Journal of Applied Analysis and Computation
Volume 15, Number 2, April 2025, 1039–1067
Website:http://www.jaac-online.com
DOI:10.11948/20240246

ANALYSIS OF FRACTIONAL ORDER SCHRÖDINGER EQUATION WITH SINGULAR AND NON-SINGULAR KERNEL DERIVATIVES VIA NOVEL HYBRID SCHEME

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Abstract In the present study, a novel semi-analytic scheme is proposed to obtain exact and approximate series solutions for the time fractional linear and non-linear Schrödinger equation. This hybrid scheme employs the general bivariate transform followed by the homotopy perturbation method to formulate the recurrence relation. The recurrence relation leads to a system of linear differential equations that associates with the desired components of the series solution. To characterize the considered model with memory effects, the fractional temporal order is considered in the Caputo, Caputo-Fabrizio, and Atangana-Baleanu in Caputo senses. The adapted scheme appears efficient and competent in identifying a diverse collection of trigonometric, wave, and soliton solutions with the availability of initial data. Configurational variations in the governing phenomena with alterations in the fractional order are addressed through graphical illustrations. The potential of the developed regime is affirmed through the uniqueness and convergence analysis of the acquired results. Numerical results are found to be in accordance with existing results in terms of absolute error norms. The main highlight of the proposed scheme is its efficacy and simplicity in constructing a series solution that rapidly converges to the exact solution.

Keywords Fractional derivative, partial differential equations, integral transform, homotopy perturbation method.

MSC(2010) 26A33, 35A22, 35C05, 35C20.

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1. Introduction

The concept of calculus of fractional operators was developed to find an answer to the question that prompted L' Hospital in 1695 to ask Leibnitz about the scenario of $n = 1/2$ for the derivative of order n i.e. $\frac{d^n f}{d\zeta^n}$. An essential role of the fractional operators is that it furnishes the real-world models with an arbitrary order of differentiation and integrals for characterizing memory and hereditary effects. These are increasingly employed in the disciplines of applied sciences to decode the non-local physical appeal of complex systems [2, 8, 9, 14, 24, 56]. As a result of the usefulness of the fractional calculus in the mathematical analysis its practical and theoretical aspects have emerged as an enthralling topic in recent decades. For instance, mathematical modeling of viscoelastic materials frequently makes use of fractional derivatives [50]. A differential equation of diffusion that does not use integers can be used to describe anomalous diffusion phenomena in non-homogeneous media [12]. Fractance, an electrical circuit with non-integer order impedance that possesses both resistance and capacitance qualities, is another example of an element with fractional order [36]. Additionally, it has been demonstrated that fractional order calculus can be used to more accurately simulate the dynamic process of heat conduction [22]. It has been demonstrated in biology that biological cell membranes exhibit fractal order conductance and are classified as non-integer order systems [21, 26]. Some financial systems are known to exhibit fractional order dynamics in economics [37]. Due to the wide range of multidisciplinary applications, the piqued attention of researchers glorified the literature with innumerable definitions of differentiation and integration of arbitrary order. The capacity to choose the most advantageous definition increased the ability of mathematical simulation to adapt the given facts. Some of the renowned definitions are Riemann-Liouville derivative, Caputo derivative, Caputo-Fabrizio derivative, Atangana Baleanu derivative, Grunwald-Letnikov derivative and Riesz derivative etc [15, 19, 33, 44, 45]. Despite the abundance of literature on fractional calculus, the optimal definition of a fractional derivative is still lacking. Researchers have reported the setbacks of prevailing non-integer derivatives in several aspects. R-L derivative have singular kernel and results in variable function even for derivative of constant function. All existing fractional derivatives just inherited the linearity property from the classical derivative. All fractional derivatives do not follow the product rule, quotient rule, chain rule, Rolle's theorem and mean value theorem. Most of the fractional derivatives except the Caputo derivative do not cope up with zero for constant function. After discarding the R-L derivative the Caputo derivative is considered as most preferable operator by incorporating the limitations of the R-L derivative. Since, mathematical model within the framework of the Caputo derivative possess the classical initial conditions, the behavior of complex evolution processes is widely investigated with the aid of the Caputo derivative and its extensions.

Experimental studies regarding wave-particle duality witnessed the wave motion of particles in microscopic systems instead of obeying Newtonian laws of motion governed in macroscopic systems. To specify the quantitative relation between the behavior of a particle and its associated wave function, prompted by de Broglie postulate, Schrödinger's theory developed as a generalization of Newton's theory of the motion of particles in macroscopic systems. The mathematical model evolved from Schrödinger's theory entitled as Schrödinger equation (SG) has emerged as a fundamental narrator of the aspects of the particle for its specific potential energy $\nu(\zeta,\tau)$, through the behavior of associated wave function $\psi(\zeta,\tau)$. Potential energy is usually a function of ζ only and possibly can be of τ also. However, for the constant value potential energy, SG equation describes the phenomenon of free particles. The presence of imaginary unit i at a fundamental level in SG equation led to the relation of the first power of total energy to the second power of momentum. Furthermore, it associates the wave function with two real functions simultaneously, its real part and imaginary part. This is in contrast to a wave function from classical mechanics. Taking into account, the uncertainty principal of the microscopic world known as quantum mechanics, SG equation expresses the particle-wave relation in terms of probability density, first stated in 1926 by Max Born [18] as:

$$\mathcal{P}(\zeta,\tau) = |\psi(\zeta,\tau)|^2$$

This quantity indicates the probability of finding the particle close to the coordinate ζ at time τ , in per unit length of the ζ-axis. With the development of non-linear science, the non-linear SG equation established as a multidisciplinary model interpreting non-linear wave evolution phenomena frequently arise in quantum mechanics, chemical kinetics, optics, hydrodynamics, bio-genetics, plasma physics, and so forth [11,16,20,34,54,55]. The integrable cubic non-linear SG equation has special interest in revolutionized communication systems fabricated with optical fibers [17]. It exhibits the basic features of optical wave propagation in Kerr media but leaves certain finer details, such as spin which is a purely quantum phenomena [35]. For being general, non-linear SG equation incorporates a variety of non-linearities including, octic-septic-quintic-cubic non-linearity, Kerr non-linearity, logarithmic non-linearity, dual-power law non-linearity etc [6,30,43,51–53].

The Feynman path integrals is generated by quantum dynamics with Brownian-like attributes. However, commonly known as $\psi(\tau) \sim \tau^{1/2}$, square root law of governing Brownian motion regarding change in location over time has not been spotted for many complex quanta and classical physics [38–40]. To analyze complex quantum dynamic processes, broader evolution law $\psi(\tau) \sim \tau^{1/\varpi}$ has been confined with general scaling $1/\varpi$; $0 < \varpi \leq 2$, known as Levy flights. As a fundamental result of extending the Feynman path integrals to Levy-like quantum mechanical path, Nick Lasin discovered the eminent fractional Schrödinger equation interpreting the mechanics of fractional quantum [38–40]. To extract the dynamical information of variants of fractional SG equation, various analytical and numerical studies have been organized with different preeminent techniques [3–5,7,10,27,29,31,32,42,47].

In present study, to characterize the SG equation with the memory effect, the fractional temporal operator has been adapted with three terminologies such as Caputo(C), Caputo-Fabrizio (CF) and Atangana Baleanu in Caputo sense (ABC) in the following form:

$$iC^{\rho}_{\tau}\psi(\zeta,\tau) = \beta\psi_{\zeta\zeta}(\zeta,\tau) + \nu(\zeta)\psi(\zeta,\tau) + \lambda\psi(\zeta,\tau)|\psi(\zeta,\tau)|^2. \tag{1.1}$$

Initially,

$$\psi(\zeta,0) = \phi(\zeta), \tag{1.2}$$

where $i = \sqrt{-1}, \psi(\zeta,\tau)$ is complex-valued function while ζ and τ are spatial and temporal variables, respectively. The parameter ρ describes the order of fractional temporal derivative. β stands for coefficient of group velocity dispersion (GVD) and λ represents the coefficient of non-linearity. The term $\psi(\zeta,\tau)|\psi(\zeta,\tau)|^2$ represents

Kerr-law non-linearity which implies direct proportional dependence of refractive index of light on its intensity. $\nu(\zeta)$ is the trapping potential relies on the description of loss and dispersion management schemes. $\phi(\zeta)$ is the known initial function.

The paper comprises seven sections. Section 1 is about the general introduction of the problem. Section 2 consists of fundamental terminologies of fractional derivatives and general bivariate (GB) transform. Section 3 gives brief description of the general bivariate homotopy perturbation method (GB-HPM). Sufficient conditions for uniqueness and convergence of the solution are presented in Section 4. Several numerical examples are discussed in Section 5. Section 6 briefs the results and thorough observations. Finally, conclusions are given in Section 7.

2. Basic definitions

This section covers basic concepts and definitions of Caputo, CF, ABC fractional derivatives and the GB-transform that will be used in this work.

Definition 2.1. [44] In Caputo's terminology the fractional operator over function $\varphi(\tau)$ is defined as:

$$ {}_0^C C_\tau^\rho(\varphi(\tau)) = \begin{cases} \dfrac{1}{\Gamma(n-\rho)} \displaystyle\int_0^\tau \dfrac{\varphi^n(\xi)}{(\tau-\xi)^{\rho+1-n}} d\xi, & n-1<\rho<n, \\ \dfrac{d^n}{d\tau^n}\varphi(\tau), & \rho = n. \end{cases} \tag{2.1} $$

By replacing the singular power law kernel and multiplier of intergral in Caputo's fractional derivative fractional operators with non-singular kernels are defined as:

Definition 2.2. [14, 19] In CF's terminology the fractional operator of order ρ over function $\varphi(\tau)$ is defined as:

$$ {}_0^{CF} C_\tau^\rho(\varphi(\tau)) = \begin{cases} \dfrac{B(\rho)}{1-\rho} \displaystyle\int_0^\tau \exp\left(-\dfrac{\rho(\tau-\xi)}{1-\rho}\right)\varphi'(\xi)\, d\xi, & 0<\rho<1, \\ \dfrac{d}{d\tau}\varphi(\tau), & \rho = 1, \end{cases} \tag{2.2} $$

where $B(\rho)$ is the normalization function with $B(0) = B(1) = 1$.

Definition 2.3. [2, 15] In ABC's terminology the fractional operator of order ρ over function $\varphi(\tau)$ is defined as:

$$ {}_0^{ABC} C_\tau^\rho(\varphi(\tau)) = \begin{cases} \dfrac{B(\rho)}{1-\rho} \displaystyle\int_0^\tau E_\rho\left(-\dfrac{\rho(\tau-\xi)^\rho}{1-\rho}\right)\varphi'(\xi)\, d\xi, & 0<\rho<1, \\ \dfrac{d}{d\tau}\varphi(\tau), & \rho = 1, \end{cases} \tag{2.3} $$

where $B(\rho)$ is the normalization function and $E_\rho(.)$ is the Mittag-Leffler function.

In the present study, $B(\rho) = 1$ is considered for simplicity.

These terminologies of fractional calculus have been widely explored to trace the non-local behavior of numerous physical systems. In search of efficient tools, integral transforms have emerged as competent tools to construct the solution of fractional models. However, the efficiency of these transforms is totally problem oriented. The present study directs to the capabilities of the recently introduced GB transform [13] for fractional models framed with different Caputo's type fractional derivatives. GB transform reduces to ARA [48] and Formable transform [49] for specific values of its accounted parameters.

Definition 2.4. [13] The GB transform of function $\varphi(\tau)$ is given by:

$$\mathscr{A}_m(\varphi(\tau)) = \mathscr{P}_m(s,\gamma) = \frac{s}{\gamma^m}\int_0^\infty \tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\varphi(\tau)\,d\tau, \tag{2.4}$$

where m is order, s and γ are transformation variable.

Definition 2.5. [13] If $\mathscr{P}_m(s,\gamma)$ is the GB transform of function $\varphi(\tau)$, then the inverse of $\mathscr{P}_m(s,\gamma)$ is the $\varphi(\tau)$ is such that

$$\mathscr{A}_m^{-1}(\mathscr{P}_m(s,\gamma)) = \varphi(\tau). \tag{2.5}$$

Lemma 2.1. [13] *If $\mathscr{A}_m(\varphi_1(\tau)) = \mathscr{P}_m(s,\gamma)$ and $\mathscr{A}_m(\varphi_2(\tau)) = \mathscr{Q}_m(s,\gamma)$ then*

$$\mathscr{A}_m(\lambda_1\varphi_1(\tau)+\lambda_2\varphi_2(\tau)) = \lambda_1\mathscr{P}_m(s,\gamma)+\lambda_2\mathscr{Q}_m(s,\gamma), \tag{2.6}$$

where λ_1 and λ_2 are constants.

Lemma 2.2. [13] *(Shifting in n-domain)*

$$\mathscr{A}_m(\tau^n\varphi(\tau)) = \gamma^n\mathscr{A}_{m+n}(\varphi(\tau)), \tag{2.7}$$

where $m+n-1\geq 0$.

Theorem 2.1. *The GB transform for fractional operator defined in Caputo's sense is given by:*

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^C C_\tau^\rho(\varphi(\tau))\right]\\ =&\frac{1}{\Gamma(n-\rho)}\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma(n+m-\rho-r)\frac{\gamma^{n-\rho}}{s^{n+m-\rho-r}}\mathscr{A}_r\left[\varphi^n(\tau)\right],\end{aligned} \tag{2.8}$$

where $n-1<\rho\leq n$.

Proof. By using the definition of GB transform and Caputo's derivative one can have

$$\mathscr{A}_m\left[{}_0^C C_\tau^\rho(\varphi(\tau))\right] = \frac{s}{\gamma^m}\int_0^\infty\tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\left(\frac{1}{\Gamma(n-\rho)}\int_0^\tau\frac{\varphi^n(\xi)}{(\tau-\xi)^{\rho-n+1}}d\xi\right)d\tau. \tag{2.9}$$

Altering the order of integration in Eq. (2.9), gives

$$\mathscr{A}_m\left[{}_0^C C_\tau^\rho(\varphi(\tau))\right] = \frac{s}{\gamma^m\Gamma(n-\rho)}\int_0^\infty\varphi^n(\xi)\int_\xi^\infty\frac{\tau^{m-1}}{(\tau-\xi)^{\rho-n+1}}\exp\left(-\frac{s}{\gamma}\tau\right)d\tau\,d\xi. \tag{2.10}$$

Letting $\lambda=\tau-\xi$ in Eq. (2.10) leads to

$$\mathscr{A}_m\left[{}_0^C C_\tau^\rho(\varphi(\tau))\right]$$

$$=\frac{s}{\gamma^{m}\Gamma\left(n-\rho\right)}\int_{0}^{\infty}\varphi^{n}\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma}\lambda\right)\left(\lambda+\xi\right)^{m-1}\lambda^{n-\rho-1}d\lambda\,d\xi. \tag{2.11}$$

Using the binomial formula, Eq. (2.11) gives

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^C C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\ =&\frac{s}{\gamma^{m}\Gamma\left(n-\rho\right)}\int_{0}^{\infty}\varphi^{n}\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\int_{0}^{\infty}\exp\left(-\frac{s}{\gamma}\lambda\right)\sum_{r=1}^{m}\binom{m-1}{r-1}\\ &\times\lambda^{n+m-\rho-r-1}\xi^{r-1}d\lambda\,d\xi.\end{aligned} \tag{2.12}$$

Using the definition of GB transform of order one, Eq. (2.12) implies

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^C C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\ =&\frac{1}{\gamma^{m-1}\Gamma\left(n-\rho\right)}\\ &\times\int_{0}^{\infty}\varphi^{n}(\xi)\exp\left(-\frac{s}{\gamma}\xi\right)\mathscr{A}_1\left[\sum_{r=1}^{m}\binom{m-1}{r-1}\lambda^{n+m-\rho-r-1}\right]\xi^{r-1}d\xi\\ =&\frac{1}{\Gamma\left(n-\rho\right)}\\ &\times\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma\left(n+m-\rho-r\right)\frac{\gamma^{n-\rho-r}}{s^{n+m-\rho-r-1}}\int_{0}^{\infty}\varphi^{n}\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\xi^{r-1}d\xi.\end{aligned} \tag{2.13}$$

Using the definition of GB transform of order r, Eq. (2.13) implies

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^C C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\ =&\frac{1}{\Gamma\left(n-\rho\right)}\sum_{r=1}^{m}\binom{m-1}{r-1}\Gamma\left(n+m-\rho-r\right)\frac{\gamma^{n-\rho}}{s^{n+m-\rho-r}}\mathscr{A}_r\left[\varphi^{n}\left(\tau\right)\right].\end{aligned}$$

□

Remark 2.1. (Special Cases of Eq. (2.8))

i. For m=1:

$$\mathscr{A}_1[{}_0^C C_\tau^\rho\left(\varphi\left(\tau\right)\right)]=\frac{\gamma^{n-\rho}}{s^{n-\rho}}\mathscr{A}_1[\varphi^n(\tau)],\quad n-1<\rho\leq n.$$

ii. For m=2:

$$\mathscr{A}_2[{}_0^C C_\tau^\rho\left(\varphi\left(\tau\right)\right)]=(n-\rho)\frac{\gamma^{n-\rho}}{s^{n-\rho+1}}\mathscr{A}_1[\varphi^n(\tau)]+\frac{\gamma^{n-\rho}}{s^{n-\rho}}\mathscr{A}_2[\varphi^n(\tau)],\quad n-1<\rho\leq n.$$

For $\gamma=1$, the resulting equations of GB transform for Caputo's operator are in alignment with existing equations of ARA transform for Caputo's operator [46].

Theorem 2.2. *The GB transform for fractional operator defined in Caputo-Fabrizio's sense is given by*

$$\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=B\left(\rho\right)\sum_{r=1}^{m}\binom{m-1}{r-1}\frac{(m-r)!}{(1-\rho)}\frac{\gamma}{\left(s+\frac{\rho}{1-\rho}\gamma\right)^{m-r+1}}\mathscr{A}_r\left[\varphi'\left(\tau\right)\right], \tag{2.14}$$

where $0<\rho\leq 1$.

Proof. By using the definition of GB transform and Caputo-Fabrizio's derivative one can have

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\&=\frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^m}\int_0^\infty\tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\left[\int_0^\tau\varphi'\left(\xi\right)\exp\left(-\frac{\rho\left(\tau-\xi\right)}{1-\rho}\right)d\xi\right]d\tau.\end{aligned} \tag{2.15}$$

Altering the order of integration in Eq. (2.15), gives

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\&=\frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\int_\xi^\infty\exp\left(-\frac{s}{\gamma}\tau\right)\exp\left(-\frac{\rho\left(\tau-\xi\right)}{1-\rho}\right)\tau^{m-1}d\tau\,d\xi.\end{aligned} \tag{2.16}$$

Letting $\lambda=\tau-\xi$ in Eq. (2.16) leads to

$$\begin{aligned}\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=&\frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\int_0^\infty\exp\left(-\frac{s}{\gamma}\left(\lambda+\xi\right)\right)\exp\left(-\frac{\rho\lambda}{1-\rho}\right)\\&\times\left(\lambda+\xi\right)^{m-1}d\lambda\,d\xi.\end{aligned} \tag{2.17}$$

Using the binomial formula, Eq. (2.17) gives

$$\begin{aligned}\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=&\frac{B\left(\rho\right)s}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\int_0^\infty\exp\left(-\frac{s}{\gamma}\left(\lambda+\xi\right)\right)\exp\left(-\frac{\rho\lambda}{1-\rho}\right)\\&\times\sum_{r=1}^{m}\binom{m-1}{r-1}\lambda^{m-r}\xi^{r-1}d\lambda\,d\xi.\end{aligned} \tag{2.18}$$

From the definition of GB transform of order one, one can get

$$\begin{aligned}\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=&\frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}}\int_0^\infty\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\mathscr{A}_1\left[\sum_{r=1}^{m}\binom{m-1}{r-1}\right.\\&\left.\times\exp\left(-\frac{\rho\lambda}{1-\rho}\right)\lambda^{m-r}\right]\xi^{r-1}d\xi\\=&\frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}}\sum_{r=1}^{m}\binom{m-1}{r-1}\left(m-r\right)!\frac{s\gamma^{m-r}}{\left(s+\frac{\rho}{1-\rho}\gamma\right)^{m-r+1}}\\&\times\int_0^\infty\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\xi^{r-1}d\xi.\end{aligned} \tag{2.19}$$

Using the definition of GB transform of order r, Eq. (2.19) implies

$$\mathscr{A}_m\left[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right] = B\left(\rho\right)\sum_{r=1}^{m}\binom{m-1}{r-1}\frac{(m-r)!}{1-\rho}\frac{\gamma}{\left(s+\frac{\rho}{1-\rho}\gamma\right)^{m-r+1}}\mathscr{A}_r\left[\varphi'\left(\tau\right)\right].$$

□

Remark 2.2. (Special Cases of Eq. (2.14))

i. For $m = 1$:

$$\mathscr{A}_1[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)] = \frac{B\left(\rho\right)}{1-\rho+\rho\frac{\gamma}{s}}\frac{\gamma}{s}\mathscr{A}_1[\varphi'(\tau)], \quad 0<\rho\le 1.$$

ii. For $m = 2$:

$$\begin{aligned}&\mathscr{A}_2[{}_0^{CF}C_\tau^\rho\left(\varphi\left(\tau\right)\right)]\\&=\frac{B\left(\rho\right)\gamma(1-\rho)}{[s(1-\rho)+\rho\gamma]^2}\mathscr{A}_1[\varphi'(\tau)]+\frac{B\left(\rho\right)\gamma}{s(1-\rho)+\rho\gamma}\mathscr{A}_2[\varphi'(\tau)], \quad 0<\rho\le 1.\end{aligned}$$

Theorem 2.3. *The GB transform for fractional operator defined in Atangana Baleanu in Caputo sense is given by*

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\&=\frac{B\left(\rho\right)}{(1-\rho)}\sum_{r=1}^{m}\binom{m-1}{r-1}\frac{1}{s^{m-r}}\sum_{k=0}^{\infty}\left(-\frac{\rho}{1-\rho}\right)^k\frac{1}{\Gamma\left(\rho k+1\right)}\left(\rho k+m-r\right)!\\&\quad\times\frac{\gamma^{\rho k+1}}{s^{\rho k+1}}\mathscr{A}_r\left[\varphi'\left(\tau\right)\right],\end{aligned}\tag{2.20}$$

where $0<\rho\le 1$.

Proof. By using the definition of GB transform and Atangana Baleanu in Caputo sense derivative one can have

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\&=\frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^m}\int_0^\infty\tau^{m-1}\exp\left(-\frac{s}{\gamma}\tau\right)\left[\int_0^\tau\varphi'\left(\xi\right)E_\rho\left(-\frac{\rho\left(\tau-\xi\right)^\rho}{1-\rho}\right)d\xi\right]d\tau.\end{aligned}\tag{2.21}$$

Altering the order of integration in Eq. (2.21), gives

$$\begin{aligned}&\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\&=\frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\int_\xi^\infty\exp\left(-\frac{s}{\gamma}\tau\right)E_\rho\left(-\frac{\rho\left(\tau-\xi\right)^\rho}{1-\rho}\right)\tau^{m-1}d\tau\, d\xi.\end{aligned}\tag{2.22}$$

Letting $\lambda=\tau-\xi$ in Eq. (2.22) leads to

$$\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right] = \frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\int_0^\infty\exp\left(-\frac{s}{\gamma}\left(\lambda+\xi\right)\right)E_\rho\left(-\frac{\rho\lambda^\rho}{1-\rho}\right)$$

$$\times\left(\lambda+\xi\right)^{m-1}d\lambda\,d\xi. \tag{2.23}$$

Using the binomial formula, Eq. (2.23) gives

$$\begin{aligned}
&\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\
&=\frac{sB\left(\rho\right)}{\left(1-\rho\right)\gamma^m}\int_0^\infty\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\left[\int_0^\infty\exp\left(-\frac{s}{\gamma}\lambda\right)E_\rho\left(-\frac{\rho\lambda^\rho}{1-\rho}\right)\right.\\
&\quad\left.\times\sum_{r=1}^m\binom{m-1}{r-1}\lambda^{m-r}d\lambda\right]\xi^{r-1}d\xi.
\end{aligned}\tag{2.24}$$

From the definition of GB transform of order one, one can get

$$\begin{aligned}
\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=&\frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}}\int_0^\infty\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\\
&\times\mathscr{A}_1\left[\sum_{r=1}^m\binom{m-1}{r-1}E_\rho\left(-\frac{\rho\lambda^\rho}{1-\rho}\right)\lambda^{m-r}\right]\xi^{r-1}d\xi.
\end{aligned}\tag{2.25}$$

By using series expansion of $E_\rho(.)$ [2, 45], one gets

$$\begin{aligned}
&\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]\\
&=\frac{B\left(\rho\right)}{\left(1-\rho\right)\gamma^{m-1}}\sum_{r=1}^m\binom{m-1}{r-1}\frac{1}{s^{m-r}}\sum_{k=0}^\infty\left(-\frac{\rho}{1-\rho}\right)^k\frac{1}{\Gamma\left(\rho k+1\right)}\\
&\quad\times\left(\rho k+m-r\right)!\frac{\gamma^{\rho k+m-r}}{s^{\rho k}}\int_0^\infty\varphi'\left(\xi\right)\exp\left(-\frac{s}{\gamma}\xi\right)\xi^{r-1}d\xi.
\end{aligned}\tag{2.26}$$

Using the definition of GB transform of order r, Eq. (2.26) implies

$$\begin{aligned}
\mathscr{A}_m\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]=&\frac{B\left(\rho\right)}{\left(1-\rho\right)}\sum_{r=1}^m\binom{m-1}{r-1}\frac{1}{s^{m-r}}\\
&\times\sum_{k=0}^\infty\left(-\frac{\rho}{1-\rho}\right)^k\frac{1}{\Gamma\left(\rho k+1\right)}\left(\rho k+m-r\right)!\frac{\gamma^{\rho k+1}}{s^{\rho k+1}}\mathscr{A}_r\left[\varphi'\left(\tau\right)\right].
\end{aligned}$$

□

Remark 2.3. (Special Case of Eq. (2.20) for $m=1$)

$$\begin{aligned}
\mathscr{A}_1\left[{}_0^{ABC}C_\tau^\rho\left(\varphi\left(\tau\right)\right)\right]&=\frac{B(\rho)}{(1-\rho)}\sum_{k=0}^\infty(-1)^k\left[\frac{\rho}{1-\rho}\left(\frac{\gamma}{s}\right)^\rho\right]^k\frac{\gamma}{s}\mathscr{A}_1[\varphi'(\tau)]\\
&=\frac{B(\rho)}{(1-\rho)}\left[1+\frac{\rho}{1-\rho}\left(\frac{\gamma}{s}\right)^\rho\right]^{-1}\frac{\gamma}{s}\mathscr{A}_1[\varphi'(\tau)]\\
&=\frac{B(\rho)}{\left[1-\rho+\rho\left(\frac{\gamma}{s}\right)^\rho\right]}\frac{\gamma}{s}\mathscr{A}_1[\varphi'(\tau)],\quad 0<\rho\le 1.
\end{aligned}$$

3. Methodology

In this section, the fundamental operation of the proposed semi-analytic procedure derived from amalgamation of GB transform and HPM [1, 28] is presented to solve time fractional non-linear partial differential equations. Consider the generic non-linear non-homogeneous PDE with temporal fractional operator as:

$$C_\tau^\rho(\psi(\zeta,\tau)) = \Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau)) + f(\zeta,\tau), \tag{3.1}$$

with the initial condition

$$\psi(\zeta,0) = \phi(\zeta), \tag{3.2}$$

where C_τ^ρ is the fractional temporal operator. Λ and Θ specify the linear and non-linear operators, respectively. $f(\zeta,\tau)$ is the source term and $\phi(\zeta)$ is the known initial function.

By employing the GB transform of order one on Eq. (3.1) with consideration of fractional derivatives of Caputo type, yields

$$\frac{1}{r(s,\gamma,\rho)}\left[\mathscr{A}_1\left[\psi(\zeta,\tau)\right] - \phi(\zeta)\right] = \mathscr{A}_1\left[M(\zeta,\tau)\right], \tag{3.3}$$

where,

$$M(\zeta,\tau) = \Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau)) + f(\zeta,\tau), \tag{3.4}$$

and the function $r(s,\gamma,\rho)$ emerges in the following forms:

i. (**GB-HPM**$_C$) For Caputo's terminology:

$$r(s,\gamma,\rho) = \left(\frac{\gamma}{s}\right)^\rho. \tag{3.5}$$

ii. (**GB-HPM**$_{CF}$) For CF's terminology:

$$r(s,\gamma,\rho) = 1-\rho+\rho\left(\frac{\gamma}{s}\right). \tag{3.6}$$

iii. (**GB-HPM**$_{ABC}$) For ABC's terminology:

$$r(s,\gamma,\rho) = \frac{1-\rho+\rho\left(\frac{\gamma}{s}\right)^\rho}{B(\rho)}. \tag{3.7}$$

By operating inverse of GB transform, Eq. (3.3) can be re-written as

$$\psi(\zeta,\tau) = \phi(\zeta) + \mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[M(\zeta,\tau)]]. \tag{3.8}$$

Embedding the homotopy parameter i.e., $q \in [0,1]$ constructs the perturbation equation as

$$\psi(\zeta,\tau) = \phi(\zeta) + q\mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[M(\zeta,\tau)]]. \tag{3.9}$$

The solution of Eq. (3.1) is presumed in the form of series expanded with respect to the small parameter q as

$$\psi(\zeta,\tau) = \sum_{l=0}^{\infty} q^l \psi_l(\zeta,\tau). \tag{3.10}$$

Decompose the non-linear operator $\Theta(\psi(\zeta,\tau))$ as

$$\Theta(\psi(\zeta,\tau)) = \sum_{l=0}^{\infty} \mathscr{H}_l, \tag{3.11}$$

where the $\mathscr{H}_l$ represent the He's polynomials [25]:

$$\mathscr{H}_l(\psi_0,\psi_1,\ldots,\psi_l) = \frac{1}{l!}\frac{\partial^l}{\partial q^l}\left[\Theta\left(\sum_{j=0}^{\infty} q^j \psi_j(\zeta,\tau)\right)\right]_{q=0}, \quad l = 0,1,2\ldots. \tag{3.12}$$

Expression (3.12) can be expanded as:

$$\begin{aligned}
\mathscr{H}_0 &= \Theta(\psi_0),\\
\mathscr{H}_1 &= \psi_1\Theta'(\psi_0),\\
\mathscr{H}_2 &= \psi_2\Theta'(\psi_0) + \frac{1}{2!}\psi_1^2\Theta''(\psi_0), \ \ldots.
\end{aligned} \tag{3.13}$$

By inserting the Eq. (3.10) and Eq. (3.11) into Eq. (3.9), one obtains the following system of linear equations corresponding to the components of desired series solution as:

$$\begin{aligned}
\psi_0(\zeta,\tau) &= \phi(\zeta) + \mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[f(\zeta,\tau)]],\\
\psi_1(\zeta,\tau) &= \mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[\Lambda(\psi_0(\zeta,\tau)) + \mathscr{H}_0]],\\
&\ \vdots\\
\psi_{l+1}(\zeta,\tau) &= \mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[\Lambda(\psi_l(\zeta,\tau)) + \mathscr{H}_l]],\ l = 0,1,2,\ldots.
\end{aligned} \tag{3.14}$$

By substituting calculated components from system (3.14) into Eq. (3.10) with $q \to 1$ gives the solution of Eq. (3.1) as

$$\psi(\zeta,\tau) = \psi_0(\zeta,\tau) + \psi_1(\zeta,\tau) + \psi_2(\zeta,\tau) + \psi_3(\zeta,\tau) + ..., \tag{3.15}$$

where truncated series of n components termed as n^{th} order approximate solution and closed form represents analytic solution.

4. Analysis of GB-HPM

Theorem 4.1. [2, 56] *Let $\mathfrak{B} = (\mathcal{C}[\Omega], \|\ .\ \|)$ be the Banach space that incorporates $\|\ \psi\ \| = \max_{(\zeta,\tau)\in\Omega}|\psi(\zeta,\tau)|$, for all continuous functions defined on Ω. Λ and Θ (linear and non-linear operators in GB-HPM scheme) are Lipschitz operators, that is*

$$|\Lambda[\psi] - \Lambda[\psi^*]| < \kappa_1|\psi - \psi^*|, \tag{4.1}$$

and

$$|\Theta[\psi] - \Theta[\psi^*]| < \kappa_2|\psi - \psi^*|, \tag{4.2}$$

where κ_1 and κ_2 are Lipschitz constants. Then GB-HPM solution is convergent if $\exists\, \kappa \in (0,1)$ such that *i.*

$$\kappa=(\kappa_1+\kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)} \text{ for GB-HPM}_C \text{ solution.} \tag{4.3}$$

ii.

$$\kappa=(\kappa_1+\kappa_2)(1-\rho+\rho\tau) \text{ for GB-HPM}_{CF} \text{ solution.} \tag{4.4}$$

iii.

$$\kappa=(\kappa_1+\kappa_2)\left(1-\rho+\rho\frac{\tau^\rho}{\Gamma(\rho+1)}\right) \text{ for GB-HPM}_{ABC} \text{ solution.} \tag{4.5}$$

Proof. Let $\Psi_m=\sum_{r=0}^m \psi_r$. To show that Ψ_m is a Cauchy sequence in $\mathfrak{B}$. Consider,

$$\begin{aligned}
&\parallel \Psi_m-\Psi_n \parallel\\
&=\max_{(\zeta,\tau)\in\Omega}|\Psi_m-\Psi_n|\\
&=\max_{(\zeta,\tau)\in\Omega}\Big|\sum_{r=n+1}^{m}\psi_r(\zeta,\tau)\Big|,\ n=1,2,3...\\
&\leq \max_{(\zeta,\tau)\in\Omega}\left|\mathscr{A}_1^{-1}\left\{r(s,\gamma,\rho)\mathscr{A}_1\left[\sum_{r=n+1}^{m}(\Lambda(\psi_{r-1}(\zeta,\tau))+\Theta(\psi_{r-1}(\zeta,\tau)))\right]\right\}\right|\\
&\leq \max_{(\zeta,\tau)\in\Omega}\left|\mathscr{A}_1^{-1}\left\{r(s,\gamma,\rho)\mathscr{A}_1\left[\sum_{r=n}^{m-1}(\Lambda(\psi_r(\zeta,\tau))+\Theta(\psi_r(\zeta,\tau)))\right]\right\}\right|\\
&\leq \max_{(\zeta,\tau)\in\Omega}\left|\mathscr{A}_1^{-1}\left\{r(s,\gamma,\rho)\mathscr{A}_1\left[\Lambda(\psi_{m-1})-\Lambda(\psi_{n-1})+\Theta(\psi_{m-1})-\Theta(\psi_{n-1})\right]\right\}\right|,
\end{aligned} \tag{4.6}$$

i. **GB-HPM**$_C$ solution: Using Eq. (3.5), Eq. (4.6) drives

$$\leq(\kappa_1+\kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)}\parallel \psi_{m-1}-\psi_{n-1}\parallel.$$

Let $m=n+1$, then

$$\parallel \psi_{n+1}-\psi_n\parallel\leq\kappa\parallel\psi_n-\psi_{n-1}\parallel\leq\kappa^2\parallel\psi_{n-1}-\psi_{n-2}\parallel\leq\ldots\leq\kappa^n\parallel\psi_1-\psi_0\parallel,$$

where $\kappa=(\kappa_1+\kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)}$.

In the same manner,

$$\begin{aligned}
\parallel \psi_{m-1}-\psi_{n-1}\parallel&\leq\parallel\psi_{n+1}-\psi_n\parallel+\parallel\psi_{n+2}-\psi_{n+1}\parallel+\ldots+\parallel\psi_m-\psi_{m-1}\parallel\\
&\leq\left(\kappa^n+\kappa^{n+1}+\ldots+\kappa^{m-1}\right)\parallel\psi_1-\psi_0\parallel\\
&\leq\kappa^n\left(\frac{1-\kappa^{m-n}}{1-\kappa}\right)\parallel\psi_1\parallel.
\end{aligned}$$

As $0<\kappa<1$ implies $1-\kappa^{m-n}<1$. Therefore

$$\parallel\Psi_m-\Psi_n\parallel\leq\frac{\kappa^n}{1-\kappa}\parallel\psi_1\parallel.$$

Since $\parallel \psi_1 \parallel < \infty$. $\parallel \Psi_m - \Psi_n \parallel \to_0$ when $n \to \infty$. Hence, Ψ_m is a Cauchy sequence in $\mathfrak{B}$ which is sufficient to prove the convergence of series solution.

ii. **GB-HPM**$_{CF}$ solution: Using Eq. (3.6), Eq. (4.6) gives

$$\leq (\kappa_1 + \kappa_2)(1 - \rho + \rho\tau) \parallel \psi_{m-1} - \psi_{n-1} \parallel .$$

Further proof is analogous to the (i) part.

iii. **GB-HPM**$_{ABC}$ solution: Using Eq. (3.7), Eq. (4.6) gives

$$\leq (\kappa_1 + \kappa_2)\left(1 - \rho + \rho\frac{\tau^\rho}{\Gamma(\rho+1)}\right) \parallel \psi_{m-1} - \psi_{n-1} \parallel .$$

Further proof is analogous to the (i) part. □

Theorem 4.2. [2, 56] *Presume the conditions of Theorem 4.1 then the solution obtained with the aid of GB-HPM solution is unique whenever* $0 < \kappa < 1$, *where*

i.

$$\kappa = (\kappa_1 + \kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)} \text{ for GB-HPM}_C \text{ solution.} \tag{4.7}$$

ii.

$$\kappa = (\kappa_1 + \kappa_2)(1 - \rho + \rho\tau) \text{ for GB-HPM}_{CF} \text{ solution.} \tag{4.8}$$

iii.

$$\kappa = (\kappa_1 + \kappa_2)\left(1 - \rho + \rho\frac{\tau^\rho}{\Gamma(\rho+1)}\right) \text{ for GB-HPM}_{ABC} \text{ solution.} \tag{4.9}$$

Proof. Eq. (3.8) gives

$$\psi(\zeta,\tau) = \phi(\zeta) + \mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau))\right]\right].$$

If possible, let ψ and ψ^* be the two distinct function values. Then

$$\begin{aligned}
\parallel \psi - \psi^* \parallel &= \max\left|\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\Lambda(\psi(\zeta,\tau)) + \Theta(\psi(\zeta,\tau))\right]\right]\right. \\
&\quad \left. - \mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\Lambda(\psi^*(\zeta,\tau)) + \Theta(\psi^*(\zeta,\tau))\right]\right]\right| \\
&\leq \max\left|\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\kappa_1|\psi - \psi^*|\right]\right]\right| \\
&\quad + \max\left|\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\kappa_2|\psi - \psi^*|\right]\right]\right| \\
&\leq (\kappa_1 + \kappa_2)\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1 \parallel \psi - \psi^* \parallel\right],
\end{aligned} \tag{4.10}$$

i. **(GB-HPM$_C$)** For Caputo's terminology Eq. (4.10) reduces to

$$\begin{aligned}
&= (\kappa_1 + \kappa_2)\mathscr{A}_1^{-1}\left[\left(\frac{\gamma}{s}\right)^\rho \mathscr{A}_1 \parallel \psi - \psi^* \parallel\right], \\
&\parallel \psi - \psi^* \parallel \leq (\kappa_1 + \kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)} \parallel \psi - \psi^* \parallel, \\
&(1 - \kappa) \parallel \psi - \psi^* \parallel \leq 0.
\end{aligned} \tag{4.11}$$

Since $0 < \kappa = (\kappa_1 + \kappa_2)\frac{\tau^\rho}{\Gamma(\rho+1)} < 1$, then $\parallel \psi - \psi^* \parallel = 0$ which implies $\psi = \psi^*$. This proves the uniqueness of GB-HPM$_C$ solution.

ii. **(GB-HPM$_{CF}$)** For CF's terminology:
Using $r(s,\gamma,\rho) = 1-\rho+\rho\left(\frac{\gamma}{s}\right)$ uniqueness condition for GB-HPM$_{CF}$ solution can be derived as part (i).
iii. **(GB-HPM$_{ABC}$)** For ABC's terminology:
Using $r(s,\gamma,\rho) = \frac{1-\rho+\rho\left(\frac{\gamma}{s}\right)^{\rho}}{B(\rho)}$ uniqueness condition for GB-HPM$_{ABC}$ solution can be derived as part (i). □

5. Numerical experiments

This section provides the demonstration of the developed method with error analysis for considered examples.

Example 5.1. Consider the time fractional linear SG equation as follows:

$$C_{\tau}^{\rho}\psi + i\psi_{\zeta\zeta} = 0, \quad 0 < \rho \leq 1, \tag{5.1}$$

with initial condition

$$\psi(\zeta,0) = e^{3i\zeta}. \tag{5.2}$$

The exact solution for Eq. (5.1) is established as $\psi(\zeta,\tau) = e^{3i(\zeta+3\tau)}$ for $\rho = 1$. Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta,\tau)) = -i\psi_{\zeta\zeta}$ and $\Theta(\psi(\zeta,\tau)) = 0$ yields the recurrent connection in the following form:

$$\begin{aligned}
&\psi_0 = \psi(\zeta,0),\\
&\psi_{l+1}(\zeta,\tau) = -i\mathscr{A}_1^{-1}[r(s,\gamma,\rho)\mathscr{A}_1[(\psi_l)_{\zeta\zeta}]], \quad l = 0,1,2,\dots\,.
\end{aligned} \tag{5.3}$$

i. **GB-HPM$_C$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.5) for Caputo's terminology, the solution of the above system of successive iterative components can be written in the form

$$\begin{aligned}
\psi_0(\zeta,\tau) &= e^{3i\zeta},\\
\psi_1(\zeta,\tau) &= \frac{9i\tau^{\rho}}{\Gamma(1+\rho)}e^{3i\zeta},\\
\psi_2(\zeta,\tau) &= \frac{(9i\tau^{\rho})^2}{\Gamma(1+2\rho)}e^{3i\zeta},\\
\psi_3(\zeta,\tau) &= \frac{(9i\tau^{\rho})^3}{\Gamma(1+3\rho)}e^{3i\zeta}, \ \dots\,.
\end{aligned}$$

Consequently, the series of the GB-HPM$_C$ solution is represented as

$$\psi^c(\zeta,\tau) = e^{3i\zeta} + \frac{9i\tau^{\rho}}{\Gamma(1+\rho)}e^{3i\zeta} + \frac{(9i\tau^{\rho})^2}{\Gamma(1+2\rho)}e^{3i\zeta} + \frac{(9i\tau^{\rho})^3}{\Gamma(1+3\rho)}e^{3i\zeta} + \dots, \tag{5.4}$$

$$= e^{3i\zeta}E_n(9it^{\rho}). \tag{5.5}$$

where $E_n(.)$ is Mittag-Leffler function.

ii. **GB-HPM$_{CF}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.6) for CF's terminology, the solution of the above system of successive iterative components can be written in the form

$$\psi_0(\zeta,\tau) = e^{3i\zeta},$$

$$\psi_1(\zeta,\tau) = 9ie^{3i\zeta}\left[\rho(\tau-1)+1\right],$$

$$\psi_2(\zeta,\tau) = \frac{(9i)^2}{2!}e^{3i\zeta}\left[\rho^2\left(\tau^2-4\tau+2\right)+4\rho(\tau-1)+2\right],$$

$$\psi_3(\zeta,\tau) = \frac{(9i)^3}{3!}e^{3i\zeta}\left[\rho^3\left(\tau^3-9\tau^2+18\tau-6\right)\right.$$
$$\left.+\rho^2\left(9\tau^2-36\tau+18\right)+18\rho(\tau-1)+6\right],\ldots .$$

Consequently, the series of the GB-HPM$_{CF}$ solution is represented as

$$\begin{aligned}\psi^{CF}(\zeta,\tau) =& e^{3i\zeta}\Big\{1+9i\left[\rho(\tau-1)+1\right]+\frac{(9i)^2}{2!}\left[\rho^2\left(\tau^2-4\tau+2\right)\right.\\&\left.+4\rho(\tau-1)+2\right]+\frac{(9i)^3}{3!}\left[\rho^3\left(\tau^3-9\tau^2+18\tau-6\right)\right.\\&\left.+\rho^2\left(9\tau^2-36\tau+18\right)+18\rho(\tau-1)+6\right],\ \ldots\Big\}.\end{aligned} \tag{5.6}$$

iii. **GB-HPM$_{ABC}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.7) for ABC's terminology, the solution of the above system of successive iterative components can be written in the form

$$\psi_0(\zeta,\tau) = e^{3i\zeta},$$

$$\psi_1(\zeta,\tau) = \frac{9ie^{3i\zeta}}{\Gamma(\rho+1)}\left[\rho\tau^\rho-(\rho-1)\Gamma(\rho+1)\right],$$

$$\psi_2(\zeta,\tau) = e^{3i\zeta}\left(\frac{162\rho^2\tau^\rho}{\Gamma(\rho+1)}-\frac{162\rho\tau^\rho}{\Gamma(\rho+1)}-\frac{81\rho^2\tau^{2\rho}}{\Gamma(2\rho+1)}\right.$$
$$\left.-81-81\rho^2+162\rho\right),\ \ldots .$$

Consequently, the series of the GB-HPM$_{ABC}$ solution is represented as

$$\begin{aligned}&\psi^{ABC}(\zeta,\tau)\\&=e^{3i\zeta}\Bigg\{1+\frac{9ie^{3i\zeta}}{\Gamma(\rho+1)}\left[\rho\tau^\rho-(\rho-1)\Gamma(\rho+1)\right]\\&\quad+\left(\frac{162\rho^2\tau^\rho}{\Gamma(\rho+1)}-\frac{162\rho\tau^\rho}{\Gamma(\rho+1)}-\frac{81\rho^2\tau^{2\rho}}{\Gamma(2\rho+1)}-81-81\rho^2+162\rho\right)+\ldots\Bigg\}.\end{aligned} \tag{5.7}$$

By considering $\rho = 1$, the GB-HPM$_{ABC}$, GB-HPM$_{CF}$ and GB-HPM$_{ABC}$ solutions become

$$\psi(\zeta,\tau) = e^{3i\zeta}\sum_{n=0}^{\infty}\frac{(9i\tau)^n}{n!} = e^{3i(\zeta+3\tau)}, \tag{5.8}$$

which represents the exact solution for the linear SG equation (5.1) with classical derivative.

Example 5.2. Consider $\beta = -\frac{1}{2}$, $\nu(\zeta) = 0$ and $\lambda = -1$ then time fractional non-linear SG equation (1.1) becomes

$$iC_\tau^\rho\psi+\frac{1}{2}\psi_{\zeta\zeta}+|\psi|^2\psi = 0, \quad 0<\rho\leq 1, \tag{5.9}$$

with initial condition

$$\psi(\zeta,0) = e^{i\zeta}. \tag{5.10}$$

The exact solution for Eq. (5.9) is established as $\psi(\zeta,\tau) = e^{i(\zeta+\tau/2)}$ in the case of $\rho = 1$. Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta,\tau)) = i\frac{1}{2}\psi_{\zeta\zeta}$ and $\Theta(\psi(\zeta,\tau)) = i|\psi|^2\psi$ yields the recurrent connection in the following form

$$\begin{aligned}
&\psi_0 = \psi(\zeta,0),\\
&\psi_{l+1}(\zeta,\tau) = i\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\,\mathscr{A}_1\left[\left(\frac{1}{2}\psi_l\right)_{\zeta\zeta} + \mathscr{H}_l\right]\right], \quad l = 0,1,2,\dots,
\end{aligned} \tag{5.11}$$

where non-linear term written as $\Theta(\psi(\zeta,\tau)) = i|\psi|^2\psi = i\psi^2\ \bar{\psi}$ is represented in terms of He's polynomials (3.12) in following form:

$$\begin{aligned}
&\mathscr{H}_0 = i\psi_0^2\bar{\psi}_0, \quad \mathscr{H}_1 = i\psi_0\left(\psi_0\bar{\psi}_1 + 2\bar{\psi}_0\psi_1\right),\\
&\mathscr{H}_2 = i[2\psi_0\left(\bar{\psi}_0\psi_2 + \psi_1\bar{\psi}_1\right) + \psi_1^2\psi_0 + \psi_0^2\bar{\psi}_2], \quad \dots.
\end{aligned} \tag{5.12}$$

i. **GB-HPM$_C$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.5) for Caputo's terminology, the series solution of 3^{rd}-order approximation is obtained as following:

$$\begin{aligned}
&\psi_0(\zeta,\tau) = e^{i\zeta},\\
&\psi_1(\zeta,\tau) = \frac{i}{2}\frac{\tau^\rho}{\Gamma(\rho+1)}e^{i\zeta},\\
&\psi_2(\zeta,\tau) = -\frac{1}{4}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}e^{i\zeta},\\
&\psi_3(\zeta,\tau) = -\frac{i}{8}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\left(5 - 2\frac{\Gamma(2\rho+1)}{(\Gamma(\rho+1))^2}\right)e^{i\zeta}, \quad \dots.
\end{aligned}$$

Similarly, one can get more components for different values of l in Eq. (5.11). Then the GB-HPM$_C$ solution can be written in following series

$$\begin{aligned}
\psi^C(\zeta,\tau) = e^{i\zeta}\Bigg[&1 + \frac{i}{2}\frac{\tau^\rho}{\Gamma(\rho+1)} - \frac{1}{4}\frac{\tau^{2\rho}}{\Gamma(2\rho+1)}\\
&-\frac{i}{8}\frac{\tau^{3\rho}}{\Gamma(3\rho+1)}\left(5 - 2\frac{\Gamma(2\rho+1)}{(\Gamma(\rho+1))^2}\right) + \dots\Bigg].
\end{aligned} \tag{5.13}$$

ii. **GB-HPM$_{CF}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.6) for CF's terminology, the series solution of 3^{rd}-order approximation is obtained as following:

$$\begin{aligned}
&\psi_0(\zeta,\tau) = e^{i\zeta},\\
&\psi_1(\zeta,\tau) = ie^{i\zeta}(\rho(\tau-1)+1),\\
&\psi_2(\zeta,\tau) = -\frac{i}{2}e^{i\zeta}\left[\rho^2(\tau^2 - 4\tau + 2) + 4\rho(\tau-1) + 2\right],\\
&\psi_3(\zeta,\tau) = -\frac{i}{6}e^{i\zeta}\left[\rho^3\left(\tau^3 - 21\tau^2 + 54\tau + 18\right)\right.
\end{aligned}$$

$$+\rho^2\left(2\tau^2-108\tau+54\right)+54\rho(\tau-1)+18\right], \ \ldots .$$

Similarly, one can get more components for different values of l in Eq. (5.11). Then the GB-HPM$_{CF}$ solution can be written in following series

$$\begin{aligned}\psi^{CF}(\zeta,\tau)=&e^{i\zeta}\Bigg\{1+\ i\left[\rho(\tau-1)+1\right]-\frac{i}{2}\left[\rho^2\left(\tau^2-4\tau+2\right)+4\rho(\tau-1)+2\right]\\&-\frac{i}{6}e^{i\zeta}\left[\rho^3\left(\tau^3-21\tau^2+54\tau+18\right)+\rho^2\left(2\tau^2-108\tau+54\right)\right.\\&\left.+54\rho(\tau-1)+18\right]+\ldots\Bigg\}.\end{aligned} \tag{5.14}$$

iii. **GB-HPM$_{ABC}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.7) for ABC's terminology, the approximate series solution is obtained as following:

$$\begin{aligned}&\psi_0(\zeta,\tau)=e^{i\zeta},\\&\psi_1(\zeta,\tau)=\frac{ie^{i\zeta}}{2\Gamma(\rho+1)}\left[\rho\tau^{\rho}-(\rho-1)\Gamma(\rho+1)\right],\\&\psi_2(\zeta,\tau)=e^{i\zeta}\left(-\frac{1}{4}-\frac{\rho^2}{4}+\frac{\rho}{2}+\frac{\rho^2\tau^{\rho}}{2\Gamma(\rho+1)}-\frac{\rho\tau^{\rho}}{2\Gamma(\rho+1)}-\frac{\rho^2\tau^{2\rho}}{4\Gamma(2\rho+1)}\right), \ \ldots .\end{aligned}$$

Similarly, one can get more components for different values of l in Eq. (5.11). Then the GB-HPM$_{ABC}$ solution can be written in following series

$$\begin{aligned}&\psi^{ABC}(\zeta,\tau)\\&=e^{i\zeta}\Bigg\{1+\frac{ie^{i\zeta}}{2\Gamma(\rho+1)}\left[\rho\tau^{\rho}-(\rho-1)\Gamma(\rho+1)\right]\\&\quad+\left(-\frac{1}{4}-\frac{\rho^2}{4}+\frac{\rho}{2}+\frac{\rho^2\tau^{\rho}}{2\Gamma(\rho+1)}-\frac{\rho\tau^{\rho}}{2\Gamma(\rho+1)}-\frac{\rho^2\tau^{2\rho}}{4\Gamma(2\rho+1)}\right)+\ldots\Bigg\}.\end{aligned} \tag{5.15}$$

As $\rho \rightarrow 1$, the GB-HPM$_{C}$, GB-HPM$_{CF}$ and GB-HPM$_{ABC}$ solutions become

$$\psi(\zeta,\tau)=e^{i\zeta}\sum_{n=0}^{\infty}\frac{\left(\frac{i\tau}{2}\right)^n}{n!}=e^{i\left(\zeta+\frac{\tau}{2}\right)}, \tag{5.16}$$

which represents the exact solution for the non-linear SG equation (5.9) with classical derivatives.

Example 5.3. Consider $\beta=-\frac{1}{2},\ \nu(\zeta)=-\cos^2(\zeta)$ and $\lambda=1$ then time fractional non-linear SG equation (1.1) becomes

$$iC_{\tau}^{\rho}\psi+\frac{1}{2}\psi_{\zeta\zeta}-\cos^2(\zeta)\psi-|\psi|^2\psi=0, \qquad 0<\rho\leq 1, \tag{5.17}$$

with initial condition

$$\psi(\zeta,0)=\sin(\zeta). \tag{5.18}$$

The exact solution for Eq. (5.17) is given as $\psi(\zeta,\tau)=e^{-3i\tau/2}\sin(\zeta)$ when $\rho=1$.

Proceeding as developed methodology with choice of $\Lambda(\psi(\zeta,\tau)) = i\left(\frac{1}{2}\psi_{\zeta\zeta} - \cos^2(\zeta)\psi\right)$ and $\Theta(\psi(\zeta,\tau)) = -i|\psi|^2\psi$ yields the recurrent connection in the following form

$$\psi_0 = \psi(\zeta,0),$$
$$\psi_{l+1}(\zeta,\tau) = i\mathscr{A}_1^{-1}\left[r(s,\gamma,\rho)\mathscr{A}_1\left[\left(\frac{1}{2}\psi_l\right)_{\zeta\zeta} - \cos^2(\zeta)\psi_l - \mathscr{H}_l\right]\right],\ l = 0,1,2\dots \tag{5.19}$$

where in order to deal with existing non-linearity $\Theta(\psi(\zeta,\tau)) = -i|\psi|^2\psi = -i\psi^2\bar{\psi}$, He's polynomials (3.12) are used in following form:

$$\mathscr{H}_0 = -i\psi_0^2\bar{\psi}_0,\quad \mathscr{H}_1 = -i\psi_0\left(\psi_0\bar{\psi}_1 + 2\bar{\psi}_0\psi_1\right),$$
$$\mathscr{H}_2 = -i[2\psi_0\left(\bar{\psi}_0\psi_2 + \psi_1\bar{\psi}_1\right) + \psi_1^2\psi_0 + \psi_0^2\bar{\psi}_2],\ \dots . \tag{5.20}$$

i. **GB-HPM$_C$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.5) for Caputo's terminology, the series solution of 3^{rd}-order approximation is obtained as following:

$$\psi_0(\zeta,\tau) = \sin(\zeta),$$
$$\psi_1(\zeta,\tau) = -\frac{3i\tau^\rho\sin(\zeta)}{2\Gamma(\rho+1)},$$
$$\psi_2(\zeta,\tau) = -\frac{9\tau^{2\rho}\sin(\zeta)}{4\Gamma(2\rho+1)},$$
$$\psi_3(\zeta,\tau) = \frac{9i\tau^{3\rho}\sin(\zeta)[4(\Gamma(\rho+1))^2\sin^2(\zeta) - 2\Gamma(2\rho+1)\sin^2(\zeta) + 3(\Gamma(\rho+1))^2]}{8(\Gamma(\rho+1))^2\Gamma(3\rho+1)},\ \dots .$$

Similarly, one can get more components for different values of l in Eq. (5.19). Then the GB-HPM$_C$ solution can be written in following series

$$\psi^C(\zeta,\tau) = \sin(\zeta)\left\{1 - \frac{3i\tau^\rho}{2\Gamma(\rho+1)} - \frac{9\tau^{2\rho}}{4\Gamma(2\rho+1)}\right.$$
$$\left.\frac{9i\tau^{3\rho}[4(\Gamma(\rho+1))^2\sin^2(\zeta) - 2\Gamma(2\rho+1)\sin^2(\zeta) + 3(\Gamma(\rho+1))^2]}{8(\Gamma(\rho+1))^2\Gamma(3\rho+1)},\dots\right\}. \tag{5.21}$$

ii. **GB-HPM$_{CF}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.6) for CF's terminology, the series solution of 3^{rd}-order approximation is obtained as following:

$$\psi_0(\zeta,\tau) = \sin(\zeta),$$

$$\psi_1(\zeta,\tau) = -\frac{3i\sin(\zeta)}{2}\left[\rho(\tau-1)+1\right],$$
$$\psi_2(\zeta,\tau) = -\frac{9\sin(\zeta)}{8}\left[\rho^2\left(\tau^2-4\tau+2\right)+4\rho(\tau-1)+2\right], \ \dots .$$

Similarly, one can get more components for different values of l in Eq. (5.19). Then the GB-HPM$_{CF}$ solution can be written in following series

$$\psi^{CF}(\zeta,\tau) = \sin(\zeta)\Bigg\{1-\frac{3i}{2}[\rho(\tau-1)+1]-\frac{9}{8}[\rho^2(\tau^2-4\tau+2) + 4\rho(\tau-1)+2]+\dots\Bigg\}. \tag{5.22}$$

iii. **GB-HPM$_{ABC}$** solution: Using explicit form of $r(s,\gamma,\rho)$ described in Eq. (3.7) for ABC's terminology, the approximate series solution is obtained as following:

$$\psi_0(\zeta,\tau) = \sin(\zeta),$$
$$\psi_1(\zeta,\tau) = -\frac{3i\sin(\zeta)}{2\Gamma(\rho+1)}\left(\rho\tau^\rho-(\rho-1)\Gamma(\rho+1)\right),$$
$$\psi_2(\zeta,\tau) = -\frac{9\sin(\zeta)}{4\Gamma(\rho+1)\Gamma(2\rho+1)}\Bigg(\Gamma(\rho+1)\Gamma(2\rho+1) - 2\rho\Gamma(\rho+1)\Gamma(2\rho+1)+2\rho\tau^\rho\Gamma(2\rho+1) + \rho^2\Gamma(\rho+1)\Gamma(2\rho+1)-2\rho^2\tau^\rho\Gamma(2\rho+1)+\rho^2\tau^{2\rho}\Gamma(\rho+1)\Bigg),$$
$$\dots .$$

Similarly, one can get more components for different values of l in Eq. (5.19). Then the GB-HPM$_{ABC}$ solution can be written in following series

$$\begin{aligned}\psi^{ABC}(\zeta,\tau) &= \sin(\zeta)\Bigg\{1-\frac{3i\sin(\zeta)}{2\Gamma(\rho+1)}\left(\rho\tau^\rho-(\rho-1)\Gamma(\rho+1)\right)-\frac{9\sin(\zeta)}{4\Gamma(\rho+1)\Gamma(2\rho+1)}\\ &\times\Big(\Gamma(\rho+1)\Gamma(2\rho+1)-2\rho\Gamma(\rho+1)\Gamma(2\rho+1)+2\rho\tau^\rho\Gamma(2\rho+1)\\ &+\rho^2\Gamma(\rho+1)\Gamma(2\rho+1)-2\rho^2\tau^\rho\Gamma(2\rho+1)+\rho^2\tau^{2\rho}\Gamma(\rho+1)\Big)+\dots\Bigg\}.\end{aligned} \tag{5.23}$$

Considering $\rho=1$, the GB-HPM$_C$, GB-HPM$_{CF}$ and GB-HPM$_{ABC}$ solutions gives

$$\psi(\zeta,\tau) = \sin(\zeta)\sum_{n=0}^{\infty}\frac{\left(\frac{-3i\tau}{2}\right)^n}{n!} = \sin(\zeta)e^{-\frac{3i\tau}{2}}, \tag{5.24}$$

which represents the exact solution for the non-linear SG equation (5.17) with classical derivative.

6. Results and discussions

The advantageous feature about this scheme is that, unlike the variables separation strategy and most of numerical schemes, which requires the use of both initial and boundary conditions, it solves for an analytic solution by using only the initial condition. In Example 5.1 the GB-HPM solution is found in accordance with existing solutions obtained using Sumudu transform iterative method (STIM) [31] and differential transform method (DTM) [23] for Caputo's derivative. In Figure 1, two-dimensional graphs are provided to compare approximate real and imaginary part of the GB-HPM solution for Caputo, CF and ABC terminologies when $\rho = 0.5, 0.7, 0.9$ and 1 at $\tau = 0.5$ for Example 5.1. Resulting in waves with higher amplitude displays the impact of variation in fractional order. Table 1. describes the performance of present scheme in terms of absolute error for approximate solution of Example 5.1. It demonstrates the efficiency of suggested method with maximum error of order 10^{-5}.

Table 1. Computed absolute errors for GB-HPM$_{20^{th}-order}$ solutions for Example 5.1 when ρ=1.

ζ	τ	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$	|Exact|
-5	0.1	4.7123×10^{-16}	4.7123×10^{-16}	4.7123×10^{-16}	1.0000
	0.3	2.2250×10^{-11}	2.2250×10^{-11}	2.2250×10^{-11}	1.0000
	0.6	4.5730×10^{-5}	4.5730×10^{-5}	4.5730×10^{-5}	1.0000
-1	0.1	1.1102×10^{-16}	1.1102×10^{-16}	1.1102×10^{-16}	1.0000
	0.3	2.2250×10^{-11}	2.2250×10^{-11}	2.2250×10^{-11}	1.0000
	0.6	4.5730×10^{-5}	4.5730×10^{-5}	4.5730×10^{-5}	1.0000
1	0.1	1.1102×10^{-16}	1.1102×10^{-16}	1.1102×10^{-16}	1.0000
	0.3	2.2249×10^{-11}	2.2249×10^{-11}	2.2249×10^{-11}	1.0000
	0.6	4.5730×10^{-5}	4.5730×10^{-5}	4.5730×10^{-5}	1.0000
5	0.1	2.9894×10^{-16}	2.9894×10^{-16}	2.9894×10^{-16}	1.0000
	0.3	2.2250×10^{-11}	2.2250×10^{-11}	2.2250×10^{-11}	1.0000
	0.6	4.5730×10^{-5}	4.5730×10^{-5}	4.5730×10^{-5}	1.0000

Figure 2 depicts the graphical presentation of solutions obtained for distinct fractional order values ($\rho = 0.5, 0.7, 0.9$ and 1) of Example 5.2 with Caputo, CF and ABC operators in computational domain $-5 \leq \zeta \leq 5$ at $\tau = 0.5$. Figure 2 demonstrates that each of the subfigures behave almost in identical manner, are in comparable nature and have good precision alignment. For Example 5.2, the absolute errors of present scheme are compared to the existing results of modified generalized Mittag-Leffler function method (MGMLFM) [10] which verify the superiority of present scheme for non-linear differential equations. Table 3 exhibits the fast convergence of approximate series solutions to exact solution with evaluation of more components of series solutions.

The wave form behavior of numerical results obtained for Example 5.3 with Caputo, CF and ABC derivatives at different values of fractional order ($\rho = 0.5, 0.7, 0.9$ and 1) is presented in Figure 3 for limited spatial domain of $-5 \leq \zeta \leq 5$ at $\tau = 0.5$. In Table 4, the tabulated results of Example 5.3 again indicates the better perfor-

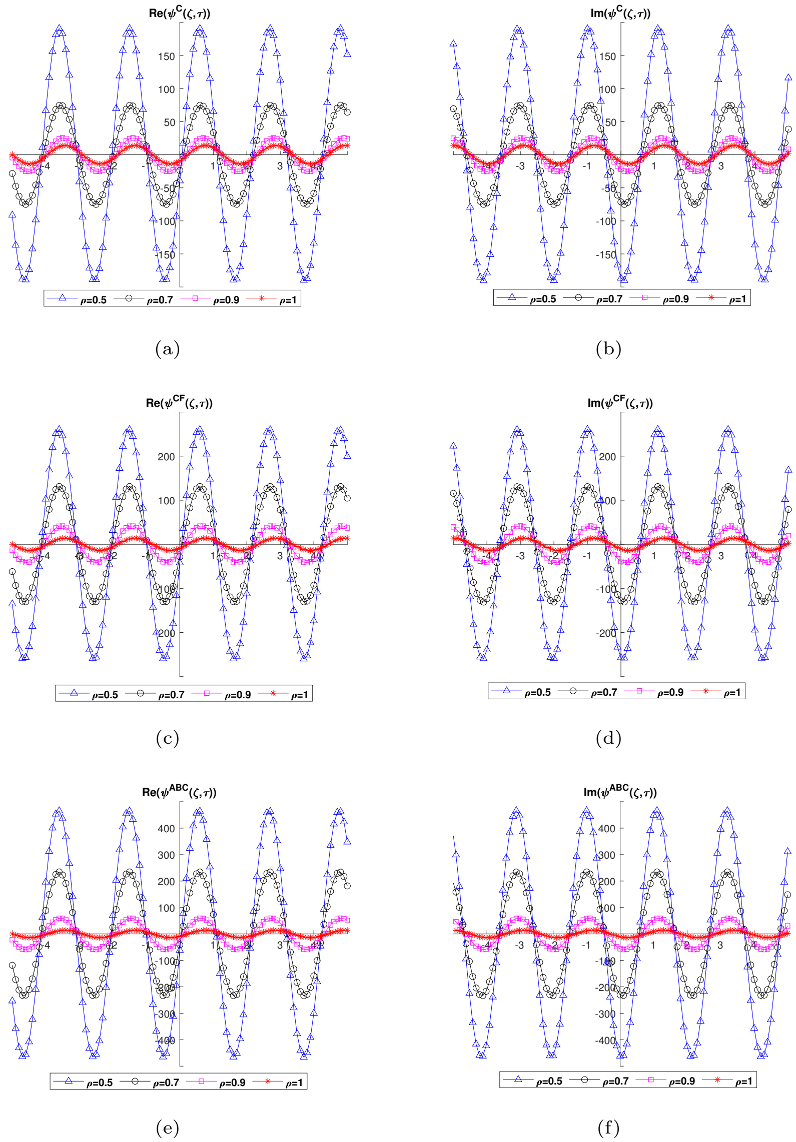

Figure 1. Waveform behavior of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.

mance of present scheme in comparison of MGMLFM [10] results. Table 5 ensures that the required precision can be attained through the assessment of additional components of series solutions.

In present study it is observed that however the CF and ABC operators with non-singular kernels are considered extensions of Caputo operator but due to their high sensitivity with respect to the small change in fractional order, Caputo operator still holds its legacy. GB-HPM$_C$ solutions are found more consistent with respect to the small variations in ρ. For $\rho = 1$ GB-HPM solution for Caputo, CF and ABC operators coincides and results in a rapidly convergent series solution. Finding the

Table 2. Comparison of the absolute errors of obtained solutions using GB-HPM$_{3^{rd}-order}$ and modified generalized Mittag-Leffler function method (MGMLFM) for Example 5.2 when $\rho = 1$.

ζ	τ	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$	MGMLFM [10]
-5	0.01	2.6042×10^{-11}	2.6042×10^{-11}	2.6042×10^{-11}	8.3335×10^{-8}
	0.03	2.1094×10^{-9}	2.1094×10^{-9}	2.1094×10^{-9}	2.2506×10^{-6}
	0.06	3.3750×10^{-8}	3.3750×10^{-8}	3.3750×10^{-8}	1.8020×10^{-5}
-1	0.01	2.6042×10^{-11}	2.6042×10^{-11}	2.6042×10^{-11}	8.3335×10^{-8}
	0.03	2.1094×10^{-9}	2.1094×10^{-9}	2.1094×10^{-9}	2.2506×10^{-6}
	0.06	3.3750×10^{-8}	3.3750×10^{-8}	3.3750×10^{-8}	1.8020×10^{-5}
1	0.01	2.6042×10^{-11}	2.6042×10^{-11}	2.6042×10^{-11}	8.3335×10^{-8}
	0.03	2.1094×10^{-9}	2.1094×10^{-9}	2.1094×10^{-9}	2.2506×10^{-6}
	0.06	3.3750×10^{-8}	3.3750×10^{-8}	3.3750×10^{-8}	1.8020×10^{-5}
5	0.01	2.6042×10^{-11}	2.6042×10^{-11}	2.6042×10^{-11}	8.3335×10^{-8}
	0.03	2.1094×10^{-9}	2.1094×10^{-9}	2.1094×10^{-9}	2.2506×10^{-6}
	0.06	3.3750×10^{-8}	3.3750×10^{-8}	3.3750×10^{-8}	1.8020×10^{-5}

Table 3. Comparison of the absolute errors of obtained GB-HPM solutions for Example 5.2 when $\rho = 1$ and $\zeta = 5$.

	3^{rd}-order solutions			5^{th}-order solutions		
τ	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$
0.1	2.6041×10^{-7}	2.6041×10^{-7}	2.6041×10^{-7}	2.1701×10^{-11}	2.1701×10^{-11}	2.1701×10^{-11}
0.2	4.1661×10^{-6}	4.1661×10^{-6}	4.1661×10^{-6}	1.3888×10^{-9}	1.3888×10^{-9}	1.3888×10^{-9}
0.3	2.1087×10^{-5}	2.1087×10^{-5}	2.1087×10^{-5}	1.5818×10^{-8}	1.5818×10^{-8}	1.5818×10^{-8}
0.4	6.6631×10^{-5}	6.6631×10^{-5}	6.6631×10^{-5}	8.8862×10^{-8}	8.8862×10^{-8}	8.8862×10^{-8}
0.5	1.6262×10^{-4}	1.6262×10^{-4}	1.6262×10^{-4}	3.3892×10^{-7}	3.3892×10^{-7}	3.3892×10^{-7}
0.6	3.3710×10^{-4}	3.3710×10^{-4}	3.3710×10^{-4}	1.0118×10^{-6}	1.0118×10^{-6}	1.0118×10^{-6}
0.7	6.2424×10^{-4}	6.2424×10^{-4}	6.2424×10^{-4}	2.5508×10^{-6}	2.5508×10^{-6}	2.5508×10^{-6}
0.8	1.0644×10^{-3}	1.0644×10^{-3}	1.0644×10^{-3}	5.6819×10^{-6}	5.6819×10^{-6}	5.6819×10^{-6}
0.9	1.7040×10^{-3}	1.7040×10^{-3}	1.7040×10^{-3}	1.1515×10^{-5}	1.1515×10^{-5}	1.1515×10^{-5}
1	2.5955×10^{-3}	2.5955×10^{-3}	2.5955×10^{-3}	2.1660×10^{-5}	2.1660×10^{-5}	2.1660×10^{-5}

closed form of a series solution allows one to obtain the desired exact solution for considered problem.

7. Conclusion

In this study, approximate semi-analytic solutions for time fractional linear and non-linear Schrödinger equations are investigated using three types of fractional derivatives: Caputo, Caputo-Fabrizio (CF), and Atangana-Baleanu-Caputo (ABC). These solutions are obtained through the recently proposed GB transform followed by the HPM. The study establishes the application of the GB transform to the considered fractional derivatives and demonstrates the GB-HPM procedure for general time fractional non-linear equations.

Compared to the traditional HPM scheme, the suggested GB-HPM scheme

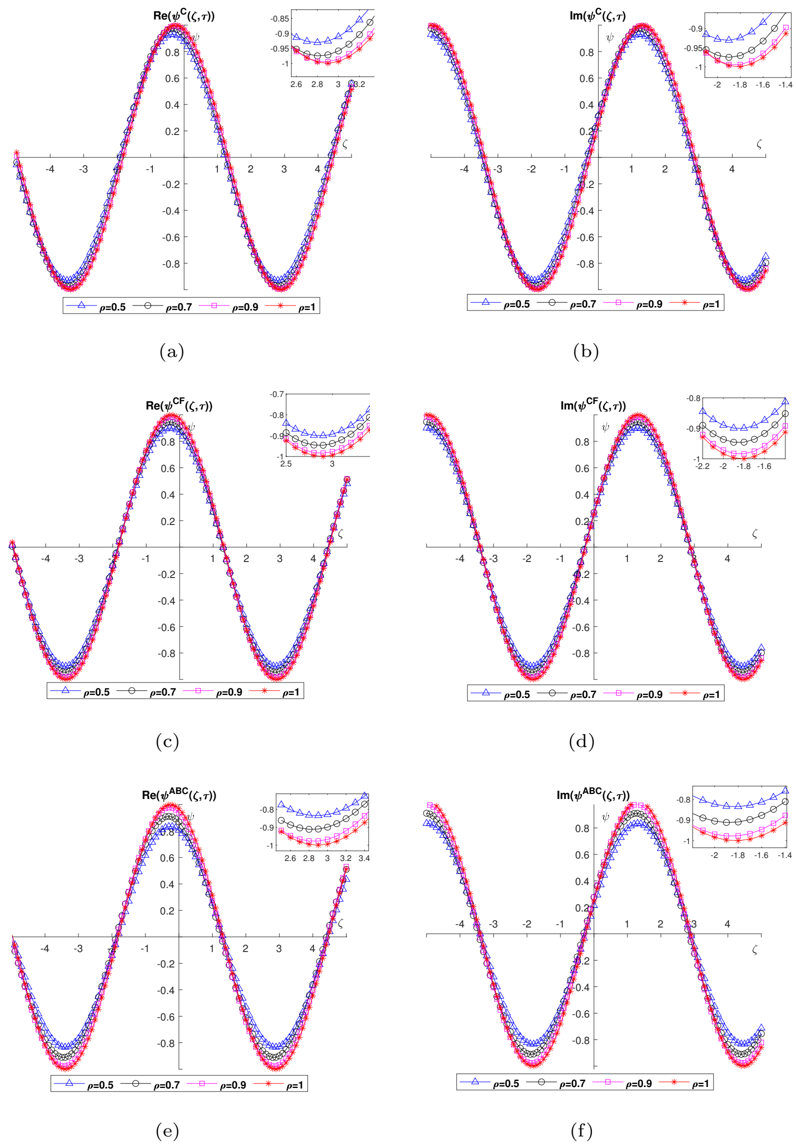

Figure 2. Waveform behavior of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.

streamlines the sequence estimation process by avoiding the need to compute fractional derivatives and fractional integrals in the recursive mechanism. Numerical results of the GB-HPM solutions are displayed in figures and tables for different values of the parameter ρ across all three fractional terminologies. Graphical simulations help in understanding the impact of Caputo, CF, and ABC operators on the complex internal structure of wave propagation.

A comparative error analysis of the GB-HPM solution with previously published results reflects the supremacy and consistency of the proposed scheme. The numer-

Table 4. Comparison of the absolute errors of obtained solutions using GB-HPM$_{3^{rd}-order}$ and modified generalized Mittag-Leffler function method (MGMLFM) for Example 5.3 when $\rho = 1$.

ζ	τ	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$	MGMLFM [10]
-5	0.01	2.0227×10^{-9}	2.0227×10^{-9}	2.0227×10^{-9}	6.6173×10^{-7}
	0.03	1.6384×10^{-7}	1.6384×10^{-7}	1.6384×10^{-7}	1.7956×10^{-5}
	0.06	2.6212×10^{-6}	2.6212×10^{-6}	2.6212×10^{-6}	1.4613×10^{-4}
-1	0.01	1.7750×10^{-9}	1.7750×10^{-9}	1.7750×10^{-9}	4.4717×10^{-7}
	0.03	1.4377×10^{-7}	1.4377×10^{-7}	1.4377×10^{-7}	1.2139×10^{-5}
	0.06	2.3001×10^{-6}	2.3001×10^{-6}	2.3001×10^{-6}	9.8906×10^{-5}
1	0.01	1.7750×10^{-9}	1.7750×10^{-9}	1.7750×10^{-9}	4.4717×10^{-7}
	0.03	1.4377×10^{-7}	1.4377×10^{-7}	1.4377×10^{-7}	1.1395×10^{-5}
	0.06	2.3001×10^{-6}	2.3001×10^{-6}	2.3001×10^{-6}	9.8906×10^{-5}
5	0.01	2.0227×10^{-9}	2.0227×10^{-9}	2.0227×10^{-9}	6.6173×10^{-7}
	0.03	1.6384×10^{-7}	1.6384×10^{-7}	1.6384×10^{-7}	1.7956×10^{-5}
	0.06	2.6212×10^{-6}	2.6212×10^{-6}	2.6212×10^{-6}	1.4613×10^{-4}

Table 5. Comparison of the absolute errors of obtained GB-HPM solutions for Example 5.3 when $\rho = 1$ and $\zeta = 5$.

	3^{rd}-order solutions			5^{th}-order solutions		
τ	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$	GB-HPM$_C$	GB-HPM$_{CF}$	GB-HPM$_{ABC}$
0.1	2.0221×10^{-5}	2.0221×10^{-5}	2.0221×10^{-5}	1.5168×10^{-8}	1.5168×10^{-8}	1.5168×10^{-8}
0.2	3.2325×10^{-4}	3.2325×10^{-4}	3.2325×10^{-4}	9.7024×10^{-7}	9.7024×10^{-7}	9.7024×10^{-7}
0.3	1.6340×10^{-3}	1.6340×10^{-3}	1.6340×10^{-3}	1.1042×10^{-5}	1.1042×10^{-5}	1.1042×10^{-5}
0.4	5.1534×10^{-3}	5.1534×10^{-3}	5.1534×10^{-3}	6.1967×10^{-5}	6.1967×10^{-5}	6.1967×10^{-5}
0.5	1.2548×10^{-2}	1.2548×10^{-2}	1.2548×10^{-2}	2.3602×10^{-4}	2.3602×10^{-4}	2.3602×10^{-4}
0.6	2.5933×10^{-2}	2.5933×10^{-2}	2.5933×10^{-2}	7.0343×10^{-4}	7.0343×10^{-4}	7.0343×10^{-4}
0.7	4.7858×10^{-2}	4.7858×10^{-2}	4.7858×10^{-2}	1.7698×10^{-3}	1.7698×10^{-3}	1.7698×10^{-3}
0.8	8.1279×10^{-2}	8.1279×10^{-2}	8.1279×10^{-2}	3.9334×10^{-3}	3.9334×10^{-3}	3.9334×10^{-3}
0.9	1.2953×10^{-1}	1.2953×10^{-1}	1.2953×10^{-1}	7.9510×10^{-3}	7.9510×10^{-3}	7.9510×10^{-3}
1	1.9632×10^{-1}	1.9632×10^{-1}	1.9632×10^{-1}	1.4913×10^{-2}	1.4913×10^{-2}	1.4913×10^{-2}

ical results demonstrate high accuracy, even when only a few components of the series solution are considered. The straightforward GB-HPM scheme is effective in resolving non-linear fractional differential equations and has enormous potential for identifying hidden mechanisms in dynamic mathematical systems.

Acknowledgements

We would like to thank the worthy reviewers for their valuable feedback which helped us to strengthen the manuscript.

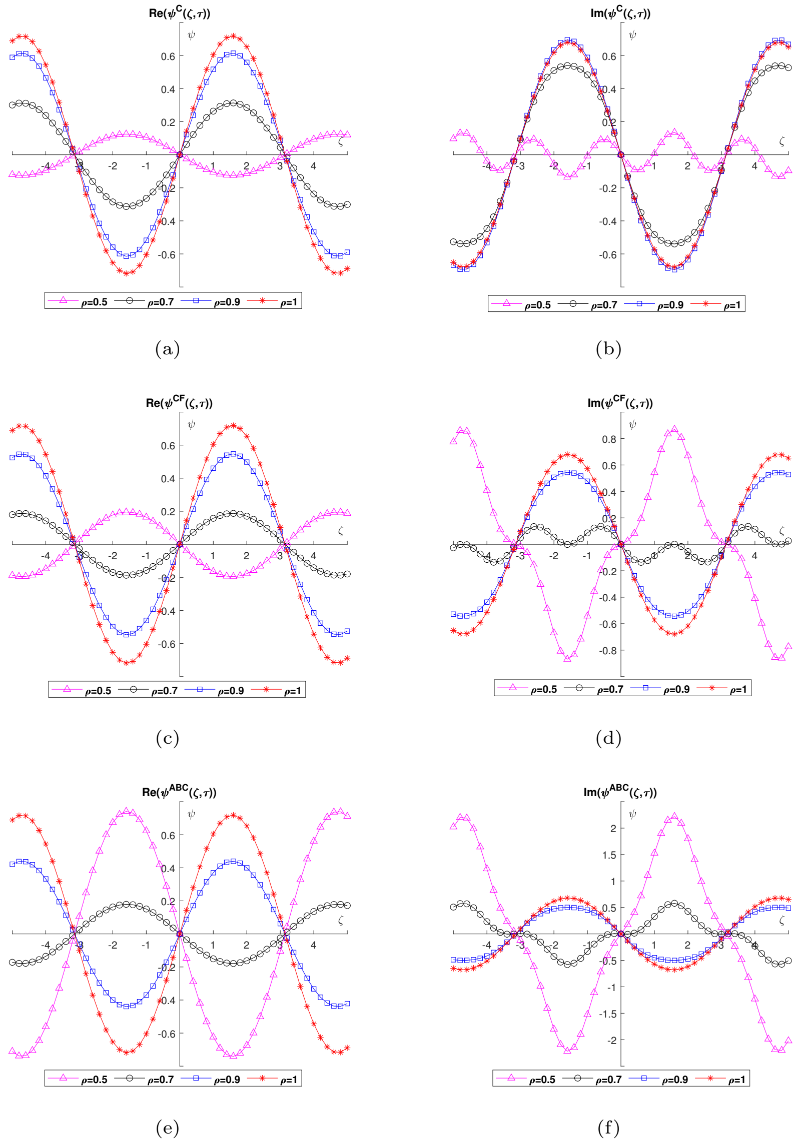

Figure 3. Waveform behavior of real (Re) and imaginary (Im) part of GB-HPM solution for Caputo, CF and ABC operators at $\tau = 0.5$.

References

[1] S. Abbasbandy, *Iterated He's homotopy perturbation method for quadratic Riccati differential equation*, Applied Mathematics and Computation, 2006, 175(1), 581–589.

[2] R. K. Adivi Sri Venkata, A. Kirubanandam and R. Kondooru, *Numerical solutions of time fractional Sawada Kotera Ito equation via natural transform decomposition method with singular and non-singular kernel derivatives*, Mathematical Methods in the Applied Sciences, 2021, 44(18), 14025–14040.

[3] L. Akinyemi, K. S. Nisar, C. A. Saleel, H. Rezazadeh, P. Veeresha, M. M. Khater and M. Inc, *Novel approach to the analysis of fifth-order weakly nonlocal fractional Schrödinger equation with Caputo derivative*, Results in Physics, 2021, 31, 104958.

[4] G. Akram, M. Sadaf and I. Zainab, *Effect of a new local derivative on space-time fractional non-linear Schrödinger equation and its stability analysis*, Optical and Quantum Electronics, 2023, 55(9), 834.

[5] A. Ali, J. Ahmad and S. Javed, *Exploring the dynamic nature of soliton solutions to the fractional coupled non-linear Schrödinger model with their sensitivity analysis*, Optical and Quantum Electronics, 2023, 55(9), 810.

[6] A. Ali, A. R. Seadawy and D. Lu, *Soliton solutions of the non-linear Schrödinger equation with the dual power law non-linearity and resonant non-linear Schrödinger equation and their modulation instability analysis*, Optik, 2017, 145, 79–88.

[7] K. K. Ali and M. Maneea, *Optical solitons using optimal homotopy analysis method for time-fractional (1+1)-dimensional coupled non-linear Schrödinger equations*, Optik, 2023, 283, 170907.

[8] K. K. Ali, M. Maneea and M. S. Mohamed, *Solving nonlinear fractional models in superconductivity using the q-homotopy analysis transform method*, Journal of Mathematics, 2023, 2023, 6647375.

[9] S. A. Alsallami, M. Maneea, E. M. Khalil, S. Abdel-Khalek and K. K. Ali, *Insights into time fractional dynamics in the Belousov-Zhabotinsky system through singular and non-singular kernels*, Scientific Reports, 2023, 13(1), 22347.

[10] I. G. Ameen, R. O. A. Taie and H. M. Ali, *Two effective methods for solving non-linear coupled time-fractional Schrödinger equations*, Alexandria Engineering Journal, 2023, 70, 331–347.

[11] V. D. Ao, D. V. Tran, K. T. Pham, D. M. Nguyen, H. D. Tran, T. K. Do, V. H. Do and T. V. Phan, *A Schrödinger equation for evolutionary dynamics*, Quantum Reports, 2023, 5(4), 659–682.

[12] V. E. Arkhincheev, *Anomalous diffusion in inhomogeneous media: Some exact results*, Modelling Measurement and Control a General Physics Electronics and Electrical Engineering, 1993, 49, 11.

[13] S. Arora and A. Pasrija, *A novel integral transform operator and its applications*, Iranian Journal of Numerical Analysis and Optimization, 2023, 13(3), 553–575.

[14] S. Arshad, I. Saleem, A. Akgül, J. Huang, Y. Tang and S. M. Eldin, *A novel numerical method for solving the Caputo-Fabrizio fractional differential equation*, AIMS Math., 2023, 8(4), 9535–9556.

[15] A. Atangana and D. Baleanu, *New fractional derivatives with non-local and non-singular kernel*, Thermal Science, 2016, 20(2), 763–769.

[16] R. P. Bell, *The application of quantum mechanics to chemical kinetics*, Proceedings of the Royal Society of London, 1933, 139(838), 466–474.

[17] M. Bilal, J. Ren, M. Inc and R. K. Alhefthi, *Optical soliton and other solutions to the non-linear dynamical system via two efficient analytical mathematical schemes*, Optical and Quantum Electronics, 2023, 55(11), 938.

[18] M. Born, *Quantenmechanik der stoßvorgänge*, Zeitschrift für Physik, 1926, 38(11), 803–827.

[19] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation & Applications, 2015, 1(2), 73–85.

[20] A. Chabchoub and R. H. Grimshaw, *The hydrodynamic non-linear Schrödinger equation: Space and time*, Fluids, 2016, 1(3), 23.

[21] K. S. Cole, *Electric Conductance of Biological Systems*, Cold Spring Harbor Laboratory Press, NY, USA, 1933.

[22] V. D. Djordjevic and T. M. Atanackovic, *Similarity solutions to non-linear heat conduction and Burgers/Korteweg–deVries fractional equations*, Journal of Computational and Applied Mathematics, 2008, 222(2), 701–714.

[23] S. O. Edeki, G. O. Akinlabi and S. A. Adeosun, *Analytic and numerical solutions of time-fractional linear Schrödinger equation*, Communications in Mathematics and Applications, 2016, 7(1), 1–10.

[24] Z. Y. Fan, K. K. Ali, M. Maneea, M. Inc and S. W. Yao, *Solution of time fractional Fitzhugh–Nagumo equation using semi analytical techniques*, Results in Physics, 2023, 51, 106679.

[25] A. Ghorbani, *Beyond Adomian polynomials: He polynomials*, Chaos, Solitons & Fractals, 2009, 39(3), 1486–1492.

[26] W. G. Glöckle and T. F. Nonnenmacher, *A fractional calculus approach to self-similar protein dynamics*, Biophysical Journal, 1995, 68(1), 46–53.

[27] S. H. Hamed, E. A. Yousif and A. I. Arbab, *Analytic and approximate solutions of the space-time fractional Schrödinger equations by homotopy perturbation Sumudu transform method*, in Abstract and Applied Analysis, 2014, 2014, 863015.

[28] J. H. He, *Homotopy perturbation method: A new non-linear analytical technique*, Applied Mathematics and Computation, 2003, 135(1), 73–79.

[29] B. Hong, J. Wang and C. Li, *Analytical solutions to a class of fractional coupled non-linear Schrödinger equations via Laplace-HPM technique*, AIMS Math., 2023, 8(7), 15670–15688.

[30] Z. Z. Kang, T. C. Xia and W. X. Ma, *Riemann–Hilbert approach and N-soliton solution for an eighth-order non-linear Schrödinger equation in an optical fiber*, Advances in Difference Equations, 2019, 2019, 188.

[31] M. Kapoor, *Analytical approach for solution of linear and non-linear time-fractional Schrödinger equations by employing Sumudu transform iterative method*, International Journal of Applied and Computational Mathematics, 2023, 9(3), 38.

[32] A. Khan, A. Ali, S. Ahmad, S. Saifullah, K. Nonlaopon and A. Akgül, *Non-linear Schrödinger equation under non-singular fractional operators: A computational study*, Results in Physics, 2022, 43, 106062.

[33] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, the Netherlands, 2006.

[34] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, USA, 2003.

[35] N. M. Kocherginsky, *Interpretation of Schrödinger equation based on classical mechanics and spin*, Quantum Studies: Mathematics and Foundations, 2021, 8(2), 217–227.

[36] B. T. Krishna, *Studies on fractional order differentiators and integrators: A survey*, Signal Processing, 2011, 91(3), 386–426.

[37] N. Laskin, *Fractional market dynamics*, Physica A: Statistical Mechanics and its Applications, 2000, 287(3–4), 482–492.

[38] N. Laskin, *Fractional quantum mechanics and Lévy path integrals*, Physics Letters A, 2000, 268(4–6), 298–305.

[39] N. Laskin, *Fractional quantum mechanics*, Physical Review E, 2000, 62(3), 3135.

[40] N. Laskin, *Fractional Schrödinger equation*, Physical Review E, 2002, 66(5), 056108.

[41] M. Lechelon, Y. Meriguet, M. Gori, S. Ruffenach, I. Nardecchia, E. Floriani, D. Coquillat, S. Teppe, S. Mailfert, D. Marguet and P. Ferrier, *Experimental evidence for long-distance electrodynamic intermolecular forces*, Science Advances, 2022, 8(7), eabl5855.

[42] E. K. Lenzi, E. C. Gabrick, E. Sayari, A. S. de Castro, J. Trobia and A. M. Batista, *Anomalous relaxation and three-level system: A fractional Schrödinger equation approach*, Quantum Reports, 2023, 5(2), 442–458.

[43] W. X. Ma and M. Chen, *Direct search for exact solutions to the non-linear Schrödinger equation*, Applied Mathematics and Computation, 2009, 215(8), 2835–2842.

[44] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.

[45] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*, Academic Press, USA, 1998.

[46] A. Qazza, A. Burqan and R. Saadeh, *A new attractive method in solving families of fractional differential equations by a new transform*, Mathematics, 2021, 9(23), 3039.

[47] S. Z. Rida, H. M. El-Sherbiny and A. A. M. Arafa, *On the solution of the fractional non-linear Schrödinger equation*, Physics Letters A, 2008, 372(5), 553–558.

[48] R. Saadeh, A. Qazza and A. Burqan, *A new integral transform: ARA transform and its properties and applications*, Symmetry, 2020, 12(6), 925.

[49] R. Z. Saadeh and B. F. A. Ghazal, *A new approach on transforms: Formable integral transform and its applications*, Axioms, 2021, 10(4), 332.

[50] L. J. Shen, *Fractional derivative models for viscoelastic materials at finite deformations*, International Journal of Solids and Structures, 2020, 190, 226–237.

[51] H. Tariq, M. Sadaf, G. Akram, H. Rezazadeh, J. Baili, Y. P. Lv and H. Ahmad, *Computational study for the conformable non-linear Schrödinger equation with cubic–quintic–septic non-linearities*, Results in Physics, 2021, 30, 104839.

[52] A. M. Wazwaz and G. Q. Xu, *Bright, dark and Gaussons optical solutions for fourth-order Schrödinger equations with cubic–quintic and logarithmic nonlinearities*, Optik, 2020, 202, 163564.

[53] Y. Xie, Z. Yang and L. Li, *New exact solutions to the high dispersive cubic–quintic non-linear Schrödinger equation*, Physics Letters A, 2018, 382(36), 2506–2514.

[54] T. Xu, B. Tian, L. L. Li, X. Lü and C. Zhang, *Dynamics of Alfvén solitons in inhomogeneous plasmas*, Physics of Plasmas, 2008, 15(10), 102307.

[55] N. Zettili, *Quantum Mechanics: Concepts and Applications*, 2nd Ed., John Wiley & Sons, Chichester, UK, 2009.

[56] M. X. Zhou, A. R. Kanth, K. Aruna, K. Raghavendar, H. Rezazadeh, M. Inc and A. A. Aly, *Numerical solutions of time fractional Zakharov-Kuznetsov equation via natural transform decomposition method with non-singular kernel derivatives*, Journal of Function Spaces, 2021, 2021, 9884027.