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BIFURCATIONS OF CODIMENSION THREE IN A LESLIE-GOWER TYPE PREDATOR-PREY SYSTEM WITH HERD BEHAVIOR AND PREDATOR HARVESTING*

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Abstract A Leslie-Gower type predator-prey system with herd behavior in prey and constant harvesting in predators is considered in this paper. It is shown that there are two non-hyperbolic equilibria, one is a nilpotent cusp of codimension at most three and the other one is a weak focus of multiplicity also at most three. A complete analysis on bifurcations with codimension three is given as the bifurcation parameters vary, which includes a Bogdanov-Takens bifurcation of codimension three and a degenerate Hopf bifurcation of codimension three. The results indicate that the Leslie-Gower type system exhibits richer bifurcations than the classic Leslie-Gower model and also reveal the complexity of the interaction between the prey, predators and humans.

Keywords Leslie-Gower type system, herd behavior, predator harvesting, Bogdanov-Takens bifurcation of codimension three, degenerate Hopf bifurcation of codimension three.

MSC(2010) 34C23, 92D25.

1. Introduction

Individuals of one population usually gather together in herd for the purposes of foraging and defense in the ecosystem such as the cooperative hunting and the group defense. For the defensive purpose, the weakest prey individuals occupy the interior of the herd and the stronger ones stay at the border of the herd when the attack arises. As a ecological consequence of the herd behavior in prey, it is mostly the prey individuals at the border that suffer the attack from the predators. As a mathematical consequence of the herd behavior in prey, a series of nonlinear functional responses is proposed to account for the assumption that the interaction only occurs along the border [1, 3, 8, 9, 14, 22]. For instance, Ajraldi [1] proposed the square root functional response $a\sqrt{x}$ and He and Li [9] considered the following

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Leslie-Gower type system with the square root functional response

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(1 - \dfrac{x}{k}) - a\sqrt{x}y, \\ \dfrac{dy}{dt} = r_2 y(1 - \dfrac{y}{px}), \end{cases} \tag{1.1}$$

where x and y are the densities of prey and predator, r_1 and r_2 are the intrinsic growth rates of prey and predator, k and px represent the carrying capacities of prey and predator, $a\sqrt{x}$ is the per-unit predator extraction rate of prey, respectively. They obtained that the unique positive equilibrium of system (1.1) is either globally asymptotically stable or unstable and at most one stable limit cycle is induced by the Hopf bifurcation. Therefore, the dynamics of system (1.1) turn out to be richer than that of the classic Leslie-Gower model because the functional response takes the square root of density of prey rather than simply the density of prey [10, 15].

Humans usually harvest some populations for the commercial purpose, which provides a direct motivation to model the harvesting behavior in the predator-prey systems [19–21, 24, 27]. The most basic type of harvesting is the constant harvesting, whose influence on the dynamics of predator-prey systems has received great attention [11, 12, 18, 25, 26]. Huang and Gong [12] considered the following Leslie-Gower type system with constant predator harvesting and performed detailed analyses of dynamics

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(1 - \dfrac{x}{k}) - axy, \\ \dfrac{dy}{dt} = r_2 y(1 - \dfrac{y}{px}) - H, \end{cases} \tag{1.2}$$

where H represents the constant predator harvesting. Their results showed that the system has a weak focus of multiplicity 2 and a cusp of codimension 3 for suitable parameter values and exhibits various kinds of bifurcations including the saddle-node bifurcation, the Hopf bifurcations and the Bogdanov-Takens bifurcation of codimension 2 as the values of parameters vary. Therefore, the dynamics of system (1.2) are more complex than that of the classic Leslie-Gower model because the term of constant predator harvesting was considered.

In this paper, the following Leslie-Gower type predator-prey system is considered, in which the prey exhibits herd behavior and the predators are continuously harvested at the constant harvesting rate

$$\begin{cases} \dfrac{dx}{dt} = r_1 x(1 - \dfrac{x}{k}) - a\sqrt{x}y, \\ \dfrac{dy}{dt} = r_2 y(1 - \dfrac{y}{px}) - H. \end{cases} \tag{1.3}$$

For mathematical simplicity, we nondimensionalize model (1.3) by

$$\bar{x} := \frac{x}{k}, \quad \bar{y} := \frac{ay}{r_1\sqrt{k}}, \quad \bar{t} := r_1 t, \quad s := \frac{r_2}{r_1}, \quad n := \frac{ap\sqrt{k}}{r_1}, \quad h := \frac{aH}{r_1^2\sqrt{k}}$$

and drop the bars. Then system (1.3) takes the form

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x) - \sqrt{x}y, \\ \dfrac{dy}{dt} = sy(1 - \dfrac{y}{nx}) - h, \end{cases} \tag{1.4}$$

where we still denote $\bar{x}$, $\bar{y}$ and $\bar{t}$ as x, y and t respectively. Due to the biological significance and not well-defined at $x = 0$ of system (1.4), we restrict our attention to system (1.4) in $\Omega := \{(x, y) : x > 0, y \geq 0\}$. In order to study the orbits of system (1.4) near $x = 0$ in Ω, it is necessary to discuss the dynamics of the system at $x = 0$. By the time rescaling $\tau := nxt$, system (1.4) is changed into the following topologically equivalent system

$$\begin{cases} \dfrac{dx}{dt} = nx\{x(1-x) - \sqrt{x}y\} = P_1(x, y), \\ \dfrac{dy}{dt} = sy(nx - y) - hnx = P_2(x, y), \end{cases} \tag{1.5}$$

where τ is still denoted as t. Although the square root functional response is non-differentiable at $x = 0$, we can claim that system (1.5) is Lipschitzian in $\bar{\Omega} := \Omega \cup \{(x, y) : x = 0, y \geq 0\}$. In fact,

$$|P_1(x_1, y_1) - P_1(x_2, y_2)| \leq n\{|x_1^2(1 - x_1) - x_2^2(1 - x_2)| + |x_1\sqrt{x_1}y_1 - x_2\sqrt{x_2}y_2|\}$$

with (x_1, y_1) and (x_2, y_2) in $\bar{\Omega}$. Let $f_1(x) := x^2(1 - x)$ and $f_2(x, y) := x\sqrt{x}y$, which are C^1. Therefore, there exist constants K_1 and K_2 such that

$$\begin{aligned} |P_1(x_1, y_1) - P_1(x_2, y_2)| &\leq K_1|x_1 - x_2| + K_2(|x_1 - x_2| + |y_1 - y_2|) \\ &\leq K \parallel (x_1, y_1) - (x_2, y_2) \parallel \end{aligned}$$

with $K := \max(K_1 + K_2, K_2)$, which implies that $P_1(x, y)$ is Lipschitzian. Analogously, we also obtain that $P_2(x, y)$ is Lipschitzian. Therefore, system (1.5) is Lipschitzian in $\bar{\Omega}$, which means that the uniqueness of solution of system (1.5) holds in $\bar{\Omega}$. The equilibrium $(0, 0)$ of system (1.5) is degenerate, whose associated Jacobian matrix is identically null. We need to perform the desingularization of the equilibrium $(0, 0)$ by the blow up technique [2]. By applying the y-directional blow up and the x-directional blow up sequentially, we get the local phase portrait of system (1.5) near $(0, 0)$ in $\bar{\Omega}$, i.e., $x = 0$ is the unique solution approaches $(0, 0)$ in $\bar{\Omega}$ as $t \to +\infty$ and the other solutions with the initial values near $(0, 0)$ in $\bar{\Omega}$ cross the x-axis as the time increases (see Figure 1).

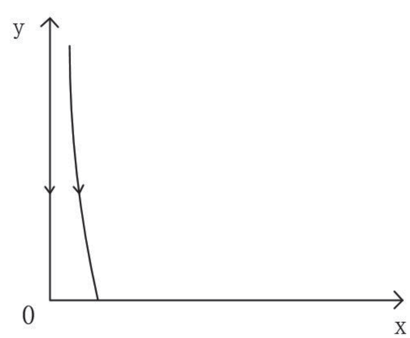

Figure 1. The local phase portrait of system (1.5) near $(0, 0)$ in $\bar{\Omega}$.

The purpose of this paper is to discuss the dynamics of system (1.4) in Ω and clarify how the herd behavior in prey and the constant harvesting in predators affect the dynamics of system (1.4), thereby revealing the effects of which on the densities of prey and predator in ecology. The layout of this paper is as follows. In Section 2, the number, type and stability of equilibria in system (1.4) are discussed. In Section 3, the degenerate Bogdanov-Takens and Hopf bifurcations of codimension 3 around the cusp and the weak focus are shown in detail. The paper ends with a brief discussion in Section 4.

2. Equilibria

In this section, we firstly consider the number, type and stability of equilibria in system (1.4), which are presented in the following result. Let $z := \sqrt{x}$ and

$$\begin{aligned} F(z) &:= z^4 + nz^3 - 2z^2 - nz + \frac{hn}{s} + 1, \\ T(z) &:= -3nz^3 + 4sz^2 + n(2s+1)z - 4s, \\ h_1 &:= -\frac{s}{n}(z_0^4 + nz_0^3 - 2z_0^2 - nz_0 + 1), \end{aligned}$$

where function $F(z)$ has a double positive root z_* and two simple positive roots z_1 and z_2 with $z_1 < z_2$, and z_0 is the unique positive root of the first derivative of function $F(z)$, i.e.,

$$F'(z) := 4z^3 + 3nz^2 - 4z - n.$$

Theorem 2.1. *System (1.4) has at most two equilibria. Concretely, system (1.4) has*
(i) no equilibrium if $h > h_1$;
(ii) a unique equilibrium $E_* := (z_*^2, z_*(1 - z_*^2))$ if $h = h_1$, which is degenerate;
(iii) two equilibria $E_1 := (z_1^2, z_1(1 - z_1^2))$ and $E_2 := (z_2^2, z_2(1 - z_2^2))$ if $0 < h < h_1$, where E_2 is a saddle and E_1 is either an unstable node or focus if $T(z_1) > 0$ or center type if $T(z_1) = 0$ or a stable node or focus if $T(z_1) < 0$.

Proof. Equilibria of system (1.4) are determined by the following nullclines

$$\begin{cases} x(1-x) - \sqrt{x}y = 0, \\ sy(1 - \frac{y}{nx}) - h = 0. \end{cases} \tag{2.1}$$

The first equation of (2.1) has one positive root $y = \sqrt{x}(1-x)$ with $0 < x < 1$ in Ω. Substituting $y = \sqrt{x}(1-x)$ in the second equation of (2.1), we obtain the quartic equation $F(z)$ given just above Theorem 2.1. In what follows, we discuss the positive roots of $F(z)$ in the interval $(0, 1)$ by analyzing the monotonic interval partition and the signs of $F(z)$ at endpoints of interval indirectly rather than by the formulae of quartic roots directly. The second derivative of $F(z)$ is $F''(z) := 6z^2 + 3nz - 2$, which is strictly increasing on the interval $(0, 1)$ and has one positive root

$$\tilde{z}_0 := \frac{-3n + \sqrt{9n^2 + 48}}{12}$$

in the interval $(0, 1)$. Clearly, the first derivative $F'(z)$ given just above Theorem 2.1 is strictly decreasing on the interval $(0, \tilde{z}_0)$ and strictly increasing on the interval

$(\tilde{z}_0, 1)$. That the monotonic interval partition of $F'(z)$ combined with the signs of $F'(z)$ at endpoints of interval (i.e., $F'(0) < 0$ and $F'(1) > 0$) indicates that $F'(z)$ has no positive root in the interval $(0, \tilde{z}_0)$ and one positive root z_0 in the interval $(\tilde{z}_0, 1)$. Therefore, function $F(z)$ is strictly decreasing on the interval $(0, z_0)$ and strictly increasing on the interval $(z_0, 1)$. Similarly, the monotonic interval partition of $F(z)$ combined with $F(0) > 0$ and $F(1) > 0$ indicates that function $F(z)$ has no positive root if $F(z_0) > 0$, one positive root z_* if $F(z_0) = 0$ and two positive roots z_1 and z_2 with $z_1 < z_0 < z_2$ if $F(z_0) < 0$. It follows from the expression of $F(z)$ that $F(z_0) > 0$, $=$ and < 0 if and only if $h > h_1$, $h = h_1$ and $0 < h < h_1$, respectively, where the expression of h_1 is given just above Theorem 2.1. Incidentally, we can claim that $h_1 > 0$ by applying the successive pseudo-division to h_1 and $F'(z_0)$. Correspondingly, system (1.4) has no equilibrium if $h > h_1$, one equilibrium $E_* = (z_*^2, z_*(1 - z_*^2))$ if $h = h_1$ and two equilibria $E_1 = (z_1^2, z_1(1 - z_1^2))$ and $E_2 = (z_2^2, z_2(1 - z_2^2))$ if $0 < h < h_1$.

Next we study types of equilibria of system (1.4). The Jacobian matrix of system (1.4) at positive equilibrium $E = (z^2, z(1 - z^2))$ is given by

$$J(E) := \begin{pmatrix} \dfrac{1 - 3z^2}{2} & -z \\ \dfrac{s(1 - z^2)^2}{nz^2} & \dfrac{s(2z^2 + nz - 2)}{nz} \end{pmatrix}.$$

Then the determinant and trace of $J(E)$ are

$$\mathrm{Det}(J(E)) := -\frac{sF'(z)}{2n}, \quad \mathrm{Tr}(J(E)) := \frac{T(z)}{2nz},$$

respectively, where $F'(z)$ and $T(z)$ are given just above Theorem 2.1. It implies that E_2 is a saddle since $F'(z_2) > 0$, E_* is a degenerate equilibrium since $F'(z_*) = 0$ and E_1 is either an unstable node or focus if $T(z_1) > 0$ or center type if $T(z_1) = 0$ or a stable node or focus if $T(z_1) < 0$ since $F'(z_1) < 0$. The proof of Theorem 2.1 is completed. $\square$

Theorem 2.1 shows that equilibria E_1 and E_2 coalesce into a unique double equilibrium E_* when $h = h_1$. In what follows, we need to consider the detailed types of degenerate equilibrium E_* further. Let

$$\begin{aligned}
S_1 &:= \Big\{(s, n, h) \in \mathbb{R}_+^3 : h = \frac{s(1 - z_*^4)}{4z_*}, n = \frac{4z_*(1 - z_*^2)}{3z_*^2 - 1}, s \neq \frac{z_*^2(3z_*^2 - 1)}{1 - z_*^2}, \\
&\qquad \frac{1}{\sqrt{3}} < z_* < 1\Big\}, \\
S_{21} &:= \Big\{(s, n, h) \in \mathbb{R}_+^3 : h = \frac{(z_*^2 + 1)(3z_*^2 - 1)z_*}{4}, n = \frac{4z_*(1 - z_*^2)}{3z_*^2 - 1}, s = \frac{z_*^2(3z_*^2 - 1)}{1 - z_*^2}, \\
&\qquad \frac{1}{\sqrt{3}} < z_* < z_3\Big\}, \\
S_{22} &:= \Big\{(s, n, h) \in \mathbb{R}_+^3 : h = \frac{(z_*^2 + 1)(3z_*^2 - 1)z_*}{4}, n = \frac{4z_*(1 - z_*^2)}{3z_*^2 - 1}, s = \frac{z_*^2(3z_*^2 - 1)}{1 - z_*^2}, \\
&\qquad z_3 < z_* < 1\Big\}, \\
S_3 &:= \Big\{(s, n, h) \in \mathbb{R}_+^3 : h = \frac{(z_*^2 + 1)(3z_*^2 - 1)z_*}{4}, n = \frac{4z_*(1 - z_*^2)}{3z_*^2 - 1}, s = \frac{z_*^2(3z_*^2 - 1)}{1 - z_*^2}, \\
&\qquad z_* = z_3\Big\},
\end{aligned}$$

where

$$z_3 = \sqrt{\frac{1}{3}\{\sqrt[3]{4\sqrt{107}i - 212} - \sqrt[3]{4\sqrt{107}i + 212} - 5\}} \doteq 0.7427$$

is the maximal positive root of function

$$g(z) := 3z^6 + 15z^4 - 11z^2 + 1.$$

Theorem 2.2. *Equilibrium E_* is a saddle-node if $(s, n, h) \in S_1$, a cusp of codimension 2 if $(s, n, h) \in S_{21} \cup S_{22}$, and a cusp of codimension 3 if $(s, n, h) \in S_3$. The phase portraits of cusp are shown in Figure 2.*

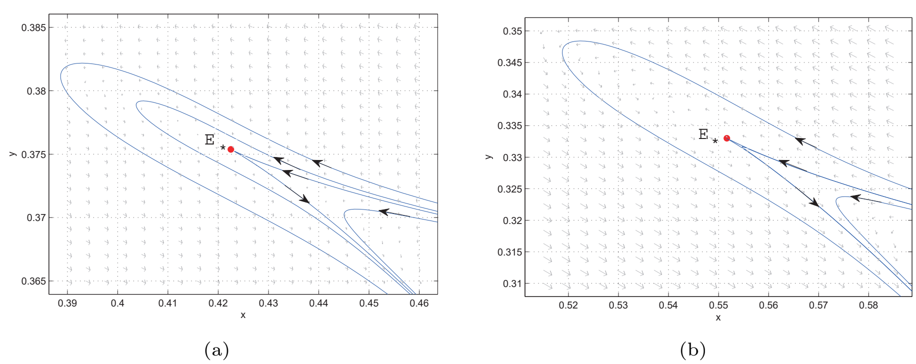

Figure 2. The phase portraits of cusp E_*: (a) cusp of codimension 2 when $s = 0.1957$, $n = 5.6131$ and $h = 0.0618$; (b) cusp of codimension 3 when $s = 0.8056$, $n = 2.0341$ and $h = 0.1887$.

Proof. For $(s, n, h) \in S_1$, equilibrium E_* is degenerate with one simple zero eigenvalue. Applying transformation

$$x = x_1 + \frac{2z_*^3}{s(z_*^2 - 1)}y_1 + z_*^2, \quad y = \frac{1 - 3z_*^2}{2z_*}x_1 + y_1 + z_*(1 - z_*^2)$$

to translate E_* to the origin and diagonalize the linear part of system (1.4), we obtain

$$\left\{\begin{aligned}
\frac{dx_1}{dt} &= \frac{s(3z_*^4 + 1)}{4(1 - z_*^2)\{3z_*^4 + (s-1)z_*^2 - s\}}x_1^2 \\
&\quad - \frac{21z_*^6 + (14s - 5)z_*^4 + 3s(s-2)z_*^2 - s^2}{2s(z_*^2 - 1)\{3z_*^4 + (s-1)z_*^2 - s\}}y_1^2 \\
&\quad + \frac{(s+5)z_*^4 + (2s-1)z_*^2 - s}{z_*(z_*^2 - 1)\{3z_*^4 + (s-1)z_*^2 - s\}}x_1y_1 + O(\| (x_1, y_1) \|^3), \\
\frac{dy_1}{dt} &= -\frac{3z_*^4 + (s-1)z_*^2 - s}{2z_*^2}y_1 \\
&\quad + \frac{s(3z_*^2 - 1)\{-3z_*^6 + (s+2)z_*^4 + (2s+1)z_*^2 + s\}}{16z_*^5\{3z_*^4 + (s-1)z_*^2 - s\}}x_1^2 \\
&\quad + \frac{(3z_*^2 - 1)\{-9z_*^8 + (2s+1)z_*^6 + 2s(2s-1)z_*^4 + s^2(s-4)z_*^2 - s^3\}}{4z_*^3 s(z_*^2 - 1)\{3z_*^4 + (s-1)z_*^2 - s\}}y_1^2 \\
&\quad - \frac{(3z_*^2 - 1)\{6z_*^6 + 3z_*^4 s + s(s+1)z_*^2 + s^2\}}{4z_*^4\{3z_*^4 + (s-1)z_*^2 - s\}}x_1y_1 + O(\| (x_1, y_1) \|^3).
\end{aligned}\right. \tag{2.2}$$

Because the coefficient of x_1^2 in the first equation of system (2.2) satisfies

$$\frac{s(3z_*^4+1)}{4(1-z_*^2)\{3z_*^4+(s-1)z_*^2-s\}} \neq 0$$

for $(s,n,h) \in S_1$, Theorem 7.1 in [29] shows that the origin of system (2.2) is a saddle-node. Therefore, degenerate equilibrium E_* is a saddle-node if $(s,n,h) \in S_1$.

For the other cases, equilibrium E_* is degenerate with one double zero eigenvalue. Applying transformation

$$x = -\frac{2z_*}{3z_*^2-1}x_1 + y_1 + z_*^2, \quad y = y_1 + z_*(1-z_*^2), \quad t = \frac{4z_*}{(3z_*^2-1)^2}\tau$$

to translate E_* to the origin and normalize the linear part of system (1.4), we obtain

$$\begin{cases} \dfrac{dx_1}{d\tau} = y_1 + a_{20}x_1^2 + a_{02}y_1^2 + a_{11}x_1y_1 + a_{30}x_1^3 + a_{21}x_1^2y_1 + a_{12}x_1y_1^2 + a_{03}y_1^3 \\ \qquad + a_{40}x_1^4 + a_{31}x_1^3y_1 + a_{22}x_1^2y_1^2 + a_{13}x_1y_1^3 + a_{04}y_1^4 + O(\| (x_1,y_1) \|^5), \\ \dfrac{dy_1}{d\tau} = b_{20}x_1^2 + b_{02}y_1^2 + b_{11}x_1y_1 + b_{30}x_1^3 + b_{21}x_1^2y_1 + b_{12}x_1y_1^2 + b_{03}y_1^3 \\ \qquad + b_{40}x_1^4 + b_{31}x_1^3y_1 + b_{22}x_1^2y_1^2 + b_{13}x_1y_1^3 + b_{04}y_1^4 + O(\| (x_1,y_1) \|^5), \end{cases} \tag{2.3}$$

where

$$\begin{aligned}
&a_{20} := -\frac{(z_*^2+1)^2}{(z_*^2-1)^2(3z_*^2-1)^2}, \quad a_{11} := -\frac{2(z_*^2+1)}{z_*(3z_*^2-1)(z_*^2-1)}, \quad a_{03} := \frac{1}{z_*^4}, \\
&b_{04} := -\frac{197z_*^2-69}{32z_*^5(3z_*^2-1)^2}, \quad a_{30} := -\frac{2(z_*^2+1)^2}{z_*(z_*^2-1)^2(3z_*^2-1)^3}, \\
&a_{21} := -\frac{(3z_*^2-5)(z_*^2+1)}{z_*^2(3z_*^2-1)^2(z_*^2-1)^2}, \quad a_{12} := \frac{4}{z_*^3(3z_*^2-1)(z_*^2-1)}, \\
&a_{40} := -\frac{4(z_*^2+1)^2}{z_*^2(z_*^2-1)^2(3z_*^2-1)^4}, \quad a_{31} := -\frac{4(z_*^2-3)(z_*^2+1)}{z_*^3(z_*^2-1)^2(3z_*^2-1)^3}, \\
&a_{22} := \frac{3z_*^4+6z_*^2-13}{z_*^4(3z_*^2-1)^2(z_*^2-1)^2}, \quad a_{13} := \frac{2(z_*^2-3)}{z_*^5(3z_*^2-1)(z_*^2-1)}, \\
&a_{04} := -\frac{1}{z_*^6}, \quad b_{20} := -\frac{4z_*^3(3z_*^4+1)}{(z_*^2-1)^2(3z_*^2-1)^4}, \quad b_{02} := -\frac{21z_*^2-5}{2z_*(3z_*^2-1)^2}, \\
&b_{11} := -\frac{4(5z_*^2-1)}{(3z_*^2-1)^3(z_*^2-1)}, \quad b_{30} := -\frac{4(2z_*^6+7z_*^4-1)}{(z_*^2-1)^2(3z_*^2-1)^5}, \\
&b_{21} := -\frac{21z_*^6-23z_*^4-25z_*^2+11}{z_*(3z_*^2-1)^4(z_*^2-1)^2}, \quad b_{12} := \frac{(5z_*-3)(5z_*+3)}{z_*^2(3z_*^2-1)^3(z_*^2-1)}, \\
&b_{03} := \frac{25z_*^2-9}{4z_*^3(3z_*^2-1)^2}, \quad b_{40} := -\frac{41z_*^6+93z_*^4+11z_*^2-17}{2z_*(z_*^2-1)^2(3z_*^2-1)^6}, \\
&a_{02} := -\frac{1}{z_*^2}, \quad b_{31} := -\frac{2(14z_*^6-31z_*^4-28z_*^2+13)}{z_*^2(z_*^2-1)^2(3z_*^2-1)^5}, \\
&b_{22} := \frac{75z_*^6+123z_*^4-375z_*^2+113}{4z_*^3(z_*^2-1)^2(3z_*^2-1)^4}, \quad b_{13} := \frac{25z_*^4-83z_*^2+26}{2z_*^4(3z_*^2-1)^3(z_*^2-1)}.
\end{aligned}$$

Then, using the near-identity transformation

$$\begin{aligned} x_2 :=& x_1, \\ y_2 :=& y_1 + a_{20}x_1^2 + a_{02}y_1^2 + a_{11}x_1y_1 + a_{30}x_1^3 + a_{21}x_1^2y_1 + a_{12}x_1y_1^2 \\ &+ a_{03}y_1^3 + a_{40}x_1^4 + a_{31}x_1^3y_1 + a_{22}x_1^2y_1^2 + a_{13}x_1y_1^3 + a_{04}y_1^4, \end{aligned}$$

where the right hand side of the second equation is the fourth order truncation of the right hand side of the first equation in system (2.3), system (2.3) takes the Kukles form

$$\begin{cases} \dfrac{dx_2}{d\tau} = y_2, \\ \dfrac{dy_2}{d\tau} = c_{20}x_2^2 + c_{02}y_2^2 + c_{11}x_2y_2 + c_{30}x_2^3 + c_{21}x_2^2y_2 + c_{12}x_2y_2^2 + c_{03}y_2^3 \\ \qquad + c_{40}x_2^4 + c_{31}x_2^3y_2 + c_{22}x_2^2y_2^2 + c_{13}x_2y_2^3 + c_{04}y_2^4 + O(\| (x_2, y_2) \|^5), \end{cases} \tag{2.4}$$

where

$$\begin{aligned} &c_{20} := -\frac{4z_*^3(3z_*^4+1)}{(z_*^2-1)^2(3z_*^2-1)^4}, \quad c_{02} := -\frac{33z_*^4-18z_*^2+1}{2z_*(3z_*^2-1)^2(z_*^2-1)}, \\ &c_{11} := -\frac{2(3z_*^6+15z_*^4-11z_*^2+1)}{(3z_*^2-1)^3(z_*^2-1)^2}, \quad c_{30} := \frac{16z_*^6}{(z_*^2-1)^2(3z_*^2-1)^5}, \\ &c_{21} := -\frac{2(21z_*^8+12z_*^6-24z_*^4+8z_*^2-1)}{z_*(3z_*^2-1)^4(z_*^2-1)^3}, \quad c_{12} := -\frac{51z_*^6-48z_*^4+15z_*^2-2}{z_*^2(3z_*^2-1)^3(z_*^2-1)^2}, \\ &c_{03} := \frac{z_*^2-1}{4z_*^3(3z_*^2-1)^2}, \quad c_{40} := \frac{24z_*^7}{(3z_*^2-1)^6(z_*^2-1)^3}, \\ &c_{31} := -\frac{2(69z_*^{10}-3z_*^8-66z_*^6+42z_*^4-11z_*^2+1)}{z_*^2(3z_*^2-1)^5(z_*^2-1)^4}, \\ &c_{22} := -\frac{153z_*^8-198z_*^6+99z_*^4-24z_*^2+2}{z_*^3(3z_*^2-1)^4(z_*^2-1)^3}, \quad c_{13} := \frac{3}{2z_*^2(3z_*^2-1)^3}, \\ &c_{04} := \frac{3(z_*^2-1)}{32z_*^5(3z_*^2-1)^2}. \end{aligned}$$

Further, using the transformation $x_3 := x_2$, $y_3 := y_2 - c_{02}x_2y_2$ and time rescaling $t := (1 + c_{02}x_3)\tau$ to eliminate the term of y_2^2 in system (2.4), we obtain

$$\begin{cases} \dfrac{dx_3}{dt} = y_3, \\ \dfrac{dy_3}{dt} = d_{20}x_3^2 + d_{11}x_3y_3 + d_{30}x_3^3 + d_{21}x_3^2y_3 + d_{12}x_3y_3^2 + d_{03}y_3^3 + d_{40}x_3^4 \\ \qquad + d_{31}x_3^3y_3 + d_{22}x_3^2y_3^2 + d_{13}x_3y_3^3 + d_{04}y_3^4 + O(\| (x_3, y_3) \|^5), \end{cases} \tag{2.5}$$

where

$$d_{20} := c_{20},\ d_{11} := c_{11},\ d_{30} := c_{30} - 2c_{02}c_{20},\ d_{21} := c_{21} - c_{02}c_{11},\ d_{12} := c_{12} - c_{02}^2,$$

$$d_{03} := c_{03},\ d_{40} := 2c_{02}^2c_{20} - 2c_{02}c_{30} + c_{40},\ d_{31} := c_{31} - c_{02}c_{21},\ d_{22} := c_{22} - c_{02}^3,$$

$$d_{13} := c_{02}c_{03} + c_{13},\ d_{04} := c_{04}.$$

Because $d_{20} < 0$ and $d_{11} \neq 0$ (reps. =0) for $(s,n,h) \in S_{21} \cup S_{22}$ (reps. S_3), Theorem 7.3 in [29] shows that the origin of system (2.5) is a cusp of codimension 2 (reps. at least 3). Therefore, degenerate equilibrium E_* is a cusp of codimension 2 (reps. at least 3) if $(s,n,h) \in S_{21} \cup S_{22}$ (reps. S_3).

Next, we determine the exact codimension of cusp E_* for $(s,n,h) \in S_3$. The transformation

$$\begin{aligned} x_3 &= x_4, \\ y_3 &= y_4 + d_{03}x_4y_4^2 + \tfrac{1}{2}d_{13}x_4^2y_4^2 + d_{04}x_4y_4^3 \end{aligned}$$

and time rescaling $\tau := (1 + d_{03}x_4y_4 + \frac{1}{2}d_{13}x_4^2y_4 + d_{04}x_4y_4^2)t$ bring system (2.5) into the form

$$\begin{cases} \dfrac{dx_4}{d\tau} = y_4, \\ \dfrac{dy_4}{d\tau} = d_{20}x_4^2 + d_{30}x_4^3 + d_{21}x_4^2y_4 + d_{12}x_4y_4^2 + d_{40}x_4^4 + (d_{31} - 3d_{20}d_{03})x_4^3y_4 \\ \qquad\quad + d_{22}x_4^2y_4^2 + O(\| (x_4, y_4) \|^5). \end{cases} \tag{2.6}$$

Further, reducing the coefficient of term of x_4^2 in system (2.6) to 1 by the transformation $x_4 = -x_5$, $y_4 = -\sqrt{-d_{20}}y_5$ and time rescaling $t := \sqrt{-d_{20}}\tau$, we obtain

$$\begin{cases} \dfrac{dx_5}{dt} = y_5, \\ \dfrac{dy_5}{dt} = x_5^2 + e_{30}x_5^3 + e_{12}x_5y_5^2 + e_{21}x_5^2y_5 + e_{40}x_5^4 + e_{22}x_5^2y_5^2 + e_{31}x_5^3y_5 \\ \qquad\quad + O(\| (x_5, y_5) \|^5), \end{cases} \tag{2.7}$$

where

$$\begin{aligned} e_{30} &:= -\frac{d_{30}}{d_{20}}, \; e_{12} := d_{12}, \; e_{21} := \frac{d_{21}}{\sqrt{-d_{20}}}, \\ e_{40} &:= \frac{d_{40}}{d_{20}}, \; e_{22} := -d_{22}, \; e_{31} := \frac{3d_{03}d_{20} - d_{31}}{\sqrt{-d_{20}}}. \end{aligned}$$

Proposition 5.3 in [16] shows that system (2.7) is equivalent to the system

$$\begin{cases} \dfrac{dx_6}{dt} = y_6, \\ \dfrac{dy_6}{dt} = x_6^2 + Gx_6^3y_6 + O(\| (x_6, y_6) \|^5), \end{cases}$$

where

$$\begin{aligned} G &:= e_{31} - e_{30}e_{21} \\ &= \frac{3(3231z_*^{16} + 12240z_*^{14} - 23696z_*^{12} + 19440z_*^{10} - 13514z_*^8 + 7216z_*^6 - 2104z_*^4 + 272z_*^2 - 13)}{2z_*(z_*^2-1)^3(3z_*^2-1)^5(3z_*^4+1)\sqrt{z_*(3z_*^4+1)}} \\ &\doteq 1.6457 \end{aligned}$$

for $(s,n,h) \in S_3$. Therefore, degenerate equilibrium E_* is a cusp of codimension 3 if $(s,n,h) \in S_3$ and there is no cusp with codimension larger than 3 for system (1.4). The proof of Theorem 2.2 is completed. □

3. Bifurcations

In this section, we discuss the possible bifurcations in system (1.4) around the nonhyperbolic equilibria E_* and E_1. From Theorem 2.2, we can see that system (1.4) may undergo a degenerate Bogdanov-Takens bifurcation of codimension 3 around E_*, which is first displayed in the following result.

Theorem 3.1. *System (1.4) undergoes a degenerate Bogdanov-Takens bifurcation of codimension 3 in a small neighborhood of equilibrium E_* as parameter (s, n, h) varies near S_3. Hence, system (1.4) can exhibit the existence of two limit cycles or one limit cycle and one homoclinic loop.*

Proof. Introducing small $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ into system (1.4), we obtain the unfolding system

$$\begin{cases} \dfrac{dx}{dt} = x(1-x) - \sqrt{x}y, \\ \dfrac{dy}{dt} = (s+\epsilon_1)y(1 - \dfrac{y}{(n+\epsilon_2)x}) - (h+\epsilon_3). \end{cases} \tag{3.1}$$

Following the procedures given by Li *et al.* [17] (see also [13]), we make a series of transformations transform system (3.1) to the versal unfolding of Bogdanov-Takens singularity of codimension 3

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = \gamma_1 + \gamma_2 y + \gamma_3 xy + x^2 + x^3 y + R(x, y, \epsilon), \end{cases} \tag{3.2}$$

where

$$R(x, y, \epsilon) = y^2 O(|x, y|^2) + O(|x, y|^5) + O(\epsilon)(O(y^2) + O(|x, y|^3)) + O(\epsilon^2)O(|x, y|),$$

and check

$$\left| \frac{\partial(\gamma_1, \gamma_2, \gamma_3)}{\partial(\epsilon_1, \epsilon_2, \epsilon_3)} \right|_{\epsilon=0} \neq 0.$$

Firstly, transforming equilibrium E_* to the origin when $\epsilon = 0$ by $x = x_1 + z_3^2$, $y = y_1 + z_3(1 - z_3^2)$ and using Taylor expansion, then system (3.1) becomes

$$\begin{cases} \dfrac{dx_1}{dt} = \tilde{a}_{10}x_1 + \tilde{a}_{01}y_1 + \tilde{a}_{20}x_1^2 + \tilde{a}_{11}x_1y_1 + \tilde{a}_{30}x_1^3 + \tilde{a}_{21}x_1^2y_1 + \tilde{a}_{31}x_1^3y_1 \\ \qquad\quad + \tilde{a}_{40}x_1^4 + O(|x_1, y_1|^5), \\ \dfrac{dy_1}{dt} = \tilde{b}_{00} + \tilde{b}_{10}x_1 + \tilde{b}_{01}y_1 + \tilde{b}_{20}x_1^2 + \tilde{b}_{11}x_1y_1 + \tilde{b}_{02}y_1^2 + \tilde{b}_{30}x_1^3 + \tilde{b}_{21}x_1^2y_1 \\ \qquad\quad + \tilde{b}_{12}x_1y_1^2 + \tilde{b}_{40}x_1^4 + \tilde{b}_{31}x_1^3y_1 + \tilde{b}_{22}x_1^2y_1^2 + O(|x_1, y_1|^5), \end{cases} \tag{3.3}$$

where

$$\tilde{a}_{10} := \frac{1 - 3z_3^2}{2},\ \tilde{a}_{01} := -z_3,\ \tilde{a}_{20} := \frac{1 - 9z_3^2}{8z_3^2},\ \tilde{a}_{11} := -\frac{1}{2z_3},\ \tilde{a}_{30} := \frac{z_3^2 - 1}{16z_3^4},$$
$$\tilde{a}_{21} := \frac{1}{8z_3^3},\ \tilde{a}_{31} := -\frac{1}{16z_3^5},\ \tilde{a}_{40} := \frac{5(1 - z_3^2)}{128z_3^6},$$

$$\tilde{b}_{10} := \frac{(3z_3^2-1)(z_3^2-1)(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)}{(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^2},$$

$$\tilde{b}_{00} := -\frac{-4z_3(3z_3^2-1)(z_3^2-1)\epsilon_1\epsilon_2+4(z_3^2+1)(z_3^2-1)^2\epsilon_1+4(1-3z_3^2)\epsilon_2\epsilon_3+z_3(3z_3^2-1)^3\epsilon_2+16z_3(z_3^2-1)\epsilon_3}{4(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)},$$

$$\tilde{b}_{01} := \frac{(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(2z_3^4+3z_3^3\epsilon_2-4z_3^2-z_3\epsilon_2+2)}{z_3(z_3^2-1)(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)},$$

$$\tilde{b}_{20} := -\frac{(3z_3^2-1)(z_3^2-1)(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)}{(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^4},$$

$$\tilde{b}_{11} := -\frac{2(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{z_3^3(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)},$$

$$\tilde{b}_{02} := -\frac{(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{(z_3^2-1)(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^2},$$

$$\tilde{b}_{30} := \frac{(3z_3^2-1)(z_3^2-1)(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)}{(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^6},$$

$$\tilde{b}_{21} := \frac{2(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{z_3^5(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)},$$

$$\tilde{b}_{12} := \frac{(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{(z_3^2-1)(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^4},$$

$$\tilde{b}_{40} := -\frac{(3z_3^2-1)(z_3^2-1)(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)}{(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^8},$$

$$\tilde{b}_{31} := -\frac{2(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{z_3^7(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)},$$

$$\tilde{b}_{22} := -\frac{(3z_3^4-z_3^2\epsilon_1-z_3^2+\epsilon_1)(3z_3^2-1)}{(z_3^2-1)(4z_3^3-3z_3^2\epsilon_2-4z_3+\epsilon_2)z_3^6},$$

in which parameters h, n and s have been eliminated by equations given in S_3.

Secondly, reducing system (3.3) to the Kukles form by the near-identity transformation

$$\begin{aligned}
x_2 &= x_1, \\
y_2 &= \tilde{a}_{10}x_1 + \tilde{a}_{01}y_1 + \tilde{a}_{20}x_1^2 + \tilde{a}_{11}x_1y_1 + \tilde{a}_{30}x_1^3 + \tilde{a}_{21}x_1^2y_1 + \tilde{a}_{31}x_1^3y_1 \\
&\quad + \tilde{a}_{40}x_1^4 + O(|x_1, y_1|^5),
\end{aligned}$$

then we obtain

$$\begin{cases}
\dfrac{dx_2}{dt} = y_2, \\
\dfrac{dy_2}{dt} = \tilde{c}_{00} + \tilde{c}_{10}x_2 + \tilde{c}_{01}y_2 + \tilde{c}_{20}x_2^2 + \tilde{c}_{11}x_2y_2 + \tilde{c}_{02}y_2^2 + \tilde{c}_{30}x_2^3 + \tilde{c}_{21}x_2^2y_2 \\
\qquad + \tilde{c}_{12}x_2y_2^2 + \tilde{c}_{40}x_2^4 + \tilde{c}_{31}x_2^3y_2 + \tilde{c}_{22}x_2^2y_2^2 + O(|x_2, y_2|^5),
\end{cases} \tag{3.4}$$

where

$$\tilde{c}_{00} := \tilde{a}_{01}\tilde{b}_{00}, \ \tilde{c}_{10} := \tilde{a}_{01}\tilde{b}_{10} - \tilde{a}_{10}\tilde{b}_{01} + \tilde{a}_{11}\tilde{b}_{00}, \ \tilde{c}_{01} := \tilde{b}_{01} + \tilde{a}_{10},$$

$$\begin{aligned}
\tilde{c}_{20} :=& \frac{1}{\tilde{a}_{01}}\{\tilde{b}_{20}\tilde{a}_{01}^2 - \tilde{b}_{11}\tilde{a}_{10}\tilde{a}_{01} + (\tilde{a}_{11}\tilde{b}_{10} - \tilde{a}_{20}\tilde{b}_{01} + \tilde{a}_{21}\tilde{b}_{00})\tilde{a}_{01} + \tilde{b}_{02}\tilde{a}_{10}^2\},\\
\tilde{c}_{11} :=& \frac{1}{\tilde{a}_{01}}(2\tilde{a}_{01}\tilde{a}_{20} + \tilde{a}_{01}\tilde{b}_{11} - \tilde{a}_{10}\tilde{a}_{11} - 2\tilde{a}_{10}\tilde{b}_{02}),\ \tilde{c}_{02} := \frac{1}{\tilde{a}_{01}}(\tilde{a}_{11} + \tilde{b}_{02}),\\
\tilde{c}_{30} :=& \frac{1}{\tilde{a}_{01}^2}\{\tilde{b}_{30}\tilde{a}_{01}^3 + (-\tilde{a}_{10}\tilde{b}_{21} + \tilde{a}_{11}\tilde{b}_{20} - \tilde{a}_{20}\tilde{b}_{11} + \tilde{a}_{21}\tilde{b}_{10} - \tilde{a}_{30}\tilde{b}_{01} + \tilde{a}_{31}\tilde{b}_{00})\tilde{a}_{01}^2\\
&+ \tilde{a}_{10}(\tilde{a}_{10}\tilde{b}_{12} + 2\tilde{a}_{20}\tilde{b}_{02})\tilde{a}_{01} - \tilde{a}_{10}^2\tilde{a}_{11}\tilde{b}_{02}\},\\
\tilde{c}_{21} :=& \frac{1}{\tilde{a}_{01}^2}\{(3\tilde{a}_{30} + \tilde{b}_{21})\tilde{a}_{01}^2 - (2\tilde{a}_{10}\tilde{a}_{21} + 2\tilde{a}_{10}\tilde{b}_{12} + \tilde{a}_{11}\tilde{a}_{20} + 2\tilde{a}_{20}\tilde{b}_{02})\tilde{a}_{01}\\
&+ \tilde{a}_{10}\tilde{a}_{11}(\tilde{a}_{11} + 2\tilde{b}_{02})\},\\
\tilde{c}_{12} :=& \frac{1}{\tilde{a}_{01}^2}(2\tilde{a}_{01}\tilde{a}_{21} + \tilde{a}_{01}\tilde{b}_{12} - \tilde{a}_{11}^2 - \tilde{a}_{11}\tilde{b}_{02}),\\
\tilde{c}_{40} :=& \frac{1}{\tilde{a}_{01}^3}\{\tilde{b}_{40}\tilde{a}_{01}^4 + (-\tilde{a}_{10}\tilde{b}_{31} + \tilde{a}_{11}\tilde{b}_{30} - \tilde{a}_{20}\tilde{b}_{21} + \tilde{a}_{21}\tilde{b}_{20} - \tilde{a}_{30}\tilde{b}_{11} + \tilde{a}_{31}\tilde{b}_{10}\\
&- \tilde{a}_{40}\tilde{b}_{01})\tilde{a}_{01}^3 + (\tilde{a}_{10}^2\tilde{b}_{22} + 2\tilde{a}_{10}\tilde{a}_{20}\tilde{b}_{12} + 2\tilde{a}_{10}\tilde{a}_{30}\tilde{b}_{02} + \tilde{a}_{20}^2\tilde{b}_{02})\tilde{a}_{01}^2\\
&- \tilde{a}_{10}(\tilde{a}_{10}\tilde{a}_{11}\tilde{b}_{12} + \tilde{a}_{10}\tilde{a}_{21}\tilde{b}_{02} + 2\tilde{a}_{11}\tilde{a}_{20}\tilde{b}_{02})\tilde{a}_{01} + \tilde{a}_{10}^2\tilde{a}_{11}^2\tilde{b}_{02}\},\\
\tilde{c}_{31} :=& \frac{1}{\tilde{a}_{01}^3}\{(4\tilde{a}_{40} + \tilde{b}_{31})\tilde{a}_{01}^3 - (3\tilde{a}_{10}\tilde{a}_{31} + 2\tilde{a}_{10}\tilde{b}_{22} + \tilde{a}_{11}\tilde{a}_{30} + 2\tilde{a}_{20}\tilde{a}_{21} + 2\tilde{a}_{20}\tilde{b}_{12}\\
&+ 2\tilde{a}_{30}\tilde{b}_{02})\tilde{a}_{01}^2 + (3\tilde{a}_{10}\tilde{a}_{11}\tilde{a}_{21} + 2\tilde{a}_{10}\tilde{a}_{11}\tilde{b}_{12} + 2\tilde{a}_{10}\tilde{a}_{21}\tilde{b}_{02} + \tilde{a}_{11}^2\tilde{a}_{20}\\
&+ 2\tilde{a}_{11}\tilde{a}_{20}\tilde{b}_{02})\tilde{a}_{01} - \tilde{a}_{10}\tilde{a}_{11}^2(\tilde{a}_{11} + 2\tilde{b}_{02})\},\\
\tilde{c}_{22} :=& \frac{1}{\tilde{a}_{01}^3}\{(3\tilde{a}_{31} + \tilde{b}_{22})\tilde{a}_{01}^2 - (3\tilde{a}_{11}\tilde{a}_{21} + \tilde{a}_{11}\tilde{b}_{12} + \tilde{a}_{21}\tilde{b}_{02})\tilde{a}_{01} + \tilde{a}_{11}^2(\tilde{a}_{11} + \tilde{b}_{02})\}.
\end{aligned}$$

Thirdly, removing the term of y_2^2 from system (3.4) by the near-identity transformation $x_2 = x_3 + \frac{\tilde{c}_{02}}{2}x_3^2$, $y_2 = y_3 + \tilde{c}_{02}x_3y_3$, then system (3.4) is transformed to

$$\begin{cases}
\dfrac{dx_3}{dt} = y_3,\\
\dfrac{dy_3}{dt} = \tilde{d}_{00} + \tilde{d}_{10}x_3 + \tilde{d}_{01}y_3 + \tilde{d}_{20}x_3^2 + \tilde{d}_{11}x_3y_3 + \tilde{d}_{30}x_3^3 + \tilde{d}_{21}x_3^2y_3\\
\qquad + \tilde{d}_{12}x_3y_3^2 + \tilde{d}_{40}x_3^4 + \tilde{d}_{31}x_3^3y_3 + \tilde{d}_{22}x_3^2y_3^2 + O(|x_3, y_3|^5),
\end{cases} \tag{3.5}$$

where

$$\begin{aligned}
&\tilde{d}_{00} := \tilde{c}_{00},\ \tilde{d}_{10} := \tilde{c}_{10} - \tilde{c}_{00}\tilde{c}_{02},\ \tilde{d}_{01} := \tilde{c}_{01},\ \tilde{d}_{11} := \tilde{c}_{11},\ \tilde{d}_{21} := \frac{1}{2}(\tilde{c}_{02}\tilde{c}_{11} + 2\tilde{c}_{21}),\\
&\tilde{d}_{20} := \frac{1}{2}(2\tilde{c}_{00}\tilde{c}_{02}^2 - \tilde{c}_{02}\tilde{c}_{10} + 2\tilde{c}_{20}),\ \tilde{d}_{30} := -\frac{1}{2}(2\tilde{c}_{00}\tilde{c}_{02}^3 - \tilde{c}_{02}^2\tilde{c}_{10} - 2\tilde{c}_{30}),\\
&\tilde{d}_{12} := 2\tilde{c}_{02}^2 + \tilde{c}_{12},\ \tilde{d}_{40} := \frac{1}{4}(4\tilde{c}_{00}\tilde{c}_{02}^4 - 2\tilde{c}_{02}^3\tilde{c}_{10} + \tilde{c}_{02}^2\tilde{c}_{20} + 2\tilde{c}_{02}\tilde{c}_{30} + 4\tilde{c}_{40}),\\
&\tilde{d}_{31} := \tilde{c}_{02}\tilde{c}_{21} + \tilde{c}_{31},\ \tilde{d}_{22} := -\frac{1}{2}(2\tilde{c}_{02}^3 - 3\tilde{c}_{02}\tilde{c}_{12} - 2\tilde{c}_{22}).
\end{aligned}$$

Fourthly, removing the term of $x_3y_3^2$ from system (3.5) by the near-identity transformation $x_3 = x_4 + \frac{\tilde{d}_{12}}{6}x_4^3$, $y_3 = y_4 + \frac{\tilde{d}_{12}}{2}x_4^2 y_4$, then we obtain

$$\begin{cases} \dfrac{dx_4}{dt} = y_4, \\ \dfrac{dy_4}{dt} = \tilde{e}_{00} + \tilde{e}_{10}x_4 + \tilde{e}_{01}y_4 + \tilde{e}_{20}x_4^2 + \tilde{e}_{11}x_4y_4 + \tilde{e}_{30}x_4^3 + \tilde{e}_{21}x_4^2y_4 \\ \qquad + \tilde{e}_{40}x_4^4 + \tilde{e}_{31}x_4^3y_4 + \tilde{e}_{22}x_4^2y_4^2 + O(|x_4, y_4|^5), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} &\tilde{e}_{00} := \tilde{d}_{00},\ \tilde{e}_{10} := \tilde{d}_{10},\ \tilde{e}_{01} := \tilde{d}_{01},\ \tilde{e}_{11} := \tilde{d}_{11},\ \tilde{e}_{21} := \tilde{d}_{21}, \\ &\tilde{e}_{20} := -\tfrac{1}{2}(\tilde{d}_{00}\tilde{d}_{12} - 2\tilde{d}_{20}),\ \tilde{e}_{31} := \tfrac{1}{6}(\tilde{d}_{11}\tilde{d}_{12} + 6\tilde{d}_{31}),\ \tilde{e}_{22} := \tilde{d}_{22}, \\ &\tilde{e}_{30} := -\tfrac{1}{3}(\tilde{d}_{10}\tilde{d}_{12} - 3\tilde{d}_{30}),\ \tilde{e}_{40} := \tfrac{1}{12}(3\tilde{d}_{00}\tilde{d}_{12}^2 - 2\tilde{d}_{12}\tilde{d}_{20} + 12\tilde{d}_{40}). \end{aligned}$$

Fifthly, removing the terms of x_4^3 and x_4^4 from system (3.6) when $\epsilon = 0$ by the near-identity transformation and time rescaling

$$\begin{aligned} &x_4 = x_5 - \frac{\tilde{e}_{30}}{4\tilde{e}_{20}}x_5^2 + \frac{15\tilde{e}_{30}^2 - 16\tilde{e}_{20}\tilde{e}_{40}}{80\tilde{e}_{20}^2}x_5^3, \quad y_4 = y_5, \\ &\tau = \{1 + \frac{\tilde{e}_{30}}{2\tilde{e}_{20}}x_5 + \frac{48\tilde{e}_{20}\tilde{e}_{40} - 25\tilde{e}_{30}^2}{80\tilde{e}_{20}^2}x_5^2 + \frac{\tilde{e}_{30}(48\tilde{e}_{20}\tilde{e}_{40} - 35\tilde{e}_{30}^2)}{80\tilde{e}_{20}^3}x_5^3\}t, \end{aligned}$$

where $\tilde{e}_{20} = \frac{(3z_3^2-1)(3z_3^4+1)}{8(z_3^2-1)^2} \doteq 0.7788$ for $\epsilon = 0$, then system (3.6) becomes

$$\begin{cases} \dfrac{dx_5}{d\tau} = y_5, \\ \dfrac{dy_5}{d\tau} = \tilde{f}_{00} + \tilde{f}_{10}x_5 + \tilde{f}_{01}y_5 + \tilde{f}_{20}x_5^2 + \tilde{f}_{11}x_5y_5 + \tilde{f}_{30}x_5^3 + \tilde{f}_{21}x_5^2y_5 \\ \qquad + \tilde{f}_{40}x_5^4 + \tilde{f}_{31}x_5^3y_5 + \tilde{f}_{22}x_5^2y_5^2 + O(|x_5, y_5|^5), \end{cases} \tag{3.7}$$

where

$$\begin{aligned} &\tilde{f}_{00} := \tilde{e}_{00},\ \tilde{f}_{01} := \tilde{e}_{01},\ \tilde{f}_{22} := \tilde{e}_{22},\ \tilde{f}_{10} := \frac{1}{2\tilde{e}_{20}}(2\tilde{e}_{10}\tilde{e}_{20} - \tilde{e}_{00}\tilde{e}_{30}), \\ &\tilde{f}_{20} := \frac{1}{80\tilde{e}_{20}^2}\{(45\tilde{e}_{30}^2 - 48\tilde{e}_{20}\tilde{e}_{40})\tilde{e}_{00} - 20\tilde{e}_{20}(3\tilde{e}_{10}\tilde{e}_{30} - 4\tilde{e}_{20}^2)\}, \\ &\tilde{f}_{11} := \frac{1}{2\tilde{e}_{20}}(2\tilde{e}_{11}\tilde{e}_{20} - \tilde{e}_{01}\tilde{e}_{30}),\ \tilde{f}_{30} := \frac{\tilde{e}_{10}}{40\tilde{e}_{20}^2}(35\tilde{e}_{30}^2 - 32\tilde{e}_{20}\tilde{e}_{40}), \\ &\tilde{f}_{21} := \frac{1}{80\tilde{e}_{20}^2}\{(45\tilde{e}_{30}^2 - 48\tilde{e}_{20}\tilde{e}_{40})\tilde{e}_{01} - 20\tilde{e}_{20}(3\tilde{e}_{11}\tilde{e}_{30} - 4\tilde{e}_{20}\tilde{e}_{21})\}, \\ &\tilde{f}_{31} := \frac{1}{40\tilde{e}_{20}^2}\{(35\tilde{e}_{30}^2 - 32\tilde{e}_{20}\tilde{e}_{40})\tilde{e}_{11} + 40\tilde{e}_{20}(\tilde{e}_{20}\tilde{e}_{31} - \tilde{e}_{21}\tilde{e}_{30})\}, \\ &\tilde{f}_{40} := \frac{1}{6400\tilde{e}_{20}^4}\{(2304\tilde{e}_{20}^2\tilde{e}_{40}^2 - 1440\tilde{e}_{20}\tilde{e}_{30}^2\tilde{e}_{40} - 275\tilde{e}_{30}^4)\tilde{e}_{00} \\ &\qquad + 100\tilde{e}_{10}\tilde{e}_{20}\tilde{e}_{30}(16\tilde{e}_{20}\tilde{e}_{40} - 15\tilde{e}_{30}^2)\}. \end{aligned}$$

Sixthly, removing the term of $x_5^2y_5$ from system (3.7) by the near-identity transformation and time rescaling

$$x_5 = x_6, \quad y_5 = y_6 + \frac{\tilde{f}_{21}}{3\tilde{f}_{20}}y_6^2 + \frac{\tilde{f}_{21}^2}{36\tilde{f}_{20}^2}y_6^3, \quad t = \{1 + \frac{\tilde{f}_{21}}{3\tilde{f}_{20}}y_6 + \frac{\tilde{f}_{21}^2}{36\tilde{f}_{20}^2}y_6^2\}\tau,$$

where $\tilde{f}_{20}=\frac{(3z_*^2-1)(3z_*^4+1)}{8(z_*^2-1)^2}\doteq 0.7788$ for $\epsilon=0$, then system (3.7) is changed into

$$\begin{cases}\dfrac{dx_6}{dt}=y_6,\\ \dfrac{dy_6}{dt}=\tilde{g}_{00}+\tilde{g}_{10}x_6+\tilde{g}_{01}y_6+\tilde{g}_{20}x_6^2+\tilde{g}_{11}x_6y_6+\tilde{g}_{31}x_6^3y_6+R_1(x_6,y_6,\epsilon),\end{cases}\tag{3.8}$$

where

$$\begin{aligned}&\tilde{g}_{00}:=\tilde{f}_{00},\ \tilde{g}_{10}:=\tilde{f}_{10},\ \tilde{g}_{20}:=\tilde{f}_{20},\ \tilde{g}_{01}:=\tfrac{1}{\tilde{f}_{20}}(\tilde{f}_{01}\tilde{f}_{20}-\tilde{f}_{00}\tilde{f}_{21}),\\ &\tilde{g}_{11}:=\tfrac{1}{\tilde{f}_{20}}(\tilde{f}_{11}\tilde{f}_{20}-\tilde{f}_{10}\tilde{f}_{21}),\ \tilde{g}_{31}:=\tfrac{1}{\tilde{f}_{20}}(\tilde{f}_{20}\tilde{f}_{31}-\tilde{f}_{21}\tilde{f}_{30}),\end{aligned}$$

and $R_1(x_6,y_6,\epsilon)$ has the property of $R(x,y,\epsilon)$.

Seventhly, Changing $\tilde{g}_{20}$ and $\tilde{g}_{31}$ to 1 by rescalings

$$x_6=\tilde{g}_{20}^{\frac{1}{5}}\tilde{g}_{31}^{-\frac{2}{5}}x_7,\quad y_6=\tilde{g}_{20}^{\frac{4}{5}}\tilde{g}_{31}^{-\frac{3}{5}}y_7,\quad \tau=\tilde{g}_{20}^{\frac{3}{5}}\tilde{g}_{31}^{-\frac{1}{5}}t$$

since $\tilde{g}_{20}=\frac{(3z_3^2-1)(3z_3^4+1)}{8(z_3^2-1)^2}\doteq 0.7788$ and

$$\begin{aligned}&\tilde{g}_{31}\\ =&\frac{-543z_3^{22}+5019z_3^{20}+3041z_3^{18}-7917z_3^{16}+5134z_3^{14}-6702z_3^{12}+5942z_3^{10}-2374z_3^{8}+513z_3^{6}-61z_3^{4}-7z_3^{2}+3}{160(z_3^2-1)^6z_3^6(3z_3^4+1)^2}\\ \doteq&0.1874\end{aligned}$$

for $\epsilon=0$, then we can transform system (3.8) to

$$\begin{cases}\dfrac{dx_7}{d\tau}=y_7,\\ \dfrac{dy_7}{d\tau}=\tilde{h}_{00}+\tilde{h}_{10}x_7+\tilde{h}_{01}y_7+\tilde{h}_{11}x_7y_7+x_7^2+x_7^3y_7+R_2(x_7,y_7,\epsilon),\end{cases}\tag{3.9}$$

where

$$\tilde{h}_{00}:=\tilde{g}_{00}\tilde{g}_{31}^{\frac{4}{5}}\tilde{g}_{20}^{-\frac{7}{5}},\ \tilde{h}_{10}:=\tilde{g}_{10}\tilde{g}_{31}^{\frac{2}{5}}\tilde{g}_{20}^{-\frac{6}{5}},\ \tilde{h}_{01}:=\tilde{g}_{01}\tilde{g}_{31}^{\frac{1}{5}}\tilde{g}_{20}^{-\frac{3}{5}},\ \tilde{h}_{11}:=\tilde{g}_{11}\tilde{g}_{20}^{-\frac{2}{5}}\tilde{g}_{31}^{-\frac{1}{5}},$$

and $R_2(x_7,y_7,\epsilon)$ has the property of $R(x,y,\epsilon)$.

Eighthly, removing the term of x_7 from system (3.9) by $x_7=x_8-\frac{\tilde{h}_{10}}{2}$, $y_7=y_8$, then system (3.9) can be put in the form of

$$\begin{cases}\dfrac{dx_8}{d\tau}=y_8,\\ \dfrac{dy_8}{d\tau}=\gamma_1+\gamma_2y_8+\gamma_3x_8y_8+x_8^2+x_8^3y_8+R_3(x_8,y_8,\epsilon),\end{cases}\tag{3.10}$$

where

$$\gamma_1:=\tilde{h}_{00}-\tfrac{1}{4}\tilde{h}_{10}^2,\ \gamma_2:=\tilde{h}_{01}-\tfrac{1}{8}(\tilde{h}_{10}^3+4\tilde{h}_{10}\tilde{h}_{11}),\ \gamma_3:=\tilde{h}_{11}+\tfrac{3}{4}\tilde{h}_{10}^2,$$

and $R_3(x_8,y_8,\epsilon)$ has the property of $R(x,y,\epsilon)$.

Furthermore, the direct computation shows that

$$\left|\frac{\partial(\gamma_1,\gamma_2,\gamma_3)}{\partial(\epsilon_1,\epsilon_2,\epsilon_3)}\right|_{\epsilon=0} \doteq -0.2236.$$

So system (3.10) is the versal unfolding of cusp of codimension 3. Hence, system (1.4) undergoes a Bogdanov-Takens bifurcation of codimension 3 in a small neighborhood of equilibrium E_* as parameter (s,n,h) varies near S_3. The proof of Theorem 3.1 is completed. □

We next describe the bifurcation diagram of system (3.10) as in Dumortier *et al.* [6] and Chow *et al.* [5]. System 3.10 obviously has no equilibrium for $\gamma_1 > 0$. $\gamma_1 = 0$ is a saddle-node bifurcation plane, i.e., the saddle and the node or focus are created as γ_1 crosses the plane to $\gamma_1 < 0$. The other bifurcation surfaces are located in the half space $\gamma_1 < 0$. Each bifurcation surface is a cone with vertex at the origin, which can best be visualized by drawing its trace on the half sphere

$$S := \{(\gamma_1,\gamma_2,\gamma_3)|\gamma_1^2+\gamma_2^2+\gamma_3^2 = r^2, \gamma_1 \le 0, r > 0 \text{ sufficiently small}\}.$$

There are three bifurcation curves on S, i.e., Hopf bifurcation curve H, homoclinic bifurcation curve C and saddle-node bifurcation curve of limit cycles L. To see the trace on the half sphere clearly, Figure 3 gives the projection of trace on the (γ_1,γ_2) plane. Bifurcation curves H and C are tangent to the boundary ∂S at points b_1 and b_2 and cross each other at point d. Bifurcation curve L is tangent to curves H and C at points h_2 and c_2, respectively. Various bifurcations can be describe as below.

The saddle-node bifurcation occurs along boundary ∂S except for points b_1 and b_2, while the Bogdanov-Takens bifurcations of codimension 2 occur at points b_1 and b_2.

The Hopf bifurcation of codimension 1 occurs along the curve H except for the point h_2, i.e., the subcritical Hopf bifurcation occurs when ϵ crosses the arc b_1h_2 of curve H from the right to the left, which induces an unstable limit cycle, the supercritical Hopf bifurcation occurs when ϵ crosses the arc h_2b_2 of curve H from the left to the right, which induces a stable limit cycle, while the Hopf bifurcation point of codimension 2 occurs at point h_2.

The homoclinic bifurcation of codimension 1 occurs along curve C except for point c_2, i.e., the separatrices of saddle coincide and an unstable limit cycle appears when ϵ crosses the arc b_1c_2 of curve C from the left to the right, the separatrices of saddle coincide and a stable limit cycle appears when ϵ crosses the arc c_2b_2 of curve C from the right to the left, while the homoclinic bifurcation of codimension 2 occurs at point c_2.

The Hopf bifurcation of order 1 and the homoclinic bifurcation of order 1 occur simultaneously at point d.

The saddle-node bifurcation of limit cycles occurs along curve L, i.e., two limit cycles (the inner one is stable and the outer one is unstable) appear when ϵ crosses curve L from the right to the left triangle dh_2c_2, and two limit cycles coalesce into a semistable limit cycle on arc L.

From Theorem 2.1, we can see that equilibrium E_1 may be a weak focus if $(s,n,h) \in D_0 := D_{01} \cup D_{02} \cup D_{03}$, where

$$D_{01} := \{(s,n,h) \in \mathbb{R}_+^3 : h = \frac{\sqrt{3}}{9}s, n = \frac{4}{\sqrt{3}}, z_1 = \frac{1}{\sqrt{3}}\},$$

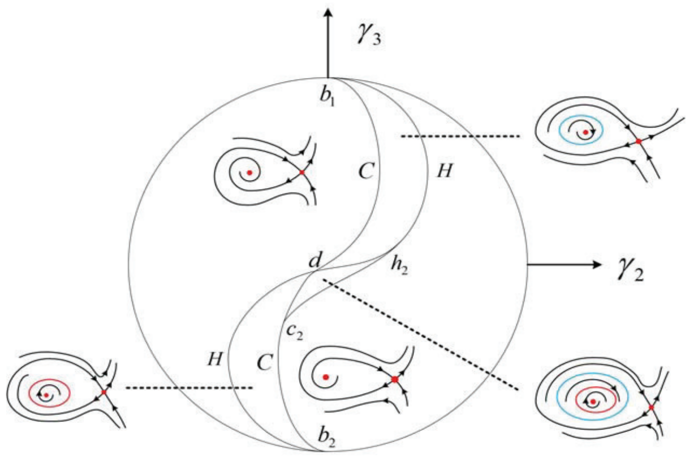

Figure 3. Bifurcation diagram for system (1.4) on S.

$$D_{02} := \{(s,n,h) \in \mathbb{R}^3_+ : h = h_3, s = s_1, \frac{1-z_1^2}{z_1} < n < \frac{2(1-z_1^2)}{z_1}, 0 < z_1 < \frac{1}{\sqrt{3}}\},$$

$$D_{03} := \{(s,n,h) \in \mathbb{R}^3_+ : h = h_3, s = s_1, \frac{2(1-z_1^2)}{z_1} < n < \frac{4z_1(1-z_1^2)}{3z_1^2-1}, \frac{1}{\sqrt{3}} < z_1 < 1\}$$

with

$$h_3 := \frac{(1-z_1^2)(z_1^2+nz_1-1)(3z_1^2-1)z_1}{2(2z_1^2+nz_1-2)}, \quad s_1 := \frac{nz_1(3z_1^2-1)}{2(2z_1^2+nz_1-2)}.$$

In the following, we devoted to exploring the final multiplicity of weak focus E_1 and determining the exact codimension of Hopf bifurcation around E_1.

Theorem 3.2. *For $(s,n,h) \in D_0$, equilibrium E_1 of system (1.4) is a weak focus of multiplicity at most 3. More exactly,*
(i) E_1 is a weak focus of multiplicity 1 if $(s,n,h) \in D_1 := D_{11} \cup D_{12} \cup D_{13}$ with

$$D_{11} := \{(s,n,h) \in D_{01} : s \neq \frac{\sqrt{6}}{3}\}, \quad D_{12} := \{(s,n,h) \in D_{02} : L_1 \neq 0\},$$
$$D_{13} := \{(s,n,h) \in D_{03} : L_1 \neq 0\},$$

(ii) E_1 is a weak focus of multiplicity 2 if $(s,n,h) \in D_2 := D_{21} \cup D_{22} \cup D_{23}$ with

$$D_{21} := D_{01} \backslash D_{11}, \quad D_{22} := D_{02} \backslash D_{12}, \quad D_{23} := \{(s,n,h) \in D_{03} \backslash D_{13} : L_2 \neq 0\},$$

(iii) E_1 is a weak focus of multiplicity 3 if $(s,n,h) \in D_3 := D_{03} \backslash (D_{13} \cup D_{23})$, where

$$L_1 := 3z_1^4(3z_1^2+1)n^3 - 2z_1(6z_1^6+9z_1^4+1)n^2 - 2(z_1^2-1)(33z_1^6+3z_1^4-5z_1^2+1)n$$
$$- 16z_1(3z_1^4-1)(z_1^2-1)^2$$

and L_2 is given in the Appendix.

Proof. Translating E_1 to the origin and making the linear transformation

$$x = \frac{nz_1^2}{s(z_1^2-1)^2}u - \frac{(3z_1^2-1)nz_1^2}{2s(z_1^2-1)^2 w}v, \quad y = \frac{1}{w}v$$

and the time rescaling $\tau := wt$ with $w := \sqrt{\mathrm{Det}(J(E_1))}$ system (1.4) takes the form

$$\begin{cases} \dfrac{du}{d\tau} = -v + \displaystyle\sum_{i+j=2}^{7} \hat{a}_{ij}u^i v^j + O(|(u,v)|^8), \\ \dfrac{dv}{d\tau} = u + \displaystyle\sum_{i+j=2}^{7} \hat{b}_{ij}u^i v^j + O(|(u,v)|^8), \end{cases} \tag{3.11}$$

where the coefficients $\hat{a}_{ij}$ and $\hat{b}_{ij}$ with $(s,n,h) \in D_0$ are given in the Appendix.

In what follows, we compute the focal values of weak focus E_1 by the method of successive function [29] and prove whether they have common zeros for $(s,n,h) \in D_0$ so as to show that E_1 is weak focus of multiplicity at most 3.

For $(s,n,h) \in D_{01}$, the first two focal values $\tilde{L}_1$ and $\tilde{L}_2$ are given by

$$\tilde{L}_1 = \frac{27\sqrt{3}}{32}s^{-\frac{5}{2}}(3s^2-2), \quad \tilde{L}_2 = \frac{54675\sqrt[4]{216} - 275562\sqrt[4]{6}}{16384}. \tag{3.12}$$

It is obvious that $\tilde{L}_1 = 0$ and $\tilde{L}_2 < 0$ if $s = \frac{\sqrt{6}}{3}$ otherwise $\tilde{L}_1 \neq 0$ in D_{01}. Therefore, E_1 is a weak focus of multiplicity at most 2 for $(s,n,h) \in D_{01}$. More exactly, E_1 is a weak focus of multiplicity 1 if $(s,n,h) \in D_{11}$ and E_1 is a weak focus of multiplicity 2 if $(s,n,h) \in D_{21}$.

For $(s,n,h) \in D_{02} \cup D_{03}$, the first three focal values $\tilde{L}_i (i=1,2,3)$ are given by

$$\begin{aligned} \tilde{L}_1 &= \frac{-n^2 z_1(3z_1^2-1)^2 L_1}{512(z_1^2-1)^2(2z_1^2+nz_1-2)^4 s_1^2 w^5}, \\ \tilde{L}_2 &= \frac{n^4 z_1^2(3z_1^2-1)^4 L_2}{12582912(z_1^2-1)^6(2z_1^2+nz_1-2)^9 s_1^4 w^{11}}, \\ \tilde{L}_3 &= \frac{-n^6 z_1^3(3z_1^2-1)^6 L_3}{37108517437440(z_1^2-1)^{10}(2z_1^2+nz_1-2)^{14} s_1^6 w^{17}}, \end{aligned} \tag{3.13}$$

where $L_i (i=1,2,3)$ are listed in Theorem 3.2 and Appendix. Since the other factors in the numerators and denominators of $L_i (i=1,2,3)$ are all positive, the zeros of $\tilde{L}_i (i=1,2,3)$ are determined by $L_i (i=1,2,3)$, respectively

We first claim that E_1 is a weak focus of multiplicity at most 3 for $(s,n,h) \in D_{02} \cup D_{03}$. By Lemma 2 in [4], we have the following decomposition of algebraic variety

$$V(L_1,L_2,L_3) = V(L_1,L_2,L_3,\mathrm{lcoeff}(L_1,n)) \cup V\left(\frac{L_1,L_2,L_3,r_1(2),r_1(3)}{\mathrm{lcoeff}(L_1,n)}\right), \tag{3.14}$$

where lcoeff(L_1,n) denotes the leading coefficient of L_1 with respect to the variable n, $r_1(2)$ and $r_1(3)$ are Sylvester resultants [7]

$$\begin{aligned} r_1(2) &:= \mathrm{res}(L_1,L_2,n) = c_1 r_0 r_1 r_2^2, \\ r_1(3) &:= \mathrm{res}(L_1,L_3,n) = c_2 z_1^9(3z_1^4+1)(3z_1^2-1)^6(z_1^2-1)^{10} r_0 r_2^2 r_3, \end{aligned} \tag{3.15}$$

where c_i $(i = 1, 2)$ are nonzero constants and

$$
\begin{aligned}
r_0 &:= z_1^{22}(3z_1^2+1)(z_1^2+1)(3z_1^4+1)^2(3z_1^2-1)^{11}(z_1^2-1)^{18},\\
r_1 &:= 228927z_1^{22} + 1650930z_1^{20} + 4445478z_1^{18} + 5460554z_1^{16} + 2513088z_1^{14}\\
&\quad - 884090z_1^{12} - 1040730z_1^{10} - 164290z_1^{8} + 60705z_1^{6} + 16272z_1^{4} + 1044z_1^{2} + 112,\\
r_2 &:= 3z_1^6 + 15z_1^4 - 11z_1^2 + 1,\\
r_3 &:= 95214674781205887953 7z_1^{58} + 1655836798420404825969 6z_1^{56}\\
&\quad + 10851133563595038896553 6z_1^{54} + 32791947601001901987632 4z_1^{52}\\
&\quad + 41497570031980102957931 4z_1^{50} - 31501440547934684294916z_1^{48}\\
&\quad - 53425006139366791877718 0z_1^{46} - 22697371614327942982482 0z_1^{44}\\
&\quad + 35163411862747300204671 9z_1^{42} + 20267484337367878596541 2z_1^{40}\\
&\quad - 13786218368531914165119 6z_1^{38} - 66609031213834294956984z_1^{36}\\
&\quad + 14373894746278711902588z_1^{34} + 16146925396460767595288z_1^{32}\\
&\quad + 2882275291034243971208z_1^{30} - 3666368186160886521384z_1^{28}\\
&\quad - 804492483513977076097z_1^{26} + 717504696640659860216z_1^{24}\\
&\quad - 42260280396861425 84z_1^{22} - 7452429117688338702 0z_1^{20}\\
&\quad + 1766375400414366942 6z_1^{18} + 2533990992640113580z_1^{16}\\
&\quad - 198486707993967607 6z_1^{14} + 164118424014051612z_1^{12}\\
&\quad + 57575503826637745z_1^{10} - 8891156550630028z_1^{8} + 353944700440404z_1^{6}\\
&\quad - 61689385722720z_1^{4} + 3576597966400z_1^{2} + 221457920000.
\end{aligned}
$$

Since $\operatorname{lcoeff}(L_1, n) = 3z_1^4(3z_1^2+1)$, r_0 and the first four factors of r_{13} do not vanish, it immediately follows that

$$
\begin{aligned}
V(L_1,L_2,L_3)\cap(D_{02}\cup D_{03}) &= V(L_1,L_2,L_3,r_1r_2,r_2r_3)\cap(D_{02}\cup D_{03})\\
&= V_1\cup V_2,
\end{aligned}
\tag{3.16}
$$

where

$$
V_1 := V(L_1,L_2,L_3,r_1,r_3)\cap(D_{02}\cup D_{03}),\quad V_2 := V(L_1,L_2,L_3,r_2)\cap(D_{02}\cup D_{03}).
$$

Next, we prove that varieties V_1 and V_2 are both empty. It follows that $V_1 = \emptyset$ since the resultant $r_2(3) := \operatorname{res}(r_1, r_3, z_1)$ is a nonzero constant. In order to prove that V_2 is also empty, we start to consider zeros of the single-variable function r_2. By the formulae of cubic roots, we see that r_2 has exactly two positive real zeros $z_3 \in (\frac{1}{\sqrt{3}}, 1)$ given before Theorem 2.2 and

$$
z_4 := \sqrt{\frac{1}{3}\{\frac{(\sqrt{3}i-1)^2}{4}\sqrt[3]{4\sqrt{107}i-212} - \frac{\sqrt{3}i-1}{2}\sqrt[3]{4\sqrt{107}i+212} - 5\}}
$$

$$\doteq 0.3268 \in (0, \frac{1}{\sqrt{3}}).$$

Moreover, applying the successive pseudo-division to equations $L_1 = 0$ and $L_2 = 0$ we get $\tilde{r}_1 := \mathrm{prem}(L_1, \tilde{r}_2, n) = g_1(z_1)n + g_2(z_1)$, where

$$\begin{aligned}
g_1(z_1) :=& -322486272z_1^{42}(54860818455z_1^{44} + 191256189930z_1^{42} - 695417174496z_1^{40}\\
&- 1869469732926z_1^{38} + 1107216015201z_1^{36} + 2083168072188z_1^{34}\\
&- 1080044334108z_1^{32} - 732537168696z_1^{30}\\
&+ 452206117494z_1^{28} + 83371542876z_1^{26}\\
&- 90652681152z_1^{24} + 7461668988z_1^{22} + 7797647922z_1^{20} - 2620184704z_1^{18}\\
&+ 103188776z_1^{16} + 154423848z_1^{14} - 40456509z_1^{12} + 1640426z_1^{10} + 750656z_1^{8}\\
&- 189742z_1^{6} + 26909z_1^{4} - 2220z_1^{2} + 84)(9z_1^4 - 1)^5(3z_1^2 + 1)^2(z_1^2 - 1)^9,\\
g_2(z_1) :=& -2579890176z_1^{43}(8972419005z_1^{42} + 39081041919z_1^{40} - 62229028491z_1^{38}\\
&- 276057764469z_1^{36} + 39576279564z_1^{34}\\
&+ 316654253208z_1^{32} - 43399997964z_1^{30}\\
&- 140520561588z_1^{28} + 28863674838z_1^{26} + 28568713986z_1^{24} - 8841136842z_1^{22}\\
&- 2292826998z_1^{20} + 1293682212z_1^{18} - 94033244z_1^{16} - 59582300z_1^{14}\\
&+ 24075036z_1^{12} - 3192171z_1^{10} - 484841z_1^{8} + 183997z_1^{6} - 23933z_1^{4} + 1960z_1^{2}\\
&- 84)(9z_1^4 - 1)^5(3z_1^2 + 1)^2(z_1^2 - 1)^{10},
\end{aligned}$$

$\tilde{r}_2 := \mathrm{prem}(L_2, L_1, n)$ and $\mathrm{prem}(\alpha, \beta, x)$ denotes the pseudo-remainder [23] of $\alpha(x)$ divided by $\beta(x)$. It is clear that $L_1 = L_2 = 0$ if and only if $\tilde{r}_1 = \tilde{r}_2 = 0$. From the equation $\tilde{r}_1 = 0$, we obtain the dependence of n on z_1

$$n = n_1 := -\frac{g_2(z_1)}{g_1(z_1)}.$$

For the case $z_1 = z_4$, we have $n = n_1 = -1.717978052 \notin I_1 \doteq (2.7333, 5.4665)$ with

$$I_1 := (\frac{1 - z_1^2}{z_1}, \frac{2(1 - z_1^2)}{z_1})$$

implying that L_1 and L_2 have no common zeros in the interval I_1. For the other case $z_1 = z_3$, the number of the zeros of polynomial L_1 in the interval

$$I_2 := (\frac{2(1 - z_1^2)}{z_1}, \frac{4z_1(1 - z_1^2)}{3z_1^2 - 1})$$

is equal to the number of the positive zeros of

$$\begin{aligned}
\Phi(k) :=& (1 + k)^3 L_1(\frac{2(1 - z_1^2)(3kz_1^2 + 2z_1^2 - k)}{(3z_1^2 - 1)(1 + k)z_1})\\
=& \frac{4(z_1^2 - 1)^2 k}{z_1(3z_1^2 - 1)^2}\{-(z_1^2 + 1)(3z_1^2 - 1)^4 k^2 - 4z_1^2(z_1^2 + 2)(3z_1^2 - 1)^3 k - 39z_1^{10}\\
&- 147z_1^8 + 114z_1^6 - 30z_1^4 + 5z_1^2 + 1\}
\end{aligned}$$

as indicated in [28]. We can check that all the coefficients of k in the bracket $\{\cdots\}$ are negative for $z_1 = z_3$ implying that $\Phi(k)$ has no positive zeros. Thus, L_1 has no zeros in the interval I_2 for $z_1 = z_3$. Summarily, we see that $V_2 = \emptyset$. Since the two varieties V_1 and V_2 are proved to be empty, we see that $V(L_1, L_2, L_3) \cap (D_{02} \cup D_{03}) = \emptyset$ from (3.16), which implies the claimed result that E_1 is a weak focus of multiplicity at most 3 for $(s, n, h) \in D_{02} \cup D_{03}$.

Then, we further claim that the multiplicity of weak focus E_1 can be up to 3 for $(s, n, h) \in D_{02} \cup D_{03}$. Using Lemma 2 in [4] again we similarly decompose the algebraic variety

$$V(L_1, L_2) \cap (D_{02} \cup D_{03}) = V(L_1, L_2, r_1 r_2) \cap (D_{02} \cup D_{03}) = V_3 \cup V_4,$$

where

$$V_3 := V(L_1, L_2, r_2) \cap (D_{02} \cup D_{03}), \quad V_4 := V(L_1, L_2, r_1) \cap (D_{02} \cup D_{03}).$$

Next, we can prove that variety V_3 is empty and variety V_4 is not empty, or more specifically, $V(L_1, L_2, r_1) \cap D_{03}$ is not empty. It immediately shows that $V_3 = \emptyset$ from the above proof of $V_2 = \emptyset$. Polynomial r_1 has two positive zeros $z_5 \doteq 0.5491 \in (0, \frac{1}{\sqrt{3}})$ and $z_6 \doteq 0.6815 \in (\frac{1}{\sqrt{3}}, 1)$. For $z_1 = z_5$, we obtain $n = n_1 \doteq -0.9026 \notin I_1 \doteq (1.2722, 2.5444)$ implying that L_1 and L_2 have no common zeros. Thus, $V(L_1, L_2, r_1) \cap D_{02} = \emptyset$. For $z_1 = z_6$, we obtain that $n = n_1 \doteq 2.0755 \in I_2 \doteq (1.5720, 3.7138)$ is the common zero of $\tilde{r}_1$ and $\tilde{r}_2$ implying that L_1 and L_2 have common zeros. In fact, substituting the expression of n_1 into $\tilde{r}_2$ we just obtain that r_1 is one of the factor of the numerator of $\tilde{r}_2$. Thus, $V(L_1, L_2, r_1) \cap D_{03} \neq \emptyset$, which actually is the set D_3 defined in Theorem 3.2. Consequently, the claimed result is proved. The proof of Theorem 3.2 is completed. □

4. Discussions

The basic idea of modeling is that while the predators are assumed to be harvested by humans and live independently of others, the prey instead gathers in herds. So that a Leslie-Gower type predator-prey system with herd behavior in prey and constant harvesting in predators is considered in this paper. We present the complete analysis on qualitative properties of equilibria and bifurcations around the non-hyperbolic ones in system (1.4). It is shown that system (1.4) undergoes a Bogdanov-Takens bifurcation of codimension three and a degenerate Hopf bifurcation of codimension three.

From the viewpoint of mathematics, the dynamics of system (1.4) are much more complex than the dynamics of the classic Leslie-Gower model because the latter only has a unique globally asymptotically stable positive equilibrium. The reason for this difference in dynamics between them undoubtedly comes down to the additional herd behavior and constant predator harvesting. By comparing the dynamics of systems (1.4) and (1.1), both of which take the herd behavior into account, we obtain that the constant predator harvesting is responsible for the more positive equilibria, the Bogdanov-Takens bifurcation and the degenerate Hopf bifurcation since the latter only has one positive equilibrium and undergoes the Hopf bifurcation. By continuing to compare the dynamics of systems (1.4) and (1.3), both of which take the constant predator harvesting into account, we find that the herd behavior is what causes the degenerate Hopf bifurcation of codimension three since the latter has a weak focus with multiplicity at most two. The above series of dynamic comparisons show that the constant predator harvesting has a greater impact on the dynamics than the herd behavior, because the constant predator harvesting is the cause of the more positive equilibria and the Bogdanov-Takens and degenerate Hopf bifurcations, while the herd behavior merely increases the multiplicity of the weak focus. From the ecological viewpoint, the herd behavior and the constant predator harvesting actually affect the coexistence of the prey and the predators. When the intensity of the predator harvesting is relatively high, i.e., $h > h_1$, the predators go extinct since system (1.4) has no positive equilibrium, which results in all solutions of system (1.4) cross the x-axis and leave the region Ω in finite time. This phenomenon results from the overexploitation of predator population. When the intensity of the predator harvesting is not too high, i.e., $h < h_1$, system (1.4) has the stable positive equilibrium and even stable periodic solutions. Therefore, there exist some parameter values and initial values such that the predators and prey can coexist. Moreover, the herd behavior in prey can increase the probability of coexistence of predators and prey because the herd behavior can increase the number of limit cycle induced by the Hopf bifurcation. Some numerical examples are given to illustrate the rich dynamics of system (1.4) and exhibit the coexistence of predators and prey (see Figure 4). System (1.4) has an unstable focus E_1 surrounded by a stable limit cycle when $s = 0.78$, $n = 3.308$ and $h = 0.0807$ (see Figure 4 (a)). System (1.4) has a stable focus E_1 surrounded by two limit cycles (the inner one is unstable and the outer one is stable) when $s = 0.78$, $n = 4.995$ and $h = 0.0403$ (see Figure 4 (b)). System (1.4) has an unstable focus E_1 surrounded by a homoclinic orbit when $s = 0.78$, $n = 2.09799$ and $h = 0.18$ (see Figure 4 (c)). System (1.4) has a stable focus E_1 surrounded by an unstable limit cycle and a homoclinic orbit when $s = 0.78$, $n = 5.0475$ and $h = 0.0403$ (see Figure 4 (d)).

The research results show that system (1.4) has richer dynamics compared to the system without constant predator harvesting and different dynamics compared to the system without herd behavior in prey. However, the too great predator harvesting intensity can lead to the extinction of predators due to the overharvesting except for the moderate harvesting intensity. The complex dynamics indicate the coexistence of positive equilibria, limit cycles or homoclinic orbit, which also reveal the complexity of the interaction between the prey, predators and humans.

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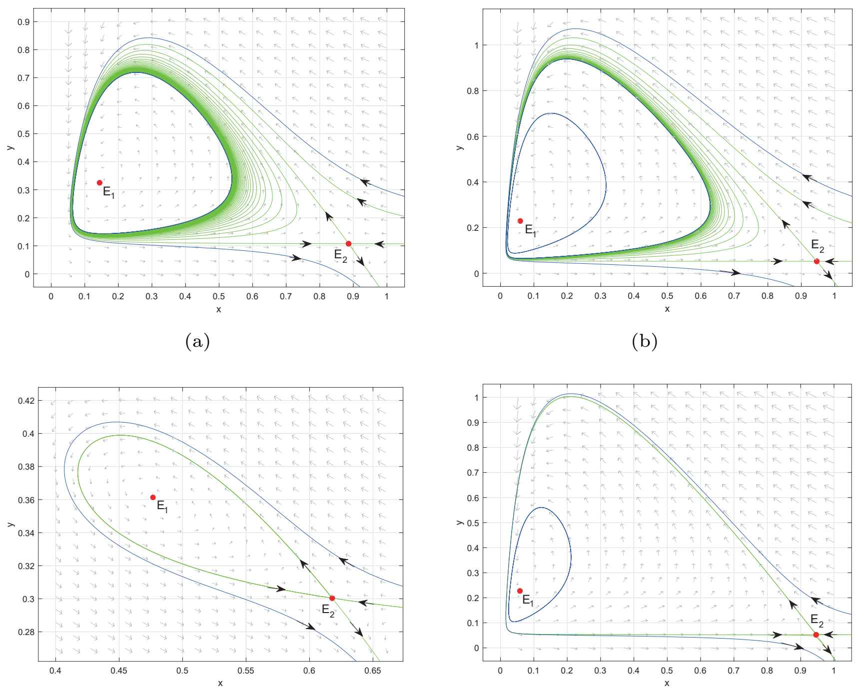
Appendix

Coefficients $\hat{a}_{ij}$ and $\hat{b}_{ij}$ in system (3.11) are defined as follows:

$$\hat{a}_{70} := \frac{n^6(3105z_1^2 - 1057)}{2048s^6(z_1^2-1)^{12}w}, \quad \hat{a}_{60} := -\frac{n^5(1557z_1^2 - 533)}{1024s^5(z_1^2-1)^{10}w}, \quad \hat{a}_{50} := \frac{n^4(391z_1^2 - 135)}{256s^4(z_1^2-1)^8 w},$$

$$\hat{a}_{61} := -\frac{n^5\{7z_1(3z_1^2-1)(3105z_1^2-1057)n - 4s(z_1^2-1)(3093z_1^2-1045)\}}{4096z_1s^6w^2(z_1^2-1)^{12}},$$



(a) (b)

(c) (d)

Figure 4. Limit cycles and homoclinic orbit of system (1.4): (a) a stable limit cycle when $s = 0.78$, $n = 3.308$ and $h = 0.0807$; (b) two limit cycles (the inner one is unstable and the outer one is stable) when $s = 0.78$, $n = 4.995$ and $h = 0.0403$; (c) a homoclinic orbit when $s = 0.78$, $n = 2.09799$ and $h = 0.18$; (d) an unstable limit cycle and a homoclinic orbit when $s = 0.78$, $n = 5.0475$ and $h = 0.0403$.

$$\hat{a}_{20} := -\frac{n(21z_1^2-5)}{8s(z_1^2-1)^2w},$$

$$\hat{a}_{52} := \frac{n^4(3z_1^2-1)\{21z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-24sz_1(z_1^2-1)(3093z_1^2-1045)n+4096s^2(z_1^2-1)^2\}}{8192s^6w^3(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{43} := -\frac{5n^4(3z_1^2-1)^2\{7z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-12sz_1(z_1^2-1)(3093z_1^2-1045)n+4096s^2(z_1^2-1)^2\}}{16384s^6w^4(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{34} := \frac{5n^4(3z_1^2-1)^3\{7z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-16sz_1(z_1^2-1)(3093z_1^2-1045)n+8192s^2(z_1^2-1)^2\}}{32768s^6w^5(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{25} := -\frac{n^4(3z_1^2-1)^4\{21z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-60sz_1(z_1^2-1)(3093z_1^2-1045)n+40960s^2(z_1^2-1)^2\}}{65536s^6w^6(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{16} := \frac{n^4(3z_1^2-1)^5\{7z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-24sz_1(z_1^2-1)(3093z_1^2-1045)n+20480s^2(z_1^2-1)^2\}}{131072s^6w^7(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{07} := -\frac{n^4(3z_1^2-1)^6\{z_1^2(3z_1^2-1)(3105z_1^2-1057)n^2-4sz_1(z_1^2-1)(3093z_1^2-1045)n+4096s^2(z_1^2-1)^2\}}{262144s^6w^8(z_1^2-1)^{12}z_1^2},$$

$$\hat{a}_{51} := \frac{n^4\{3z_1(3z_1^2-1)(1557z_1^2-533)n-4s(z_1^2-1)(775z_1^2-263)\}}{1024z_1s^5w^2(z_1^2-1)^{10}},$$

$$\hat{a}_{30} := \frac{n^2(25z_1^2-9)}{16s^2(z_1^2-1)^4w},$$

$$\hat{a}_{42} := -\frac{n^3(3z_1^2-1)\{15z_1^2(3z_1^2-1)(1557z_1^2-533)n^2-40sz_1(z_1^2-1)(775z_1^2-263)n+2048s^2(z_1^2-1)^2\}}{4096s^5w^3(z_1^2-1)^{10}z_1^2},$$

$$\hat{a}_{33} := \frac{n^3(3z_1^2-1)^2\{5z_1^2(3z_1^2-1)(1557z_1^2-533)n^2-20sz_1(z_1^2-1)(775z_1^2-263)n+2048s^2(z_1^2-1)^2\}}{2048s^5w^4(z_1^2-1)^{10}z_1^2},$$

$$\hat{a}_{24} := -\frac{n^3(3z_1^2-1)^3\{15z_1^2(3z_1^2-1)(1557z_1^2-533)n^2-80sz_1(z_1^2-1)(775z_1^2-263)n+12288s^2(z_1^2-1)^2\}}{16384s^5w^5(z_1^2-1)^{10}z_1^2},$$

$$\hat{a}_{15} := \frac{n^3(3z_1^2-1)^4\{3z_1^2(3z_1^2-1)(1557z_1^2-533)n^2-20sz_1(z_1^2-1)(775z_1^2-263)n+4096s^2(z_1^2-1)^2\}}{16384s^5w^6(z_1^2-1)^{10}z_1^2},$$

$$\hat{a}_{06} := -\frac{n^3(3z_1^2-1)^5\{z_1^2(3z_1^2-1)(1557z_1^2-533)n^2-8sz_1(z_1^2-1)(775z_1^2-263)n+2048s^2(z_1^2-1)^2\}}{65536s^5w^7(z_1^2-1)^{10}z_1^2},$$

$$\hat{a}_{41} := -\frac{n^3\{5z_1(391z_1^2-135)(3z_1^2-1)n-4s(z_1^2-1)(389z_1^2-133)\}}{512z_1s^4w^2(z_1^2-1)^8},$$

$$\hat{a}_{40} := -\frac{n^3(197z_1^2-69)}{128s^3(z_1^2-1)^6w},$$

$$\hat{a}_{32} := \frac{n^2(3z_1^2-1)\{5z_1^2(391z_1^2-135)(3z_1^2-1)n^2-8sz_1(z_1^2-1)(389z_1^2-133)n+256s^2(z_1^2-1)^2\}}{512s^4w^3(z_1^2-1)^8z_1^2},$$

$$\hat{a}_{23} := -\frac{n^2(3z_1^2-1)^2\{5z_1^2(391z_1^2-135)(3z_1^2-1)n^2-12sz_1(z_1^2-1)(389z_1^2-133)n+768s^2(z_1^2-1)^2\}}{1024s^4w^4(z_1^2-1)^8z_1^2},$$

$$\hat{a}_{14} := \frac{n^2(3z_1^2-1)^3\{5z_1^2(391z_1^2-135)(3z_1^2-1)n^2-16sz_1(z_1^2-1)(389z_1^2-133)n+1536s^2(z_1^2-1)^2\}}{4096s^4w^5(z_1^2-1)^8z_1^2},$$

$$\hat{a}_{05} := -\frac{n^2(3z_1^2-1)^4\{z_1^2(391z_1^2-135)(3z_1^2-1)n^2-4sz_1(z_1^2-1)(389z_1^2-133)n+512s^2(z_1^2-1)^2\}}{8192s^4w^6(z_1^2-1)^8z_1^2},$$

$$\hat{a}_{31} := \frac{n^2\{z_1(3z_1^2-1)(197z_1^2-69)n-4s(z_1^2-1)(49z_1^2-17)\}}{64z_1s^3w^2(z_1^2-1)^6},$$

$$\hat{b}_{61} := -\frac{n^5(21nz_1^3-4sz_1^2-7nz_1+4s)}{2s^6(z_1^2-1)^{12}wz_1},$$

$$\hat{a}_{22} := -\frac{n(3z_1^2-1)\{3z_1^2(3z_1^2-1)(197z_1^2-69)n^2-24sz_1(z_1^2-1)(49z_1^2-17)n+128s^2(z_1^2-1)^2\}}{256s^3w^3(z_1^2-1)^6z_1^2},$$

$$\hat{a}_{13} := \frac{n(3z_1^2-1)^2\{z_1^2(3z_1^2-1)(197z_1^2-69)n^2-12sz_1(z_1^2-1)(49z_1^2-17)n+128s^2(z_1^2-1)^2\}}{256s^3w^4(z_1^2-1)^6z_1^2},$$

$$\hat{a}_{04} := -\frac{n(3z_1^2-1)^3\{z_1^2(3z_1^2-1)(197z_1^2-69)n^2-16sz_1(z_1^2-1)(49z_1^2-17)n+256s^2(z_1^2-1)^2\}}{2048s^3w^5(z_1^2-1)^6z_1^2},$$

$$\hat{a}_{21} := -\frac{(9nz_1^3-4sz_1^2-3nz_1+4s)n(25z_1^2-9)}{32z_1s^2w^2(z_1^2-1)^4},$$

$$\hat{a}_{11} := \frac{z_1(21z_1^2-5)(3z_1^2-1)n-4s(z_1^2-1)(7z_1^2-3)}{8z_1sw^2(z_1^2-1)^2},$$

$$\hat{a}_{12} := \frac{(3z_1^2-1)\{3z_1^2(25z_1^2-9)(3z_1^2-1)n^2-8sz_1(z_1^2-1)(25z_1^2-9)n+32s^2(z_1^2-1)^2\}}{64s^2w^3(z_1^2-1)^4z_1^2},$$

$$\hat{a}_{03} := -\frac{(3z_1^2-1)^2\{z_1^2(25z_1^2-9)(3z_1^2-1)n^2-4sz_1(z_1^2-1)(25z_1^2-9)n+32s^2(z_1^2-1)^2\}}{128s^2w^4(z_1^2-1)^4z_1^2},$$

$$\hat{a}_{02} := -\frac{(3z_1^2-1)\{z_1^2(21z_1^2-5)(3z_1^2-1)n^2-8sz_1(z_1^2-1)(7z_1^2-3)n+16s^2(z_1^2-1)^2\}}{32sw^3(z_1^2-1)^2z_1^2n},$$

$$\hat{b}_{70} := \frac{n^6}{s^6(z_1^2-1)^{12}},$$

$$\hat{b}_{52} := \frac{n^4\{21z_1^2(3z_1^2-1)^2n^2-24sz_1(z_1^2-1)(3z_1^2-1)n+4s^2(z_1^2-1)^2\}}{4s^6w^2(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{43} := -\frac{(5(3z_1^2-1))n^4\{7z_1^2(3z_1^2-1)^2n^2-12sz_1(z_1^2-1)(3z_1^2-1)n+4s^2(z_1^2-1)^2\}}{8s^6w^3(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{34} := \frac{5(3z_1^2-1)^2n^4\{7z_1^2(3z_1^2-1)^2n^2-16sz_1(z_1^2-1)(3z_1^2-1)n+8s^2(z_1^2-1)^2\}}{16s^6w^4(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{25} := -\frac{(3z_1^2-1)^3n^4\{21z_1^2(3z_1^2-1)^2n^2-60sz_1(z_1^2-1)(3z_1^2-1)n+40s^2(z_1^2-1)^2\}}{32s^6w^5(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{16} := \frac{(3z_1^2-1)^4n^4(3nz_1^3-2sz_1^2-nz_1+2s)(21nz_1^3-10sz_1^2-7nz_1+10s)}{64s^6w^6(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{60} := -\frac{n^5}{s^5(z_1^2-1)^{10}},$$

$$\hat{b}_{42} := -\frac{n^3\{15z_1^2(3z_1^2-1)^2n^2-20sz_1(z_1^2-1)(3z_1^2-1)n+4s^2(z_1^2-1)^2\}}{4s^5w^2(z_1^2-1)^{10}z_1^2},$$

$$\hat{b}_{33} := \frac{(3z_1^2-1)n^3\{5z_1^2(3z_1^2-1)^2n^2-10sz_1(z_1^2-1)(3z_1^2-1)n+4s^2(z_1^2-1)^2\}}{2s^5w^3(z_1^2-1)^{10}z_1^2},$$

$$\hat{b}_{24} := -\frac{(3z_1^2-1)^2n^3\{15z_1^2(3z_1^2-1)^2n^2-40sz_1(z_1^2-1)(3z_1^2-1)n+24s^2(z_1^2-1)^2\}}{16s^5w^4(z_1^2-1)^{10}z_1^2},$$

$$\hat{b}_{15} := \frac{(3z_1^2-1)^3n^3(3nz_1^3-2sz_1^2-nz_1+2s)(9nz_1^3-4sz_1^2-3nz_1+4s)}{16s^5w^5(z_1^2-1)^{10}z_1^2},$$

$$\hat{b}_{06} := -\frac{(3z_1^2-1)^4n^3(3nz_1^3-2sz_1^2-nz_1+2s)^2}{64s^5w^6(z_1^2-1)^{10}z_1^2},$$

$$\hat{b}_{41} := -\frac{n^3(15nz_1^3-4sz_1^2-5nz_1+4s)}{2s^4(z_1^2-1)^8wz_1}, \quad \hat{b}_{51} := \frac{n^4(9nz_1^3-2sz_1^2-3nz_1+2s)}{s^5(z_1^2-1)^{10}(z_1+1)^{10}wz_1},$$

$$\hat{b}_{32} := \frac{n^2\{5z_1^2(3z_1^2-1)^2n^2-8sz_1(z_1^2-1)(3z_1^2-1)n+2s^2(z_1^2-1)^2\}}{2s^4w^2(z_1^2-1)^8z_1^2},$$

$$\hat{b}_{23} := -\frac{(3z_1^2-1)n^2\{5z_1^2(3z_1^2-1)^2n^2-12sz_1(z_1^2-1)(3z_1^2-1)n+6s^2(z_1^2-1)^2\}}{4s^4w^3(z_1^2-1)^8z_1^2},$$

$$\hat{b}_{14} := \frac{(3z_1^2-1)^2n^2(15nz_1^3-6sz_1^2-5nz_1+6s)(3nz_1^3-2sz_1^2-nz_1+2s)}{16s^4w^4(z_1^2-1)^8z_1^2},$$

$$\hat{b}_{50} := \frac{n^4}{s^4(z_1^2-1)^8}, \quad \hat{b}_{05} := -\frac{(3z_1^2-1)^3n^2(3nz_1^3-2sz_1^2-nz_1+2s)^2}{32s^4w^5(z_1^2-1)^8z_1^2},$$

$$\hat{b}_{07} := -\frac{(3z_1^2-1)^5n^4(3nz_1^3-2sz_1^2-nz_1+2s)^2}{128s^6w^7(z_1^2-1)^{12}z_1^2},$$

$$\hat{b}_{22} := -\frac{n\{3z_1^2(3z_1^2-1)^2n^2-6sz_1(z_1^2-1)(3z_1^2-1)n+2s^2(z_1^2-1)^2\}}{2s^3w^2(z_1^2-1)^6z_1^2},$$

$$\hat{b}_{40} := -\frac{n^3}{s^3(z_1^2-1)^6},$$

$$\hat{b}_{13} := \frac{(3z_1^2-1)n(3nz_1^3-sz_1^2-nz_1+s)(3nz_1^3-2sz_1^2-nz_1+2s)}{2s^3w^3(z_1^2-1)^6z_1^2},$$

$$\hat{b}_{31} := \frac{2n^2(3nz_1^3-sz_1^2-nz_1+s)}{s^3(z_1^2-1)^6wz_1},$$

$$\hat{b}_{04} := -\frac{(3z_1^2-1)^2n(3nz_1^3-2sz_1^2-nz_1+2s)^2}{16s^3w^4(z_1^2-1)^6z_1^2},$$

$$\hat{b}_{30} := \frac{n^2}{s^2(z_1^2-1)^4}, \quad \hat{b}_{21} := -\frac{n(9nz_1^3-4sz_1^2-3nz_1+4s)}{2s^2(z_1^2-1)^4wz_1},$$

$$\hat{b}_{12} := \frac{(3nz_1^3 - 2sz_1^2 - nz_1 + 2s)(9nz_1^3 - 2sz_1^2 - 3nz_1 + 2s)}{4w^2(z_1^2-1)^4 z_1^2 s^2},$$

$$\hat{b}_{03} := -\frac{(3z_1^2-1)(3nz_1^3 - 2sz_1^2 - nz_1 + 2s)^2}{8s^2w^3(z_1^2-1)^4 z_1^2}, \quad \hat{b}_{20} := -\frac{n}{s(z_1^2-1)^2},$$

$$\hat{b}_{11} := \frac{3nz_1^3 - 2sz_1^2 - nz_1 + 2s}{z_1(z_1^2-1)^2 ws}, \quad \hat{b}_{02} := -\frac{(3nz_1^3 - 2sz_1^2 - nz_1 + 2s)^2}{4w^2nz_1^2(z_1^2-1)^2 s}.$$

Functions L_i $(i = 2, 3)$ used in system (3.13) are defined as follows:

$$\begin{aligned}
L_2 :=\, & 108z_1^8(3z_1^2+1)(143z_1^4 - 95z_1^2 + 16)(3z_1^2-1)^2n^8 + 18z_1^5(3z_1^2-1)(31548z_1^{12} \\
& - 72169z_1^{10} + 36444z_1^8 - 2982z_1^6 - 912z_1^4 - 185z_1^2 + 64)n^7 - 3z_1^4(z_1^2-1) \\
& \times (141597z_1^{14} + 2493762z_1^{12} - 2814297z_1^{10} + 1219352z_1^8 - 331845z_1^6 + 98042z_1^4 \\
& - 22047z_1^2 + 1868)n^6 - 2z_1^3(8188425z_1^{14} - 1456839z_1^{12} - 5695419z_1^{10} \\
& + 4824021z_1^8 - 2037109z_1^6 + 581659z_1^4 - 95513z_1^2 + 5911)(z_1^2-1)^2n^5 \\
& - 2z_1^2(24953319z_1^{14} - 16197615z_1^{12} - 4200453z_1^{10} + 8978389z_1^8 - 4436395z_1^6 \\
& + 1159491z_1^4 - 150647z_1^2 + 6935)(z_1^2-1)^3n^4 - 4z_1(21080169z_1^{14} \\
& - 15344343z_1^{12} - 942051z_1^{10} + 5922861z_1^8 - 2926549z_1^6 \\
& + 661771z_1^4 - 65201z_1^2 + 1951)(z_1^2-1)^4n^3 - 8(11011059z_1^{14} \\
& - 7491447z_1^{12} - 567501z_1^{10} + 2649841z_1^8 - 1172031z_1^6 + 212515z_1^4 - 14119z_1^2 \\
& + 163)(z_1^2-1)^5n^2 - 96z_1(545049z_1^{12} - 312258z_1^{10} - 70389z_1^8 + 116420z_1^6 \\
& - 41857z_1^4 + 5246z_1^2 - 163)(z_1^2-1)^6n - 256z_1^2(52083z_1^{10} - 22347z_1^8 - 12768z_1^6 \\
& + 9200z_1^4 - 2491z_1^2 + 163)(z_1^2-1)^7, \\
L_3 :=\, & 5184z_1^{12}(3z_1^2+1)(12458469z_1^8 - 19967454z_1^6 + 11284531z_1^4 - 2725616z_1^2 \\
& + 240310)(3z_1^2-1)^4n^{13} + 864z_1^9(3317163309z_1^{16} - 12238490586z_1^{14} \\
& + 14130886647z_1^{12} - 6192615540z_1^{10} + 624151491z_1^8 + 242248838z_1^6 \\
& - 45161631z_1^4 - 4294488z_1^2 + 961240)(3z_1^2-1)^3n^{12} \\
& - 36z_1^8(z_1^2-1)(989070410907z_1^{18} - 47023537680z_1^{16} - 2975856806592z_1^{14} \\
& + 3005913369720z_1^{12} - 1228301999562z_1^{10} + 282290712576z_1^8 - 64702055832z_1^6 \\
& + 17353749848z_1^4 - 2825698681z_1^2 + 174322496)(3z_1^2-1)^2n^{11} \\
& - 24z_1^7(45695865082014z_1^{20} - 103941957376728z_1^{18} + 58740703027767z_1^{16} \\
& + 9869253552918z_1^{14} - 24043414935042z_1^{12} + 12652798787838z_1^{10} \\
& - 4247095158576z_1^8 + 1074838478274z_1^6 - 188164131372z_1^4 + 18744599298z_1^2 \\
& - 762674551)(3z_1^2-1)(z_1^2-1)^2n^{10} - 3z_1^6(2587195409565429z_1^{22} \\
& - 10291960983438126z_1^{20} + 11585581627742181z_1^{18} - 5358763410353568z_1^{16} \\
& + 496981774162314z_1^{14} + 660280202775468z_1^{12} - 409680382037742z_1^{10} \\
& + 134913752252584z_1^8 - 28786741669663z_1^6 + 3835647297810z_1^4 \\
& - 278282259543z_1^2 + 7944498536)(z_1^2-1)^3n^9 + 2z_1^5(4433154447117708z_1^{22} \\
& + 17142916834885635z_1^{20} - 34009861529459646z_1^{18} + 23688703580530635z_1^{16} \\
& - 8203163972842944z_1^{14} + 1219544166962502z_1^{12} + 187778010226332z_1^{10}
\end{aligned}$$

$$
\begin{aligned}
&- 148214936757714z_1^8 + 36900369153604z_1^6 - 4608055410025z_1^4 \\
&+ 257739136274z_1^2 - 3569553369) \times (z_1^2-1)^4 n^8 + 2z_1^4(48282047584509717z_1^{22} \\
&- 37067008721622003z_1^{20} - 19862618155792497z_1^{18} + 38225865970680327z_1^{16} \\
&- 23420248140748158z_1^{14} + 8370454699108530z_1^{12} - 1915895036120946z_1^{10} \\
&+ 277501229287678z_1^8 - 25723827700295z_1^6 + 2046643459985z_1^4 \\
&- 179219152749z_1^2 + 8107116715) \times (z_1^2-1)^5 n^7 + 8z_1^3(34285798924318017z_1^{22} \\
&- 42900228139283514z_1^{20} + 11750464748697921z_1^{18} + 10449223300971288z_1^{16} \\
&- 10785848993793006z_1^{14} + 4760663238173652z_1^{12} - 1261013585350374z_1^{10} \\
&+ 215989248067544z_1^8 - 24929669786547z_1^6 + 2011537619862z_1^4 \\
&- 105506070523z_1^2 + 2400207328) \times (z_1^2-1)^6 n^6 + 8z_1^2(56547709919403147z_1^{22} \\
&- 78844225140026685z_1^{20} + 31187999490255009z_1^{18} + 9538280097146505z_1^{16} \\
&- 14376772806062082z_1^{14} + 6565577529675502z_1^{12} - 1688816097432430z_1^{10} \\
&+ 272149619344802z_1^8 - 28521699091033z_1^6 + 1933756677119z_1^4 \\
&- 74089523043z_1^2 + 993689125) \times (z_1^2-1)^7 n^5 + 32z_1(15212813518875267z_1^{22} \\
&- 21789630186493770z_1^{20} + 8963372703489111z_1^{18} + 2174380511542128z_1^{16} \\
&- 3477698258666058z_1^{14} + 1494291910928996z_1^{12} - 348650238490282z_1^{10} \\
&+ 49315385127672z_1^8 - 4348677019065z_1^6 + 229630652310z_1^4 - 5830020141z_1^2 \\
&+ 36446952)(z_1^2-1)^8 n^4 + 32(10890025177930371z_1^{22} - 15507309608572419z_1^{20} \\
&+ 5995880738452629z_1^{18} + 1785617243840691z_1^{16} - 2352140417467554z_1^{14} \\
&+ 896777998461922z_1^{12} - 180586236274678z_1^{10} + 21134723181958z_1^8 \\
&- 1451651768593z_1^6 + 52932322449z_1^4 - 684859759z_1^2 + 1234775)(z_1^2-1)^9 n^3 \\
&+ 512z_1(314099653794597z_1^{20} - 437290158832848z_1^{18} + 148473464303313z_1^{16} \\
&+ 64801162978632z_1^{14} - 65081138021202z_1^{12} + 21080940656096z_1^{10} \\
&- 3462527130682z_1^8 + 308596444712z_1^6 - 14500231955z_1^4 + 278764416z_1^2 \\
&- 1234775)(z_1^2-1)^{10} n^2 + 512(84824406877563z_1^{18} - 114432526193517z_1^{16} \\
&+ 31711081166172z_1^{14} + 21869748125172z_1^{12} - 16775797792374z_1^{10} \\
&+ 4405249129618z_1^8 - 537081258772z_1^6 + 30975169844z_1^4 - 737987149z_1^2 \\
&+ 6173875)z_1^2(z_1^2-1)^{11} n + 4096z_1^3(1274725515645z_1^{16} - 1661125718682z_1^{14} \\
&+ 337134836556z_1^{12} + 399122967042z_1^{10} - 236485845318z_1^8 \\
&+ 46584553930z_1^6 - 3304302188z_1^4 + 81000718z_1^2 - 1234775)(z_1^2-1)^{12}.
\end{aligned}
$$

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