# ORDERING GRAPHS WITH FIXED SIZE AND GIRTH BY THEIR $A_{\alpha}$ -SPECTRAL RADIUS

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Abstract For a graph G and real number  $\alpha \in [0, 1]$ , the  $A_{\alpha}$ -spectral radius of G is the largest eigenvalue of  $A_{\alpha}(G) := \alpha D(G) + (1 - \alpha)A(G)$ , where A(G)and D(G) are the adjacency matrix and the diagonal degree matrix of G, respectively. Recently, for  $\alpha \in [\frac{1}{2}, 1]$ , Chen, Li and Huang [Discrete Appl. Math., 2023, 340, 350–362], as well as Ye, Guo and Zhang [Discrete Appl. Math., 2024, 342, 286–294] independently identified the graph with maximum  $A_{\alpha}$ -spectral radius among all graphs in  $\mathcal{G}(m, g)$ , the class of connected graphs on m edges with girth g. In this paper, we further determine the second to the  $(\lfloor \frac{g}{2} \rfloor + 2)$ th largest  $A_{\alpha}$ -spectral radius of graphs in  $\mathcal{G}(m, g)$ . Moreover, for  $\alpha \in [\frac{1}{2}, 1]$ , we also determine the first to the  $(\lfloor \frac{g}{2} \rfloor + 3)$ th largest  $A_{\alpha}$ -spectral radius of graphs in  $\mathcal{G}(m, \geq g)$ , the class of connected graphs on m edges with girth no less than g, which generalizes the recent result of Hu, Lou and Huang (2022) on  $\alpha = \frac{1}{2}$ .

**Keywords**  $A_{\alpha}$ -spectral radius, size, girth, order.

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## 1. Introduction

All graphs considered in this paper are undirected and simple (*i.e.*, without loops or multiple edges). Let G = (V(G), E(G)) be a graph of order n and size m. For  $v \in V(G)$ , let  $d_G(v)$  and  $N_G(v)$  (or d(v) and N(v) for short) be the degree and the set of neighbors of v, respectively. And let  $N_G[v] = N_G(v) \cup \{v\}$ . The maximum degree of G is denoted by  $\Delta(G)$  (or  $\Delta$  for short). The girth of a graph G, denoted by g, is the length of the shortest cycle in G. Let A(G) and D(G) be the adjacency matrix and the diagonal degree matrix of a graph G, respectively. For  $\alpha \in [0, 1]$ , Nikiforov [14] defined the  $A_{\alpha}(G)$ -matrix of G as

$$A_{\alpha}(G) = \alpha D(G) + (1 - \alpha)A(G).$$

Clearly,  $A_0(G) = A(G)$ ,  $A_1(G) = D(G)$  and  $A_{\frac{1}{2}}(G) = \frac{1}{2}Q(G)$ , where Q(G) is known as the signless Laplacian matrix of G. Note that  $A_{\alpha}(G)$  is a non-negative and real symmetric matrix. Then the eigenvalues of  $A_{\alpha}(G)$  (also called the  $A_{\alpha}$ -eigenvalues

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of G) are real. The largest eigenvalue of  $A_{\alpha}(G)$ , denoted by  $\rho_{\alpha}(G)$ , is called the  $A_{\alpha}$ -spectral radius of G. Obviously,  $2\rho_{\frac{1}{2}}(G)$  is the signless Laplacian spectral radius of G (the largest eigenvalue of Q(G)).

Cvetković et al. [5] proposed some possible directions for further investigations on graph spectra. One of which is how to order graphs according to their spectral invariants. Hence ordering graphs with various properties by their spectra, specially by their spectral radius (the largest eigenvalues of various matrices associated with graphs), becomes an attractive topic and has received a lot of attentions in recent years (see [1, 3, 6-11, 13, 15] for details).

Let  $\mathcal{G}(m,g)$   $(\mathcal{G}(m, \geq g))$  be the set of connected graphs with size m and girth g(girth at least g). Moreover, let  $C_g = u_0 u_1 \dots u_{g-2} u_{g-1} u_o$  be a cycle of length g, and denote by  $G_i \in \mathcal{G}(m,g)$  the graph obtained from  $C_g$  by attaching m - g - 1pendent vertices to a vertex  $u_0$  and a pendent vertex w being adjacent to  $u_i$  in  $C_g$ , where  $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$ . Clearly,  $G_0$  is the graph obtained by identifying a vertex  $u_0$ of  $C_g$  and the central vertex of  $K_{1,m-g}$ . Let  $G^*$  be the graph obtained from  $C_g$  by attaching m - g - 2 pendent edges and a  $P_3$ , respectively, to  $u_0$ , where  $P_3 = u_0 w_1 v_1$ . The above mentioned three graphs are shown in Figure 1.



**Figure 1.** The graphs  $G_0$ ,  $G_i$  and  $G^*$ .

Among all graphs in  $\mathcal{G}(m, g)$ , Chen, Wang and Zhai [4] identified the graph with maximum signless Laplacian spectral radius. Very recently, Hu, Lou and Huang [8] further determined the second to the  $(\lfloor \frac{g}{2} \rfloor + 2)$ th largest graphs according to their signless Laplacian spectral radius. Their results can be combined into the following theorem.

**Theorem 1.1** ( [4,8]). Among all graphs in  $\mathcal{G}(m,g)$  with  $m \geq 3g \geq 12$ , the order of the first  $(\lfloor \frac{g}{2} \rfloor + 2)$ th largest graphs according to their signless Laplacian spectral radius is given by:

$$G_0, G_1, G^*, G_2, G_3, \ldots, G_{\lfloor \frac{g}{2} \rfloor}$$

Let  $G_g^i$  and  $G_g^*$  instead of  $G_i$   $(0 \le i \le \lfloor \frac{g}{2} \rfloor)$  and  $G^*$ , respectively. Hu, Lou and Huang [8] also determined the first to the  $(\lfloor \frac{g}{2} \rfloor + 3)$ th largest graphs according to their signless Laplacian spectral radius among all graphs in  $\mathcal{G}(m, \ge g)$  as follows.

**Theorem 1.2** ([8]). Among all graphs in  $\mathcal{G}(m, \geq g)$  with  $m \geq 3g \geq 12$ , the order of the first  $(\lfloor \frac{g}{2} \rfloor + 3)$ th largest graphs according to their signless Laplacian spectral radius is given by:

$$G_g^0, G_g^1, G_g^*, G_g^2, G_g^3, \dots, G_g^{\lfloor \frac{g}{2} \rfloor}, G_{g+1}^0.$$

Very recently, for  $\alpha \in [\frac{1}{2}, 1)$ , Chen, Li and Huang [2], as well as Ye, Guo and Zhang [18] independently determined the following graph with maximum  $A_{\alpha}$ spectral radius among all graphs in  $\mathcal{G}(m, g)$ , which generalizes the result of Chen, Wang and Zhai [4] on  $\alpha = \frac{1}{2}$ .

**Theorem 1.3** ( [2,18]). For any  $G \in \mathcal{G}(m,g)$  and  $\alpha \in [\frac{1}{2},1)$ , we have  $\rho_{\alpha}(G) \leq \rho_{\alpha}(G_0)$ . Moreover, the equality holds if and only if  $G \cong G_0$ .

In this paper, for  $\alpha \in [\frac{1}{2}, 1)$ , we further determine the second to the  $(\lfloor \frac{g}{2} \rfloor + 2)$ th largest graphs in  $\mathcal{G}(m, g)$  according to their  $A_{\alpha}$ -spectral radius as follows.

**Theorem 1.4.** For  $\alpha \in [\frac{1}{2}, 1)$ , the first  $\lfloor \frac{g}{2} \rfloor + 1$  graphs in  $\mathcal{G}(m, g) \setminus \{G_0\}$  with  $m \geq 3g \geq 12$  according to their  $A_{\alpha}$ -spectral radius are as follows:

$$G_1, G^*, G_2, G_3, \ldots, G_{\lfloor \frac{g}{2} \rfloor}.$$

**Remark 1.1.** Combining Theorem 1.3 and Theorem 1.4, we know that among all graphs in  $\mathcal{G}(m,g)$ , the first to  $\left(\lfloor \frac{g}{2} \rfloor + 2\right)$ th largest graphs according to their  $A_{\alpha}$ -spectral radius for  $\alpha \in \lfloor \frac{1}{2}, 1$ ) are determined, which generalizes Theorem 1.1.

Moreover, we further consider the first to the  $(\lfloor \frac{g}{2} \rfloor + 3)$ th largest graphs according to their  $A_{\alpha}$ -spectral radius among all graphs in  $\mathcal{G}(m, \geq g)$  and extend Theorem 1.2 as follows.

**Theorem 1.5.** For  $\alpha \in [\frac{1}{2}, 1)$ , the first  $\lfloor \frac{g}{2} \rfloor + 3$  graphs in  $\mathcal{G}(m, \geq g)$  with  $m \geq 3g \geq 12$  according to their  $A_{\alpha}$ -spectral radius are as follows:

$$G_g^0, G_g^1, G_g^*, G_g^2, G_g^3, \dots, G_g^{\lfloor \frac{g}{2} \rfloor}, G_{g+1}^0.$$

Clearly, Theorem 1.2 follow from Theorem 1.5 if we let  $\alpha = \frac{1}{2}$ .

# 2. Preliminaries

In this section, we present some preliminary results and lemmas which are useful.

Recall that  $A_{\alpha}(G)$  is a nonnegative and real symmetric matrix. Then there is a non-negative unit eigenvector **x** of  $A_{\alpha}(G)$  corresponding to  $\rho_{\alpha}(G)$  such that

$$\rho_{\alpha}(G) = \mathbf{x}^{T} A_{\alpha}(G) \mathbf{x} = (2\alpha - 1) \sum_{u \in V(G)} d(u) x_{u}^{2} + (1 - \alpha) \sum_{uv \in E(G)} (x_{u} + x_{v})^{2}, \quad (2.1)$$

where  $x_u$  is the entry of  $\mathbf{x}$  corresponding to the vertex u. We call such eigenvector  $\mathbf{x}$  the *Perron vector* of  $A_{\alpha}(G)$ . In addition, if G is connected, then  $A_{\alpha}(G)$  is irreducible and thus its Perron vector is a positive unit eigenvector. Clearly, the Perron vector of  $A_{\alpha}(G)$  satisfies the *eigenvalue equation*  $A_{\alpha}(G)\mathbf{x} = \rho_{\alpha}(G)\mathbf{x}$ , that is

$$\rho_{\alpha}(G)x_u = \alpha d(u)x_u + (1-\alpha)\sum_{v \in N(u)} x_v.$$

$$(2.2)$$

Consider an  $n \times n$  real symmetric matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix}$$

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whose rows and columns are partitioned according to a partitioning  $X_1, X_2, \ldots, X_m$ of  $\{1, 2, \ldots, n\}$ . The quotient matrix  $\mathcal{B}$  of the matrix M is the  $m \times m$  matrix whose entries are the average row sums of the blocks  $M_{i,j}$  of M. The partition is equitable if each block  $M_{i,j}$  of M has constant row (and column) sum.

**Lemma 2.1** ([19]). Let M be a square matrix with an equitable partition  $\pi$  and let  $M_{\pi}$  be the corresponding quotient matrix. Then every eigenvalue of  $M_{\pi}$  is an eigenvalue of M. Furthermore, if M is nonnegative and  $M_{\pi}$  is irreducible, then the largest eigenvalues of M and  $M_{\pi}$  are equal.

Let  $S_{n,3}$  be a graph on *n* vertices obtained from  $K_{1,n-7}$  by attaching three pendant paths of length 2 at the center vertex of  $K_{1,n-7}$ , and let  $H_0$  be an unicycle graph of order *n* and girth 4. The graphs  $S_{n,3}$  and  $H_0$  are shown in Figure 2.



**Figure 2.** The graphs  $S_{n,3}$  and  $H_0$ .

**Lemma 2.2.** For  $\alpha \in [0,1]$ ,  $\rho_{\alpha}(S_{n,3})$  and  $\rho_{\alpha}(H_0)$  are the largest root of  $f(\lambda, n) = 0$ and  $g(\lambda, n) = 0$ , respectively, where

$$f(\lambda, n) = \lambda^4 - \alpha n \lambda^3 + \left[ (3\alpha^2 + 2\alpha - 1)n - 6\alpha - 8\alpha^2 + 3 \right] \lambda^2$$
$$- \left[ (\alpha^3 + 8\alpha^2 - 4\alpha)n - 36\alpha^2 + 18\alpha \right] \lambda$$
$$+ 2\alpha^3 n + 3\alpha^2 n - 4\alpha n + n - 2\alpha^3 - 27\alpha^2 + 28\alpha - 7$$

and

$$\begin{split} g(\lambda,n) \\ =& x^6 - (n+5)\alpha x^5 + [(7\alpha^2 + 2\alpha - 1)n + \alpha^2]x^4 \\ &- [(15\alpha^3 + 20\alpha^2 - 10\alpha)n - 17\alpha^3 - 40\alpha^2 + 20\alpha]x^3 \\ &+ [(10\alpha^4 + 50\alpha^3 - 13\alpha^2 - 12\alpha + 3)n - 8\alpha^4 - 132\alpha^3 + 18\alpha^2 + 48\alpha - 12]x^2 \\ &- [(2\alpha^5 + 28\alpha^4 + 34\alpha^3 - 48\alpha^2 + 12\alpha)n - 44\alpha^4 - 194\alpha^3 + 216\alpha^2 - 54\alpha]x \end{split}$$

$$+ (4\alpha^5 + 14\alpha^4 - 20\alpha^2 + 12\alpha - 2)n - 48\alpha^4 - 48\alpha^3 + 132\alpha^2 - 72\alpha + 12.$$

**Proof.** We partition  $V(S_{n,3})$  as  $V(S_{n,3}) = \{u_1, u_2, u_3\} \cup \{w_1, w_2, w_3\} \cup \{u_0\} \cup \{w_4, \ldots, w_{n-4}\}$ . Then the corresponding quotient matrix of  $A_{\alpha}(S_{n,3})$  is

$$\mathcal{B}_{1} = \begin{pmatrix} \alpha & 1 - \alpha & 0 & 0 \\ 1 - \alpha & 2\alpha & 1 - \alpha & 0 \\ 0 & 3(1 - \alpha) & (n - 4)\alpha & (n - 7)(1 - \alpha) \\ 0 & 0 & 1 - \alpha & \alpha \end{pmatrix}.$$

It is easy to versify that the characteristic polynomial of  $\mathcal{B}_1$  is  $f(\lambda, n)$ . Note that the partition is equitable. Then Lemma 2.1 implies that  $\rho_{\alpha}(S_{n,3})$  is the largest root of  $f(\lambda, n) = 0$ .

Similarly, we partition  $V(H_0)$  as  $V(H_0) = \{v_1\} \cup \{w_1\} \cup \{w_2, \dots, w_{n-5}\} \cup \{u_0\} \cup \{u_1, u_3\} \cup \{u_2\}$ . Then the corresponding quotient matrix of  $A_{\alpha}(H_0)$  is

$$\mathcal{B}_{2} = \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 & 0 & 0 \\ 1-\alpha & 2\alpha & 0 & 1-\alpha & 0 & 0 \\ 0 & 0 & \alpha & 1-\alpha & 0 & 0 \\ 0 & 1-\alpha & (n-6)(1-\alpha) & (n-3)\alpha & 2(1-\alpha) & 0 \\ 0 & 0 & 0 & 1-\alpha & 2\alpha & 1-\alpha \\ 0 & 0 & 0 & 0 & 2(1-\alpha) & 2\alpha \end{pmatrix}$$

It is easy to versify that the characteristic polynomial of  $\mathcal{B}_2$  is  $g(\lambda, n)$ . Note that the partition is equitable. Then Lemma 2.1 implies that  $\rho_{\alpha}(H_0)$  is the largest root of  $g(\lambda, n) = 0$ .

**Lemma 2.3** ([14]). If G is a connected graph and H is a proper subgraph of G, then we have  $\rho_{\alpha}(H) < \rho_{\alpha}(G)$ .

**Lemma 2.4** ([17]). For a connected graph G and  $u, v \in V(G)$ , let  $W \subseteq N(v) \setminus (N(u) \cup \{u\})$ . Let  $G' = G - \{vw : w \in W\} + \{uw : w \in W\}$  and **x** be the Perron vector of  $A_{\alpha}(G)$ . If  $x_u \geq x_v$  and  $W \neq \emptyset$ , then  $\rho_{\alpha}(G') > \rho_{\alpha}(G)$  for  $\alpha \in [0, 1)$ .

The following lemma can be derived directly from lemma 2.3 in Ref. [12].

**Lemma 2.5** ([12]). Let G' be a graph obtained from a connected graph G by a local switching of edges  $u_1v_1$  and  $u_2v_2$  to the positions of non-edges  $u_1v_2$  and  $v_1u_2$ . Let **x** be the Perron vector of  $A_{\alpha}(G)$ . If  $(u_1 - u_2)(v_1 - v_2) \ge 0$ , then  $\rho_{\alpha}(G') \ge \rho_{\alpha}(G)$  for  $\alpha \in [0, 1)$ , with equality if and only if  $u_1 = u_2$  and  $v_1 = v_2$ .

**Lemma 2.6** ([14,16]). Let G be a graph of order n with maximum degree  $\Delta$ . Then

$$\rho_{\alpha}(G) \geq \begin{cases} \alpha(\Delta+1), & \text{for } \alpha \in [0, \frac{1}{2}], \\ \alpha\Delta+1-\alpha, & \text{for } \alpha \in [\frac{1}{2}, 1). \end{cases}$$

**Lemma 2.7** ([14]). Let G be a graph without isolated vertices. Then for  $\alpha \in [0, 1)$ , we have

$$\rho_{\alpha}(G) \leq \max_{u \in V(G)} \left\{ \alpha d(u) + (1 - \alpha)m(u) \right\},\,$$

where  $m(u) = m_G(u) = \frac{1}{d(u)} \sum_{v \in N(u)} d(v)$ . If  $\alpha \in (\frac{1}{2}, 1)$  and G is connected, then the equality holds if and only if G is regular.

**Lemma 2.8.** For any connected graph G with size  $m \ge 5$  and maximum degree  $\Delta$ . If  $\Delta \le s$  and  $s \ge \frac{2m}{3}$ , then for  $\alpha \in [\frac{1}{2}, 1)$ , we have

$$\rho_{\alpha}(G) \le \alpha s + 2(1-\alpha).$$

**Proof.** Let  $z \in V(G)$  such that

$$\alpha d(z) + (1 - \alpha) \frac{\sum_{v \in N(z)} d(v)}{d(z)} = \max_{u \in V(G)} \left\{ \alpha d(u) + (1 - \alpha) \frac{\sum_{v \in N(u)} d(v)}{d(u)} \right\}.$$

If d(z) = 1, then by Lemma 2.7,  $\rho_{\alpha}(G) \leq \alpha d(z) + (1-\alpha) \frac{\sum_{v \in N(z)} d(v)}{d(z)} \leq \alpha + (1-\alpha)\Delta \leq \alpha + (1-\alpha)s$ . If d(z) = 2, then by Lemma 2.7,  $\rho_{\alpha}(G) \leq \alpha d(z) + (1-\alpha) \frac{\sum_{v \in N(z)} d(v)}{d(z)} \leq 2\alpha + (1-\alpha)\Delta \leq 2\alpha + (1-\alpha)s$ . Next, we only need to consider  $d(z) \geq 3$ . Then by Lemma 2.7, we have

$$\rho_{\alpha}(G) \leq \alpha d(z) + (1-\alpha) \frac{\sum_{v \in N(z)} d(v)}{d(z)}$$
$$\leq \alpha d(z) + (1-\alpha) \frac{2m - d(z)}{d(z)}$$
$$= \alpha d(z) + (1-\alpha) \frac{2m}{d(z)} - 1 + \alpha.$$

Let  $f(x) = \alpha x + \frac{2m(1-\alpha)}{x}$ . Clearly,  $f(x) \ge f_{min} = f\left(\sqrt{\frac{2m(1-\alpha)}{\alpha}}\right)$  for x > 0. Note that  $3 \le x \le \Delta \le s$ ,  $\sqrt{\frac{2m(1-\alpha)}{\alpha}} \le \sqrt{2m} < \frac{2m}{3}$  and

$$f\left(\frac{2m}{3}\right) - f(3) = \frac{2m}{3}(2\alpha - 1) + 3(1 - 2\alpha) \ge \frac{1}{3}(2\alpha - 1) \ge 0.$$

Then for any  $s \geq \frac{2m}{3}$ , we have

$$\rho_{\alpha}(G) \le f(d(z)) - 1 + \alpha \le \alpha s + (1 - \alpha)\frac{2m}{s} - 1 + \alpha \le \alpha s + 2(1 - \alpha).$$

On the other hand, note that  $\max\{2\alpha + (1-\alpha)s, \alpha + (1-\alpha)s\} \le \alpha s + 2(1-\alpha)$ . Thus,  $\rho_{\alpha}(G) \le \alpha s + 2(1-\alpha)$ , as desired.

Note that if m = g, then  $\mathcal{G}(m,g) = C_g$ . If m = g+1, then  $\mathcal{G}(m,g) = C_g^+$ , where  $C_g^+$  is a graph obtained from  $C_g$  by attaching a pendant edge at some vertex of  $C_g$ . In what follows, we consider  $m \ge g+2$  and the corresponding  $|\mathcal{G}(m,g)| \ge 2$ . For  $g \ge 3$  and  $m \ge g+2$ , let  $\mathcal{H}(m,g)$  be the set of graphs in  $\mathcal{G}(m,g)$  with maximum degree  $\Delta = m - g + 1$ . Hu, Lou and Huang [8] obtained the following result.

**Lemma 2.9** ([8]).  $\mathcal{H}(m,g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G^*\}, \text{ where } g \geq 4 \text{ and } m \geq g+2.$ 

By simple observation, we see that  $G_0$  is the unique graph among  $\mathcal{G}(m, \geq g)$  with maximum degree  $\Delta(G_0) = m - g + 2$ . Then from Lemma 2.8, we have the following result.

**Corollary 2.1.** Let  $G \in \mathcal{G}(m, \geq g)$  with  $m \geq 3g - 3$ . Then  $\alpha \in [\frac{1}{2}, 1)$ , we have  $\rho_{\alpha}(G) \leq \rho_{\alpha}(G_0)$ , with equality holds if and only if  $G \cong G_0$ .

**Proof.** For any graph  $G \in \mathcal{G}(m, \geq g) \setminus \{G_0\}$ , we have  $\Delta(G) \leq m - g + 1$ . Note that  $m - g + 1 \geq \frac{2m}{3}$  since  $m \geq 3g - 3$ . Then by Lemma 2.8, we have  $\rho_{\alpha}(G) \leq \alpha(m-g+1)+2(1-\alpha) = \alpha(m-g-1)+2$ . On the other hand, since  $\Delta(G_0) = m-g+2$  and  $K_{1,\Delta(G_0)}$  is a proper subgraph of  $G_0$ , then by Lemmas 2.3 and 2.6, we have

$$\rho_{\alpha}(G_0) > \alpha(m - g + 2) + 1 - \alpha = \alpha(m - g + 1) + 1 \ge \alpha(m - g - 1) + 2 \ge \rho_{\alpha}(G),$$

as desired.

By Lemma 2.8, we can compare the  $A_{\alpha}$ -spectral radius of graphs with distinct girths and maximum degrees, respectively.

**Corollary 2.2.** Let G and H respectively be graph with the maximum  $A_{\alpha}$ -spectral radius in  $\mathcal{G}(m,g)$  and  $\mathcal{G}(m,g')$ . If g < g' and  $m \ge 3g' - 3$ , then  $\rho_{\alpha}(G) > \rho_{\alpha}(H)$ .

**Proof.** Since  $m \ge 3g' - 3$  and g' > g, we have  $m \ge 3g - 3$ . Then Corollary 2.1 implies that  $\Delta(G) = m - g + 2$  and  $\Delta(H) = m - g' + 2$ . Note that  $\Delta(H) = m - g' + 2 \ge \frac{2m}{3}$  since  $m \ge 3g' - 6$ . Then by Lemma 2.8, we have  $\rho_{\alpha}(H) \le \alpha(m - g' + 2) + 2(1 - \alpha) = \alpha(m - g') + 2$ . On the other hand, since  $\Delta(G) = m - g + 2$  and  $K_{1,\Delta(G)}$  is a proper subgraph of G, then by Lemmas 2.3 and 2.6, we have

$$\rho_{\alpha}(G) > \alpha(m - g + 2) + 1 - \alpha \ge \alpha(m - g' + 2) + 1 \ge \alpha(m - g') + 2 \ge \rho_{\alpha}(H),$$

as desired.

**Corollary 2.3.** Let G and H be graphs with size  $m \ge 5$  and maximum degree  $\Delta(G)$  and  $\Delta(H)$ , respectively. If  $\Delta(G) > \Delta(H) \ge \frac{2m}{3}$ , then  $\rho_{\alpha}(G) > \rho_{\alpha}(H)$ .

**Proof.** Since  $K_{1,\Delta(G)}$  is a proper subgraph of G, then by Lemmas 2.3 and 2.6, we have  $\rho_{\alpha}(G) > \alpha \Delta(G) + 1 - \alpha \ge \alpha \Delta(H) + 1$ . On the other hand, since  $\Delta(H) \ge \frac{2m}{3}$ , then by Lemma 2.6, we have  $\rho_{\alpha}(H) \le \alpha \Delta(H) + 2(1 - \alpha) \le \alpha \Delta(H) + 1 < \rho_{\alpha}(G)$ , as desired.

## 3. Proofs of Theorems 1.4 and 1.5

Before giving the proofs of Theorems 1.4 and 1.5, we need some necessary lemmas.

**Lemma 3.1.** Let  $G_i \in \mathcal{H}(m,g)$  be the graph as shown in Figure 1, where  $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$ . Then

$$\rho_{\alpha}(G_1) > \rho_{\alpha}(G_2) > \cdots > \rho_{\alpha}(G_{\lfloor \frac{g}{2} \rfloor}).$$

**Proof.** For any  $2 \leq i \leq \lfloor \frac{g}{2} \rfloor$ , let **x** be the Perron vector of  $\rho_{\alpha}(G_i)$ . It suffices to show  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G_{i-1})$ . To prove our result, first we give the following claim.

Claim 1. If there exists  $1 \leq j \leq i-1$  such that  $x_{u_{i-j}} < x_{u_{i+j-1}}$  and  $x_{u_{i-j-1}} \geq x_{u_{i+j}}$ , then  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G_{i-1})$ .

**Proof.** Let

$$G' = G_i - \{u_{i-j-1}u_{i-j}, u_{i+j-1}, u_{i+j}\} + \{u_{i-j-1}u_{i+j-1}, u_{i-j}u_{i+j}\},\$$

where  $G_i$  and G' are shown in Figure 3. Clearly,  $G' \cong G_{i-1}$ . Note that  $x_{u_{i-j}} < x_{u_{i+j-1}}$  and  $x_{u_{i-j-1}} \ge x_{u_{i+j}}$ . Then by Lemma 2.5, we have  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G') = \rho_{\alpha}(G_{i-1})$ .

We start to prove by firstly assuming  $x_{u_{i-1}} \ge x_{u_i}$ . Now we construct

$$G'' = G_i - \{wu_i\} + \{wu_{i-1}\}$$

from  $G_i$ , where G'' is shown in Figure 3. Clearly,  $G'' \cong G_{i-1}$ . By Lemma 2.4, we have  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G'') = \rho_{\alpha}(G_{i-1})$ . Otherwise  $x_{u_{i-1}} < x_{u_i}$ , if  $x_{u_{i-2}} \ge x_{u_{i+1}}$  then from Claim 1 we get  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G_{i-1})$  by taking j = 1. Otherwise  $x_{u_{i-2}} < x_{u_{i+1}}$ , if  $x_{u_{i-3}} \ge x_{u_{i+2}}$  then from Claim 1 we get  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G_{i-1})$  by taking j = 2. Repeating i steps we come to the assumption  $x_{u_0} < x_{u_{2i-1}}$  for j = i. Note that  $N_{G_i}(u_0) = \{u_1, u_{g-1}, w_1, \dots, w_{m-g-1}\}$ . Let

$$G''' = G_i - \{u_0 w_s | 1 \le s \le m - g - 1\} + \{u_{2i-1} w_t | 1 \le t \le m - g - 1\}.$$

Clearly,  $G''' \cong G_{i-1}$ . By Lemma 2.4, we have  $\rho_{\alpha}(G_i) < \rho_{\alpha}(G'') = \rho_{\alpha}(G_{i-1})$ , as desired.



Figure 3. The graphs  $G_i$ , G' and G'', where the edge with "X" represents it is deleted.

**Lemma 3.2.** Let  $G_i \in \mathcal{H}(m, g)$  be graphs as shown in Figure 1, where  $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$ . If  $m \geq g+3$  and  $\frac{1}{2} \leq \alpha < 1$ , then

$$\alpha(m-g) + 1 < \rho_{\alpha}(G_i) \le \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1} < \alpha(m-g) + 2 - \alpha.$$

**Proof.** Since  $\Delta(G_i) = m - g + 1$  and  $K_{1,\Delta(G_i)}$  is a proper subgraph of  $G_i$ , then by Lemmas 2.3 and 2.6, we have  $\rho_{\alpha}(G_i) > \rho_{\alpha}(K_{1,\Delta(G_i)}) = \alpha(m - g + 1) + 1 - \alpha = \alpha(m - g) + 1$ . On the other hand, let  $z \in V(G_i)$  such that

$$\alpha d(z) + (1 - \alpha) \frac{\sum_{v \in N_{G_i}(z)} d(v)}{d(z)} = \max_{u \in V(G_i)} \left\{ \alpha d(u) + (1 - \alpha) \frac{\sum_{v \in N_{G_i}(u)} d(v)}{d(u)} \right\}$$

If  $z = u_0$ , then by Lemma 2.7, we have

$$\rho_{\alpha}(G_i) \le \alpha(m-g+1) + (1-\alpha)\frac{m-g+3}{m-g+1} = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1} := f(\alpha).$$

If  $z = u_i$ , then by Lemma 2.7 and Wolfram Mathematica, we have

$$\rho_{\alpha}(G_i) \leq \alpha d(u_i) + (1-\alpha) \frac{d(w) + d(u_{i-1}) + d(u_0)}{d(u_i)} < \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1}$$

If d(z) = 1, then by Lemma 2.7, we have

$$\rho_{\alpha}(G_i) \le \alpha d(z) + (1-\alpha)d(u_0) = \alpha + (1-\alpha)(m-g+1) := g(\alpha).$$

Now let  $\varphi(\alpha) = f(\alpha) - g(\alpha) = (2\alpha - 1)(m - g) + \frac{2(1-\alpha)}{m - g + 1}$ . Then we have

$$\varphi'(\alpha) = 2(m-g) - \frac{2}{m-g+1} = 2\left(m-g - \frac{1}{m-g+1}\right) > 0$$

Thus  $\varphi(\alpha)$  is a monotonically increasing function on  $\alpha \geq \frac{1}{2}$ . Hence,  $\varphi(\alpha) \geq \varphi\left(\frac{1}{2}\right) = \frac{1}{m-g+1} > 0$ . It follows that

$$\rho_{\alpha}(G_i) \le g(\alpha) < f(\alpha) = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1}.$$

If  $z \in V(G) \setminus \{u_0, u_i\}$  is not a pendent vertex, then d(z) = 2 and by Lemma 2.7, we have

$$\rho_{\alpha}(G_i) \le 2\alpha + (1-\alpha)\frac{d(u_0) + d(u_i)}{2} = 2\alpha + (1-\alpha)\frac{m-g+4}{2} := \phi(\alpha).$$

Now let  $\psi(\alpha) = f(\alpha) - \phi(\alpha)$ , it follows that

$$\begin{split} \psi(\alpha) &= f(\alpha) - \phi(\alpha) \\ &= \alpha(m-g-1) + (1-\alpha)\frac{m-g+3}{m-g+1} - (1-\alpha)\frac{m-g+4}{2} \\ &= \frac{1}{2}\bigg((3\alpha-1)(m-g) + \frac{4(1-\alpha)}{m-g+1} - 2\bigg). \end{split}$$

Then we have

$$\psi'(\alpha) = \frac{1}{2} \left( 3(m-g) - \frac{4}{m-g+1} \right) = \frac{3(m-g)(m-g+1) - 4}{2(m-g+1)} > 0.$$

Thus  $\psi(\alpha)$  is a monotonically increasing function on  $\alpha \geq \frac{1}{2}$ . Hence

$$\psi(\alpha) \ge \psi\left(\frac{1}{2}\right) = \frac{m-g}{4} + \frac{1}{m-g+1} - 1 \ge 0.$$

It follows that

$$\rho_{\alpha}(G_i) \le \psi(\alpha) \le f(\alpha) = \alpha(m-g) + 1 + \frac{2(1-\alpha)}{m-g+1},$$

as desired. This completes the proof of Lemma 3.2.

**Lemma 3.3.** For  $1 \le i \le \lfloor \frac{g}{2} \rfloor$  and  $m \ge g+4$ , let **x** be the Perron vector of  $\rho_{\alpha}(G_i)$ . Then for  $\frac{1}{2} \le \alpha < 1$ , we have  $x_{u_0} = \max_{u \in V(G_i)} x_u$ .

**Proof.** By Lemma 3.2, we have  $\rho_{\alpha}(G_i) > \alpha(m-g) + 1 \ge 3$ . On the other hand, let  $x_{u^*} = \max_{u \in V(G_i)} x_u$ . We then assert that  $d_{G_i}(u^*) \ge 4$ . Suppose to the contrary that  $d_{G_i}(u^*) \le 3$ . Then, by (2.2), we have

$$\rho_{\alpha}(G_{i})x_{u^{*}} = \alpha d_{G_{i}}(u^{*})x_{u^{*}} + (1-\alpha)\sum_{u \in N_{G_{i}}(u^{*})}x_{u^{*}}$$
$$\leq \alpha d_{G_{i}}(u^{*})x_{u^{*}} + (1-\alpha)d_{G_{i}}(u^{*})x_{u^{*}}$$
$$\leq 3x_{u^{*}},$$

which implies that  $\rho_{\alpha}(G_i) \leq 3$ . However, recall that  $\rho_{\alpha}(G_i) > 3$ , a contradiction. Thus,  $d_{G_i}(u^*) \geq 4$ . It follows that  $u^* = u_0$ , as desired.

**Lemma 3.4.** If  $g \ge 4$ ,  $\frac{1}{2} \le \alpha < 1$  and  $m \ge g+7$ , then  $\rho_{\alpha}(G_1) > \rho_{\alpha}(G^*) > \rho_{\alpha}(G_2)$ .

**Proof.** We first prove  $\rho_{\alpha}(G_1) > \rho_{\alpha}(G^*)$ . Let **x** be the Perron vector of  $A_{\alpha}(G^*)$ . The vertices  $w_1$  and  $v_1$  of  $G^*$  are shown in Fig. 1. By (2.2), we have

$$\rho_{\alpha}(G^{*})x_{v_{1}} = \alpha x_{v_{1}} + (1-\alpha)x_{w_{1}}, \quad \rho_{\alpha}(G^{*})x_{w_{1}} = 2\alpha x_{w_{1}} + (1-\alpha)(x_{v_{1}} + x_{u_{0}}),$$
  

$$\rho_{\alpha}(G^{*})x_{u_{0}} = \alpha(m-g+1)x_{u_{0}} + \frac{(1-\alpha)^{2}(m-g-2)}{\rho_{\alpha}(G^{*}) - \alpha}x_{u_{0}}$$
  

$$+ (1-\alpha)(+x_{w_{1}} + x_{u_{1}} + x_{u_{g-1}}).$$

Note that  $x_{u_1} = x_{u_{g-1}}$  due to the symmetry of  $G^*$ . From the above equalities, we have

$$\begin{cases} x_{w_1} = \frac{(1-\alpha)(\rho_{\alpha}(G^*) - \alpha)}{(\rho_{\alpha}(G^*) - 2\alpha)(\rho_{\alpha}(G^*) - \alpha) - (1-\alpha)^2} x_{u_0}, \\ x_{u_1} = \frac{1}{2} \left( \frac{\rho_{\alpha}(G^*) - \alpha(m-g+1)}{1-\alpha} - \frac{(1-\alpha)(m-g-2)}{\rho_{\alpha}(G^*) - \alpha} - \frac{(\rho_{\alpha}(G^*) - \alpha)(1-\alpha)}{(\rho_{\alpha}(G^*) - 2\alpha)(\rho_{\alpha}(G^*) - \alpha) - (1-\alpha)^2} \right) x_{u_0}. \end{cases}$$

Let

$$\begin{split} f(x) \\ =& \frac{1}{2} \left( \frac{x - \alpha(m - g + 1)}{1 - \alpha} - \frac{(1 - \alpha)(m - g - 2)}{x - \alpha} \right) - \frac{3(x - \alpha)(1 - \alpha)}{2(x - 2\alpha)(x - \alpha) - (1 - \alpha)^2} \\ =& \frac{(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)((x - \alpha(m - g + 1))(x - \alpha) - (1 - \alpha)^2(m - g - 2))}{2(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)(x - \alpha)(1 - \alpha)} \\ -& \frac{3(x - \alpha)^2(1 - \alpha)^2}{2(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)(x - \alpha)(1 - \alpha)}. \end{split}$$

Then  $x_{u_1} - x_{w_1} = f(\rho_{\alpha}(G^*))x_{u_0}$ . On the other hand, by Wolfram Mathematica, we have

$$f(x, m - g + 5)$$
  
= $x^4 - (m - g + 5)\alpha x^3 + ((3m - 3g + 7)\alpha^2 + 2(m - g + 2)\alpha - m + g - 2)x^2$   
+ $\alpha (4m - (m - g + 5)\alpha^2 - 4(2m - 2g + 1)\alpha - 4g + 2)x + (m - g + 5)$ 

$$(2\alpha^3 + 3\alpha^2 - 4\alpha) + m - g - 2\alpha^3 - 27\alpha^2 + 28\alpha - 2$$
  
= $(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)((x - \alpha(m - g + 1))(x - \alpha) - (1 - \alpha)^2(m - g - 2))$   
 $- 3(x - \alpha)^2(1 - \alpha)^2,$ 

where f(x, m - g + 5) is defined in Lemma 2.2. Then

$$f(x) = \frac{f(x, m - g + 5)}{2(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)(x - \alpha)(1 - \alpha)}.$$
(3.1)

If  $g \geq 5$ , then  $S_{m-g+5,3}$  is a proper subgraph of  $G^*$ , where  $S_{m-g+5,3}$  is shown in Figure 2. Therefore,  $\rho_{\alpha}(G^*) > \rho_{\alpha}(S_{m-g+5,3})$ . Note that  $\Delta(G^*) = m - g + 1$  and  $K_{1,\Delta(G^*)}$  is a proper subgraph of  $G^*$ . By Lemma 2.6, we have  $\rho_{\alpha}(G^*) > \alpha(m-g)+1$ . It is easy to versify that  $x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha > 0$  for  $x > 7\alpha + 1$ . Thus  $f(\rho_{\alpha}(G^*)) > 0$ , which implies that  $x_{u_1} > x_{w_1}$ . If g = 4, then  $G^* \cong H_0$ , where  $H_0$  is shown in Fig.2. Thus, (3.1) becomes

$$f(x) = \frac{1}{2(x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha)(x - \alpha)(1 - \alpha)} \cdot \frac{g(x, m) + 2(\alpha - 1)^4(x - \alpha)x}{x^2 - 4\alpha x + 2(\alpha^2 + 2\alpha - 1)},$$

where g(x,m) is defined by Lemma 2.2. Clearly,  $g(\rho_{\alpha}(G^*),m) = 0$ . On the other hand, we have  $x^2 - 3\alpha x + \alpha^2 - 1 + 2\alpha > 0$  and  $x^2 - 4\alpha x + 2(\alpha^2 + 2\alpha - 1) > 0$ for  $x > 7\alpha + 1$ . Recall that  $\rho_{\alpha}(G^*) > 7\alpha + 1$ . Thus  $f(\rho_{\alpha}(G^*)) > 0$ , it follows that  $x_{u_1} > x_{w_1}$ . Let  $G' = G^* - \{w_1v_1\} + \{u_1v_1\}$ . Clearly,  $G' \cong G_1$ . By Lemma 2.4, we have  $\rho_{\alpha}(G^*) < \rho_{\alpha}(G') = \rho_{\alpha}(G_1)$ .

Next we will prove  $\rho_{\alpha}(G^*) > \rho_{\alpha}(G_2)$ . Let **y** be the Perron vector of  $A_{\alpha}(G_2)$ . By Lemma 3.3, we have  $y_{u_0} = \max_{u \in V(G_2)} y_u$ . Let  $u \in V(G_2)$  with  $d_{G_2}(u) = 2$ , by (2.2), we have

$$\rho_{\alpha}(G_2)y_u = 2\alpha y_u + (1-\alpha) \sum_{v \in N_{G_2}(u)} y_v \le 2\alpha y_u + 2(1-\alpha)y_{u_0}.$$

It follows that  $y_u \leq \frac{2(1-\alpha)}{\rho_{\alpha}(G_2)-2\alpha}y_{u_0}$ . Using (2.2) again, we have

 $\rho_{\alpha}(G_2)y_w = \alpha y_w + (1-\alpha)y_{u_2}, \quad \rho_{\alpha}(G_2)y_{u_2} = 3\alpha y_{u_2} + (1-\alpha)(y_w + y_{u_1} + y_{u_3}).$ 

Then

$$\rho_{\alpha}(G_2)(y_{u_2} - y_w) = (4\alpha - 1)y_{u_2} + (1 - 2\alpha)y_w + (1 - \alpha)(y_{u_1} + y_{u_3}).$$

Thus

$$\frac{\rho_{\alpha}(G_2)(\rho_{\alpha}(G_2)-1)}{\rho_{\alpha}(G_2)-\alpha}y_{u_2} = \frac{(4\alpha-1)(\rho_{\alpha}(G_2)-\alpha)+(1-2\alpha)(1-\alpha)}{\rho_{\alpha}(G_2)-\alpha}y_{u_2} + (1-\alpha)(y_{u_1}+y_{u_3}),$$

which implies that

$$y_{u_2} \le \frac{\rho_{\alpha}(G_2) - \alpha}{\rho_{\alpha}^2(G_2) - 4\alpha\rho_{\alpha}(G_2) + 2\alpha(\alpha+1) - 1} \cdot \frac{4(1-\alpha)^2}{\rho_{\alpha}(G_2) - 2\alpha} y_{u_0}.$$
 (3.2)

Now let  $f_1(x) = 4(1-\alpha)^2(x-a)^2$ ,  $g_1(x) = (1-\alpha)(x-2\alpha)(x^2-4\alpha x+2\alpha(\alpha+1)-1)$ ,  $h(x) = g_1(x) - f_1(x)$  and  $x > 7\alpha + 1$ . Then we have  $h(x) = (1-\alpha)[x^3 - 2(\alpha+2)x^2 + (2\alpha(\alpha+5) - 1)x + 2\alpha(1-4\alpha)] := (1-\alpha)\zeta(x)$ .

It follows that

$$\begin{aligned} \zeta'(x) &= 3x^2 - 4(\alpha + 2)x + 2\alpha(\alpha + 5) - 1\\ &> 3(7\alpha + 1)^2 - 4(\alpha + 2)(7\alpha + 1) + 2\alpha(\alpha + 5) - 1\\ &= 121\alpha^2 - 8\alpha - 6\\ &> 0. \end{aligned}$$

Thus  $\zeta(x)$  is a monotonically increasing function on  $x > 7\alpha + 1$ . Hence,  $h(x) = (1-\alpha)\zeta(x) > (1-\alpha)\zeta(7\alpha+1) = (1-\alpha)(259\alpha^3 - 13\alpha^2 - 32\alpha - 4) > 0$ . It follows that  $g_1(x) > f_1(x)$  and

$$\frac{x-\alpha}{x^2-4\alpha x+2\alpha(\alpha+1)-1}\cdot\frac{4(1-\alpha)^2}{x-2\alpha}<\frac{1-\alpha}{x-\alpha}$$

Note that  $\rho_{\alpha}(G_2) > 7\alpha + 1$  and  $y_v = \frac{1-\alpha}{\rho_{\alpha}(G_2)-\alpha}y_{u_0}$  for any  $v \in \{w_1, \ldots, w_{m-g-1}\}$ , then by (3.2), we have

$$y_{u_{2}} \leq \frac{\rho_{\alpha}(G_{2}) - \alpha}{\rho_{\alpha}^{2}(G_{2}) - 4\alpha\rho_{\alpha}(G_{2}) + 2\alpha(\alpha + 1) - 1} \cdot \frac{4(1 - \alpha)^{2}}{\rho_{\alpha}(G_{2}) - 2\alpha} y_{u_{0}}$$
  
$$< \frac{1 - \alpha}{\rho_{\alpha}(G_{2}) - \alpha} y_{u_{0}}$$
  
$$= y_{w_{1}}.$$

Hence,  $y_{u_2} < y_{w_1}$ . Let  $G'' = G_2 - \{wu_2\} + \{w_1w\}$ . Clearly,  $G'' \cong G^*$ . By Lemma 2.4, we have  $\rho_{\alpha}(G_2) < \rho_{\alpha}(G'') = \rho_{\alpha}(G^*)$ .

Now we can give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Notice that  $g \ge 4$  and  $m \ge g + 7$  since  $m \ge 3g \ge 12$ . Then by Lemmas 3.1 and 3.4, we have

$$\rho_{\alpha}(G_1) > \rho_{\alpha}(G^*) > \rho_{\alpha}(G_2) > \dots > \rho_{\alpha}(G_{\lfloor \frac{g}{2} \rfloor})$$

Set  $\mathcal{G}_{m-g}(m,g) = \mathcal{G}(m,g) \setminus (\{G_0\} \cup \mathcal{H}(m,g))$ . Note that  $G_0$  is a unique graph with maximum degree m - g + 2 among  $\mathcal{G}(m,g)$ . By Lemma 2.9,  $\mathcal{H}(m,g) = \{G_1, G_2, \ldots, G_{\lfloor \frac{g}{2} \rfloor}, G^*\}$  is the set of graphs in  $\mathcal{G}(m,g)$  with maximum degree m - g + 1. Then for any  $G' \in \mathcal{G}_{m-g}(m,g)$ , we have  $\Delta(G') \leq m - g$ . On the other hand, we have  $\Delta(G_{\lfloor \frac{g}{2} \rfloor}) = m - g + 1 > m - g \geq \Delta(G') \geq \frac{2m}{3}$  since  $m \geq 3g$ . By

Corollary 2.3, we obtain  $\rho_{\alpha}(G_{\lfloor \frac{g}{2} \rfloor}) > \rho_{\alpha}(G')$ . Thus, the second to the  $\left( \lfloor \frac{g}{2} \rfloor + 2 \right)$ th largest graphs in  $\mathcal{G}(m,g) \setminus \{G_0\}$  by their  $A_{\alpha}$ -spectral radius is given by

$$\rho_{\alpha}(G_1) > \rho_{\alpha}(G^*) > \rho_{\alpha}(G_2) > \dots > \rho_{\alpha}(G_{\lfloor \frac{g}{2} \rfloor}).$$

This completes the proof of Theorem 1.4.

Moreover, we further consider the first to the  $(\lfloor \frac{g}{2} \rfloor + 3)$ th largest graphs according to their  $A_{\alpha}$ -spectral radius among all graphs in  $\mathcal{G}(m, \geq g)$ . For  $0 \leq i \leq \lfloor \frac{g}{2} \rfloor$ , we use  $G_g^i$  and  $G_g^*$  instead of  $G_i$  and  $G^*$  to distinguish the girth of the graphs in the following proof. Now, we give the proof of Theorem 1.5.

**Proof of Theorem 1.5.** Denote by M and N are the sets of all graphs in  $\mathcal{G}(m, \geq g)$  with maximum degree at least m - g + 1 and at most m - g, respectively. It follows that  $\mathcal{G}(m, \geq g) = M \cup N$ . It is easy to check that M =

 $\{G_g^0, G_g^1, G_g^*, G_g^2, \dots, G_g^{\lfloor \frac{g}{2} \rfloor}, G_{g+1}^0\}$ , where the maximum degree of  $G_g^1, G_g^*, G_g^2, \dots$ ,  $G_g^{\lfloor \frac{g}{2} \rfloor}$  and  $G_{g+1}^0$  is m-g+1 and  $\Delta(G_g^0) = m-g+2$ . For any  $G \in M$ , note that  $\Delta(G) \ge m-g+1$  and  $K_{1,(G)}$  is a proper subgraph of G. By Lemma 2.6, we have  $\rho_{\alpha}(G) > \alpha(m-g) + 1$ . For any  $G' \in N$ , since  $\Delta(G') \leq m-g$  and  $m-g \geq \frac{2m}{3}$ , then by Lemma 2.8, we have

$$\rho_{\alpha}(G') \le \alpha(m-g) + 2(1-\alpha).$$

Since  $\alpha(m-g) + 2(1-\alpha) \leq \alpha(m-g) + 1$ , each  $A_{\alpha}$ -spectral radius of the graph in M is more than that of the graph in N. By Theorem 1.4, to complete the proof it remains to show  $\rho_{\alpha}(G_g^{\lfloor \frac{g}{2} \rfloor}) > \rho_{\alpha}(G_{g+1}^0)$ . Let **x** be the Perron vector of  $\rho_{\alpha}(G_{g+1}^0)$ . Note that

$$\rho_{\alpha}(G^{0}_{g+1}) > \alpha(m-g) + 1 = 8\alpha + 1$$

since  $m \ge 3g \ge 12$ . If g is even, by the symmetry of  $G_{g+1}^0$ , then  $x_{u_{\lfloor \frac{g}{2} \rfloor}} = x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$ (see Figure 4). If g is odd, by the symmetry of  $G_{g+1}^0$ , then  $x_{u_{\lfloor \frac{g}{g} \rfloor}} = x_{u_{\lfloor \frac{g}{g} \rfloor+2}}$  (see Figure 4). By (2.2), we have

$$\begin{split} \rho_{\alpha}(G_{g+1}^{0})x_{u_{\lfloor \frac{g}{2} \rfloor+1}} &= 2\alpha x_{u_{\lfloor \frac{g}{2} \rfloor+1}} + (1-\alpha)(x_{u_{\lfloor \frac{g}{2} \rfloor}} + x_{u_{\lfloor \frac{g}{2} \rfloor+2}}) \\ &= 2\alpha x_{u_{\lfloor \frac{g}{2} \rfloor+1}} + 2(1-\alpha)x_{u_{\lfloor \frac{g}{2} \rfloor}}, \end{split}$$

which implies that

$$x_{u_{\lfloor \frac{g}{2} \rfloor}} = \frac{\rho_{\alpha}(G_{g+1}^{0}) - 2\alpha}{2(1-\alpha)} x_{u_{\lfloor \frac{g}{2} \rfloor+1}} > \frac{6\alpha + 1}{2(1-\alpha)} x_{u_{\lfloor \frac{g}{2} \rfloor+1}} \ge 4x_{u_{\lfloor \frac{g}{2} \rfloor+1}}.$$

It follows that  $x_{u_{\lfloor \frac{g}{2} \rfloor}} > x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$ . Thus, we have  $x_{u_{\lfloor \frac{g}{2} \rfloor}} \ge x_{u_{\lfloor \frac{g}{2} \rfloor+1}}$ . Let

$$G' = G_{g+1}^0 - \{ u_{\lfloor \frac{g}{2} \rfloor + 1} u_{\lfloor \frac{g}{2} \rfloor + 2} \} + \{ u_{\lfloor \frac{g}{2} \rfloor} u_{\lfloor \frac{g}{2} \rfloor + 2} \}.$$

Clearly,  $G' \cong G_g^{\lfloor \frac{d}{2} \rfloor}$  (see Figure 4). By Lemma 2.4, we have  $\rho_{\alpha}(G_{g+1}^0) < \rho_{\alpha}(G') =$  $\rho_{\alpha}(G_q^{\lfloor \frac{g}{2} \rfloor})$ . This completes the proof of Theorem 1.5. 



Figure 4. The graphs  $G_{g+1}^0$  and G', where the edge with "X" represents it is deleted.

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