1$. Figure 1 represents the dynamical behavior of an integer-order Lotka- Volterra predator-prey system with a time-varying delay component taken to be $\tau(p) = \frac{80e^p}{1+e^p}$. Based on this Figure, it is easy to observe that the oscillations take a long time to be damped out.

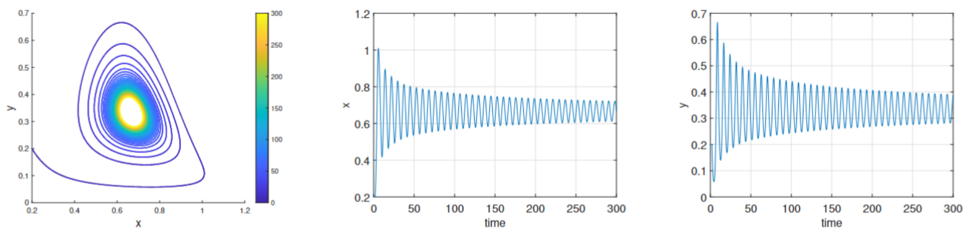
The Lotka-Volterra predator-prey model, when considered with a constant fractional-order $\vartheta = 0.95$ (an arbitrary choice), is simulated in Figure 2 having a time-varying delay component $\tau(p) = \frac{80e^p}{1+e^p}$. Figure 2a represents the chaotic nature of the predator-prey model, Figure 2b shows the phase portrait plotted for the state variable $x(p)$ vs time. Figure 2c displays the phase portraits for the state variable $y(p)$ vs time where $p \in [0, 300]$. According to this Figure, the oscillations begin to vanish when $p \to 100$. Such behavior was not possible to observe with the integer-order version of the model under consideration.

In Figure 3, we have shown the variable-order version of the Lotka- Volterra predator-prey model with a constant delay as $\tau(p) = 0.75$ while the variable-order $\vartheta(p)$ is taken to be $\vartheta(p) = 0.98 + 0.04\cos(\frac{p}{10})$. This Figure shows large oscillations in time series plots and the same behavior is depicted by the phase portrait, where the system has quite complex and chaotic behavior. One of the reasons could be the absence of the negative exponential terms in $\tau(p)$.

Figure 4, represents the variable-order version of the Lotka- Volterra predator-prey model with a constant delay as $\tau(p) = 0.6$ while the variable-order $\vartheta(p)$ is taken to be $\vartheta(p) = 0.98 + 0.04\cos(\frac{p}{10})$. This figure shows comparatively small oscillations in time series plots and the same behavior is depicted by the phase portrait, where the system has reasonable chaotic behavior. One of the reasons could be the absence of the negative exponential terms in $\tau(p)$ with a smaller magnitude.

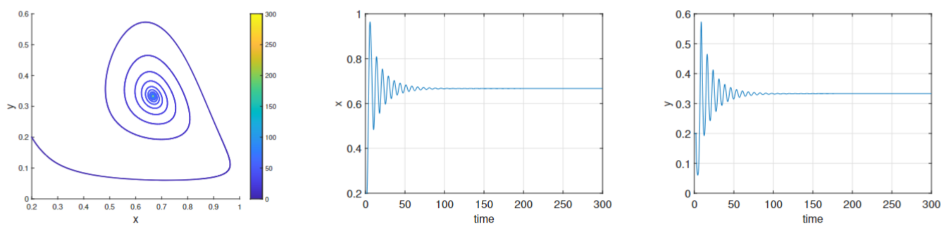
This time, Figure 5 is obtained with variable-order $\vartheta(p) = 0.98 + 0.04\cos(\frac{p}{10})$ and time-varying delay $\tau(p) = \frac{80e^p}{1+e^p}$. Once again, larger oscillations with complex chaotic behavior are noted. Both the fractional variable-order and the time-delay component are taken to be functions. Strange chaotic patterns, impossible to obtain when the model is classical, are observed as shown in Figure 5a. The same is true for Figure 5b and Figure 5c.

In Figures 6 and 7, we have shown that the fractional variable-order function $\vartheta(p)$ with time-varying delay component $\tau(p)$ plays an important role in the dynamics of the Lotka-Volterra predator-prey system. Same fractional variable-order function $\vartheta(p) = 0.98 + 0.04\cos(\frac{p}{10})$ is chosen for both figures while $\tau(p) = 30 + 30.5|\sin(p)|$ is chosen for Figure 6 and $\tau(p) = 30 + 30.5|\sin(p)|$ is for Figure 7. It can be seen in Figure 6 that when the oscillatory function $\sin(p)$ is chosen to be absolutely continuous then the oscillations are controlled to a large extent while, on the other hand, an opposite behavior can be observed in Figure 7.



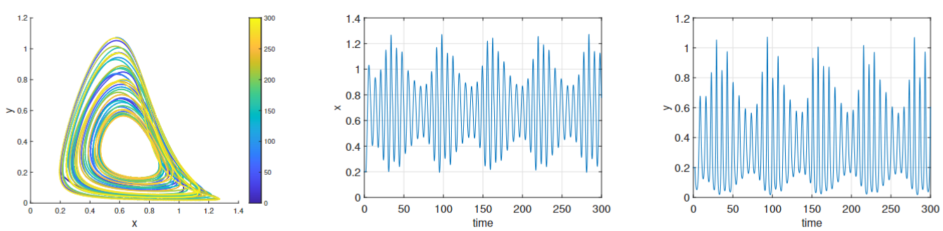

(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 1.** Dynamical response of the integer-order version of the Lotka-Volterra predator-prey model with time-varying delay function $\tau(p) = \frac{80e^p}{1+e^p}$.



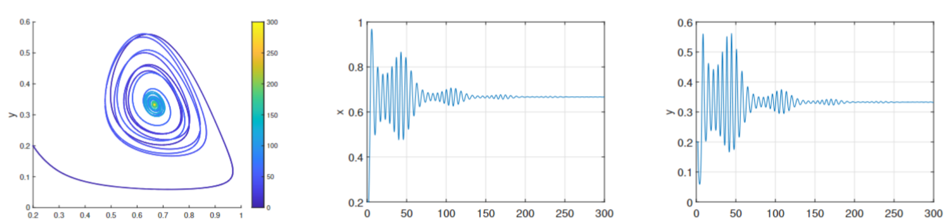

(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 2.** Dynamical response of the fractional-order ($\vartheta = 0.95$) version of the Lotka-Volterra predator-prey model with time-varying delay function $\tau(p) = \frac{80e^p}{1+e^p}$.



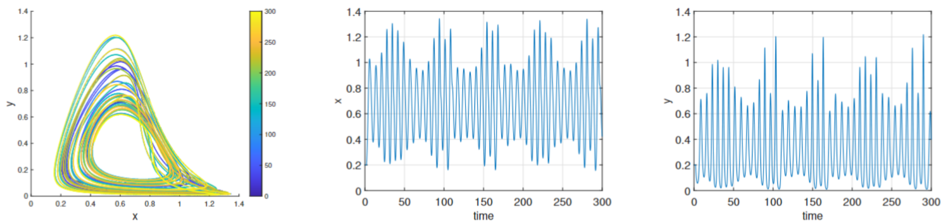

(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 3.** Variable order constant delay with $\tau(p) = 0.75$.



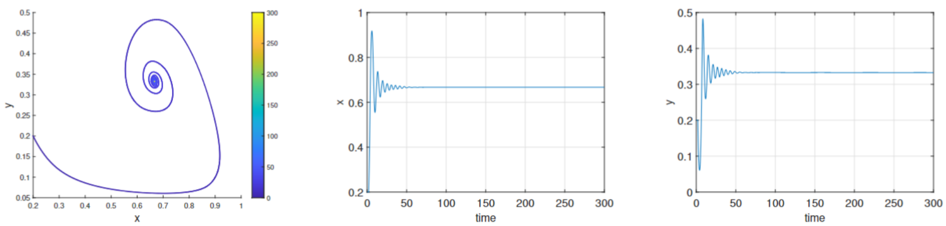

(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 4.** Variable order constant delay $\tau(p) = 0.60$.



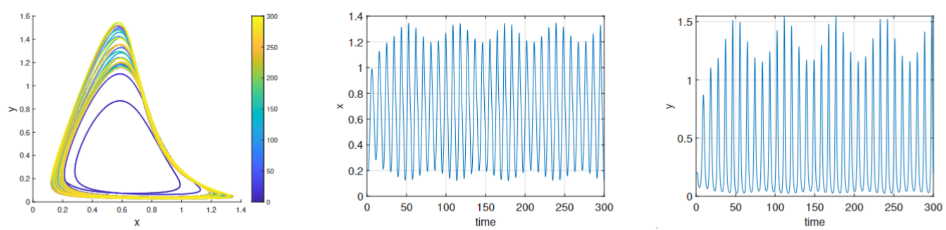

(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 5.** Variable order constant delay $\tau(p) = 0.60$.




(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 6.** Variable order constant delay $\tau(p) = 0.60$.




(a) Chaotic behavior of $(x, y)$ (b) Time responses of $x$ (c) Time responses of $y$

**Figure 7.** Variable order constant delay $\tau(p) = 0.60$.

# 6. Conclusion

This study rigorously examines the existence and uniqueness of a fractional variable-order $\vartheta(p)$ Lotka-Volterra predator-prey model, incorporating a time-varying delay component $\tau(p)$. The present work examines the comparison between distinct constant and time delays of integer-, constant fractional-, and variable orders. The efficacy of the established technique is verified by conducting several simulations using various types of fractional variable-order functions and time delays. The Adams Bashforth Moulton predictor-corrector scheme is a robust and versatile tool for numerically solving variable-order fractional differential equations. Moreover, this method has more accuracy and efficiency when compared to the existing numerical schemes. The variable order fractional derivative for the Lotka-Volterra system has long-range interactions and memory effects. We can adjust the system's order to enhance its stability and accuracy for the proposed model. It is important to mention that there are other potential applications for the approaches that utilize time delay as a bifurcation parameter. These applications include more intricate models with varying delays, as well as the study of Hopf bifurcation in higher-dimensional fractional-order systems.

The objective of the planned future research is to investigate the dynamic complexity and stability features of Lotka-Volterra predator-prey fractional variable order systems of ODEs with time delay. Future research will prioritize the integration of variable-order fractional calculus concepts with the classical Lotka-Volterra model. This will involve including temporal delays to create a more precise representation of real-world ecological and biological systems. The study aims to examine the impact of varied order and time delays on the presence and stability of equilibria, oscillatory behaviors, and the potential for chaos in predator-prey and competitive systems. Additionally, advanced mathematical models will be used to develop numerical methods for simulating the intricate dynamics of ecological and other natural systems. This will allow for a deeper understanding of their long-term behavior. This research field has promise for applications in comprehending population dynamics, modeling diseases, and managing resources, providing a more holistic framework for forecasting and overseeing natural occurrences. Future research should also explore the potential of integrating stochastic elements into the fractional variable-order Lotka-Volterra models to account for environmental uncertainties. Additionally, the application of these models to real-world data could further validate their accuracy and provide insights into the practical implications of time-varying delays in ecological and biological systems.

## Acknowledgements

We want to thank the reviewers and editors of this paper for their careful attention and constructive feedback, which helped us to improve the manuscript greatly.

## Conflict of interests

The authors declare that they have no conflicts of interest.

## References

[1] H. Abdul, K. Adnan, G. Alhamzi and A. Saeed, *A new solution of the nonlinear fractional logistic differential equations utilizing efficient techniques*, AIP Advances, 2024, 14(035134).

[2] H. Abdul, F. Mofarreh and K. Adnan, *A fractional analysis of Zakharov–Kuznetsov equations with the Liouville–Caputo operator*, Axioms, 2023, 12(6), 609.

[3] H. G. Abdul, M. Houas, M. M. AlBaidani and D. Fathima, *Coupled system of three sequential caputo fractional differential equations: Existence and stability analysis*, Mathematical Methods in the Applied Sciences, 2023, 1–14.

[4] N. Ahmed, M. Rani, S. S. Dragomir and A. Akgul, *New exact solutions to space–time fractional telegraph equation with conformable derivative*, International Journal of Modern Physics B, 2023, 2350275.

[5] J. Alidousti and M. M. Ghahfarokhi, *Stability and bifurcation for time delay fractional predator prey system by incorporating the dispersal of prey*, Applied Mathematical Modelling, 2019, 72, 385–402.

[6] N. Bairagi and D. Adak, *Switching from simple to complex dynamics in a predator–prey–parasite model: An interplay between infection rate and incubation delay*, Mathematical Biosciences, 2016, 277, 1–14.

[7] B. Bolaji, T. Onoja, C. Agbata, et al., *Dynamical analysis of hiv-tb co-infection transmission model in the presence of treatment for tb*, Bulletin of Biomathematics, 2024, 2(1), 21–56.

[8] S. Boulaaras, M. Yavuz, Y. Alrashedi, et al., *Modeling the co-dynamics of vector-borne infections with the application of optimal control theory*, Discrete and Continuous Dynamical Systems-S, 2024, 17, 1–20.

[9] R. Chinnathambi and F. A. Rihan, *Stability of fractional-order Prey–Predator system with time-delay and monod–haldane functional response*, Nonlinear Dynamics, 2018, 92(4), 1637–1648.

[10] J. Danane, M. Yavuz and M. Yıldız, *Stochastic modeling of three-species Prey–Predator model driven by lévy jump with mixed holling-ii and beddington–deangelis functional responses*, Fractal and Fractional, 2023, 7(10), 751.

[11] R. M. Eide, A. L. Krause, N. T. Fadai and R. A. Van Gorder, *Predator-prey-subsidy population dynamics on stepping-stone domains with dispersal delays*, Journal of Theoretical Biology, 2018, 451, 19–34.

[12] Z. Eskandari, P. A. Naik and M. Yavuz, *Dynamical behaviors of a discrete-time Prey-Predator model with harvesting effect on the predator*, Journal of Applied Analysis and Computation, 2024, 14(1), 283–297.

[13] T. Faria, *Stability and bifurcation for a delayed Predator–Prey model and the effect of diffusion*, Journal of Mathematical Analysis and Applications, 2001, 254(2), 433–463.

[14] M. Farman, A. Ahmad, A. Zehra, et al., *Analysis and controllability of diabetes model for experimental data by using fractional operator*, Mathematics and Computers in Simulation, 2023.

[15] A. H. Ganie, S. Noor, M. Al Huwayz, et al., *A new solution of the nonlinear numerical simulations for fractional hirota–satsuma coupled Korteweg–de vries systems*, Open Physics, 2024.

[16] M. Higazy, H. Hijaz, A. H. Ganie, et al., *Theoretical analysis and computational modeling of nonlinear fractional-order-two predators model*, Results in Physics, 2021, 17, 105139.

[17] C. Huang, X. Song, B. Fang, et al., *Modeling, analysis and bifurcation control of a delayed fractional-order Predator–Prey model*, International Journal of Bifurcation and Chaos, 2018, 28(09), 1850117.

[18] M. Izadi, A. El-Mesady and W. Adel, *A novel touchard polynomial-based spectral matrix collocation method for solving the Lotka-Volterra competition system with diffusion*, Mathematical Modelling and Numerical Simulation with Applications, 2024, 4(1), 37–65.

[19] I. Khan, R. Nawaz, A. H. Ali, et al., *Comparative analysis of the fractional order Cahn-Allen equation*, Partial Differential Equations in Applied Mathematics, 2023, 8, 100576.

[20] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Analysis: Theory, Methods & Applications, 2008, 69(8), 2677–2682.

[21] Z. Li, W. Zhang, C. Huang, et al., *Bifurcation for a fractional-order lotka-volterra predator-prey model with delay feedback control*, AIMS Mathematics, 2021, 6(1), 675–687.

[22] R. M. May, *Time-delay versus stability in population models with two and three trophic levels*, Ecology, 1973, 54(2), 315–325.

[23] P. A. Naik, M. Amer, R. Ahmed, et al., *Stability and bifurcation analysis of a discrete Predator-Prey system of ricker type with refuge effect*, Mathematical Biosciences and Engineering, 2024, 21(3), 4554–4586.

[24] P. A. Naik, Z. Eskandari and H. E. Shahraki, *Flip and generalized flip bifurcations of a two-dimensional discrete-time chemical model*, Mathematical Modelling and Numerical Simulations with Applications (MMNSA), 2021, 1(2), 95–101.

[25] P. A. Naik, Z. Eskandari, H. E. Shahkari and K. M. Owolabi, *Bifurcation analysis of a discrete-time Prey-Predator model*, Bulletin of Biomathematics, 2023, 1(2), 111–123.

[26] P. A. Naik, Z. Eskandari, M. Yavuz and Z. Huang, *Bifurcation results and chaos in a two-dimensional Predator-Prey model incorporating holling-type response function on the predator*, Discrete and Continuous Dynamical Systems-S, 2024, 17(12), 1–18.

[27] P. A. Naik, M. Yavuz, S. Qureshi, et al., *Memory impacts in hepatitis c: A global analysis of a fractional-order model with an effective treatment*, Computer Methods and Programs in Biomedicine, 2024, 254, 108306.

[28] P. A. Naik, B. M. Yeolekar, S. Qureshi, et al., *Modeling and analysis of the fractional-order epidemic model to investigate mutual influence in hiv/hcv co-infection*, Nonlinear Dynamics, 2024, 112(13), 11679–11710.

[29] S. Naveen and V. Parthiban, *Qualitative analysis of variable-order fractional differential equations with constant delay*, Mathematical Methods in the Applied Sciences, 2023, 1–12.

[30] S. Naveen, V. Parthiban and M. I. Abbas, *Qualitative analysis of rlc circuit described by hilfer derivative with numerical treatment using the lagrange polynomial method*, Fractal and Fractional, 2023, 7(11).

[31] S. Naveen, R. Srilekha, S. Suganya and V. Parthiban, *Controllability of damped dynamical systems modelled by hilfer fractional derivatives*, Journal of Taibah University for Science, 2022, 16(1), 1254–1263.

[32] S. Paul, A. Mahata, S. Mukherjee, et al., *Study of fractional order sir model with mh type treatment rate and its stability analysis*, Bulletin of Biomathematics, 2024, 2(1), 85–113.

[33] M. Peng, Z. Zhang and X. Wang, *Hybrid control of hopf bifurcation in a Lotka-Volterra Predator-Prey model with two delays*, Advances in Difference Equations, 2017, 2017(1), 1–20.

[34] I. Podlubny, *Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[35] M. Qayyum, E. Ahmad, S. T. Saeed, et al., *Traveling wave solutions of generalized seventh-order time-fractional kdv models through he-laplace algorithm*, Alexandria Engineering Journal, 2023, 70, 1–11.

[36] F. Rihan, S. Lakshmanan, A. Hashish, et al., *Fractional-order delayed Predator–Prey systems with holling type-ii functional response*, Nonlinear Dynamics, 2015, 80, 777–789.

[37] R. Shi and L. Chen, *Staged-structured Lotka–Volterra Predator–Prey models for pest management*, Applied Mathematics and Computation, 2008, 203(1), 258–265.

[38] K. White, J. D. Murray and M. Lewis, *Wolf-deer interactions: A mathematical model*, Proceedings of the Royal Society of London. Series B: Biological Sciences, 1996, 263(1368), 299–305.

[39] X.-P. Yan and W.-T. Li, *Hopf bifurcation and global periodic solutions in a delayed Predator–Prey system*, Applied Mathematics and Computation, 2006, 177(1), 427–445.

[40] X.-P. Yan and C.-H. Zhang, *Hopf bifurcation in a delayed lokta–volterra Predator–Prey system*, Nonlinear Analysis: Real World Applications, 2008, 9(1), 114–127.

[41] L. Zu, D. Jiang, D. O'Regan, et al., *Ergodic property of a Lotka–Volterra Predator–Prey model with white noise higher order perturbation under regime switching*, Applied Mathematics and Computation, 2018, 330, 93–102.