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A CLASS OF HIGHER ORDER TURNING POINT NONLINEAR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

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Abstract When a boundary layer occurs close to a turning point in a class of singularly perturbation problems with turning points, the solution manifests as a multi-layer phenomena. This paper provides a systematic solution. It focuses on a class of turning point problems and the aspects include constructing formal asymptotic solutions. It also involves establishing the existence and error estimation of the solutions, the relationship with the position of the intermediate layer and boundary layer. In addition, the numerical verification is conducted as well.

Keywords Singular perturbation, turning points, matching principle, special function, the upper and lower solution.

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1. Introduction

Differential equations are often used to describe mathematical models produced in engineering applications such as physics, chemistry and biomathematics. After a series of dimensionless processing of the parameters of these established differential equations, small parameters usually appear before the highest derivative, and such a class of problems is the singular perturbation problem.

Because of the complexity of singular perturbation problem, it is very difficult to obtain the exact solution, so it is very important to find the uniform and effective asymptotic solution. In the singular perturbation problem, various interesting boundary layer and inner layer phenomena will appear. Among them, for the problem where the stability does not change, the most effective approximate solution is the boundary layer function method [38] created by Russian scholars, who use this method to construct a consistent and effective asymptotic solution and construct a Green function corresponding to the problem itself. Then the existence proof of the solution is obtained by applying the fixed point theory and the remainder term is estimated. However, when the stability condition is not satisfied, the methods suitable for general singular perturbation problems, such as boundary layer function method, deformed coordinate method, averaging method, multi-scale method [20], etc., are not suitable for the turning point problems. And the turning point problem is the situation where the stability changes. When the original problem passes

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through the turn point, the properties of the problem will change to some extent. Therefore the turning point problems have always been the most difficult problems in singular perturbation problems.

Attention was drawn to the difficulties of such equations in a 1970 paper [30]. The solution frequently undergoes significant modifications near the turning points [37] when studying singularly perturbed boundary value issues. It is permeated with many physical difficulties, such as Stokes lines and surfaces in mathematics [9,33], shock layers in fluid and solid mechanics [21], and boundary layers in fluid mechanics [6]. Finding the exact solution to the perturbation equation with turning points is generally a difficult undertaking. Other perturbation techniques are warranted, the most crucial of which is the matching asymptotic expansion technique [12,22]. Meanwhile, the boundary layer's position is determined by the coefficient of the first derivative term. It is claimed that the problem contains multiple layers [23] in a certain location if the boundary layer's position coincides with the turning point. To address this, it is important to build the solutions for the various layers and then match them with the matching principle.

Trefethen [36] examine a linear differential equation from eight points of view, showing how it sheds light on aspects of numerical analysis, asymptotics, dynamical systems, ODE theory and so on. Nayfeh [27,28] investigated an equation having a high-order turning point that was linearly perturbed singularly. By applying the Prandtl matching principle [32], the zeroth-order asymptotic solution is obtained and fitted to the numerical solution. Based on Nayfeh's work, Chen [8] studied a class of singular perturbation problems with high-order turning points. The Prandtl matching principle was also utilized to study the problems in various situations. Fedoryuk [14] presented the main results on the asymptotic theory of ordinary linear differential equations and systems where their is a small parameter in the higher derivatives with turning points.

In this paper, we consider a class of nonlinear singularly perturbed boundary value problems with higher order turning points. And the difficulties in matching techniques are overcome and the first asymptotic solution of the original problem is obtained.

$$\begin{cases} \varepsilon^{2n+1}y'' + (x-k)^{2n+1}(y' + y^s) = \varepsilon y, \\ y(0) = \alpha, \\ y(1) = \beta, \end{cases} \tag{1.1}$$

the study of ε approaching to zero on the interval $0 \le x \le 1$. Here, s and n are integers. The two boundary conditions specified are constants and $\alpha, \beta \neq 0$. When $s \ge 2$, there is $\alpha > 0, 0 < \beta^{s-1} < \frac{1}{s-1}$.

Remark 1.1. It is difficult to obtain the first-order or the more higher order approximation. The reasons are as follows: there will be many difficulties in the matching process, such as singularity, secular terms, left and right endpoints cannot match, which make it difficult for researchers to obtain the high-order asymptotic solution.

Throughout this paper assuming that:

(H1) When $s = 1$, $0 < k < 1$, α, β are the same sign and satisfy the inequality $|\alpha| \le |\beta e|$.

(H2) When $s \geq 2$, $k = 0$, α, β are further required $\beta^{s-1} \leq \frac{1}{s}, \alpha > 0$, when $s \geq 2$, $0 < k < 1$, $\beta^{s-1} \leq \frac{1}{s}, 0 < \alpha \leq 1$.

As s changes, the original problem can be divided into two main problems, when $s = 1$, the problem is a linear problem, when $s \geq 2$, this problem is a nonlinear problem. In these two problems, different situations will be generated due to the different values of parameters k and n. When $0 \leq k \leq 1$, $x = k$ is a turning point. Meanwhile, the positive or negative value of $(x-k)^{2n+1}$ determines the position of the boundary layer. Therefore, according to the location of the multi-layer phenomena in the problem, we will discuss six situations of the two problems.

The remainder of the paper is organized as follows: Section 2 divides the original problem into six cases and provides first-order asymptotic solutions that are uniformly valid within specific regions. In particular, the solution of the nonlinear case will contain two types of special functions. Consequently, it is imperative to preprocess them before applying the matching principle. The purpose of Section 3 is to provide an estimate of the remainder and demonstrate the existence of the solution [26]. The proof procedure will be split into linear and nonlinear cases, similar to that in Section 2. The bvp4c solver in Matlab is used to generate the numerical solutions for the many specific instances presented in Section 4. In order to further highlight the significance of the work done in this research, we fit the asymptotic and the numerical solutions [13]. Finally, Section 5 indicates findings and potential directions for further study.

2. Construction and determination of asymptotic solutions

According to the linearity and nonlinearity of the problem and the location where the multi-layer phenomenon appears, we present some singular perturbation problems with turning point to solve and match respectively in this section.

The difficulty in solving this problem is that there is a turning point at $x = k$ in Eq. (1.1), where $0 \leq k \leq 1$, and that a boundary layer(s) must be introduced to satisfy one of the boundary conditions. The following two situations are considered for both the linear problem with $s = 1$, and the nonlinear problem with integers $s \geq 2$.

2.1. Linear case

Consider the linear case first, that is $s = 1$, then the Eq. (1.1) becomes

$$\varepsilon^{2n+1} y'' + (x-k)^{2n+1}(y' + y) = \varepsilon y. \tag{2.1}$$

Case 1. When $s = 1$ and $k = 0$, there is

$$\varepsilon^{2n+1} y'' + x^{2n+1}(y' + y) = \varepsilon y. \tag{2.2}$$

For the singularly perturbed boundary value problem $\varepsilon y'' = f(x, y, y')$, $y(a) = A$, $y(b) = B$, if there is a constant $k > 0$, such that $f_{y'} \leq -k < 0$, then the boundary layer position is in the neighborhood of the left endpoint [29, 35]. And the left boundary condition must be abandoned in the outer solution, and the right stability condition is satisfied. Similarly, when the left stability condition is satisfied, the boundary layer position appears in the neighborhood of the right end point, and the right boundary value will be abandoned in the outer solution. Then the boundary layer is expected to be at $x = 0$, and $x = 0$ is also a turning point. Therefore, it is speculated that the problem will have multiple layers near $x = 0$.

In this situation, the independent variable of the outer layer is x. As before, to investigate the neighborhood of the origin, we introduce the stretching transformation $\xi = \frac{x-0}{\varepsilon^v}$, i.e. $x = \varepsilon^v \xi$, where $v > 0$ and undetermined. Then substitute the stretching transformation into the Eq. (2.2), and record the inner layer solution is y^i, as the following equation is

$$\varepsilon^{2n+1-2v}\frac{d^2y^i}{d\xi^2} + (\varepsilon^v\xi)^{2n+1}(\varepsilon^{-v}\frac{dy^i}{d\xi} + y^i) = \varepsilon y^i, \tag{2.3}$$

after sorting it out, there is

$$\varepsilon^{2n+1-2v}\frac{d^2y^i}{d\xi^2} + \varepsilon^{2nv}\xi^{2n+1}\frac{dy^i}{d\xi} + (\varepsilon^{(2n+1)v}\xi^{2n+1} - \varepsilon)y^i = 0, \tag{2.4}$$

as $\varepsilon \to 0$, $\varepsilon^{(2n+1)v}$ is a small amount of ε^{2nv}. Hence, the dominant part is

$$\varepsilon^{2n+1-2v}\frac{d^2y^i}{d\xi^2} + \varepsilon^{2nv}\xi^{2n+1}\frac{dy^i}{d\xi} - \varepsilon y^i = 0. \tag{2.5}$$

It is now necessary to determine the correct balancing in the Eq. (2.5) and use the principle of least degradation to find the distinguished limit [18].

The balance between the first term and the second term was considered in Step1, in this situation, the third term should be the higher order, by calculating, $v = \frac{2n+1}{2n+2}$. This violates the original assumption that the third term is higher order, and so this balance is not possible. This condition is a trivial case and can be discarded.

In addition to the above balancing process, the following possibilities remain:

The balance between the first term and the third term was considered in Step2, and the second term is higher order, which $2n + 1 - 2v = 1$, that is $v = n$, the Eq. (2.4) becomes

$$\varepsilon\frac{d^2y^i}{d\xi^2} + \varepsilon^{2n^2}\xi^{2n+1}\frac{dy^i}{d\xi} - \varepsilon y^i + \varepsilon^{(2n+1)n}\xi^{2n+1}y^i = 0. \tag{2.6}$$

In this case, the conclusions are consistent with the original assumptions, and so this is the balancing we are looking for. This case is non-trivial and we can continue to solve it.

Finally, the second term and the third term are matched to obtain $2nv = 1$, that is $v = \frac{1}{2n}$ and this is said to be the distinguished limit for the equation. In order to distinguish from the previous case, recording $\eta = \frac{x}{\varepsilon^{\frac{1}{2n}}}$, $\xi = \frac{x}{\varepsilon^n}$, the main term of the Eq. (2.4) is

$$\varepsilon^{\frac{2n^2+n-1}{n}}\frac{d^2y^i}{d\eta^2} + \varepsilon\eta^{2n+1}\frac{dy^i}{d\eta} - \varepsilon y^i + \varepsilon^{\frac{2n+1}{2n}}\eta^{2n+1}y^i = 0, \tag{2.7}$$

this case is also non-trivial and must also be included.

Since $n > \frac{1}{2n}$, $\xi = \frac{x}{\varepsilon^n}$ describes a layer closer to the origin point and $\eta = \frac{x}{\varepsilon^{\frac{1}{2n}}}$ describes a layer between the right layer and the left layer. In the problem of viscous-inviscid-interaction [5], the three layers are called lower, middle, and upper layers.

Therefore, in this case, the variable of the outer layer is x, the variable of the boundary layer is $\xi = \frac{x}{\varepsilon^n}$, and the variable of the middle layer is $\eta = \frac{x}{\varepsilon^{\frac{1}{2n}}}$.

According to the above analysis, this paper consider a problem in which more than one distinguished limit exist in a given boundary layer. When two distinguished limits exist, the resulting expansion consists of two inner expansions in addition to the outer expansion. And the problem is a multiple-layer problem, the multi-layer phenomenon will happen close to $x = 0$. Asymptotic solution of the original problem must be constructed by the pertinent theories of the matching asymptotic expansion approach,

$$y^c = y^o + y^i + y^m - (y^o)^m - (y^i)^m. \tag{2.8}$$

The composite solution is y^c, y^o is the outer solution, y^i is the inner solution, y^m is the intermediate solution, and the common solutions of the two overlapping parts are $(y^o)^m, (y^i)^m$ respectively. We seek their expansions in the form:

$$y^o = y_0^o(x) + \varepsilon y_1^o(x) + \varepsilon^2 y_2^o(x) + \cdots, \tag{2.9}$$

$$y^i = y_0^i(\xi) + \varepsilon^n y_1^i(\xi) + \varepsilon^{2n} y_2^i(\xi) + \cdots, \tag{2.10}$$

$$y^m = y_0^m(\eta) + \varepsilon^{\frac{1}{2n}} y_1^m(\eta) + \cdots. \tag{2.11}$$

When analyzing the problem, we use the direct expansion method [15,24] to get the first term of y^o. The outer solution is solved firstly, we substitute Eq. (2.9) into Eq. (2.2), there is

$$\varepsilon^{2n+1}(\frac{d^2y_0^o}{dx^2}+\varepsilon\frac{d^2y_1^o}{dx^2}+\cdots)+x^{2n+1}(\frac{dy_0^o}{dx}+\varepsilon\frac{dy_1^o}{dx}+y_0^o+\varepsilon y_1^o+\cdots) = \varepsilon(y_0^o+\varepsilon y_1^o+\cdots),$$

since the boundary layer appears near $x = 0$, the outer solution satisfies the right boundary condition $y(1) = \beta$, that is

$$y_0^o(1) + \varepsilon y_1^o(1) + \varepsilon^2 y_2^o(1) + \cdots = \beta,$$

comparing the same power coefficients of ε on the left and right sides, it can be obtained that the equations and their asymptotic solutions are as follows, the zero-order equation and its solution are

$$x^{2n+1}(\frac{dy_0^o}{dx} + y_0^o) = 0 \Rightarrow y_0^o = \beta e^{1-x},$$

the first-order equation and its solution are

$$x^{2n+1}(\frac{dy_1^o}{dx} + y_1^o) = y_0^o \Rightarrow y_1^o = -\frac{\beta e^{1-x}}{2n}(x^{-2n} - 1).$$

Remark 2.1. The first-order solution has singular terms, but these can be eliminated in the subsequent analysis and solving step. Then the final composite solution does not have any singular part.

Subsequently, we proceed to solve the inner solution by substituting Eq. (2.10) into Eq. (2.6), thereby obtaining

$$\varepsilon(\frac{d^2y_0^i}{d\xi^2} + \varepsilon^n\frac{d^2y_1^i}{d\xi^2} + \cdots) + \varepsilon^{2n^2}\xi^{2n+1}(\frac{dy_0^i}{d\xi} + \varepsilon^n\frac{dy_1^i}{d\xi} + \cdots) - \varepsilon(y_0^i + \varepsilon^n y_1^i + \cdots)$$
$$+\varepsilon^{(2n+1)n}\xi^{2n+1}(y_0^i + \varepsilon^n y_1^i + \cdots) = 0,$$

and obtaining the zero-order equation and its inner solution are

$$\frac{d^2y_0^i}{d\xi^2} - y_0^i = 0 \Rightarrow y_0^i = a_0 e^{-\xi} + b_0 e^{\xi},$$

when $n = 1$, the first-order equation and the inner solution are

$$\begin{aligned}&\frac{d^2y_1^i}{d\xi^2} + \xi^3 \frac{dy_0^i}{d\xi} - y_1^i = 0\\ \Rightarrow &y_1^i = -\frac{1}{16} a_0 e^{-\xi}(3 + 6\xi + 6\xi^2 + 4\xi^3 + 2\xi^4) + a_1 e^{-\xi} + b_1 e^{\xi},\end{aligned}$$

when $n \geq 2$, the first-order equation and the inner solution are

$$\frac{d^2y_1^i}{d\xi^2} - y_1^i = 0 \Rightarrow y_1^i = a_1 e^{-\xi} + b_1 e^{\xi},$$

since $\xi = \frac{x}{\varepsilon^n} > 0$, the coefficients of e^{ξ} must be zero, otherwise y^i will grow exponentially with ξ and cannot perform subsequent matching, then it can be obtained $b_{0,1} = 0$. $a_{0,1}$ can be determined by the unused boundary value condition $y(0) = \alpha$ or the subsequent matching process.

So far we have considered boundary layer phenomenon for Eq. (2.2), however, the intermediate layer phenomenon can be equally addressed. The equation of middle layer can be obtained by substituting Eq.(2.11) into Eq. (2.7), then there is

$$\begin{aligned}&\varepsilon^{\frac{2n^2+n-1}{n}}\left(\frac{d^2y_0^m}{d\eta^2} + \varepsilon^{\frac{1}{2n}} \frac{d^2y_1^m}{d\eta^2} + \cdots\right) + \varepsilon\eta^{2n+1}\left(\frac{dy_0^m}{d\eta} + \varepsilon^{\frac{1}{2n}} \frac{dy_1^m}{d\eta} + \varepsilon^{\frac{1}{n}} \frac{dy_2^m}{d\eta} + \cdots\right)\\ &-\varepsilon(y_0^m + \varepsilon^{\frac{1}{2n}} y_1^m + \varepsilon^{\frac{1}{n}} y_2^m + \cdots)\varepsilon^{\frac{2n+1}{2n}}\eta^{2n+1}(y_0^m + \varepsilon^{\frac{1}{2n}} y_1^m + \cdots) = 0,\end{aligned}$$

the zeroth-order intermediate equation and the solution are

$$\eta^{2n+1}\frac{dy_0^m}{d\eta} - y_0^m = 0 \Rightarrow y_0^m = d_0 e^{-\frac{1}{2n\eta^{2n}}},$$

the first-order intermediate equation and its solution are

$$\eta^{2n+1}\frac{dy_1^m}{d\eta} + \eta^{2n+1} y_0^m - y_1^m = 0 \Rightarrow y_1^m = -d_0 \eta e^{-\frac{1}{2n\eta^{2n}}} + d_1 e^{-\frac{1}{2n\eta^{2n}}},$$

the value of $d_{0,1}$ is undetermined and will be determined by the subsequent calculation and matching process.

After finding the outer, inner and intermediate solutions, the following need to start matching these solutions to obtain the common solutions. At first, matching the inner solution and the outer solution, the inner solution is

$$y^i = \begin{cases} -\dfrac{\varepsilon}{16} a_0 e^{-\xi}(3 + 6\xi + 6\xi^2 + 4\xi^3 + 2\xi^4) \\ +a_0 e^{-\xi} + \varepsilon a_1 e^{-\xi} + \cdots, \ n = 1, \\ a_0 e^{-\xi} + \varepsilon^n a_1 e^{-\xi} + \cdots, \ n \geq 2, \end{cases} \tag{2.12}$$

the outer solution is

$$y^o = \beta e^{1-x} - \varepsilon\frac{\beta e^{1-x}}{2n}(x^{-2n} - 1) + \cdots, \tag{2.13}$$

firstly, Prandtl matching principle is used for matching (only match the first item), there is

$$(y^o)^i = \lim_{\varepsilon\to 0} \beta e^{1-x} = \lim_{\varepsilon\to 0} \beta e^{1-\varepsilon^n \xi} = \beta e,$$
$$(y^i)^o = \lim_{\varepsilon\to 0} a_0 e^{-\xi} = \lim_{\varepsilon\to 0} a_0 e^{-\frac{x}{\varepsilon^n}} = 0,$$

since $\beta \neq 0$, it does not satisfy the matching principle $(y^o)^i = (y^i)^o$.

Then Van Dyke matching principle [3,11] is used to match the first term. However, when the first term of the inner solution is expressed by the outer variable x, it cannot be expanded according to the small ε. Therefore, the inner and outer solutions cannot be directly matched.

Therefore, it is necessary to use y^m to match with y^i and y^o respectively, which is why it called the intermediate solution. Meanwhile, it produces two common solutions $(y^i)^m, (y^o)^m$ due to matching and confirms that the asymptotic solution is Eq. (2.8).

Firstly, matching the inner solution with the intermediate solution. The intermediate solution is

$$y^m = d_0 e^{-\frac{1}{2n\eta^{2n}}} + \varepsilon^{\frac{1}{2n}} e^{-\frac{1}{2n\eta^{2n}}}(-d_0\eta + d_1) + \cdots . \tag{2.14}$$

Prandtl matching principle is used to obtain $(y^i)^m = 0 = (y^m)^i$, which can satisfy the matching principle but cannot determine undetermined coefficients. After that we match the outer and intermediate solutions, but for the outer solution, the singularity appears from the second term and its singularity will be stronger and stronger. Prandtl matching principle doesn't apply from the second term, then we try to use the Van Dyke matching principle. After matching, it can be obtained $d_0 = \beta e, d_1 = 0$, then the intermediate solution can be determined. And the coincident term is

$$[y^o_{(2)}]^m_{(2)} = \beta e - \beta e x + \varepsilon\left(\frac{\beta e}{2nx^{2n-1}} - \frac{\beta e}{2nx^{2n}}\right),$$

but the exact composite solution can not be given, because the $a_{0,1}$ in the inner solution have not been determined. Although the inner solution and the intermediate solution satisfy the matching principle $(y^i)^m = (y^m)^i = 0$, $a_{0,1}$ still can not be determined. In general, in the matched asymptotic expansion method, the outer solution only satisfies the outer local boundary conditions, and the inner boundary conditions must be abandoned. Similarly, the inner solution only satisfies the local inner boundary conditions in the boundary layer problems [7]. In this case, the boundary condition $y(0) = \alpha$ should be determined by the inner solution, the intermediate solution and the common solutions [31]. When $x = 0$, there are $\xi = 0, \eta = 0$, and $\eta = 0$ causes $y^m \to 0$, since $(y^i)^m = (y^m)^i = 0$, then $y(0) = \alpha$ is only determined by the inner solution, that is $y^i(0) = \alpha$. Therefore, when $n = 1$, $a_0 = \alpha, a_1 = \frac{3}{16}\alpha$, when $n \geq 2$, $a_0 = \alpha, a_1 = 0$.

Then all undetermined coefficients have been determined, but the composite solution has singular terms, because the outer solution contains singularity. After Taylor expansion in $x = 0$, the singular terms are exactly canceled out by the common solution. Therefore, the final composite solution without singular terms can be obtained, that is

$$y^c = \begin{cases} \beta e^{1-x} - \varepsilon \dfrac{\alpha}{16} e^{-\frac{x}{\varepsilon}} [6\dfrac{x}{\varepsilon} + 6(\dfrac{x}{\varepsilon})^2 + 4(\dfrac{x}{\varepsilon})^3 + 2(\dfrac{x}{\varepsilon})^4] - \varepsilon \dfrac{\beta e}{2}(-\dfrac{x^2}{2} + x - \dfrac{1}{2}) \\ +\alpha e^{-\frac{x}{\varepsilon}} + \beta e^{1-\frac{\varepsilon}{2x^2}} - x\beta e^{1-\frac{\varepsilon}{2x^2}} - \beta e + \beta e x + \cdots, \quad n = 1, \\ \beta e^{1-x} - \varepsilon \dfrac{\beta e}{2n}(-1+x) + \alpha e^{-\frac{x}{\varepsilon^n}} + \beta e^{1-\frac{\varepsilon}{2nx^{2n}}} - x\beta e^{1-\frac{\varepsilon}{2nx^{2n}}} \\ -\beta e + \beta e x + \cdots, \quad n \geq 2. \end{cases} \tag{2.15}$$

Case 2. When $s = 1$ and $k = 1$, the problem is

$$\varepsilon^{2n+1} y'' + (x-1)^{2n+1}(y' + y^s) = \varepsilon y. \tag{2.16}$$

Since $0 \leq x \leq 1$, $(x-1)^{2n+1} \leq 0$, it can be known from the singular perturbation theory that the boundary layer appears near $x = 1$, and meanwhile $x = 1$ is still the turning point. We speculate the problem will have multi-layer phenomenon near $x = 1$ in this case. The inner layer variable is $\xi = \frac{x-1}{\varepsilon^n} < 0$, the middle layer variable is $\eta = \frac{x-1}{\varepsilon^{\frac{1}{2n}}} < 0$, and y^o, y^i, y^m can be obtained in the same way,
the outer solution is

$$y^o = \alpha e^{-x} + \varepsilon\{-\frac{\alpha}{2n} e^{-x}[(x-1)^{-2n} - 1]\} + \cdots, \tag{2.17}$$

the inner solution is

$$y^i = \begin{cases} -\dfrac{\varepsilon}{16} b_0 e^{\xi}(3 - 6\xi + 6\xi^2 - 4\xi^3 + 2\xi^4) + b_0 e^{\xi} - \varepsilon b_1 e^{\xi} + \cdots, \quad n = 1, \\ b_0 e^{\xi} + \varepsilon^n b_1 e^{\xi} + \cdots, \quad n \geq 2, \end{cases} \tag{2.18}$$

the intermediate solution is

$$y^m = d_0 e^{-\frac{1}{2n\eta^{2n}}} + \varepsilon^{\frac{1}{2n}} e^{-\frac{1}{2n\eta^{2n}}}(-d_0\eta + d_1) + \cdots, \tag{2.19}$$

by the Van Dyke matching principle, there is $d_0 = \alpha e^{-1}, d_1 = 0$. Then substitute the boundary value condition, when $n = 1$, $b_0 = \beta, b_1 = -\frac{3\beta}{16}$, when $n \geq 2$, $b_0 = \beta, b_1 = 0$, and the common soluion is

$$[y^o_{(2)}]^m_{(2)} = \alpha e^{-1} - \alpha e^{-1}(x-1) + \varepsilon(\frac{\alpha e^{-1}}{2n(x-1)^{2n-1}} - \frac{\alpha e^{-1}}{2n(x-1)^{2n}}).$$

Then the composite solution without singular terms is

$$y^c = \begin{cases} \alpha e^{-x} - \varepsilon \dfrac{\alpha e^{-1}}{2}(-\dfrac{(x-1)^2}{2} + x - \dfrac{3}{2}) + \beta e^{\frac{x-1}{\varepsilon}} \\ -\dfrac{\varepsilon}{16}\beta e^{\frac{x-1}{\varepsilon}}[6\dfrac{x-1}{\varepsilon} + 6(\dfrac{x-1}{\varepsilon})^2 - 4(\dfrac{x-1}{\varepsilon})^3 + 2(\dfrac{x-1}{\varepsilon})^4] + \alpha e^{-1} e^{-\frac{\varepsilon}{2(x-1)^2}} \\ -\varepsilon(x-1)\alpha e^{-1} e^{-\frac{\varepsilon}{2(x-1)^2}} - \alpha e^{-1} + \alpha e^{-1}(x-1) + \cdots, \quad n = 1, \\ \alpha e^{-x} - \varepsilon \dfrac{\alpha e^{-1}}{2n}(x-2) + \beta e^{\frac{x-1}{\varepsilon^n}} + \alpha e^{-1-\frac{\varepsilon}{2n(x-1)^{2n}}} \\ -(x-1)\alpha e^{-1-\frac{\varepsilon}{2n(x-1)^{2n}}} - \alpha e^{-1} + \alpha e^{-1}(x-1) + \cdots, \quad n \geq 2. \end{cases} \tag{2.20}$$

Remark 2.2. For the case of $k < 0$ and $k > 1$, the original problem is a general second-order singularly perturbed nonlinear boundary value problem without turning points. In this case, there are only boundary layer and outer layer, no intermediate layer, and no multilayer phenomenon appears. For $k < 0$, the boundary layer appears on the left and for $k > 1$, the boundary layer appears on the right. It is necessary to re-determine the extension variables and construct the asympotic solution. In these cases, the solution is $y = y^o + y^i - (y^o)^i$.

Case 3. When $s = 1$ and $0 < k < 1$, there is

$$\varepsilon^{2n+1}y'' + (x-k)^{2n+1}(y'+y) = \varepsilon y. \tag{2.21}$$

This problem is a turning point problem in a general sense. The turning point occurs at $x = k$, and the outer solution of the original problem is composed of y_L^o, y_R^o. The original interval $[0,1]$ is divided into two parts with k as the bound, and the original problem is divided into left and right problems accordingly.

The left problem is

$$\varepsilon^{2n+1}{y_L}'' + (x-k)^{2n+1}({y_L}'+y_L) = \varepsilon y_L, \tag{2.22}$$

$$y_L(0) = \alpha, \quad y_L(k) = \delta_1. \tag{2.23}$$

The right problem is

$$\varepsilon^{2n+1}{y_R}'' + (x-k)^{2n+1}({y_R}'+y_R) = \varepsilon y_R, \tag{2.24}$$

$$y_R(k) = \delta_2, \quad y_R(1) = \beta, \tag{2.25}$$

both of $\delta_{1,2}$ are unknown.

From the analysis of the above two cases, it can be seen that on the interval $[0,k)$, $(x-k)^{2n+1} < 0$, then the boundary layer appears near the right endpoint $x = k$. On the interval $(k,1]$, $(x-k)^{2n+1} > 0$, then the boundary layer appears near the left endpoint $x = k$. And $x = k$ is also the turning point of the two problems. It is assumed that there is a multi-layer phenomenon nearby $x = k$.

Consider the left problem firstly, the outer solution is

$$y_L^o = \alpha e^{-x} - \varepsilon\frac{\alpha e^{-x}}{2n}[(x-k)^{-2n} - (-k)^{2n}] + \cdots, \tag{2.26}$$

the inner solution is

$$y_L^i = \begin{cases} -\dfrac{\varepsilon}{16}b_0e^{\xi}(3-6\xi+6\xi^2-4\xi^3+2\xi^4) + b_0e^{\xi} - \varepsilon b_1e^{\xi} + \cdots, & n = 1, \\ b_0e^{\xi} + \varepsilon^n b_1 e^{\xi} + \cdots, & n \geq 2, \end{cases} \tag{2.27}$$

the intermediate solution is

$$y_L^m = d_0e^{-\frac{1}{2n\eta^{2n}}} + \varepsilon^{\frac{1}{2n}}(-d_0\eta e^{-\frac{1}{2n\eta^{2n}}} + d_1e^{-\frac{1}{2n\eta^{2n}}}) + \cdots, \tag{2.28}$$

where $b_{0,1}$, $d_{0,1}$ are undetermined, and will be solved in subsequent process.

Similarly, Van Dyke matching principle is used for matching to obtain $d_0 = \alpha e^{-k}, d_1 = 0$. By substituting $y_L^i(k) = \delta_1$, when $n = 1$, $b_0 = \delta_1, b_1 = -\frac{3\delta_1}{16}$, when

$n \geq 2$, $b_0 = \delta_1, b_1 = 0$. Thus the composite solution of the left problem without singular terms is

$$y_L^c = \begin{cases} \alpha e^{-x} - \varepsilon\dfrac{\alpha e^{-k}}{2}[\dfrac{1}{2} + (x-k)k^2 - k^2] + \delta_1 e^{\frac{x-k}{\varepsilon}} - \dfrac{\varepsilon\delta_1}{16}e^{\frac{x-k}{\varepsilon}} \\ \times[-6\dfrac{x-k}{\varepsilon} + 6(\dfrac{x-k}{\varepsilon})^2 - 4(\dfrac{x-k}{\varepsilon})^3 + 2(\dfrac{x-k}{\varepsilon})^4] + \alpha e^{-k}e^{-\frac{\varepsilon}{2(x-k)^2}} \\ -(x-k)\alpha e^{-k}e^{-\frac{\varepsilon}{2(x-k)^2}} - \alpha e^{-k} + (x-k)\alpha e^{-k} + \cdots, \quad n = 1, \\ \alpha e^{-x} - \varepsilon\dfrac{\alpha e^{-k}}{2n}(k^{2n}(x-k) - k^{2n}) + \delta_1 e^{\frac{x-k}{\varepsilon^n}} + \alpha e^{-k-\frac{\varepsilon}{2n(x-k)^{2n}}} \\ -(x-k)\alpha e^{-k-\frac{\varepsilon}{2n(x-k)^{2n}}} - \alpha e^{-k} + \alpha e^{-k}(x-k) + \cdots, \quad n \geq 2. \end{cases} \tag{2.29}$$

In the same way, the composite solution of the right problem without singular terms is

$$y_R^c = \begin{cases} \beta \mathrm{e}^{1-x} - \dfrac{\varepsilon\beta e^{1-k}}{2}((x-k)(1-k)^2 - (1-k)^2 + \dfrac{1}{2}) \\ +\delta_2 e^{-\frac{x-k}{\varepsilon}} - \beta e^{1-k} + (x-k)\beta e^{1-k} - \dfrac{\varepsilon\delta_2}{16}e^{-\frac{x-k}{\varepsilon}} \\ \times[6\dfrac{x-k}{\varepsilon} + 6(\dfrac{x-k}{\varepsilon})^2 + 4(\dfrac{x-k}{\varepsilon})^3 + 2(\dfrac{x-k}{\varepsilon})^4] + \beta e^{1-k}e^{-\frac{\varepsilon}{2(x-k)^2}} \\ -(x-k)\beta e^{1-k}e^{-\frac{\varepsilon}{2(x-k)^2}} + \cdots, \quad n = 1, \\ \beta e^{1-x} - \varepsilon\dfrac{\beta e^{1-k}}{2n}((x-k)(1-k)^{2n} - (1-k)^{2n}) \\ +\delta_2 e^{-\frac{x-k}{\varepsilon^n}} - \beta e^{1-k} + \beta e^{1-k-\frac{\varepsilon}{2n(x-k)^{2n}}} \\ -(x-k)\beta e^{1-k-\frac{\varepsilon}{2n(x-k)^{2n}}} + \beta e^{1-k}(x-k) + \cdots, \quad n \geq 2. \end{cases} \tag{2.30}$$

Finally, in order to determine the undetermined constants $\delta_{1,2}$, it can be obtained by the smooth connection of the left and right problems at $x = k$, that is

$$y_L{}^{(+)}(k) = y_R{}^{(-)}(k), \quad \frac{dy_L{}^{(+)}(k)}{dx} = \frac{dy_R{}^{(-)}(k)}{dx},$$

then we obtain $\delta_1 = \delta_2 = \varepsilon^n \frac{-\beta e^{1-k} + \alpha e^{-k}}{2}$.

Remark 2.3. Since the first term of the boundary layer starts from $O(\varepsilon^n)$. it is speculated that it may not be shown in the figures in this case, that is when the boundary layer term is small enough, it will not obvious.

2.2. Nonlinear case

When $s \geq 2$, the original problem is a nonlinear one, and the larger of s, the stronger of the nonlinear strength. The difficulty in solving this case is that we need to deal with the special functions which appear in the process of solving.

Case 1. When $s \geq 2$ and $k = 0$, there is

$$\varepsilon^{2n+1}y'' + x^{2n+1}(y' + y^s) = \varepsilon y. \tag{2.31}$$

The multi-layer phenomenon still appears near $x = 0$, and $\xi = \frac{x}{\varepsilon^n}, \eta = \frac{x}{\varepsilon^{\frac{1}{2n}}}$. The difference with $s = 1$ is the concrete form of the solution. In the same way as before, the zeroth-order outer equation and its solution can be obtained

$$x^{2n+1}(\frac{dy_0^o}{dx} + (y_0^o)^s) = 0 \Rightarrow y_0^o = [(s-1)(x-1) + \beta^{1-s}]^{\frac{1}{1-s}},$$

the first-order outer equation is

$$x^{2n+1}(\frac{dy_1^o}{dx} + s(y_0^o)^{s-1}y_1^o) = y_0^o,$$

by calculation, its solution is

$$\begin{aligned} y_1^o =&[(s-1)(x-1) + \beta^{1-s}]^{\frac{s}{1-s}} \\ &\times [\frac{s-1}{-2n+1}x^{-2n+1} - \frac{1-s+\beta^{1-s}}{2n}x^{-2n} + \frac{1-s}{-2n+1} + \frac{1-s+\beta^{1-s}}{2n}], \end{aligned}$$

then the outer solution in this case is

$$\begin{aligned} y^o =&[(s-1)(x-1) + \beta^{1-s}]^{\frac{1}{1-s}} + \varepsilon[(s-1)(x-1) + \beta^{1-s}]^{\frac{s}{1-s}} \\ &\times [\frac{s-1}{-2n+1}x^{-2n+1} - \frac{1-s+\beta^{1-s}}{2n}x^{-2n} + \frac{1-s}{-2n+1} + \frac{1-s+\beta^{1-s}}{2n}] + \cdots, \end{aligned}$$

in the same case of $s = 1$, the outer solution itself is singular.

The inner solution y^i is the same as $s = 1$, that is Eq. (2.12).

The difference is the intermediate layer solution, the zeroth-order middle layer equation and its solution are

$$\eta^{2n+1}\frac{dy_0^m}{d\eta} - y_0^m = 0 \Rightarrow y_0^m = d_0 e^{-\frac{1}{2n\eta^{2n}}},$$

when $n = 1$, the first-order intermediate equation and its solution are

$$\begin{aligned} &\eta^3\frac{dy_1^m}{d\eta} + \eta^3(y_0^m)^s - y_1^m = 0 \\ \Rightarrow& y_1^m = d_1 e^{-\frac{1}{2\eta^2}} \\ &- e^{-\frac{1}{2\eta^2}}\frac{(d_0 e^{-\frac{1}{2\eta^2}})^s(2e^{\frac{1}{2\eta^2}}\eta\sqrt{-1+s} + e^{\frac{s}{2\eta^2}}\sqrt{2\pi}(-1+s)erf(\frac{\sqrt{-1+s}}{\sqrt{2}\eta}))}{2\sqrt{-1+s}}, \end{aligned}$$

when $n \geq 2$,

$$\begin{aligned} &\eta^{2n+1}\frac{dy_1^m}{d\eta} + \eta^{2n+1}(y_0^m)^s - y_1^m = 0 \\ \Rightarrow& y_1^m = d_1 e^{-\frac{1}{2n\eta^{2n}}} - \frac{2^{-1-\frac{1}{2n}}{d_0}^s e^{-\frac{1}{2n\eta^{2n}}}\eta(\frac{-1+s}{n\eta^{2n}})^{\frac{1}{2n}}\Gamma(-\frac{1}{2n}, \frac{-1+s}{2n\eta^{2n}})}{n}. \end{aligned}$$

When $n = 1$, the intermediate solution is

$$\begin{aligned} y^m =&d_0 e^{-\frac{1}{2\eta^2}} + \varepsilon^{\frac{1}{2}}\ d_1 e^{-\frac{1}{2\eta^2}} - \varepsilon^{\frac{1}{2}}\ e^{-\frac{1}{2\eta^2}} \\ &\times \frac{(d_0 e^{-\frac{1}{2\eta^2}})^s(2e^{\frac{1}{2\eta^2}}\eta\sqrt{-1+s} + e^{\frac{s}{2\eta^2}}\sqrt{2\pi}(-1+s)erf(\frac{\sqrt{-1+s}}{\sqrt{2}\eta}))}{2\sqrt{-1+s}} + \cdots, \end{aligned} \tag{2.32}$$

when $n \geq 2$, the intermediate solution is

$$\begin{aligned} y^m =& d_0 e^{-\frac{1}{2n\eta^{2n}}} + \varepsilon^{\frac{1}{2n}}\ d_1 e^{-\frac{1}{2n\eta^{2n}}} - \varepsilon^{\frac{1}{2n}} \\ & \times \frac{2^{-1-\frac{1}{2n}} {d_0}^s e^{-\frac{1}{2n\eta^{2n}}} \eta (\frac{-1+s}{n\eta^{2n}})^{\frac{1}{2n}} \Gamma(-\frac{1}{2n}, \frac{-1+s}{2n\eta^{2n}})}{n} + \cdots . \end{aligned} \tag{2.33}$$

Just like the linear case, y^o cannot be matched with y^i directly. The matching of y^m and y^i satisfies the matching principle but cannot determine the undetermined cofficients. Before matching y^o and y^m, we need to do certain processing on them.

By Taylor's theorem, it can obtained that the decomposition of the outer solution, that is

$$\begin{aligned} &[(s-1)(x-1)+\beta^{1-s}]^{\frac{1}{1-s}} \\ =&(\beta^{1-s}-s+1)^{\frac{1}{1-s}}(1-\frac{x}{\beta^{1-s}-s+1}+\frac{sx^2}{2(\beta^{1-s}-s+1)^2}+\cdots), \\ &[(s-1)(x-1)+\beta^{1-s}]^{\frac{s}{1-s}} \\ =&(\beta^{1-s}-s+1)^{\frac{s}{1-s}}(1-\frac{sx}{\beta^{1-s}-s+1}+\frac{(2s-1)sx^2}{2(\beta^{1-s}-s+1)^2}+\cdots). \end{aligned}$$

Because there exists two special functions in y^m, that is error function and gamma function. Then need to understand the properties and do some processing on them before matching.

The definition of error function [1] is $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. The series expansion of the error function is

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}.$$

Through Van Dyke matching principle, $d_0 = (\beta^{1-s} - s + 1)^{\frac{1}{1-s}}$, and $d_1 = 0$ can be obtained. The common solution is

$$\begin{aligned} [y^o_{(2)}]^m_{(2)} =& (\beta^{1-s}-s+1)^{\frac{1}{1-s}} - x(\beta^{1-s}-s+1)^{\frac{s}{1-s}} \\ & - \varepsilon \frac{(\beta^{1-s}-s+1)^{\frac{1}{1-s}}}{2x^2} + \varepsilon(\beta^{1-s}-s+1)^{\frac{s}{1-s}} \frac{2-s}{2x}. \end{aligned}$$

When $n \geq 2$, by using a sequence of procedures, the special function [2] $\Gamma(-\frac{1}{2n}, \frac{(-1+s)\varepsilon}{2nx^{2n}})$ can be written as

$$\begin{aligned} \Gamma[-\frac{1}{2n}, \frac{(-1+s)\varepsilon}{2nx^{2n}}] =& (-2n)[(\frac{(-1+s)\varepsilon}{2nx^{2n}})^{-\frac{1}{2n}} \\ & \times (-\frac{(-1+s)\varepsilon}{(2n-1)x^{2n}} - 1 + \frac{(-1+s)\varepsilon}{2nx^{2n}} + \cdots) + \Gamma(1-\frac{1}{2n})], \end{aligned}$$

in this case, it can be determined by Van Dyke matching principle that the undeterimined coefficient $d_{0,1}$ as

$$d_0 = (\beta^{1-s} - s + 1)^{\frac{1}{1-s}},$$

$$d_1 = -2^{\frac{1}{-2n}}(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}(\frac{-1+s}{n})^{\frac{1}{2n}}\Gamma(1 - \frac{1}{2n}),$$

and the common solution is

$$\begin{aligned}
[y^m_{(2)}]^o_{(2)} =&[y^o_{(2)}]^m_{(2)} \\
=&(\beta^{1-s} - s + 1)^{\frac{1}{1-s}} - (\beta^{1-s} - s + 1)^{\frac{1}{1-s}}x - \frac{\varepsilon(\beta^{1-s} - s + 1)^{\frac{1}{1-s}}}{2nx^{2n}} \\
&- \frac{\varepsilon(-1+s)(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}}{(2n-1)x^{2n-1}} + \frac{\varepsilon s(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}}{2nx^{2n-1}}.
\end{aligned}$$

Therefore, when $n = 1, s \geq 2, k = 0$, the composite solution without singular terms is

$$\begin{aligned}
y^c =&[(s-1)(x-1) + \beta^{1-s}]^{\frac{1}{1-s}} + \varepsilon(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} \\
&\times [s - 1 + \frac{\beta^{1-s} - s + 1}{2} - \frac{s(1-s)}{\beta^{1-s} - s + 1} - \frac{s(s-1)x}{\beta^{1-s} - s + 1} \\
&- \frac{sx}{2} + \frac{s(1-s)(2s-1)x}{2(\beta^{1-s} - s + 1)^2} - \frac{s(s-1)}{4(\beta^{1-s} - s + 1)}] \\
&+ (\beta^{1-s} - s + 1)^{\frac{1}{1-s}}e^{-\frac{\varepsilon}{2x^2}} - \varepsilon^{\frac{1}{2}}\frac{e^{-\frac{\varepsilon}{2x^2}}((\beta^{1-s} - s + 1)^{\frac{1}{1-s}}e^{-\frac{\varepsilon}{2x^2}})^s}{2\sqrt{-1+s}} \\
&\times (2e^{\frac{\varepsilon}{2x^2}}\frac{x}{\varepsilon^{\frac{1}{2}}}\sqrt{-1+s} + e^{\frac{s\varepsilon}{2x^2}}\sqrt{2\pi}(-1+s)erf(\frac{\varepsilon^{\frac{1}{2}}\sqrt{-1+s}}{\sqrt{2}x})) \\
&+ \alpha e^{-\frac{x}{\varepsilon}} - \varepsilon\frac{1}{16}\alpha e^{-\frac{x}{\varepsilon}}(6\frac{x}{\varepsilon} + 6(\frac{x}{\varepsilon})^2 + 4(\frac{x}{\varepsilon})^3 + 2(\frac{x}{\varepsilon})^4) \\
&- (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} + x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} + \cdots,
\end{aligned}$$

when $n \geq 2, s \geq 2, k = 0$, the composite solution without singular terms is

$$\begin{aligned}
y^c =&[(s-1)(x-1) + \beta^{1-s}]^{\frac{1}{1-s}} + \varepsilon(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} \\
&\times [\frac{1-s}{-2n+1} + \frac{-s+1+\beta^{1-s}}{2n} - \frac{s(1-s)x}{(-s+1+\beta^{1-s})(-2n+1)} - \frac{sx}{2n}] \\
&+ \alpha e^{-\frac{x}{\varepsilon^n}} + (\beta^{1-s} - s + 1)^{\frac{1}{1-s}}e^{-\frac{\varepsilon}{2nx^{2n}}} + x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}e^{-\frac{\varepsilon}{2nx^{2n}}} \\
&\times (-\frac{\varepsilon}{(2n-1)x^{2n}} - 1 + \frac{\varepsilon}{2nx^{2n}} + \cdots) \\
&- (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} + x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} + \cdots.
\end{aligned}$$

Remark 2.4. The error function is one of the special incomplete gamma functions, we have $\Gamma(-\frac{1}{2}, \frac{-1+s}{2\eta^2}) = (-2)[\Gamma(\frac{1}{2}, \frac{-1+s}{2\eta^2}) - (\frac{-1+s}{2\eta^2})^{-\frac{1}{2}}e^{-\frac{-1+s}{2\eta^2}}]$ when $n = 1$. The gamma function and the error function have the expressions of $\Gamma(\frac{1}{2}, \frac{-1+s}{2\eta^2}) = \sqrt{\pi}(1 - erf(\sqrt{\frac{-1+s}{2\eta^2}}))$. Therefore, the intermediate solution when $n = 1$ can be included in the case of $n \geq 2$, and because the gamma function has special value $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, then through verification, the matching principle is satisfied.

In order to enhance the understanding of the related propeties of these two kinds of special functions, and to facilitate the subsequent proof of the existence of solutions, the solution and matching of other cases are still analyzed in terms of $n = 1$ and $n \geq 2$.

Case 2. When $s \geq 2$ and $k = 1$, there is

$$\varepsilon^{2n+1} y'' + (x-1)^{2n+1}(y' + y^s) = \varepsilon y. \tag{2.34}$$

In this case, using the same analysis to get the outer solution is

$$\begin{aligned} y^o =& [(s-1)x + \alpha^{1-s}]^{\frac{1}{1-s}} + \varepsilon[(s-1)x + \alpha^{1-s}]^{\frac{s}{1-s}} \\ & \times [\frac{s-1}{-2n+1}(x-1)^{-2n+1} + \frac{(s-1)+\alpha^{1-s}}{-2n}(x-1)^{-2n} \\ & - \frac{1-s}{-2n+1} + \frac{(s-1)+\alpha^{1-s}}{2n}] + \cdots, \end{aligned}$$

the inner solution is the same as $s = 1, k = 1$, the intermediate solution is the same as $s \geq 2, k = 0$, where $\xi = \frac{x-1}{\varepsilon^n} \leq 0, \eta = \frac{x-1}{\varepsilon^{\frac{1}{2n}}} \leq 0$. By using Van Dyke matching principle, it can be obtained when $n = 1$, $d_0 = (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}}$, $d_1 = 0$, the common solution is

$$\begin{aligned} [y^o_{(2)}]^m_{(2)} =& (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}} - (x-1)(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}} \\ & - \frac{\varepsilon(\alpha^{1-s} + s - 1)^{\frac{1}{1-s}}}{2(x-1)^2} + \varepsilon(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}} \frac{2-s}{2(x-1)}, \end{aligned}$$

when $n \geq 2$, there is $d_0 = (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}}$, $d_1 = -2^{\frac{1}{-2n}}(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}}$ $\times (\frac{-1+s}{n})^{\frac{1}{2n}} \Gamma(1 - \frac{1}{2n})$, and the common solution is

$$\begin{aligned} [y^m_{(2)}]^o_{(2)} =& [y^o_{(2)}]^m_{(2)} \\ =& (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}} - (\alpha^{1-s} + s - 1)^{\frac{s}{1-s}}(x-1) - \frac{\varepsilon(\alpha^{1-s} + s - 1)^{\frac{1}{1-s}}}{2n(x-1)^{2n}} \\ & - \frac{\varepsilon(-1+s)(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}}}{(2n-1)(x-1)^{2n-1}} + \frac{\varepsilon s(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}}}{2n(x-1)^{2n-1}}. \end{aligned}$$

Therefore, when $n = 1$, $s \geq 2$, $k = 1$, the composite solution without singular terms is

$$\begin{aligned} y^c =& [(s-1)x + \alpha^{1-s}]^{\frac{1}{1-s}} + \varepsilon(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}} \\ & \times [-(s-1) + \frac{\alpha^{1-s} + s - 1}{2} + \frac{s(s-1)}{\alpha^{1-s} + s - 1} + \frac{s(s-1)x}{\alpha^{1-s} + s - 1} \\ & - \frac{s(x-1)}{2} - \frac{s(s-1)(2s-1)(x-1)}{2(\alpha^{1-s} + s - 1)^2} - \frac{s(2s-1)}{4(\alpha^{1-s} + s - 1)}] \\ & - \varepsilon \frac{\beta}{16} e^{\frac{x-1}{\varepsilon}}(-6\frac{x-1}{\varepsilon} + 6(\frac{x-1}{\varepsilon})^2 - 4(\frac{x-1}{\varepsilon})^3 + 2(\frac{x-1}{\varepsilon})^4) \\ & + \beta e^{\frac{x-1}{\varepsilon}} + (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-1)^2}} \end{aligned}$$

$$
\begin{aligned}
&- \varepsilon^{\frac{1}{2}} \frac{e^{-\frac{\varepsilon}{2(x-1)^2}} ((\alpha^{1-s}+s-1)^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-1)^2}})^s}{2\sqrt{-1+s}} \\
&\times (e^{\frac{\varepsilon}{2(x-1)^2}} \frac{(x-1)2\sqrt{-1+s}}{\varepsilon^{\frac{1}{2}}} + e^{\frac{s\varepsilon}{2(x-1)^2}} \sqrt{2\pi}(-1+s) erf(\frac{\varepsilon^{\frac{1}{2}}\sqrt{-1+s}}{\sqrt{2}(x-1)})) \\
&- (\alpha^{1-s}+s-1)^{\frac{1}{1-s}} + (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}} + \cdots,
\end{aligned}
$$

when $n \geq 2, s \geq 2, k = 1$, the composite solution without singular terms is

$$
\begin{aligned}
y^c =&[(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + \varepsilon(\alpha^{1-s}+s-1)^{\frac{s}{1-s}} \\
&\times [-\frac{1-s}{-2n+1} + \frac{s-1+\alpha^{1-s}}{2n} + \frac{s(1-s)(x-1)}{(s-1+\alpha^{1-s})(-2n+1)} - \frac{s(x-1)}{2n}] \\
&+ \beta e^{\frac{x-1}{\varepsilon^n}} + (\alpha^{1-s}+s-1)^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2n(x-1)^{2n}}} \\
&+ (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}} e^{-\frac{\varepsilon}{2n(x-1)^{2n}}} \\
&\times (-\frac{\varepsilon}{(2n-1)(x-1)^{2n}} - 1 + \frac{\varepsilon}{2n(x-1)^{2n}} + \cdots) \\
&- (\alpha^{1-s}+s-1)^{\frac{1}{1-s}} + (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}} + \cdots.
\end{aligned}
$$

Case 3. When $s \geq 2$ and $0 < k < 1$, the problem is

$$
\varepsilon^{2n+1} y'' + (x-k)^{2n+1}(y' + y^s) = \varepsilon y. \tag{2.35}
$$

In the same case of $s = 1$, the problem is similarly divided into left and right problems, which have $y_L(k) = \delta_1,\ y_R(k) = \delta_2$, and undetermined. The left problem is studied onl $[0, k)$, where $\xi = \frac{x-k}{\varepsilon^n} < 0$, $\eta = \frac{x-k}{\varepsilon^{\frac{1}{2n}}} < 0$. This paper studies the right problem on $(k, 1]$, where $\xi = \frac{x-k}{\varepsilon^n} > 0$, $\eta = \frac{x-k}{\varepsilon^{\frac{1}{2n}}} > 0$, and the steps are the same as $k = 0,\ 1$, when $s \geq 2,\ n = 1$, the left composite solution is

$$
\begin{aligned}
y_L^c =&[(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + \varepsilon[(s-1)k+\alpha^{1-s}]^{\frac{s}{1-s}}[-(1-s)(-k)^{-1} \\
&+ \frac{k(s-1)+\alpha^{1-s}}{2(-k)^2} + \frac{s(s-1)}{k(s-1)+\alpha^{1-s}} + \frac{s(x-k)(1-s)(-k)^{-1}}{k(s-1)+\alpha^{1-s}} \\
&- \frac{s(x-k)}{2}(-k)^{-2}] + [(s-1)k+\alpha^{1-s}]^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-k)^2}} + \delta_1 e^{\frac{x-k}{\varepsilon}} \\
&- \varepsilon \frac{\delta_1}{16} e^{\frac{x-k}{\varepsilon}} (-6\frac{x-k}{\varepsilon} + 6(\frac{x-k}{\varepsilon})^2 - 4(\frac{x-k}{\varepsilon})^3 + 2(\frac{x-k}{\varepsilon})^4) \\
&- \varepsilon^{\frac{1}{2}} \frac{e^{-\frac{\varepsilon}{2(x-k)^2}} ([(s-1)k+\alpha^{1-s}]^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-k)^2}})^s}{2\sqrt{-1+s}} \\
&\times (2e^{\frac{\varepsilon}{2(x-k)^2}} \frac{x-k}{\varepsilon^{\frac{1}{2}}}\sqrt{-1+s} + e^{\frac{s\varepsilon}{2(x-k)^2}} \sqrt{2\pi}(-1+s) erf(\frac{\varepsilon^{\frac{1}{2}}\sqrt{-1+s}}{\sqrt{2}(x-k)})) \\
&- [(s-1)k+\alpha^{1-s}]^{\frac{1}{1-s}} + (x-k)[(s-1)k+\alpha^{1-s}]^{\frac{s}{1-s}} + \cdots,
\end{aligned}
$$

the right composite solution is

$$
y_R^c =[(s-1)(x-1)+\beta^{1-s}]^{\frac{1}{1-s}} + [(s-1)(k-1)+\beta^{1-s}]^{\frac{s}{1-s}}
$$

$$
\begin{aligned}
&\times [\frac{s-1}{1-k} + \frac{(s-1)(k-1)+\beta^{1-s}}{2(1-k)^2} + \frac{s(s-1)}{(s-1)(k-1)+\beta^{1-s}} \\
&- \frac{s(s-1)(1-k)^{-1}(x-k)}{(s-1)(k-1)+\beta^{1-s}} - \frac{s(x-k)(1-k)^{-2}}{2}] + \delta_2 e^{-\frac{x-k}{\varepsilon}} \\
&- \varepsilon\frac{\delta_2}{16} e^{-\frac{x-k}{\varepsilon}} (6\frac{x-k}{\varepsilon} + 6(\frac{x-k}{\varepsilon})^2 + 4(\frac{x-k}{\varepsilon})^3 + 2(\frac{x-k}{\varepsilon})^4) \\
&+ [(s-1)(k-1)+\beta^{1-s}]^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-k)^2}} \\
&- \varepsilon^{\frac{1}{2}} \frac{e^{-\frac{\varepsilon}{2(x-k)^2}} \left([(s-1)(k-1)+\beta^{1-s}]^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2(x-k)^2}}\right)^s}{2\sqrt{-1+s}} \\
&\times (2e^{\frac{\varepsilon}{2(x-k)^2}} \frac{x-k}{\varepsilon^{\frac{1}{2}}} \sqrt{-1+s} + e^{\frac{s\varepsilon}{2(x-k)^2}} \sqrt{2\pi}(-1+s) erf(\frac{\varepsilon^{\frac{1}{2}}\sqrt{-1+s}}{\sqrt{2}(x-k)})) \\
&- [(s-1)(k-1)+\beta^{1-s}]^{\frac{1}{1-s}} + (x-k)[(s-1)(k-1)+\beta^{1-s}]^{\frac{s}{1-s}} + \cdots,
\end{aligned}
$$

when $s \geq 2, n \geq 2$, the left composite solution is

$$
\begin{aligned}
y_L^c =& [(s-1)x + \alpha^{1-s}]^{\frac{1}{1-s}} + \varepsilon(\alpha^{1-s} + (s-1)k)^{\frac{s}{1-s}} \\
&\times [-\frac{1-s}{-2n+1}(-k)^{-2n+1} + \frac{k(s-1)+\alpha^{1-s}}{2n}(-k)^{-2n} \\
&+ \frac{s(1-s)(x-k)}{((s-1)k+\alpha^{1-s})(-2n+1)}(-k)^{-2n+1} - \frac{s(x-k)}{2n}(-k)^{-2n}] + \delta_1 e^{\frac{x-k}{\varepsilon^n}} \\
&+ (\alpha^{1-s} + (s-1)k)^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2n(x-k)^{2n}}} + (x-k)(\alpha^{1-s} + (s-1)k)^{\frac{s}{1-s}} \\
&\times e^{-\frac{\varepsilon}{2n(x-k)^{2n}}} (-\frac{\varepsilon}{(2n-1)(x-k)^{2n}} - 1 + \frac{\varepsilon}{2n(x-k)^{2n}} + \cdots) \\
&- (\alpha^{1-s} + (s-1)k)^{\frac{1}{1-s}} + (x-k)(\alpha^{1-s} + (s-1)k)^{\frac{s}{1-s}} + \cdots,
\end{aligned}
$$

the right composite solution is

$$
\begin{aligned}
y_R^c =& [(s-1)(x-1) + \beta^{1-s}]^{\frac{1}{1-s}} + \varepsilon((s-1)(k-1) + \beta^{1-s})^{\frac{s}{1-s}} \\
&\times [\frac{1-s}{-2n+1}(1-k)^{-2n+1} + \frac{(s-1)(k-1)+\beta^{1-s}}{2n}(1-k)^{-2n} \\
&- \frac{s(1-s)(x-k)}{[(s-1)(k-1)+\beta^{1-s}](-2n+1)}(1-k)^{-2n+1} - \frac{s(x-k)}{2n}(1-k)^{-2n}] \\
&+ \delta_2 e^{-\frac{x-k}{\varepsilon^n}} + ((s-1)(k-1)+\beta^{1-s})^{\frac{1}{1-s}} e^{-\frac{\varepsilon}{2n(x-k)^{2n}}} \\
&+ (x-k)((s-1)(k-1)+\beta^{1-s})^{\frac{s}{1-s}} e^{-\frac{\varepsilon}{2n(x-k)^{2n}}} \\
&\times (-\frac{\varepsilon}{(2n-1)(x-k)^{2n}} - 1 + \frac{\varepsilon}{2n(x-k)^{2n}} + \cdots) \\
&- ((s-1)(k-1)+\beta^{1-s})^{\frac{1}{1-s}} + (x-k)((s-1)(k-1)+\beta^{1-s})^{\frac{s}{1-s}} + \cdots.
\end{aligned}
$$

By the smooth continuity of the left and right solutions at $x = k$, the undetermined coefficients $\delta_1,\ \delta_2$ can be determined as

$$\delta_1 = \delta_2 = \varepsilon^n \frac{[(s-1)k + \alpha^{1-s}]^{\frac{s}{1-s}}}{2} - \varepsilon^n \frac{[(s-1)(k-1) + \beta^{1-s}]^{\frac{s}{1-s}}}{2},$$

it can be found that the boundary layer term still starts from $O(\varepsilon^n)$, then we guess when ε is small enough, the boundary layer will not be obvious.

Remark 2.5. Under certain assumptions a result similar to this holds if the Dirichlet boundary conditions for y are replaced by Robin boundary conditions.

3. Existence of a solution and estimation of the remainder

In this section, this paper will present and prove the theorem of existence for solutions to the problems addressed in this paper, Specifically, the proof of existence and estimation of remainders are conducted simultaneously using the variable-controlling approach. We set $n = 1$ in order to address both linear and nonlinear prpblems that arise with the change of s, as discussed in section 2. Therefore, when proving the existence of the solutions, it is necessary to consider both linear and nonlinear cases.

In this section, Nagumo theorem is needed to prove the existence of the solution and to estimate the remainder term. For this purpose, the generalized nagumo theorem is first introduced [16, 17, 26].

For general questions

$$\begin{cases} Ly \equiv y'' - F(y', y, t),\ 0 < t < 1, \\ y(0) = y^0,\ y(1) = y^1, \end{cases}$$

where F is defined in the following region $\bar{G}$, where

$$\bar{G} = \{(t, y, y')|0 < t < 1,\ A < y < B,\ -\infty < y' < \infty\},$$

suppose there are quadratic continuously differentiable functions $\underline{y}(t), \bar{y}(t)$ satisfy

$$\begin{aligned} &a).\ \underline{y} \le \bar{y}, \\ &b).\ L\underline{y} \ge 0,\ L\bar{y} \le 0, \\ &c).\ \underline{y}(0) \le y^0 \le \bar{y}(0),\ \underline{y}(1) \le y^1 \le \bar{y}(1), \end{aligned}$$

and the function F has continuous partial derivatives with respect to y and y' in the region and satisfies the inequality $|F(y', y, t)| < \varphi(|y'|)$, where φ is a positive continuous function satisfying the integral condition $\int_0^\infty \frac{udu}{\varphi(u)} = \infty$. There exists a solution $y(t)$ that satisfies $\underline{y}(t) \le y(t) \le \bar{y}(t)$.

If $\underline{y}, \bar{y}$ belong to C^2 only in the segment of $[a, b]$, then the above results may also be obtained. That is there exists a partition $\{t_i\}$ of $[a, b]$, which $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ such that on every subinterval, $\underline{y}, \bar{y}$ are second-order continuously differentiable (at the partition points t_{i-1} and t_i , the derivative refers to the right derivative and the left derivative respectively). It has an additional condition that for every t in $[a, b]$, $\underline{y}'(t^-) \le \underline{y}'(t^+)$, $\bar{y}'(t^-) \ge \bar{y}'(t^+)$.

For the problem studied in this paper, there are $F = \frac{\varepsilon y-(x-k)^3(y'+y^s)}{\varepsilon^3}$, and $y^o = \alpha, y^1 = \beta$. Since a differentiable function is also continuous, and the continuity of the closed interval is bounded, so y is bounded. The absolute value inequality is $|a\pm b| \le |a|+|b|$, so that there exists $M>0$, we can take $M = \frac{1}{\varepsilon^3}$, which has $|F(x,y,y')| \le M(1+|y'|)$, and it satisfies the Nagumo condition $\int_0^\infty \frac{udu}{1+|u|} = \int_0^\infty \frac{udu}{1+u} = \infty$. Then we can construct the upper and lower solutions to prove $a), b), c)$.

3.1. Linear case

Theorem 3.1. *With Nagumo condition and (H_1) hold, there exists a sufficiently small positive parameter ε_0 such that for every $0<\varepsilon\le\varepsilon_0$, the BVP (2.1) has a solution $y(x,\varepsilon)$ with the multiple layer property at $x=k$, and as ε approaches 0, the inequality $|y(x,\varepsilon)-Y_0(x,\varepsilon)|\le c\varepsilon$ holds on $[0,1]$, where*

$$Y_0(x,\varepsilon) = \begin{cases} \beta e^{1-x} + \alpha e^{-\xi} + \beta e^{1-\frac{1}{2n\eta^{2n}}} - x\beta e^{1-\frac{1}{2n\eta^{2n}}} - \beta e + x\beta e, \quad k=0, \\ \alpha e^{-x} + \beta e^{\xi} + \alpha e^{-1-\frac{1}{2n\eta^{2n}}} - (x-1)\alpha e^{-1-\frac{1}{2n\eta^{2n}}} \\ -\alpha e^{-1} + (x-1)\alpha e^{-1}, \quad k=1. \end{cases}$$

When $0<k<1$,

$$Y_0(x,\varepsilon) = \begin{cases} \alpha e^{-x} + \alpha e^{-k-\frac{1}{2n\eta^{2n}}} - (x-k)\alpha e^{-k-\frac{1}{2n\eta^{2n}}} \\ -\alpha e^{-k} + (x-k)\alpha e^{-k}, \quad 0\le x<k, \\ \beta e^{1-x} + \beta e^{1-k-\frac{1}{2n\eta^{2n}}} - (x-k)\beta e^{1-k-\frac{1}{2n\eta^{2n}}} \\ -\beta e^{1-k} + (x-k)\beta e^{1-k}, \quad k<x\le 1, \end{cases}$$

where $\xi = \frac{x-k}{\varepsilon^n}$, $\eta = \frac{x-k}{\varepsilon^{\frac{1}{2n}}}$.

Proof. Just proving that for $n=1$, when $k=0$, $\beta e^{1-k-\frac{1}{2\eta^2}} - (x-k)\beta e^{1-k-\frac{1}{2\eta^2}} - \beta e^{1-k} + (x-k)\beta e^{1-k}$ in Eq. (2.15) will change its positive or negative with the change of β. This term is less than zero when $\beta>0$, and greater than zero when $\beta<0$. Similarly, when $k=1$, $\alpha e^{-k-\frac{1}{2\eta^2}} - (x-k)\alpha e^{-k-\frac{1}{2\eta^2}} - \alpha e^{-k} + (x-k)\alpha e^{-k}$ is less than zero when $\alpha>0$, and greater than zero when $\alpha<0$. Therefore, $\bar{y}$, $\underline{y}$ can be constructed according to the positive and negative of α, β, when $\beta>0$, $k=0$,

$$\bar{y} = \beta e^{1-x} + \alpha e^{-\xi} + \gamma\varepsilon,$$
$$\underline{y} = \beta e^{1-x} + \alpha e^{-\xi} + \beta e^{1-\frac{1}{2\eta^2}} - x\beta e^{1-\frac{1}{2\eta^2}} - \beta e + x\beta e - \gamma\varepsilon,$$

when $\beta<0, k=0$,

$$\bar{y} = \beta e^{1-x} + \alpha e^{-\xi} + \beta e^{1-\frac{1}{2\eta^2}} - x\beta e^{1-\frac{1}{2\eta^2}} - \beta e + x\beta e + \gamma\varepsilon,$$
$$\underline{y} = \beta e^{1-x} + \alpha e^{-\xi} - \gamma\varepsilon.$$

It is obvious that $\underline{y}\le\bar{y}$, and after inspection, $\underline{y}(0)<\alpha<\bar{y}(0)$, $\underline{y}(1)<\beta<\bar{y}(1)$. As long as $0<\gamma<|\beta|$, there is $L\bar{y}\le 0$ and $L\underline{y}\ge 0$.

Similarly, when $\alpha>0, k=1$,

$$\bar{y} = \alpha e^{-x} + \beta e^{\xi} + \gamma\varepsilon,$$

$$\underline{y} = \alpha e^{-x} + \beta e^{\xi} + \alpha e^{-1-\frac{1}{2\eta^2}} - (x-1)\alpha e^{-1-\frac{1}{2\eta^2}} - \alpha e^{-1} + (x-1)\alpha e^{-1} - \gamma\varepsilon,$$

when $\alpha < 0, k = 1$,

$$\bar{y} = \alpha e^{-x} + \beta e^{\xi} + \alpha e^{-1-\frac{1}{2\eta^2}} - (x-1)\alpha e^{-1-\frac{1}{2\eta^2}} - \alpha e^{-1} + (x-1)\alpha e^{-1} + \gamma\varepsilon,$$
$$\underline{y} = \alpha e^{-x} + \beta e^{\xi} - \gamma\varepsilon,$$

when $k \in (0,1)$, since the boundary layer has a small parameter ε from the first term, it can be verified that the boundary layer term is very small. Therefore, when α, β are greater than zero, $\bar{y}, \underline{y}$ are respectively as follows

$$\bar{y} = \begin{cases} \alpha e^{-x} + \gamma\varepsilon, & 0 \le x < k, \\ \beta e^{1-x} + \gamma\varepsilon, & k < x \le 1, \end{cases}$$

$$\underline{y} = \begin{cases} \alpha e^{-x} + \alpha e^{-k-\frac{1}{2\eta^2}} - (x-k)\alpha e^{-k-\frac{1}{2\eta^2}} \\ -\alpha e^{-k} + (x-k)\alpha e^{-k} - \gamma\varepsilon, & 0 \le x < k, \\ \beta e^{1-x} + \beta e^{1-k-\frac{1}{2\eta^2}} - (x-k)\beta e^{1-k-\frac{1}{2\eta^2}} \\ -\beta e^{1-k} + (x-k)\beta e^{1-k} - \gamma\varepsilon, & k < x \le 1, \end{cases}$$

when $\alpha,\ \beta$ are less than zero, $\bar{y},\ \underline{y}$ are respectively as follows

$$\bar{y} = \begin{cases} \alpha e^{-x} + \alpha e^{-k-\frac{1}{2\eta^2}} - (x-k)\alpha e^{-k-\frac{1}{2\eta^2}} \\ -\alpha e^{-k} + (x-k)\alpha e^{-k} + \gamma\varepsilon, & 0 \le x < k, \\ \beta e^{1-x} + \beta e^{1-k-\frac{1}{2\eta^2}} - (x-k)\beta e^{1-k-\frac{1}{2\eta^2}} \\ -\beta e^{1-k} + (x-k)\beta e^{1-k} + \gamma\varepsilon, & k < x \le 1, \end{cases}$$

$$\underline{y} = \begin{cases} \alpha e^{-x} - \gamma\varepsilon, & 0 \le x < k, \\ \beta e^{1-x} - \gamma\varepsilon, & k < x \le 1, \end{cases}$$

since the upper and lower solutions are not smooth in this case, it is necessary to satisfy the relationship between the left and right derivatives at $x = k$. As long as (H_1) is satisfied, there is $\frac{d\underline{y}}{dx}(k^-) \le \frac{d\underline{y}}{dx}(k^+)$ and $\frac{d\bar{y}}{dx}(k^-) \ge \frac{d\bar{y}}{dx}(k^+)$. And after inspection, when $0 < |\frac{-\beta e^{1-k}+\alpha e^{-k}}{2}| < \gamma < \frac{|\beta|}{(1-k)^3}$, condirions $a), b), c)$ are satisfied. Then according to the comparison principle, there exists a solution $y(x,\varepsilon)$ of Eq. (2.1) and satisfying $\underline{y}(x,\varepsilon) \le y(x,\varepsilon) \le \bar{y}(x,\varepsilon)$, where $0 \le x \le 1, 0 < \varepsilon \ll 1$. Meanwhile, it satisfies $|y(x,\varepsilon) - Y_0(x,\varepsilon)| \le c\varepsilon$. □

Remark 3.1. When $\alpha,\ \beta$ are different signs, they can be divided into two cases of $\alpha > 0,\ \beta < 0$ and $\alpha < 0,\ \beta > 0$. In these cases, no matter how to construct the upper and lower solutions, the relationship between the left and right derivatives at the point $x = k$ is not satisfied. Therefore, the proof can only be carried out when the boundary values are the same sign and satisfiy the above additional conditions.

3.2. Nonlinear case

Theorem 3.2. *With Nagumo condition and* (H_2) *hold, there exists a sufficiently small positive parameter* ε_0 *such that for every* $0 < \varepsilon \le \varepsilon_0$*, the BVP (1.1) has a solution* $y(x, \varepsilon)$ *with the multiple layer property at* $x = k$*, and as* ε *approaches 0, the inequality* $|y(x, \varepsilon) - \bar{Y}_0(x, \varepsilon)| \le c\varepsilon$ *holds on* $[0, 1]$*, where*

$$\bar{Y}_0(x,\varepsilon) = \begin{cases} [\beta^{1-s} + (s-1)(x-1)]^{\frac{1}{1-s}} + \alpha e^{-\xi} + (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} e^{-\frac{1}{2n\eta^{2n}}} \\ -(\beta^{1-s} - s + 1)^{\frac{1}{1-s}} - x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} e^{-\frac{s}{2n\eta^{2n}}} \\ +x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}, \quad k = 0, \\ [(s-1)x + \alpha^{1-s}]^{\frac{1}{1-s}} + \beta e^{\xi} + (\alpha^{1-s} + s - 1)^{\frac{1}{1-s}} e^{-\frac{1}{2n\eta^{2n}}} \\ -(\alpha^{1-s} + s - 1)^{\frac{1}{1-s}} - (x-1)(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}} e^{-\frac{s}{2n\eta^{2n}}} \\ +(x-1)(\alpha^{1-s} + s - 1)^{\frac{s}{1-s}}, \quad k = 1, \end{cases}$$

when $0 < k < 1$*,*

$$\bar{Y}_0(x,\varepsilon) = \begin{cases} [(s-1)x + \alpha^{1-s}]^{\frac{1}{1-s}} + ((s-1)k + \alpha^{1-s})^{\frac{1}{1-s}} e^{-\frac{1}{2n\eta^{2n}}} \\ -((s-1)k + \alpha^{1-s})^{\frac{1}{1-s}} - (x-k)((s-1)k + \alpha^{1-s})^{\frac{s}{1-s}} e^{-\frac{s}{2n\eta^{2n}}} \\ +(x-k)((s-1)k + \alpha^{1-s})^{\frac{s}{1-s}}, \quad 0 \le x < k, \\ [\beta^{1-s} + (s-1)(x-1)]^{\frac{1}{1-s}} + ((s-1)(k-1) + \beta^{1-s})^{\frac{1}{1-s}} e^{-\frac{1}{2n\eta^{2n}}} \\ -((s-1)(k-1) + \beta^{1-s})^{\frac{1}{1-s}} \\ -(x-k)((s-1)(k-1) + \beta^{1-s})^{\frac{s}{1-s}} e^{-\frac{s}{2n\eta^{2n}}} \\ +(x-k)((s-1)(k-1) + \beta^{1-s})^{\frac{s}{1-s}}, \quad k < x \le 1, \end{cases}$$

where $\xi = \frac{x-k}{\varepsilon^n}, \eta = \frac{x-k}{\varepsilon^{\frac{1}{2n}}}$.

Proof. It also be proved when $n = 1$, and then $n \ge 2$ can in the same way to prove. For the nonlinear problem, it satisfies $0 < \beta^{s-1} < \frac{1}{s-1}$ and $\alpha > 0$. When $k = 0$, as long as the condition (H_2) is satisfied, this term of $(\beta^{1-s} - s + 1)^{\frac{1}{1-s}} e^{-\frac{1}{2\eta^2}} - x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} e^{-\frac{s}{2\eta^2}} - (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} + x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}}$ is always less than zero. Then the upper and lower solutions can be constructas as follows,

$$\begin{aligned} \bar{y} =& [\beta^{1-s} + (s-1)(x-1)]^{\frac{1}{1-s}} + \alpha e^{-\xi} + \gamma\varepsilon, \\ \underline{y} =& [\beta^{1-s} + (s-1)(x-1)]^{\frac{1}{1-s}} + (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} e^{-\frac{1}{2\eta^2}} \\ & - x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} e^{-\frac{s}{2\eta^2}} - (\beta^{1-s} - s + 1)^{\frac{1}{1-s}} \\ & + x(\beta^{1-s} - s + 1)^{\frac{s}{1-s}} + \alpha e^{-\xi} - \gamma\varepsilon, \end{aligned}$$

after inspection, there is $\underline{y} \le \bar{y}$, $\underline{y}(0) < \alpha < \bar{y}(0)$, $\underline{y}(1) < \beta < \bar{y}(1)$. As long as $0 < \gamma \le \frac{\beta^{2-s}}{s}$, there is $L\bar{y} \le 0$ and $L\underline{y} \ge 0$.

When $k = 1$, the $(\alpha^{1-s}+s-1)^{\frac{1}{1-s}}e^{-\frac{1}{2\eta^2}} - (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}}e^{-\frac{s}{2\eta^2}} - (\alpha^{1-s}+s-1)^{\frac{1}{1-s}} + (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}}$ is always less than zero and the upper and lower solutions in this case are as follows,

$$
\begin{aligned}
\bar{y} =& [(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + \beta e^{\xi} + \gamma\varepsilon, \\
\underline{y} =& [(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + (\alpha^{1-s}+s-1)^{\frac{1}{1-s}}e^{-\frac{1}{2\eta^2}} \\
& - (\alpha^{1-s}+s-1)^{\frac{1}{1-s}} + (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}} \\
& - (x-1)(\alpha^{1-s}+s-1)^{\frac{s}{1-s}}e^{-\frac{s}{2\eta^2}} + \beta e^{\xi} - \gamma\varepsilon.
\end{aligned}
$$

Remark 3.2. In the case of $s \geq 2$, since it satisfies $\beta^{1-s} \geq s$ and $\alpha > 0$, the main term of $L\underline{y}$ is $(x-k)^3\underline{y}'$.

In the same way of above cases, when $k \in (0,1)$, there are upper and lower solutions as follows

$$
\bar{y} = \begin{cases} [(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + \gamma\varepsilon, & 0 \leq x < k, \\ [\beta^{1-s}+(s-1)(x-1)]^{\frac{1}{1-s}} + \gamma\varepsilon, & k < x \leq 1, \end{cases}
$$

$$
\underline{y} = \begin{cases} [(s-1)x+\alpha^{1-s}]^{\frac{1}{1-s}} + ((s-1)k+\alpha^{1-s})^{\frac{1}{1-s}}e^{-\frac{1}{2\eta^2}} \\ -(x-k)((s-1)k+\alpha^{1-s})^{\frac{s}{1-s}}e^{-\frac{s}{2\eta^2}} \\ -((s-1)k+\alpha^{1-s})^{\frac{1}{1-s}} + (x-k)((s-1)k+\alpha^{1-s})^{\frac{s}{1-s}} - \gamma\varepsilon, & 0 \leq x < k, \\ [\beta^{1-s}+(s-1)(x-1)]^{\frac{1}{1-s}} + ((s-1)(k-1)+\beta^{1-s})^{\frac{1}{1-s}}e^{-\frac{1}{2\eta^2}} \\ -(x-k)((s-1)(k-1)+\beta^{1-s})^{\frac{s}{1-s}}e^{-\frac{s}{2\eta^2}} - ((s-1)(k-1)+\beta^{1-s})^{\frac{1}{1-s}} \\ +(x-k)((s-1)(k-1)+\beta^{1-s})^{\frac{s}{1-s}} - \gamma\varepsilon, & k < x \leq 1. \end{cases}
$$

If (H_2) is satisfied, and when $0 < \frac{[(s-1)(k-1)+\beta^{1-s}]^{\frac{s}{1-s}} - [(s-1)k+\alpha^{1-s}]^{\frac{s}{1-s}}}{2} \leq \gamma \leq \frac{\beta^{2-s}}{s}$, then there exists a solution $y(x,\varepsilon)$ to the problem. And according to the comparison principle, it satisfies $\underline{y}(x,\varepsilon) \leq y(x,\varepsilon) \leq \bar{y}(x,\varepsilon)$ when $0 \leq x \leq 1$, $0 < \varepsilon \ll 1$. Meanwhile, it satisfies $|y(x,\varepsilon) - \bar{Y}_0(x,\varepsilon)| \leq c\varepsilon$. □

4. Numerical examples

Bvp4c is a finite difference code that implements the three-stage Lobatto IIIa formula [25,34]. This is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth order accurate uniformly in $[a,b]$. Mesh selection and error control are based on the residual of the continuous solution.

The integral interval is divided into subintervals by the allocation method using the point grid. By solving the global group of linear algebraic equations obtained from the configuration conditions and the boundary conditions on all subintervals, the solver arrives at the numerical solution. Next, the solver calculates the numerical solution error for every subinterval. The solver modifies the grid and performs the computation again if the result does not satisfy the tolerance requirements. The original grid and a preliminary approximation of the solution at the grid points must be supplied [4,10].

In this section, this paper present several examples whose solutions disaplay some of the behavior outlined above.

4.1. Example 1

$$\begin{cases} \varepsilon^3 y'' + (x-k)^3(y'+y) = \varepsilon y, \\ y(0) = \alpha, \\ y(1) = \beta. \end{cases}$$

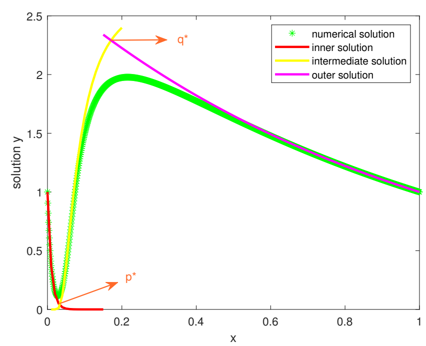

Figure 1. The analysis figure of multilayer phenomenon when $k = 0, \varepsilon = 0.01$.

When $k = 0$, the multilayer phenomenon occurs near $x = 0$ with boundary values set as $\alpha = 1$ and $\beta = 1$. Accordingly, ε is taken as 0.01. The numerical solution (green), the inner solution (red), the intermediate solution (yellow) and the outer solution (magenta) are depicted respectively in Figure 1. It is observed that while both the boundary layer and the turning point appear near $x = 0$, they consistently approach $x = 0$, making it impossible for the multi-layer phenomenon to occur at the point of $x = 0$.

The intersection of the red line and the yellow line p^* can be roughly regarded as the dividing point between the inner layer and the intermediate layer, and the intersection of the yellow line and the magenta line q^* can be roughly regarded as the dividing point between the outer layer and the intermediate layer. Particularly, the connection between the outer solution and the intermediate solution is quite different from the numerical solution is that there is no matching terms included. However, based on the aforementioned analysis, our focus lies on the points under the interval because the multilayer phenomenon could not occur at a certain point.

Remark 4.1. The multilayer phenomenon could not occur at a certain point but in a narrow interval. As we analyzed in Figure 1, we can further study the points under the interval such as p^* and q^*.

Since the asymptotic solution obtains its zeroth, first and higher approximations, we can further study the points under the interval in Remark 4.1. We suppose that if the dividing point between the inner and intermediate layers is p^* in a certain range of k, then $p^* = p_0 + \varepsilon^n p_1 + \cdots$, where p_0 is the point that the multilayer

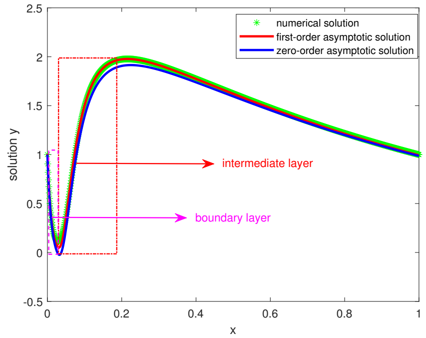

Figure 2. Comparison of asymptotic and the numerical solutions when $k = 0, \varepsilon = 0.01$.

phenomenon occurs we studied in the previous analysis and solution process, but compared with p^*, p_0 is still a little rough, so the point where the dividing point occurs will be more accurate due to the appearance of p^*. Similarly, if the dividing point between the outer and intermediate layers is q^* in a certain range of k, then $q^* = q_0 + \varepsilon^{\frac{1}{2n}} q_1 + \cdots$, where q_0 is the point that the multilayer phenomenon occurs we studied in the previous analysis and solution process. In this way we can completely determine the specific point where the multi-layer phenomenon occurs.

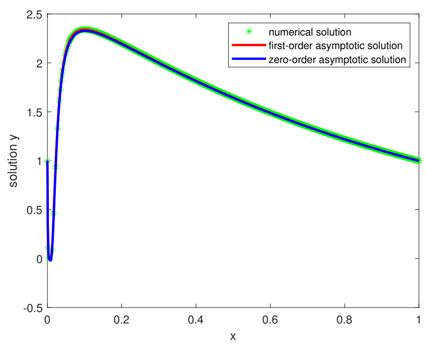

Figure 3. Comparison of asymptotic and the numerical solutions when $k = 0, \varepsilon = 0.001$.

As shown in Figure 2, we can clearly distinguish the boundary layer, the intermediate layer and the outer layer, and fitting degree between the first-order asymptotic solution and the numerical solution is better than the fitting degree between the zeroth-order asymptotic solution and the numerical solution. We adjust ε from 0.01 to 0.001 in Figure 3 to find the asymptotic solution and numerical solution have better approximate effect. Meanwhile, as ε gets smaller, the boundary layer and the intermediate layer get thinner.

When $k = 1$, the multilayer phenomenon occurs near $x = 1$ with boundary values set as $\alpha = 2$ and $\beta = 1$. Accordingly, ε is taken as 0.01 and 0.001. We can see clear multilayer phenomena in Figure 4 and Figure 5. And it is observed that the approximation degree of the first-order asymptotic solution is better than the zeroth-order asymptotic solution with the numerical solution.

Now we consider $0 < k < 1$, we select $k = 0.25$. According to the analysis and

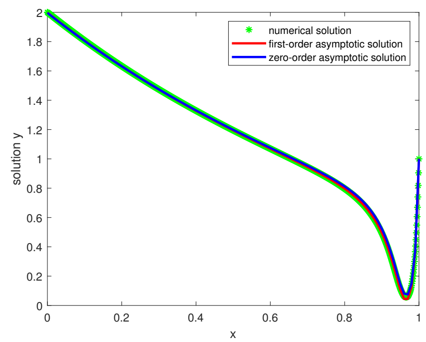

Figure 4. Comparison of asymptotic and the numerical solutions when $k = 1, \varepsilon = 0.01$.

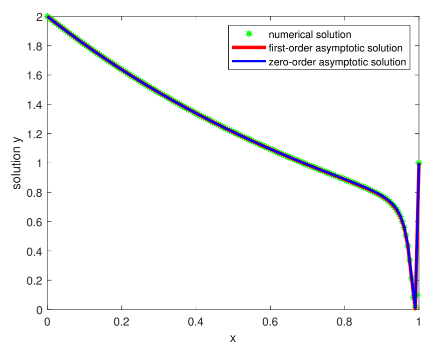

Figure 5. Comparison of asymptotic and the numerical solutions when $k = 1, \varepsilon = 0.001$.

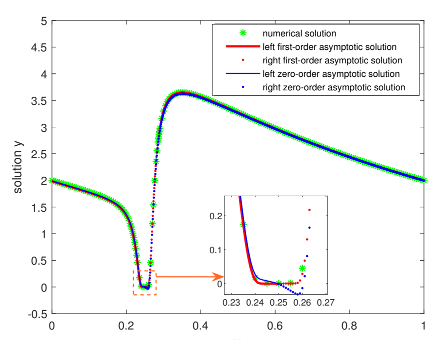

Figure 6. Comparison of asymptotic and the numerical solutions when $k = 0.25, \varepsilon = 0.001$.

solving process, the multi-layer phenomenon will occur near $x = 0.25$. As shown in Figure 6, if the figure is divided into left and right areas according to $x = 0.25$, then the left outer layer, left intermediate layer, left inner layer and right inner layer, right intermediate layer, right outer layer should appear successively in the figure. It is obvious that four layers can be clearly seen in Figure 6. We focus our attention on the vicnity of $x = 0.25$ and observe there is a narrow interval approaching zero. This is just as we suspected in Remark 2.3. Since $\varepsilon = 0.001$ in this case, the inner

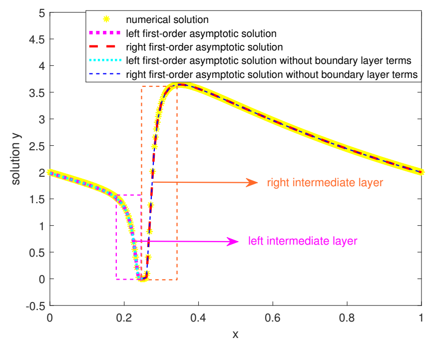

Figure 7. Comparison plot with and without boundary layer terms when $k = 0.25, \varepsilon = 0.001$.

solution starts at $O(\varepsilon)$, which is small enough compared to the other solutions. In order to further verify Remark 2.3, we draw numerical solution, the first asymptotic solution and the first asymptotic solution without inner terms respectively in Figure 7 and find their fitting degree is quite good. That is when ε is small enough, the boundary layer phenomenon is not obvious. Therefore, there is the left outer layer, left intermediate layer, right intermediate layer, and right outer layer in Figure 6.

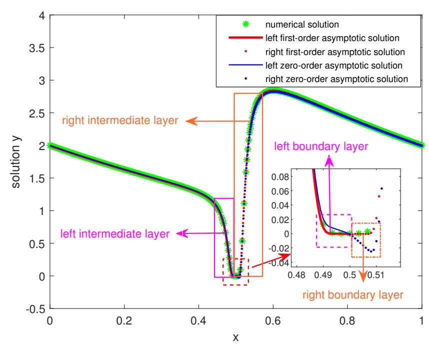

Figure 8. Comparison of asymptotic and the numerical solutions when $k = 0.5, \varepsilon = 0.001$.

When $k = 0.5$, as shown in Figure 8, the multi-layer phenomenon occurs near $x = 0.5$ and the fitting degree is well. According to the above analysis, we mark the left and right intermediate layer and the left and right boundary layer in Figure 8.

4.2. Example 2

$$\begin{cases} \varepsilon^3 y'' + (x-k)^3(y' + y^2) = \varepsilon y, \\ y(0) = \alpha, \\ y(1) = \beta. \end{cases}$$

This is a nonlinear problem, where we consider the values of k to be 0, 1, and 0.5 respectively. For $k = 0$, the initial boundary values are chosen as $\alpha = 1$ and $\beta = 0.5$, when $k = 1$, both α and β are set to be equal to 0.5. And for $k = 0.5$, the values of α and β are selected as 1 and 0.5 respectively. As shown in Figure 9, Figure 10 and Figure 11, it can also be shown that when the problem is nonlinear, the degree of fitting between the asymptotic solution and the numerical solution is quite well. Then it is sufficient to show that the formal asymptotic solution constructed previously has a good asymptotic state.

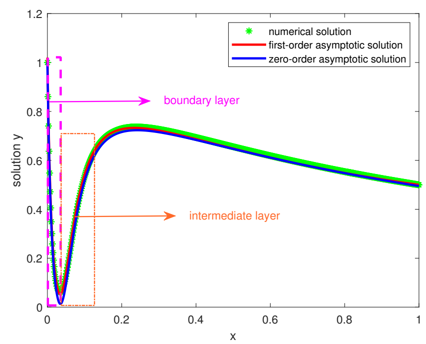

Figure 9. Comparison of asymptotic and the numerical solutions when nonlinear and $k = 0, \varepsilon = 0.01$.

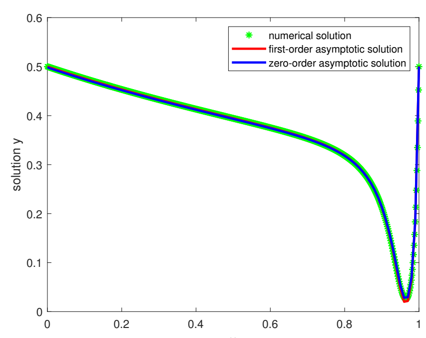

Figure 10. Comparison of asymptotic and the numerical solutions when nonlinear and $k = 1, \varepsilon = 0.01$.

Moreover, when the turning point is inside the interval, according to the previous conjecture and the linear problem in Example 1. We also make the fitting figure with and without boundary layer terms. As shown in Figure 12, when $0 < k < 1$, it can be proven that the boundary layer is not obvious if ε is small enough.

No matter from the section 2 or the above figures, we can find that with the change of k, the problem will produce multi-layer phenomenon in a certain region. In addition, as ε becomes smaller, the boundary layer and the intermediate layer also become narrower.

5. Conclusions and open problems

This study obtains the first order asymptotic solution that is uniformly valid in the appropriate interval by splitting the singularly perturbed problems with turning points into linear and nonlinear categories. We then provide numerical examples

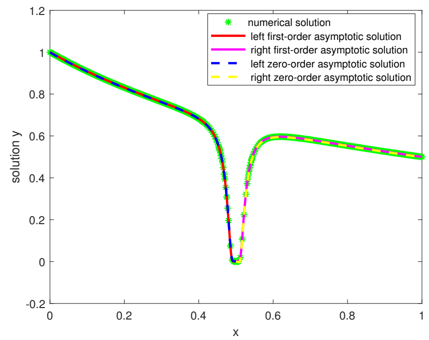

Figure 11. Comparison of asymptotic and the numerical solutions when nonlinear and $k = 0.5, \varepsilon = 0.001$.

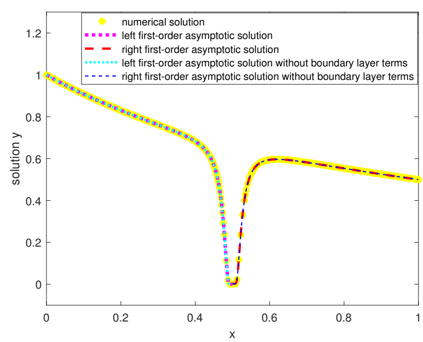

Figure 12. Comparison plot with and without boundary layer terms when nonlinear and $k = 0.5, \varepsilon = 0.001$.

and the existence proof of the solution. It embodies singular perturbation theories in their entirety.

By utilizing this method, we can proceed with solving the higher order asymptotic solution and provide further evidence for its existence. Upon obtaining the second-order asymptotic solution of the linear problem, it can be observed that an intensification of the singularity as the order increases. While it cannot be completely eliminated, the singularity can be somewhat diminished. This paper may explore different techniques in the future to eliminate its singularity. There might also be additional types of special functions and a more intricate calculation procedure for the nonlinear problem.

The aforementioned research on this issue are all quantitative in nature. This paper can do more qualitative research on this issue to better reflect the integrity and usefulness of perturbation theory and method system. Obviously, this issue is considered and resolved in a particular finite area $[0, 1]$. In the future, we wish to extend the interval and observe the solution phenomenon in order to investigate the global stability of the solution [19] for the boundary value problem with intermediate layer and boundary layer. We can expand the limited area and observe the phenomenon of its solution in a broader area, or even investigate the infinite interval. Qualitative analysis in future research will make the study of this problem more valuable.

In addition, in the future research, it can be mainly considered from two aspects: on the one hand, to improve the degree of nonlinearity of the problem. For example, we can study the problem

$$\begin{aligned}&\varepsilon^3 y'' + (x-k)^3[yy' + y^s] = \varepsilon y^2,\\ &y(0) = \alpha, y(1) = \beta.\end{aligned}$$

On the other hand, to improve the order of the original problem. For example, we can study the problem

$$\begin{aligned}&\varepsilon^{2n+1} y''' + (x-k)^{2n+1}[y' + y^s] = \varepsilon y,\\ &y(0) = A_1, y(1) = B, y'(x^*) = A_2.\end{aligned}$$

Where x^* is the position where the turning point occurs, and its value changes with the value of k.

Appendices

A. Supplement to the linear case

Since the special functions only appear in the nonlinear case, then it is easy to continue to solve the high-order asymptotic solutions of the linear case. And the singularity of the outer solution appears from the second term, we guess that the singularity will become stronger and stronger. Then there is a question that can the singularity of the problem be eliminated by matching at this point? It can be considered when $k = 0$ first.

It can be obtianed that the quadratic equation of the outer solution and its solution are

$$x^{2n+1}\left(\frac{dy_2^o}{dx} + y_2^o\right) = y_1^o \Rightarrow y_2^o = \frac{\beta e^{1-x}(x^{-4n} - 2x^{-2n} + 1)}{8n^2},$$

the singularity is definitely stronger.

When $n = 1$, the quadratic equation of the inner solution is

$$\frac{d^2 y_2^i}{d\xi^2} + \xi^3 \frac{dy_1^i}{d\xi} - y_2^i + \xi^3 y_0^i = 0,$$

and its solution is

$$\begin{aligned}y_2^i =& \frac{e^{-\xi}\left(840a_0 - 840a_1 + 1680a_0\xi - 1680a_1\xi\right)}{4480}\\ &+ \frac{e^{-\xi}(1680a_0\xi^2 - 1680a_1\xi^2 + 1120a_0\xi^3 - 1120a_1\xi^3)}{4480}\\ &+ \frac{(560a_0\xi^4 - 560a_1\xi^4 + 42a_0\xi^5 + 70a_0\xi^6 + 60a_0\xi^7 + 35a_0\xi^8)e^{-\xi}}{4480}\\ &+ a_2 e^{-\xi} + b_2 e^{\xi},\end{aligned}$$

when $n \geq 2$, the quadratic equation of the inner solution and its solution are

$$\frac{d^2 y_2^i}{d\xi^2} - y_2^i = 0 \Rightarrow y_2^i = a_2 e^{-\xi} + b_2 e^{\xi},$$

where $b_2 = 0$.

The solution of the intermediate layer produces different solution according to the value of n from the quadratic term, the quadratic equation of the intermediate solution is

$$\begin{cases} \dfrac{d^2 y_0^m}{d\eta^2} + \eta^3 \dfrac{dy_2^m}{d\eta} + \eta^3 y_1^m - y_2^m = 0, \ n = 1, \\ \eta^{2n+1} \dfrac{dy_2^m}{d\eta} + \eta^{2n+1} y_1^m - y_2^m = 0, \ n \geq 2, \end{cases}$$

the corresponding intermediate solution is

$$y_2^m = \begin{cases} e^{-\frac{1}{2\eta^2}} (\dfrac{d_0}{8\eta^8} - \dfrac{d_0}{2\eta^6} - d_1\eta + \dfrac{d_0\eta^2}{2}) + d_2 e^{-\frac{1}{2\eta^2}}, \ n = 1, \\ e^{-\frac{1}{2n\eta^{2n}}} (-d_1\eta + \dfrac{d_0}{2}\eta^2), \ n \geq 2. \end{cases}$$

When $n = 1$, through Van Dyke matching principle, there is $d_2 = \frac{\beta e}{2}$, and the common solution is

$$\begin{aligned} [y_{(3)}^o]_{(3)}^m =& [y_{(3)}^m]_{(3)}^o \\ =& \beta e - \beta e x + \frac{\beta e x^2}{2} - \frac{\varepsilon \beta e}{2x^2} + \frac{\varepsilon \beta e}{2} + \frac{\varepsilon \beta e}{2x} \\ &- \frac{\varepsilon \beta e}{4} + \frac{\beta e \varepsilon^2}{8x^4} - \frac{\beta e \varepsilon^2}{4x^2} - \frac{\beta e \varepsilon^2}{8x^3} + \frac{\beta e \varepsilon^2}{16x^2}, \end{aligned}$$

when $n \geq 2$, $d_2 = 0$, and the common solution is

$$\begin{aligned} [y_{(3)}^o]_{(3)}^m =& [y_{(3)}^m]_{(3)}^o \\ =& \beta e - \beta e x + \frac{\beta e x^2}{2} - \frac{\beta e}{2n} (\frac{\varepsilon}{x^{2n}} - \frac{\varepsilon x}{x^{2n}} + \frac{\varepsilon x^2}{2x^{2n}}) \\ &+ \frac{\beta e}{8n^2} (\frac{\varepsilon^2}{x^{4n}} - \frac{\varepsilon^2 x}{x^{4n}} + \frac{\varepsilon^2 x^2}{2x^{4n}}), \end{aligned}$$

however, it can be found that the singularity can not be completely eliminated, it only can be reduced at this point.

In the same way, when $k = 1$, we can also solve and match it, the second order outer solution is $y_2^o = \frac{\alpha e^{-x}((x-1)^{-4n} - 2(x-1)^{-2n} + 1)}{8n^2}$, and when $n = 1$, $d_2 = \frac{\alpha e^{-1}}{2}$, when $n \geq 2$, $d_2 = 0$. And when $n = 1$, the common solution is

$$\begin{aligned} [y_{(3)}^o]_{(3)}^m =& [y_{(3)}^m]_{(3)}^o \\ =& \alpha e^{-1} - \alpha e^{-1}(x-1) + \frac{\alpha e^{-1}(x-1)^2}{2} \\ &- \frac{\alpha e^{-1}}{2} (\frac{\varepsilon}{(x-1)^2} - \varepsilon - \frac{\varepsilon}{x-1} + \frac{\varepsilon}{2}) \\ &+ \frac{\alpha e^{-1}}{8} (\frac{\varepsilon^2}{(x-1)^4} - \frac{2\varepsilon^2}{(x-1)^2} - \frac{\varepsilon^2}{(x-1)^3} + \frac{\varepsilon^2}{2(x-1)^2}), \end{aligned}$$

when $n \geq 2$, the common solution is

$$[y_{(3)}^o]_{(3)}^m = [y_{(3)}^m]_{(3)}^o$$

$$
\begin{aligned}
&=\alpha e^{-1}-\alpha e^{-1}(x-1)+\frac{\alpha e^{-1}(x-1)^2}{2}\\
&\quad-\frac{\alpha e^{-1}}{2n}(\frac{\varepsilon}{(x-1)^{2n}}-\frac{\varepsilon(x-1)}{(x-1)^{2n}}+\frac{\varepsilon(x-1)^2}{2(x-1)^{2n}})\\
&\quad+\frac{\alpha e^{-1}}{8n^2}(\frac{\varepsilon^2}{(x-1)^{4n}}-\frac{\varepsilon^2(x-1)}{(x-1)^{4n}}+\frac{\varepsilon^2(x-1)^2}{2(x-1)^{4n}}).
\end{aligned}
$$

These higher order solutions can continue be solved in this way, it can be obtianed the outer solution, the inner solution and the intermediate solution and matching them. Although the singularity can not be completely eliminated at this point, it can be reduced to some extent.

As the order increases, the singularity of the problem becomes stronger. It can reduce the singularity to a certain extent by using Van Dyke matching principle, and when $s=1,\ k=0,\ n=1$, the final composite solution is

$$
\begin{aligned}
y^c=&\alpha e^{-\frac{x}{\varepsilon}}-\varepsilon\frac{\alpha}{16}e^{-\frac{x}{\varepsilon}}[6\frac{x}{\varepsilon}+6(\frac{x}{\varepsilon})^2+4(\frac{x}{\varepsilon})^3+2(\frac{x}{\varepsilon})^4]+\varepsilon^2y_2^i+\beta e^{1-x}\\
&-\frac{\beta e\varepsilon}{2}(x-\frac{x^2}{2})+\frac{\varepsilon^2\beta e}{4x}+\beta e^{1-\frac{\varepsilon}{2x^2}}-x\beta e^{1-\frac{\varepsilon}{2x^2}}\\
&+\varepsilon e^{-\frac{\varepsilon}{2x^2}}(\frac{\varepsilon^4\beta e}{8x^8}-\frac{\varepsilon^3\beta e}{2x^6}+\frac{\beta e x^2}{2\varepsilon})+\frac{\varepsilon\beta e}{2}e^{-\frac{\varepsilon}{2x^2}}-\beta e+\beta e x-\frac{\beta e x^2}{2}+\cdots,
\end{aligned}
$$

the singularity is not strong, and we can try to eliminate its singularity in other methods. In real life, the solutions of many physical models are also singular, which just reflects the authenticity of this model.

Take $k=0$ as an example to solve its third-order asymptotic solution, and obtain the solutions, when $n=1$,

$$
\begin{aligned}
y^o=&\beta e^{1-x}-\varepsilon\frac{\beta e^{1-x}}{2}(\frac{1}{x^2}-1)+\varepsilon^2\frac{\beta e^{1-x}}{8}(\frac{1}{x^4}-\frac{2}{x^2}+1)\\
&-\varepsilon^3\frac{\beta e^{1-x}}{8}(\frac{1}{6x^6}-\frac{1}{2x^4}-\frac{7}{2x^2}-\frac{23}{6})+\cdots,\\
y^m=&d_0e^{-\frac{1}{2\eta^2}}+\varepsilon^{\frac{1}{2}}(-d_0\eta e^{-\frac{1}{2\eta^2}}+d_1e^{-\frac{1}{2\eta^2}})\\
&+\varepsilon[e^{-\frac{1}{2\eta^2}}(\frac{d_0}{8\eta^8}-\frac{d_0}{2\eta^6}-d_1\eta+\frac{d_0\eta^2}{2})+d_2e^{-\frac{1}{2\eta^2}}]\\
&+\varepsilon^{\frac{3}{2}}e^{-\frac{1}{2\eta^2}}(\frac{d_1}{8\eta^8}-\frac{d_0}{8\eta^7}-\frac{d_1}{2\eta^6}+\frac{d_0}{10\eta^5}-d_2\eta+\frac{d_1\eta^2}{2}-\frac{d_0\eta^3}{6}+d_3)+\cdots,
\end{aligned}
$$

when $n\geq 2$,

$$
\begin{aligned}
y^o=&\beta e^{1-x}-\varepsilon\frac{\beta e^{1-x}}{2n}(\frac{1}{x^{2n}}-1)+\varepsilon^2\frac{\beta e^{1-x}}{8n^2}(\frac{1}{x^{4n}}-\frac{2}{x^{2n}}+1)\\
&+\varepsilon^3\frac{\beta e^{1-x}x^{-6n}(-1+x^{2n})^3}{48n^3}+\cdots,\\
y^m=&d_0e^{-\frac{1}{2n\eta^{2n}}}+\varepsilon^{\frac{1}{2n}}(-d_0\eta e^{-\frac{1}{2n\eta^{2n}}}+d_1e^{-\frac{1}{2n\eta^{2n}}})\\
&+\varepsilon^{\frac{1}{n}}[e^{-\frac{1}{2n\eta^{2n}}}(-d_1\eta+\frac{d_0\eta^2}{2})+d_2e^{-\frac{1}{2n\eta^{2n}}}]\\
&+\varepsilon^{\frac{3}{2n}}[e^{-\frac{1}{2n\eta^{2n}}}(-d_2\eta+\frac{d_1\eta^2}{2}-\frac{d_0\eta^3}{6})+d_3e^{-\frac{1}{2n\eta^{2n}}}]+\cdots,
\end{aligned}
$$

there is $d_3 = 0$ by using the Van Dyke matching principle, and the common solution is

$$[y^o_{(4)}]^m_{(4)} = [y^m_{(4)}]^o_{(4)} = \begin{cases} (\beta e - \beta e x + \frac{\beta e x^2}{2} - \frac{\beta e x^3}{6} + \frac{\beta e \varepsilon}{2} - \frac{\beta e x \varepsilon}{2}) \\ \times(1 - \frac{\varepsilon}{2x^2} + \frac{\varepsilon^2}{8x^4} - \frac{\varepsilon^3}{48x^6}),\ n = 1, \\ (\beta e - \beta e x + \frac{\beta e x^2}{2} - \frac{\beta e x^3}{6}) \\ \times(1 - \frac{\varepsilon}{2nx^{2n}} + \frac{\varepsilon^2}{8n^2x^{4n}} - \frac{\varepsilon^3}{48n^3x^{6n}}),\ n \geq 2, \end{cases}$$

then when $n = 1$, we obtain the composite solution is

$$\begin{aligned} y^c =& \alpha e^{-\frac{x}{\varepsilon}} - \varepsilon\frac{\alpha}{16}e^{-\frac{x}{\varepsilon}}[6\frac{x}{\varepsilon} + 6(\frac{x}{\varepsilon})^2 + 4(\frac{x}{\varepsilon})^3 + 2(\frac{x}{\varepsilon})^4] + \varepsilon^2 y^i_2 + \varepsilon^3 y^i_3 \\ &+ \beta e^{1-x} - \frac{\beta e \varepsilon}{2}(\frac{x}{6} - \frac{x^2}{2}) + \frac{\beta e \varepsilon^2}{8}(\frac{3}{4} - \frac{11x}{12}) + \frac{\varepsilon^3 \beta e}{8}(-\frac{7}{2x^2} - \frac{1}{4x^2}) + \beta e^{1-\frac{\varepsilon}{2x^2}} \\ &- x\beta e^{1-\frac{\varepsilon}{2x^2}} + \varepsilon e^{-\frac{\varepsilon}{2x^2}}(\frac{\varepsilon^4 \beta e}{8x^8} - \frac{\varepsilon^3 \beta e}{2x^6} + \frac{\beta e x^2}{2\varepsilon}) + \frac{\varepsilon \beta e}{2}e^{-\frac{\varepsilon}{2x^2}} \\ &+ \varepsilon^{\frac{3}{2}} e^{-\frac{\varepsilon}{2x^2}}(-\frac{\varepsilon^{\frac{7}{2}}\beta e}{8x^7} + \frac{\varepsilon^{\frac{5}{2}}\beta e}{10x^5} - \frac{x\beta e}{2\varepsilon^{\frac{1}{2}}} - \frac{\beta e x^3}{6\varepsilon^{\frac{3}{2}}}) - \beta e + \beta e x - \frac{\beta e x^2}{2} + \frac{\beta e x^3}{6} + \cdots. \end{aligned}$$

From the above analysis, it can be found that the singularity of the solution is greatly reduced from x^{-6} to x^{-2} but it still has singularity. In order to eliminate the singularity, when $n = 1$, we can obtain the common solution in the process of solving the third-order asymptotic solution and matching as follows

$$\begin{aligned} [y^o_{(3)}]^m_{(4)} =& [y^m_{(4)}]^o_{(3)} \\ =& \beta e - \beta e x + \frac{\beta e x^2}{2} - \frac{\beta e x^3}{6} \\ &- \frac{\beta e}{2}(\frac{\varepsilon}{x^2} - \varepsilon - \frac{\varepsilon}{x} + \varepsilon x + \frac{\varepsilon}{2} - \frac{\varepsilon x}{6}) \\ &+ \frac{\beta e}{8}(\frac{\varepsilon^2}{x^4} - \frac{2\varepsilon^2}{x^2} - \frac{\varepsilon^2}{x^3} + \frac{2\varepsilon^2}{x} + \frac{\varepsilon^2}{2x^2} - \frac{\varepsilon^2}{6x}), \end{aligned}$$

then the composite solution is

$$\begin{aligned} y^c_{(3,4)} =& y^o_{(3)} + y^m_{(4)} + y^i_{(4)} - [y^o_{(3)}]^m_{(4)} - [y^i_{(4)}]^m_{(4)} \\ =& \alpha e^{-\frac{x}{\varepsilon}} - \varepsilon\frac{\alpha}{16}e^{-\frac{x}{\varepsilon}}[6\frac{x}{\varepsilon} + 6(\frac{x}{\varepsilon})^2 + 4(\frac{x}{\varepsilon})^3 + 2(\frac{x}{\varepsilon})^4] + \varepsilon^2 y^i_2 + \varepsilon^3 y^i_3 \\ &+ \beta e^{1-x} - \frac{\beta e \varepsilon}{2}(\frac{x}{6} - \frac{x^2}{2}) + \frac{\varepsilon^2 \beta e}{8}(-x + \frac{x^2}{2}) + \beta e^{1-\frac{\varepsilon}{2x^2}} - x\beta e^{1-\frac{\varepsilon}{2x^2}} \\ &+ \varepsilon e^{-\frac{\varepsilon}{2x^2}}(\frac{\varepsilon^4 \beta e}{8x^8} - \frac{\varepsilon^3 \beta e}{2x^6} + \frac{\beta e x^2}{2\varepsilon}) + \frac{\varepsilon \beta e}{2}e^{-\frac{\varepsilon}{2x^2}} + \varepsilon^{\frac{3}{2}} e^{-\frac{\varepsilon}{2x^2}} \\ &\times (-\frac{\varepsilon^{\frac{7}{2}}\beta e}{8x^7} + \frac{\varepsilon^{\frac{5}{2}}\beta e}{10x^5} - \frac{x\beta e}{2\varepsilon^{\frac{1}{2}}} - \frac{\beta e x^3}{6\varepsilon^{\frac{3}{2}}}) - \beta e + \beta e x - \frac{\beta e x^2}{2} + \frac{\beta e x^3}{6} + \cdots. \end{aligned}$$

In this way, the singularity in the outer solution can be eliminated completely. This method can be used to solve other cases, because of the limitation of length, no more tautology here. But further efforts are needed to try to solve more complex situations.

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