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APPROXIMATE SOLUTION OF FRACTIONAL-ORDER FITZHUGH-NAGUMO EQUATION WITH IN NATURAL TRANSFORM

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Abstract In this paper, we use, for the first time, the Natural residual power series method (NRPSM) as a new iteration method to study the Caputo version of the Fitzhugh-Nagumo equation. The Fitzhugh-Nagumo equation is an essential mathematical model that is widely used to characterize the behavior of excitable systems, and is valuable for understanding significant physiological and biological processes. To start, we translate the Fitzhugh-Nagumo equation system into its Natural domain representation, and then we employ the NRPSM to obtain a series form result. After that, we present a new iteration methodology for improving the convergence characteristics of the series solution as well as the accuracy of the computations. In this paper, a comprehensive approach for investigating the Fitzhugh-Nagumo equation with Natural transform is developed and validated, thus can help researchers to explore the various dynamics and behaviors of the excitable systems more effectively. Based on the results obtained, we conclude that the suggested approach to the solution of DEs with the Caputo operator has a great potential for different applications in several fields of science and engineering.

Keywords Natural iterative transform method, natural residual power series method, Fitzhugh-Nagumo equation, fractional order differential equation, Caputo operator.

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1. Introduction

Fractional differential equations extend the ordinary differential equation by simply replacing the conventional derivative with the fractional derivative operator.

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Fractional derivatives, in contrast to integer-order derivatives, are nonlocal operators that also include the function history. The family of fractional derivatives, which includes the Caputo fractional derivatives and Riemann-Liouville, has a wide distribution. Fractional equations can describe complex systems in physics and engineering, just as ordinary differential equations can. In addition, they have become the only methods for solving problems with long memory, nonlocality, and anomalous diffusion, such as viscoelastic materials, porous media flow, control theory, and more. One of the key obstacles to solving fractional differential equations analytically is that it is quite difficult [5, 7, 10, 20, 28].

In most cases, numerical approaches are required to solve the problem. The scientific computing and applied mathematics fields face significant challenges when it comes to the numerical integration of fractional differential equations. Its development focuses on the rapidity and stability of the numerical schemes. Fractional calculus allows us to use the power of differential equations to solve certain equations, and as a result, it may model real-world systems more accurately. The concept of fractional differential equations comprises an interdisciplinary field that ultimately gives rise to many exciting mathematical discoveries and important applications [2, 21, 22].

Originally described to explain the behavior of spike potentials in a nerve axon, the FitzHugh-Nagumo equation simplified the Hodgkin-Huxley model. It was a 1960s invention by mathematician Richard FitzHugh and physiologist Jin-Ichi Nagumo. The FitzHugh-Nagumo model simplifies the Hodgkin-Huxley model, keeping the key but qualitative properties. The FitzHugh-Nagumo model comprises two differential equations, one for a voltage variable and another for a recovery variable. A voltage equation with cubic nonlinear dynamics will exhibit period-spiking solutions with nonlinear qualities. The recovery variable acts as a slower restorative feedback function for the oscillations. This simplification, in a sense, serves the goal of understanding the basics of excitability and oscillatory properties in nerve cell dynamics. Because of its relative simplicity compared to more complex biophysically detailed models, it is still a good prototypical model for investigating the dynamics of spiking behavior. Science and engineering also use it to study non-linear oscillatory systems and cross-field phenomena related to excitability [1, 13, 16, 27, 29]. In this study, we will look into the generalised Fitzhugh-Nagumo equation, which is stated by

$$\frac{\partial \varphi}{\partial \gamma} + \nu(\gamma)\frac{\partial \varphi}{\partial \zeta} - \mu(\gamma)\frac{\partial^2 \varphi}{\partial \zeta^2} - \eta(\gamma)\varphi(\varphi - \alpha)(1 - \varphi) = 0, \tag{1.1}$$

with the initial condition

$$\varphi(\zeta, 0) = \varphi_0$$

where $\nu(\gamma)$, $\mu(\gamma)$ and $\eta(\gamma)$ are real valued function. For $\nu(\gamma) = 0$ and $\mu(\gamma) = \eta(\gamma) = 1$ the Eq. (1.1) become Fitzhugh-Nagumo equation

$$\frac{\partial \varphi}{\partial \gamma} - \frac{\partial^2 \varphi}{\partial \zeta^2} - \varphi(\varphi - \alpha)(1 - \varphi) = 0. \tag{1.2}$$

The fractional FitzHugh-Nagumo equation is a generalization of the FitzHugh-Nagumo model, which is an alternative model to the Hodgkin-Huxley model to describe spike generation in neurons. As in the case of the Caputo derivative, the fractional FitzHugh-Nagumo equation replaces the ordinary time derivative by the fractional one. By having fractional derivatives as a dynamic feature, there is memory and hereditary in the spiking process, which this model uses. Like the classical case, the fractional FitzHugh-Nagumo model is made of two equations the temporal relationship between the voltage variable and the recovery variable. Thanks to its non-linear nature, illustrated by the graph, it enables excitability and spiking characteristics. The second order terms in the fractional derivatives lead to memory dependence and thus complicate the controllability of the spiking dynamics. A number of papers have been developed in order to understand why the fractional order terms and the model are more sensitive than the integer order models. In conclusion, the diffusional approach to the FHN model is necessary for simulation of neuronal firing and memory effects in comparison with the well established standard approach [17, 19, 23, 24].

The Residual Power Series Method (RPSM) is an analytic method for solving a peculiar class of differential orders. The task is to develop a power series solution around an ordinary point that satisfies the given differential equation. This method is the best-known example of the general process for solving them. One puts the power series into the differential equation and sets the residuals (the left-out terms) equal to zero. Formulating the recurrence relation for the coefficients implies they can be located individually. Applying the RPSM simplifies the process of finding both specific and general solutions to differential equations. It also develops a systematic method for deriving function series solutions and studying their convergence characteristics. While this method may result in many complicated algebraic steps when dealing with higher-order coefficients, it also ensures that the least squares perform very well. Science and engineering widely apply RPSM, an effective method for obtaining series answers by using differential equations, both ordinarily and partially [3, 8, 9, 11].

2. Basic definitions

Let us first provide some background on fractional calculus and other relevant facts from applied mathematics. For further information in this area of research, the reader can see references Hilfer [12], Kilbas et al. [15], Anastassiou [4], Khan and Khan [14], Silambarasn and Belgacem [26], Belgacem and Silambarasan [6].

2.1. Definition

The fractional Rieman-Liouville integral of order $p \in \mathbb{R}_+$of a function $h(\gamma) \in L([0,1],\mathbb{R})$ is expressed by [18, 25]

$$I_0^p h(\gamma) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s) ds,$$

on the assumption that the integral on the right side of the equation is convergent.

2.2. Definition

For $\mu \in \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}^+$is said to be in the space C_μ if it can be written as $f(\zeta) = \zeta^q f_1(\zeta)$ with $q > \mu$, $f_1(\zeta) \in C[0,\infty)$ and it is in space $f(\zeta) \in C_\mu^n$ if $f^{(n)} \in C_\mu$ for $n \in \mathbb{N} \cup \{0\}$ [18, 25].

2.3. Definition

The fractional Caputo derivative of a function $h \in C_{-1}^{n}$ with $n \in \mathbb{N} \cup \{0\}$ is given as [18, 25]

$$D_t^p h(t) = \begin{cases} I^{n-p} f^{(n)}, & n-1 < p \leq n, \ n \in \mathbb{N}, \\ \dfrac{d^n}{\mathrm{d}t^n} h(t), & p = n, \ n \in \mathbb{N}. \end{cases}$$

2.4. Definition

The Mittag-Leffler function (MLF) of two-parameters is expressed by [18, 25]:

$$E_{p,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(kp+\beta)}.$$

For $p = \beta = 1, E_{1,1}(t) = e^t$ and $E_{1,1}(-t) = e^{-t}$.

2.5. Definition

The NT of a function $v(\zeta, t)$ for $t \geq 0$ is defined by [18, 25]

$$\mathcal{N}[v(\zeta, t)] = R(\zeta, s, u) = \int_0^{\infty} e^{-st} v(\zeta, ut) \mathrm{d}t,$$

where s and u for the transform parameters are taken to be real and positive.

2.6. Definition

The MLF of Natural transform (NT) $E_{p,\beta}$ is given as [18, 25]

$$\mathcal{N}[v(\zeta, t)] = \int_0^{\infty} e^{-st} v(\zeta, ut) \mathrm{d}t = \sum_{k=0}^{\infty} \frac{u^{k+1} \Gamma(k+1)}{s^{k+1} \Gamma(kp+\beta)}.$$

2.7. Definition

The Miller and Ross sense of the NT of $D^p f(t)$ is expressed by the following [18, 25]:

$$\mathcal{N}\left(D^p f(t)\right) = \frac{s^p}{u^p} R(s, u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} f^{(k)}(0), n-1 < p \leq n.$$

2.8. Lemma

The NT of $\frac{\partial^p f(\zeta, t)}{\partial t^p}$ with respect to t can be defined as [18, 25]:

$$\mathcal{N}\left[\frac{\partial^p f(\zeta, t)}{\partial t^p}\right] = \frac{s^p}{u^p} R(\zeta, s, u) - \sum_{k=0}^{n-1} \frac{s^{n-k-1}}{u^{n-k}} \left[\lim_{t \to 0} \frac{\partial^p f(\zeta, t)}{\partial t^p}\right].$$

2.9. Lemma

The Natural transform of p order partial derivative of $f(\zeta, t)$ with respect to ζ is denoted by [18, 25]

$$\mathcal{N}\left[\frac{\partial^p f(\zeta, t)}{\partial \zeta^p}\right] = \frac{d^p}{d\zeta^p} R(\zeta, s, u).$$

2.10. Lemma

The dual relationship between Laplace and Natural transforms is expressed by [18, 25]

$$\mathcal{N}[f(\zeta,t)] = R(\zeta,s,u) = \frac{1}{u}\int_0^\infty e^{\frac{-u}{\zeta}} f(\zeta,t)\mathrm{d}t = \frac{1}{u}\mathcal{L}\{f(\zeta,t)\},$$

where $\mathcal{L}$ is the laplace transform. As a conclusion from the above Lemma, it can be noted that the Natural transform can be seen as an extension of both the Sumudu and Laplace transforms. In particular, when $u=1$ then the Natural transform reduces to the Laplace transform and in same way for $s=1$ the generalization lead us to Sumudu transform.

3. Outline of the suggested methodologies

3.1. First we introduce general implementation of NRPSM

Consider of fractional order's partial differential equation:

$$\begin{aligned} &D_t^p \boldsymbol{\varphi}(\zeta,t) = N_\zeta[\boldsymbol{\varphi}(\zeta,t)], \\ &\boldsymbol{\varphi}(\zeta,0) = f(\zeta) \end{aligned} \tag{3.1}$$

where N_ζ is a nonlinear function related to ζ of degree $r, \zeta \in I, t \geq 0, D_t^p$ refers to p-th fractional Caputo operator for $p \in (0,1]$, and $\boldsymbol{\varphi}(\zeta,t)$ is an unknown term.

The following steps may be taken in order to use the Natural RPSM to create the approximate solution of (3.1):

Step 1. Utilising the starting data of (3.1), use the Natural transform on each side of (3.1),

$$\begin{aligned} &\boldsymbol{\varphi}(\zeta,s) = \frac{f(\zeta)}{s} - \frac{u^p}{s^p}\mathcal{N}\left\{N_\zeta[\boldsymbol{\varphi}(\zeta,t)]\right\}, \\ &\text{where } \boldsymbol{\varphi}(\zeta,s) = \mathcal{N}[\boldsymbol{\varphi}(\zeta,t)](s), s > t. \end{aligned} \tag{3.2}$$

Step 2. Consider the following fractional expansion is the estimated solution of the equation (3.2):

$$\boldsymbol{\varphi}(\zeta,s) = \frac{f(\zeta)}{s} + \sum_{n=1}^{\infty} \frac{u^p h_n(\zeta)}{s^{np+1}}, x \in I, s > t \geq 0, \tag{3.3}$$

and the following is the form of the k-th Natural series solution:

$$\boldsymbol{\varphi}_k(\zeta,s) = \frac{f(\zeta)}{s} + \sum_{n=1}^{k} \frac{u^p h_n(\zeta)}{s^{np+1}}, \zeta \in I, s > t \geq 0. \tag{3.4}$$

Step 3. The k-th Natural residual fractional function of (3.2) is defined as:

$$\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}_k}(\zeta,s)\right) = \boldsymbol{\varphi}_k(\zeta,s) - \frac{f(\zeta)}{s} + \frac{u^p}{s^p}\mathcal{N}\left\{N_\zeta[\boldsymbol{\varphi}(\zeta,t)]\right\}, \tag{3.5}$$

and (3.2)'s Natural residual function are defined as:

$$\lim_{k\to\infty}\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}_k}(\zeta,s)\right)=\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}}(\zeta,s)\right)=\boldsymbol{\varphi}(\zeta,s)-\frac{f(\zeta)}{s}+\frac{u^p}{s^p}\mathcal{N}\left\{N_\zeta[\boldsymbol{\varphi}(\zeta,t)]\right\}. \tag{3.6}$$

Here are a few helpful Natural residual function facts that are necessary to determine the estimated solution: - $\lim_{k\to\infty}\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}_k}(\zeta,s)\right)=\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}}(\zeta,s)\right)$, for $\zeta\in I, s>t\geq 0$. - $\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}}(\zeta,s)\right)=0$, for $\zeta\in I, s>t\geq 0$. - $\lim_{s\to\infty}s^{kp+1}\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}_k}(\zeta,s)\right)=0$, for $\zeta\in I, s>t\geq 0$, and $k=1,2,3,\ldots$

Step 4. Now put the k-th Natural series solution (3.4) into the k-th Natural fractional residual function of (3.5).

Step 5. The unknown coefficients $h_k(\zeta)$, for $k=1,2,3,\ldots$, might be obtained by solving the system $\lim_{s\to\infty}s^{ka+1}\mathcal{N}\left(\mathrm{Res}_{\boldsymbol{\varphi}_k}(\zeta,s)\right)=0$. Subsequently, we gather the obtained coefficients using fractional expansion series(3.4) $\boldsymbol{\varphi}_k(\zeta,s)$.

Step 6. Applying the inverse Natural transform operator to both sides of the Natural series solution to obtain an estimated solution $\boldsymbol{\varphi}_k(\zeta,t)$, of the main Equation (3.1).

3.2. Problem 1

3.2.1. Applying NRPSM

Consider fractional nonlinear Fitzhugh-Nagumo equation

$$\mathfrak{D}_t^p\boldsymbol{\varphi}(\zeta,t)-\frac{\partial^2\boldsymbol{\varphi}(\zeta,t)}{\partial\zeta^2}-(1+\alpha)\boldsymbol{\varphi}^2(\zeta,t)+\alpha\boldsymbol{\varphi}(\zeta,t)+\boldsymbol{\varphi}^3(\zeta,t)=0 \text{ where } \ 0<p\leq 1. \tag{3.7}$$

Subjected to the following IC's:

$$\boldsymbol{\varphi}(\zeta,0)=\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}. \tag{3.8}$$

Using NT to Eq. (3.7) and Eq. (3.8), we get

$$\begin{aligned}&\boldsymbol{\varphi}(\zeta,s)-\frac{\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}}{s}-\frac{u^p}{s^p}\frac{\partial^2\boldsymbol{\varphi}(\zeta,s)}{\partial\zeta^2}-\frac{u^p(1+\alpha)}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}^2(\zeta,s)]\right]\\&+\frac{u^p\alpha}{s^p}\boldsymbol{\varphi}(\zeta,s)+\frac{u^p}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}^3(\zeta,s)]\right]=0,\end{aligned} \tag{3.9}$$

and so the k^{th}-truncated term series are

$$\boldsymbol{\varphi}(\zeta,s)=\frac{\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}}{s}+\sum_{r=1}^{k}\frac{f_r(u^p\zeta,s)}{s^{rp+1}},\quad r=1,2,3,4\cdots. \tag{3.10}$$

Natural residual function (LRF) are

$$\begin{aligned}&\mathcal{N}_tRes(\zeta,s)\\&=\boldsymbol{\varphi}(\zeta,s)-\frac{\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}}{s}-\frac{u^p}{s^p}\frac{\partial^2\boldsymbol{\varphi}(\zeta,s)}{\partial\zeta^2}-\frac{u^p(1+\alpha)}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}^2(\zeta,s)]\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{u^p\alpha}{s^p}\boldsymbol{\varphi}(\zeta,s)+\frac{u^p}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}^3(\zeta,s)]\right]\\
=&0, \qquad (3.11)
\end{aligned}
$$

and the k^{th}-NRFs as:

$$
\begin{aligned}
&\mathcal{N}_t Res_k(\zeta,s)\\
=&\boldsymbol{\varphi}_k(\zeta,s)-\frac{\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}}{s}-\frac{u^p}{s^p}\frac{\partial^2\boldsymbol{\varphi}_k(\zeta,s)}{\partial\zeta^2}-\frac{u^p(1+\alpha)}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}_k^2(\zeta,s)]\right]\\
&+\frac{u^p\alpha}{s^p}\boldsymbol{\varphi}_k(\zeta,s)+\frac{u^p}{s^p}\mathcal{N}\left[\mathcal{N}_t^{-1}[\boldsymbol{\varphi}_k^3(\zeta,s)]\right]\\
=&0. \qquad (3.12)
\end{aligned}
$$

Now, to calculate $f_r(\zeta,s)$, $r=1,2,3,\cdots$, we put the r^{th}-truncated series Eq. (3.10) into the r^{th}-NRF Eq. (3.12), multiply the solution of equation by s^{rp+1}, and then analysis recursively the relation $\lim_{s\to\infty}(s^{rp+1}\mathcal{N}_t Res_{\boldsymbol{\varphi},r}(\zeta,s))=0$, $r=1,2,3,\cdots$. Following are the first few functions:

$$
f_1(\zeta,s)=\frac{1-2\alpha}{4\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+4}, \qquad (3.13)
$$

$$
f_2(\zeta,s)=-\frac{(1-2\alpha)^2\left(\sinh\left(\sqrt{2}\zeta\right)+2\sinh\left(\frac{\zeta}{\sqrt{2}}\right)\right)}{8\left(32\left(\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+1\right)^3\cosh^6\left(\frac{\zeta}{2\sqrt{2}}\right)\right)}, \qquad (3.14)
$$

and so on.

Substituting the value of $f_r(\zeta,s)$, $r=1,2,3,\cdots$, in Eq. (3.10), we get

$$
\begin{aligned}
\boldsymbol{\varphi}(\zeta,s)=&\frac{1}{s}\left(\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}\right)+\frac{1}{s^{p+1}}\left(\frac{1-2\alpha}{4\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+4}\right)\\
&\times\frac{u^p}{s^{2p+1}}\left(-\frac{(1-2\alpha)^2\left(\sinh\left(\sqrt{2}\zeta\right)+2\sinh\left(\frac{\zeta}{\sqrt{2}}\right)\right)}{8\left(32\left(\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+1\right)^3\cosh^6\left(\frac{\zeta}{2\sqrt{2}}\right)\right)}\right)+\cdots.
\end{aligned} \qquad (3.15)
$$

Using inverse Natural Transform,we get

$$
\begin{aligned}
\boldsymbol{\varphi}(\zeta,t)=&\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}+\frac{t^p}{\Gamma(p+1)}\left(\frac{1-2\alpha}{4\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+4}\right)\\
&+\frac{t^{2p}}{\Gamma(2p+1)}\left(-\frac{(1-2\alpha)^2\left(\sinh\left(\sqrt{2}\zeta\right)+2\sinh\left(\frac{\zeta}{\sqrt{2}}\right)\right)}{8\left(32\left(\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+1\right)^3\cosh^6\left(\frac{\zeta}{2\sqrt{2}}\right)\right)}\right)+\cdots. \qquad (3.16)
\end{aligned}
$$

Table 1. The various fractional of NRPSM of Example 1 for $\alpha = 1$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.473109	0.490252	0.505185	0.505177	0.0320683	0.0149258	7.748233×10^{-6}
	0.2	0.490889	0.507976	0.522857	0.522839	0.0319508	0.0148633	0.000018038
	0.3	0.508691	0.52568	0.540473	0.540444	0.0317539	0.0147641	0.0000281783
	0.4	0.52647	0.54332	0.557987	0.557949	0.0314796	0.0146291	0.0000380834
	0.5	0.544181	0.560851	0.575359	0.575311	0.0311301	0.0144596	0.00004767
	0.6	0.56178	0.578232	0.592546	0.592489	0.0307087	0.0142571	0.0000568591
	0.7	0.579224	0.59542	0.609509	0.609444	0.0302194	0.0140236	0.0000655768
	0.8	0.596472	0.612377	0.626212	0.626138	0.0296667	0.0137611	0.0000737558
	0.9	0.613482	0.629066	0.642619	0.642538	0.0290555	0.0134721	0.0000813364
	1.	0.630219	0.645451	0.658698	0.65861	0.0283913	0.013159	0.0000882678
0.01	0.1	0.503582	0.5122	0.516422	0.516422	0.01284	0.00422149	1.008418×10^{-7}
	0.2	0.521259	0.529847	0.534053	0.534053	0.0127932	0.00420576	2.034356×10^{-7}
	0.3	0.538884	0.547419	0.551599	0.551599	0.0127148	0.00417968	3.043092×10^{-7}
	0.4	0.556412	0.564874	0.569018	0.569017	0.0126057	0.0041435	4.026150×10^{-7}
	0.5	0.5738	0.582169	0.586267	0.586267	0.0124668	0.00409756	4.975373×10^{-7}
	0.6	0.591008	0.599265	0.603308	0.603307	0.0122994	0.00404231	5.883039×10^{-7}
	0.7	0.607995	0.616122	0.620101	0.6201	0.0121052	0.00397827	6.741983×10^{-7}
	0.8	0.624725	0.632705	0.636612	0.636611	0.0118859	0.00390602	7.545714×10^{-7}
	0.9	0.641162	0.64898	0.652807	0.652806	0.0116435	0.00382622	8.288529×10^{-7}
	1.	0.657275	0.664915	0.668656	0.668655	0.0113801	0.00373956	8.965618×10^{-7}

Table 2. The various fractional of NRPSM of NRPSM of Example 1 for $\alpha = 0.2$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.544394	0.534117	0.525161	0.525156	0.0192374	0.00896085	4.282651×10^{-6}
	0.2	0.561915	0.55168	0.542759	0.542751	0.0191646	0.00892974	7.965710×10^{-6}
	0.3	0.579284	0.569116	0.560251	0.560239	0.0190449	0.00887669	0.0000115806
	0.4	0.596458	0.586381	0.577594	0.577579	0.0188795	0.00880225	0.0000150971
	0.5	0.6134	0.603437	0.594748	0.59473	0.0186701	0.00870717	0.0000184864
	0.6	0.630071	0.620245	0.611674	0.611653	0.0184188	0.00859237	0.0000217212
	0.7	0.646438	0.636769	0.628335	0.62831	0.018128	0.00845896	0.0000247763
	0.8	0.662469	0.652977	0.644696	0.644669	0.0178003	0.00830817	0.0000276292
	0.9	0.678134	0.668837	0.660726	0.660696	0.0174387	0.00814134	0.0000302599
	1.	0.693409	0.684323	0.676396	0.676363	0.0170463	0.00795992	0.000032652
0.01	0.1	0.526122	0.520952	0.518419	0.518419	0.00770276	0.00253283	3.779567×10^{-8}
	0.2	0.543716	0.538566	0.536043	0.536043	0.00767351	0.00252334	7.470726×10^{-8}
	0.3	0.561202	0.556084	0.553577	0.553577	0.00762542	0.00250764	1.109852×10^{-7}
	0.4	0.578537	0.573464	0.570978	0.570978	0.00755895	0.00248589	1.463251×10^{-7}
	0.5	0.595681	0.590664	0.588206	0.588206	0.00747477	0.0024583	1.804344×10^{-7}
	0.6	0.612594	0.607645	0.60522	0.60522	0.00737369	0.00242515	2.130364×10^{-7}
	0.7	0.62924	0.62437	0.621984	0.621983	0.00725665	0.00238673	2.438746×10^{-7}
	0.8	0.645585	0.640803	0.63846	0.63846	0.00712475	0.00234341	2.727168×10^{-7}
	0.9	0.661596	0.656913	0.654617	0.654617	0.00697917	0.00229558	2.993594×10^{-7}
	1.	0.677246	0.672668	0.670425	0.670425	0.00682115	0.00224364	3.236311×10^{-7}

3.2.2. Implementation of NITM

Using RL integral to Eq. (3.7), we get the equivalent form

$$\begin{aligned}&\boldsymbol{\varphi}(\zeta,t)\\ =&\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}-\mathcal{R}_t^p\left[\frac{\partial^2\boldsymbol{\varphi}(\zeta,t)}{\partial\zeta^2}+(1+\alpha)\boldsymbol{\varphi}^2(\zeta,t)-\alpha\boldsymbol{\varphi}(\zeta,t)-\boldsymbol{\varphi}^3(\zeta,t)\right].\end{aligned} \tag{3.17}$$

Table 3. The various fractional of NRPSM of NRPSM of Example 1 for $\alpha = -0.2$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.58001	0.556042	0.535147	0.53512	0.04489	0.0209219	0.000027362
	0.2	0.597377	0.573516	0.552707	0.552659	0.0447174	0.0208566	0.0000473396
	0.3	0.614505	0.590809	0.570136	0.570069	0.0444359	0.0207402	0.0000669075
	0.4	0.631356	0.607881	0.587393	0.587307	0.0440489	0.0205739	0.0000859039
	0.5	0.647894	0.624693	0.604437	0.604333	0.0435604	0.0203595	0.000104175
	0.6	0.664086	0.64121	0.621232	0.62111	0.0429754	0.0200991	0.000121575
	0.7	0.679902	0.657398	0.637741	0.637603	0.0422997	0.0197952	0.000137971
	0.8	0.695317	0.673228	0.653931	0.653777	0.0415393	0.0194507	0.000153246
	0.9	0.710305	0.688673	0.669772	0.669604	0.0407011	0.0190687	0.000167295
	1.	0.724848	0.703709	0.685236	0.685056	0.0397921	0.0186524	0.000180033
0.01	0.1	0.53739	0.525328	0.519418	0.519418	0.0179717	0.00590987	2.098379×10^{-7}
	0.2	0.554939	0.542925	0.537038	0.537037	0.0179021	0.00588765	4.107389×10^{-7}
	0.3	0.572353	0.560416	0.554565	0.554565	0.0177886	0.00585096	6.081510×10^{-7}
	0.4	0.58959	0.577758	0.571958	0.571958	0.0176324	0.00580017	8.004194×10^{-7}
	0.5	0.606609	0.59491	0.589175	0.589174	0.017435	0.00573578	9.859536×10^{-7}
	0.6	0.623374	0.611834	0.606177	0.606175	0.0171983	0.00565841	1.163250×10^{-6}
	0.7	0.639848	0.628492	0.622925	0.622923	0.0169247	0.00556878	1.330918×10^{-6}
	0.8	0.655999	0.64485	0.639384	0.639383	0.0166165	0.00546774	1.487696×10^{-6}
	0.9	0.671797	0.660877	0.655522	0.655521	0.0162767	0.00535617	1.632481×10^{-6}
	1.	0.687216	0.676543	0.67131	0.671308	0.015908	0.00523506	1.764345×10^{-6}

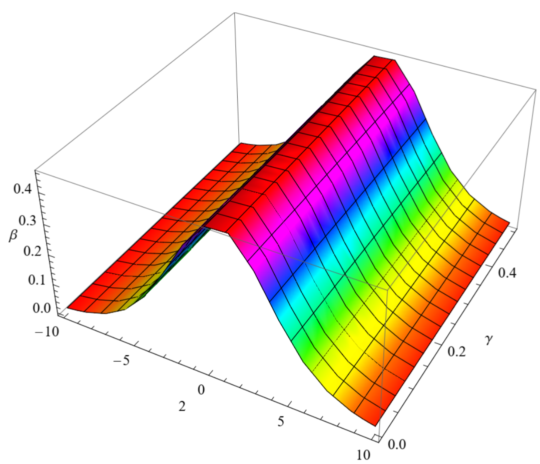

Figure 1. The fractional order $p = 1.0$ with NRPSM of Example 1.

According to NIM procedure, we obtain the following several terms

$$\boldsymbol{\varphi}_0(\zeta,t) = \frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right) + \frac{1}{2},$$

$$\boldsymbol{\varphi}_1(\zeta,t) = \frac{(1-2\alpha)t^p}{4\Gamma(p+1)\left(\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+1\right)},$$

$$\boldsymbol{\varphi}_2(\zeta,t) = \frac{1}{512}(1-2\alpha)^2 t^{2p}\mathrm{sech}^2\left(\frac{\zeta}{2\sqrt{2}}\right)\Bigg(\frac{(2\zeta-1)t^{2p}\Gamma(3p+1)\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{2}}\right)}{\Gamma(p+1)^3\Gamma(4p+1)}$$

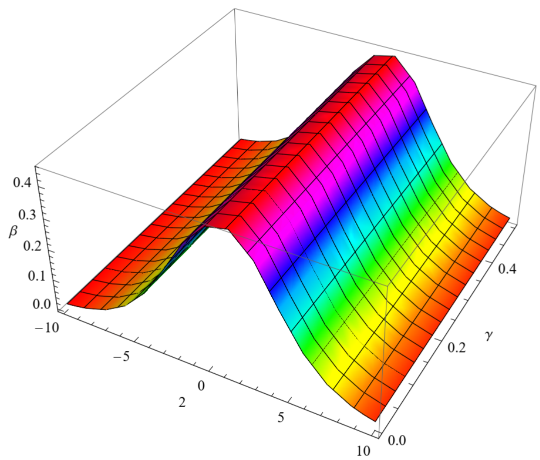

Figure 2. The fractional order $p = 0.8$ with NRPSM of Example 1.

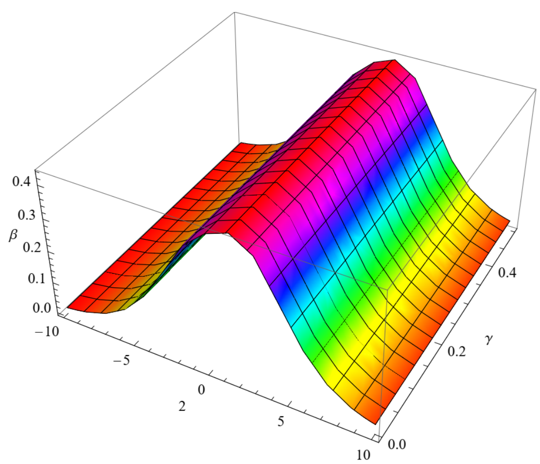

Figure 3. The fractional order $p = 0.6$ with NRPSM of Example 1.

$$+\frac{4t^p\Gamma(2p+1)\mathrm{sech}^2\left(\frac{\zeta}{2\sqrt{2}}\right)\left(2\alpha-3\tanh\left(\frac{\zeta}{2\sqrt{2}}\right)-1\right)}{\Gamma(p+1)^2\Gamma(3p+1)}-\frac{32\tanh\left(\frac{\zeta}{2\sqrt{2}}\right)}{\Gamma(2p+1)}\Big). \tag{3.18}$$

The final result by the NITM is achieved as

$$\begin{aligned}&\boldsymbol{\varphi}(\zeta,t)\\ =&\frac{1}{2}\tanh\left(\frac{\sqrt{2}\zeta}{4}\right)+\frac{1}{2}+\frac{(1-2\alpha)t^p}{4\Gamma(p+1)\left(\cosh\left(\frac{\zeta}{\sqrt{2}}\right)+1\right)}\\ &+\frac{1}{512}(1-2\alpha)^2t^{2p}\mathrm{sech}^2\Big(\frac{\zeta}{2\sqrt{2}}\Big)\Big(\frac{(2\zeta-1)t^{2p}\Gamma(3p+1)\mathrm{sech}^4\left(\frac{\zeta}{2\sqrt{2}}\right)}{\Gamma(p+1)^3\Gamma(4p+1)}\end{aligned}$$

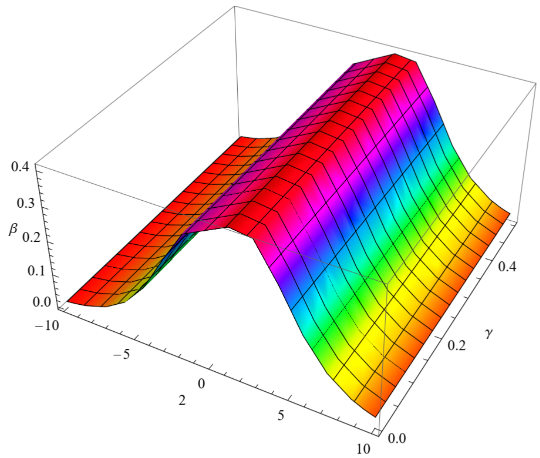

Figure 4. The fractional order $p = 0.4$ with NRPSM of Example 1.

$$+\frac{4t^p\Gamma(2p+1)\operatorname{sech}^2\left(\frac{\zeta}{2\sqrt{2}}\right)\left(2\alpha-3\tanh\left(\frac{\zeta}{2\sqrt{2}}\right)-1\right)}{\Gamma(p+1)^2\Gamma(3p+1)}-\frac{32\tanh\left(\frac{\zeta}{2\sqrt{2}}\right)}{\Gamma(2p+1)}\Bigg)+\cdots. \tag{3.19}$$

Table 4. The various fractional order NITM of Example 1 for $\alpha = 1$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.473132	0.490225	0.505177	0.505177	0.0320456	0.0149523	2.242087×10^{-7}
	0.2	0.49068	0.507879	0.522839	0.522839	0.0321597	0.0149603	4.805486×10^{-7}
	0.3	0.508253	0.525514	0.540444	0.540444	0.0321915	0.0149305	7.152781×10^{-7}
	0.4	0.525809	0.543086	0.557948	0.557949	0.0321405	0.0148633	9.247423×10^{-7}
	0.5	0.543304	0.560552	0.57531	0.575311	0.0320071	0.0147592	1.106003×10^{-6}
	0.6	0.560696	0.57787	0.592488	0.592489	0.0317926	0.0146194	1.256919×10^{-6}
	0.7	0.577945	0.594999	0.609442	0.609444	0.0314992	0.0144452	1.376182×10^{-6}
	0.8	0.595009	0.6119	0.626137	0.626138	0.0311298	0.0142382	1.463330×10^{-6}
	0.9	0.61185	0.628537	0.642536	0.642538	0.0306879	0.0140004	1.518712×10^{-6}
	1.	0.628432	0.644876	0.658609	0.65861	0.030178	0.013734	1.543435×10^{-6}
0.01	0.1	0.503568	0.512198	0.516422	0.516422	0.0128538	0.00422382	2.645001×10^{-10}
	0.2	0.521224	0.529842	0.534053	0.534053	0.0128283	0.00421075	5.162907×10^{-10}
	0.3	0.538828	0.547411	0.551599	0.551599	0.012771	0.00418728	7.463168×10^{-10}
	0.4	0.556335	0.564864	0.569017	0.569017	0.0126823	0.00415364	9.510221×10^{-10}
	0.5	0.573704	0.582157	0.586267	0.586267	0.0125632	0.00411016	1.127565×10^{-9}
	0.6	0.590892	0.59925	0.603307	0.603307	0.0124148	0.00405726	1.273896×10^{-9}
	0.7	0.607862	0.616105	0.6201	0.6201	0.0122385	0.00399544	1.388790×10^{-9}
	0.8	0.624575	0.632686	0.636611	0.636611	0.0120359	0.00392527	1.471858×10^{-9}
	0.9	0.640997	0.648959	0.652806	0.652806	0.011809	0.00384739	1.523509×10^{-9}
	1.	0.657095	0.664892	0.668655	0.668655	0.0115597	0.00376248	1.544895×10^{-9}

4. Graphical and tables discussion

The Natural Residual Power Series Method (NRPSM) for the first time to study the Caputo version of the FitzHugh-Nagumo equation. The FitzHugh-Nagumo equation is a fundamental model used to describe the behavior of excitable systems, with significant applications in physiological and biological processes. We began

Table 5. The various fractional NITM of Example 1 for $\alpha = 0.2$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.544255	0.534082	0.525156	0.525156	0.0190989	0.00892579	1.117026×10^{-7}
	0.2	0.561695	0.55162	0.54275	0.542751	0.0189438	0.00886951	2.137622×10^{-7}
	0.3	0.578983	0.569031	0.560239	0.560239	0.0187437	0.00879178	3.149955×10^{-7}
	0.4	0.596079	0.586272	0.577579	0.577579	0.0185004	0.00869339	4.136716×10^{-7}
	0.5	0.612946	0.603305	0.594729	0.59473	0.0182165	0.00857528	5.081642×10^{-7}
	0.6	0.629547	0.620091	0.611652	0.611653	0.0178945	0.00843859	5.970011×10^{-7}
	0.7	0.645848	0.636595	0.62831	0.62831	0.0175376	0.00828459	6.789035×10^{-7}
	0.8	0.661817	0.652783	0.644668	0.644669	0.0171487	0.00811465	7.528170×10^{-7}
	0.9	0.677427	0.668626	0.660695	0.660696	0.0167314	0.00793028	8.179304×10^{-7}
	1.	0.692652	0.684096	0.676362	0.676363	0.016289	0.00773302	8.736842×10^{-7}
0.01	0.1	0.526113	0.520951	0.518419	0.518419	0.00769337	0.00253179	1.016613×10^{-10}
	0.2	0.543699	0.538564	0.536043	0.536043	0.00765648	0.00252135	2.033045×10^{-10}
	0.3	0.561177	0.556081	0.553577	0.553577	0.00760087	0.00250471	3.042122×10^{-10}
	0.4	0.578505	0.57346	0.570978	0.570978	0.00752711	0.00248205	4.026591×10^{-10}
	0.5	0.595642	0.590659	0.588206	0.588206	0.0074359	0.00245358	4.970224×10^{-10}
	0.6	0.612548	0.60764	0.60522	0.60522	0.00732811	0.00241959	5.858301×10^{-10}
	0.7	0.629188	0.624364	0.621983	0.621983	0.00720476	0.00238038	6.678017×10^{-10}
	0.8	0.645527	0.640796	0.63846	0.63846	0.00706698	0.00233632	7.418786×10^{-10}
	0.9	0.661533	0.656905	0.654617	0.654617	0.00691597	0.0022878	8.072431×10^{-10}
	1.	0.677178	0.67266	0.670425	0.670425	0.00675305	0.00223524	8.633276×10^{-10}

Table 6. The various fractional NITM of Example 1 for $\alpha = -0.2$.

t	ζ	$NRPSM_{P=0.5}$	$NRPSM_{p=0.7}$	$NRPSM_{P=1}$	**Exact**	$Error_{p=0.5}$	$Error_{p=0.7}$	$Error_{p=1}$
0.1	0.1	0.578857	0.555781	0.535119	0.53512	0.0437368	0.0206615	7.121856×10^{-7}
	0.2	0.595779	0.573119	0.552658	0.552659	0.0431194	0.0204597	1.311268×10^{-6}
	0.3	0.612474	0.590279	0.570067	0.570069	0.0424051	0.0202098	1.924016×10^{-6}
	0.4	0.628908	0.607221	0.587304	0.587307	0.0416012	0.0199143	2.539556×10^{-6}
	0.5	0.645049	0.623909	0.60433	0.604333	0.0407155	0.0195761	3.147202×10^{-6}
	0.6	0.660867	0.640309	0.621107	0.62111	0.0397563	0.0191982	3.736794×10^{-6}
	0.7	0.676335	0.656387	0.637599	0.637603	0.0387321	0.0187843	4.298987×10^{-6}
	0.8	0.691429	0.672115	0.653773	0.653777	0.0376519	0.0183379	4.825497×10^{-6}
	0.9	0.706129	0.687467	0.669599	0.669604	0.0365244	0.0178628	5.309289×10^{-6}
	1.	0.720414	0.702419	0.68505	0.685056	0.0353582	0.0173628	5.744698×10^{-6}
0.01	0.1	0.537326	0.525322	0.519418	0.519418	0.0179086	0.00590369	5.727748×10^{-10}
	0.2	0.554835	0.542914	0.537037	0.537037	0.0177974	0.00587628	1.157795×10^{-10}
	0.3	0.572208	0.560399	0.554565	0.554565	0.0176432	0.0058345	1.758065×10^{-9}
	0.4	0.589405	0.577736	0.571958	0.571958	0.0174474	0.00577875	2.362949×10^{-9}
	0.5	0.606386	0.594884	0.589174	0.589174	0.017212	0.00570958	2.961948×10^{-9}
	0.6	0.623115	0.611803	0.606175	0.606175	0.0169392	0.00562764	3.545028×10^{-9}
	0.7	0.639555	0.628457	0.622923	0.622923	0.0166315	0.00553371	4.102911×10^{-9}
	0.8	0.655674	0.644811	0.639383	0.639383	0.0162917	0.00542864	4.627320×10^{-9}
	0.9	0.671444	0.660834	0.655521	0.655521	0.0159228	0.00531337	5.111168×10^{-9}
	1.	0.686836	0.676497	0.671308	0.671308	0.0155278	0.00518888	5.548689×10^{-9}

by transforming the FitzHugh-Nagumo equation into its Natural domain representation, which allowed us to apply NRPSM to obtain a series-form solution. The method was further enhanced by introducing a new iteration technique to improve the convergence properties and accuracy of the solution. This comprehensive approach not only validates the use of NRPSM but also provides a powerful tool for exploring the complex dynamics of excitable systems.

The graphical results clearly illustrate the influence of fractional orders on the behavior of the FitzHugh-Nagumo equation. Figure 1 shows the NRPSM solution for a fractional order of $p = 1.0$, which corresponds to the classical version of the

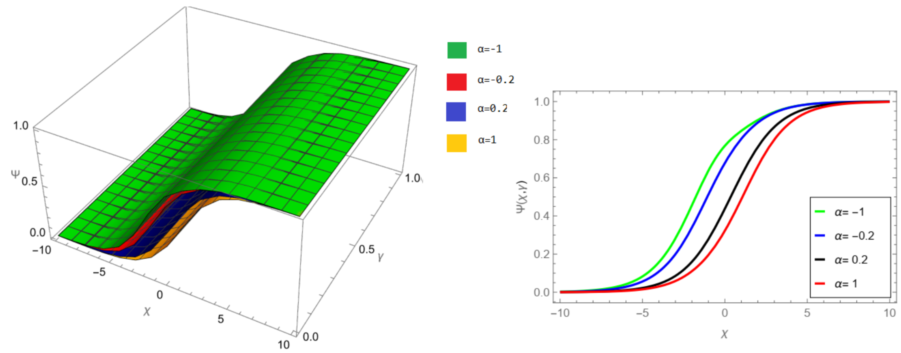

Figure 5. NITM result of Example 1 for fractional-order $p = 1$.

equation. As the fractional order decreases, as seen in Figures 2, 3 and 4 (with $p = 0.8, 0.6$, and $p = 0.4$ respectively), the dynamics of the solution become increasingly intricate. The reduction in fractional order leads to notable changes in the behavior of the system, highlighting the sensitivity of excitable systems to fractional variations. Additionally, Figure 5 presents a comparison between the NRPSM and the New Iteration Transformation Method (NITM) for $p = 1.0$, showing that both methods yield similar results, but NRPSM demonstrates a more systematic approach for fractional cases.

The tabular data further supports the effectiveness of NRPSM in solving fractional differential equations. Table 1 displays the results of various fractional orders of NRPSM for $\alpha = -0.2$, showing how the residuals and errors evolve as the fractional order decreases. This trend of increasing complexity as the fractional order reduces is consistent across other tables. Table 2 presents the fractional orders of NITM for $\alpha = 1.0$, while tables 3 and 4 provide the NITM results for $\alpha = 0.2$ and $\alpha = -0.2$, respectively. Table 5 and 6 comparison between NRPSM and NITM reveals that NRPSM offers faster convergence and more reliable accuracy, particularly for fractional orders.

Overall, the graphical and tabular analysis confirms that NRPSM is an effective method for solving the Caputo version of the FitzHugh-Nagumo equation. The method not only provides accurate solutions but also captures the intricate dynamics of excitable systems influenced by fractional orders. These findings demonstrate the potential of NRPSM for broader applications in science and engineering, particularly in fields where the modeling of fractional systems is critical.

5. Conclusion

In conclusion, this paper has presented a comprehensive investigation into the Fitzhugh-Nagumo equation using the Natural residual power series method (NRPSM) in tandem with a new iteration transform method. We have successfully transformed the equation into its Natural domain representations and applied the NRPSM to obtain a series form result, while the introduction of the novel iteration method has significantly improved the convergence properties of the solution. Through numerical examples and comparative analyses, we have demonstrated the efficacy and accuracy of our approach, showcasing its potential as a robust tool for investigating differential equations within natural transform. This research not only advances our understanding of the dynamics of excitable systems, but it also contributes to the broader field of mathematical analysis and modeling in science and engineering. The combined use of NRPSM and the new iteration technique offers a promising avenue for tackling a wide array of complex differential equations, paving the way for further advancements in the study and application of non-local and non-integer-order derivatives in various domains of research and practical problem-solving.

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