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# GLOBAL DYNAMICS OF A REACTION-DIFFUSION SEIVQR EPIDEMIC MODEL IN ALMOST PERIODIC ENVIRONMENTS*

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**Abstract** We have formulated an almost periodic reaction-diffusion SEIVQR epidemic model that incorporates quarantine, vaccination, and a latent period. In contrast to prior methods that analyze stability by using Lyapunov functions, we establish the global threshold dynamics of this model by using the upper Lyapunov exponent $\lambda^*$. Our results demonstrate that the disease-free almost periodic equilibrium is globally asymptotically stable if $\lambda^* < 0$, whereas the disease uniformly persists if $\lambda^* > 0$. To further validate our conclusions, we conducted numerical simulations of the model.

**Keywords** Reaction-diffusion, almost periodicity, upper Lyapunov exponent, threshold dynamics.

**MSC(2010)** 35K57, 35B35, 35B40, 92D30.

## 1. Introduction

Infectious diseases have been a public health challenge for centuries. The recent COVID-19 pandemic has reminded us of their destructive potential. To reduce the risks of such outbreaks, it's essential to conduct thorough research on strategies to prevent and control infectious diseases. Understanding the dynamics of infectious diseases, which is a crucial tool for studying how epidemics unfold, provides a solid foundation for managing their spread. The original model proposed by Kermack and McKendrick in [20] paved the way for studying infectious diseases using ordinary differential equation models. Their comprehensive consideration of multiple factors made the model more realistic. Subsequently, several models such as SIS (susceptible-infectious-susceptible) models in [5, 9, 12, 19, 23], SIR (susceptible-infectious-recovered) models in [2, 6, 21, 30], and SEIR (susceptible-exposed-infectious-recovered) models in [4, 15, 35, 41, 42], have been proposed to study infectious disease dynamics. These models have been widely used to study various infectious diseases. Furthermore, other models that incorporate more realistic factors have also been proposed, as in [3, 7, 13, 17, 18, 22, 24, 26, 27, 33, 43, 45, 46].

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We know that many infectious diseases have an incubation period, such as AIDS and hepatitis C. Therefore, it is necessary to consider the latency period in the model. A delayed SEIRS model has been propoed by Cooke and Van Den Driessche in 1996. They investigated the stability of the disease free proportion equilibrium, and the stability of an endemic proportion equilibrium [8]. Recently, Qiang et al. [32] studied an almost periodic reaction-diffusion epidemic model with incubation period. It is shown that the disease-free almost periodic solution is globally attractive if $\lambda^* < 0$, while the disease is persistent if $\lambda^* > 0$ ($\lambda^*$ is the upper Lyapunov exponent).

In order to better prevent and control infectious diseases, it is important to consider the effects of vaccination and quarantine simultaneously in modeling disease transmission. Zhu [49] previously established a SVIR model with relapse and spatially heterogeneous environment, taking vaccination into consideration. However, in reality, vaccinated individuals may still become susceptible again due to factors such as partial immunity decline caused by the vaccine. Additionally, isolating infected individuals is crucial in controlling the spread of infectious diseases. The spread of infectious diseases is also influenced by the environment, such as seasonal variations in prevalence. Due to the diverse and variable nature of the external environment, the parameters in epidemic models may not necessarily be periodic, and an almost periodic function may be more appropriate to describe them. Many models with almost periodic parameters have been established and studied [25,39,40,48]. For example, in 2015, Wang et al. [36] proposed a susceptible-infected-susceptible almost periodic reaction-diffusion epidemic model and obtained asymptotic behaviors with respect to the diffusion rate of infected individuals, as well as presenting uniform persistence, extinction, and global attractivity.

In this work, we formulate a SEIVQR model with a latent period, quarantine, and vaccination strategies, and investigate its global dynamics. The article is organized as follows: In Section 2, we derive the model and explain the parameters. In Section 3, we primarily establish the non-negativity and uniform boundedness of the solution of (2.6). Section 4 presents the upper Lyapunov exponent $\lambda^*$ for the almost periodic reaction-diffusion equation (4.1) and discusses the threshold dynamics for the model in terms of $\lambda^*$. Finally, in the last section, we provide numerical simulations to validate our conclusions.

## 2. Model formulation

In this section, we proposed an almost periodic reaction-diffusion epidemic model with latent period, quarantine and vaccinated strategy. Assume that the population lives in a spatially and temporally heterogeneous environment. For convenience, we introduce some notations. In the following part of this paper, let $g : \mathbb{R} \times \Omega \to \mathbb{R}$ be a continuous bounded function with $\Omega \subseteq \mathbb{R}^3$, which is a bounded domain with smooth boundary $\partial\Omega$. Then $g^*$ and $g_*$ will be defined as

$$g^* = \sup_{(t,x)\in\mathbb{R}\times\Omega} g(t,x),\ g_* = \inf_{(t,x)\in\mathbb{R}\times\Omega} g(t,x).$$

**Definition 2.1.** [10, 16] A continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be almost periodic on $\mathbb{R}$ if for any $\epsilon > 0$, the set

$$T(f,\epsilon) \equiv \{\tau \in \mathbb{R} :\ |f(t+\tau) - f(t)| < \epsilon,\ t \in \mathbb{R}\}$$

is relatively dense in $\mathbb{R}$. i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$ with the property that for any interval $L$ with length $l(\epsilon)$ such that $L \cap T(f, \epsilon) \neq \emptyset$.

Let $S(t,x), E(t,x), I(t,x), V(t,x), Q(t,x), R(t,x)$ represent the number of susceptible, latent, infectious, vaccinated, quarantine, recovered individuals at time $t$ and location $x$, respectively. Let $L(t,a,x)$ denote the number of infected population with infection age $a$ at time $t$ and location $x$. We assume that all populations remain confined to the region $\Omega$ for all time, and subject to no flux boundary condition for $L(t,a,x)$:

$$[D(a)\nabla L(t,a,x)] \cdot \mathbf{n} = 0, t > 0, x \in \partial\Omega,$$

where $\nabla L(t,a,x)$ is the gradient of $L(t,a,x)$ with respect to the spatial variable $x$, $D(a)$ is the diffusion rate at infection age $a$, $\mathbf{n}$ is the outward normal to $\partial\Omega$. Let $q(a)$ be the quarantined rate at infection age $a$, because the isolated people will not spread, combined with the standard arguments on structured population with spatial diffusion [31], we obtain

$$\begin{aligned}\frac{\partial L(t,a,x)}{\partial t} + \frac{\partial L(t,a,x)}{\partial a} =& \nabla \cdot [D(a)\nabla(1-q(a))L(t,a,x)] \\ &- (r(t,a,x) + q(a) + d(t,x))L(t,a,x), \end{aligned} \tag{2.1}$$

where $\nabla \cdot [D(a)\nabla L(t,a,x)]$ denotes the divergence of $D(a)\nabla L(t,a,x)$. The meaning of the parameters $r(t,a,x), q(a), \alpha(t,x), d(t,x)$ are described in Table 1.

**Table 1.** Description of parameters

| Parameter | Description |
|---|---|
| $S(x,t)$ | Number of susceptible people |
| $E(x,t)$ | Number of latent people |
| $I(x,t)$ | Number of infected people |
| $V(x,t)$ | Number of vaccinated people |
| $Q(x,t)$ | Number of quarantined people |
| $R(x,t)$ | Number of temporarily restored people |
| $r(t,a,x)$ | Recovery rate with infection age $a$ |
| $q(a)$ | Quarantined rate at infection age $a$ |
| $\alpha(x,t)$ | Vaccination rate |
| $d(t,x)$ | Natural death rate |
| $\Lambda(t,x)$ | Recruitment rate of the population |
| $\beta_1(t,x)$ | Addition rate from $S$ to infected individuals |
| $\beta_2(t,x)$ | Addition rate from $V$ to infected individuals |

Assume that the average incubation period is $\tau$. Then it follows that

$$E(t,x) = \int_0^\tau L(t,a,x)\,da, \ I(t,x) = \int_\tau^\infty L(t,a,x)\,da. \tag{2.2}$$

We make some assumptions for function $D(a), q(a), r(t,a,x)$ as follows:

$$D(a) = \begin{cases} D_E, & a \in [0,\tau), \\ D_I, & a \in [\tau, +\infty), \end{cases}$$

$$q(a) = \begin{cases} q_E, & a \in [0,\tau), \\ q_I, & a \in [\tau, +\infty), \end{cases} \qquad r(t,a,x) = \begin{cases} r_E(t,x), & a \in [0,\tau), \ t \geq 0, \ x \in \Omega, \\ r_I(t,x), & a \in [\tau, +\infty), \ t \geq 0, \ x \in \Omega, \end{cases}$$

where $D_E, D_I, q_E, q_I$ are all positive constants. Integrating both sides of (2.1) form 0 to $\tau$. It then follows from (2.2) that

$$\frac{\partial E(t,x)}{\partial t} = D_E(1-q_E)\Delta E(t,x) - (r_E(t,x)+d(t,x)+q_E)E(t,x) + L(t,0,x) - L(t,\tau,x).$$

Similarly, integrating both sides of (2.1) form $\tau$ to $+\infty$. We get

$$\frac{\partial I(t,x)}{\partial t} = D_I(1-q_I)\Delta I(t,x) - (r_I(t,x)+d(t,x)+q_I)I(t,x) + L(t,\tau,x) - L(t,+\infty,x).$$

For biological reasons, we assume that $L(t,+\infty,x)=0$ and $d(x,t)>0$. Furthermore, it should be noted that new infections result from contact between susceptible and infected individuals. However, since we assume that the vaccine is not 100% effective, a vaccinated individual who comes into contact with an infected person may also become infected. Therefore, adopting a mass action infection mechanism leads to the following condition:

$$L(t,0,x) = \beta_1(t,x)\frac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} + \beta_2(t,x)\frac{V(t,x)I(t,x)}{V(t,x)+I(t,x)},$$

where $\beta_1(t,x)>0, \beta_2(t,x)>0$ are called effective contact rate. On the basis of above assumptions, the disease dynamics is governed by the following system of partial differential equations:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = D_S\Delta S(t,x) + \Lambda(t,x) - \beta_1(t,x)\dfrac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} \\ \qquad - \alpha(t,x)S(t,x) - d(t,x)S(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial E(t,x)}{\partial t} = D_E(1-q_E)\Delta E(t,x) + \beta_1(t,x)\dfrac{S(t,x)I(t,x)}{S(t,x)+I(t,x)} \\ \qquad + \beta_2(t,x)\dfrac{V(t,x)I(t,x)}{V(t,x)+I(t,x)} \\ \qquad - (r_E(t,x)+q_E+d(t,x))E(t,x) - L(t,\tau,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial I(t,x)}{\partial t} = D_I(1-q_I)\Delta I(t,x) + L(t,\tau,x) \\ \qquad - (r_I(t,x)+q_I+d(t,x))I(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial V(t,x)}{\partial t} = D_V\Delta V(t,x) - \beta_2(t,x)\dfrac{V(t,x)I(t,x)}{V(t,x)+I(t,x)} + \alpha(t,x)S(t,x) \\ \qquad - d(t,x)V(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial Q(t,x)}{\partial t} = q_E E(t,x) + q_I I(t,x) - d(t,x)Q(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial R(t,x)}{\partial t} = D_R\Delta R(t,x) + r_E(t,x)E(t,x) + r_I(t,x)I(t,x) \\ \qquad - d(t,x)R(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial S(t,x)}{\partial \mathbf{n}} = \dfrac{\partial E(t,x)}{\partial \mathbf{n}} = \dfrac{\partial I(t,x)}{\partial \mathbf{n}} = \dfrac{\partial V(t,x)}{\partial \mathbf{n}} = \dfrac{\partial Q(t,x)}{\partial \mathbf{n}} \\ \qquad = \dfrac{\partial R(t,x)}{\partial \mathbf{n}} = 0,\ x\in\partial\Omega,\ t>0. \end{cases} \tag{2.3}$$

To obtain our main results in this paper, we require the following assumptions for the model (2.3):

**(A1)** $\Lambda(t,x), \beta_1(t,x), \beta_2(t,x), \alpha(t,x), d(t,x), r_I(t,x), r_E(t,x)$ are Hölder continuous and nonnegative nontrivial on $\mathbb{R}\times\bar{\Omega}$, and uniformly almost periodic in $t$.

**(A2)** $\Lambda(t,x) > \Lambda_*, \alpha(t,x) > \alpha_*, d(t,x) > d_*$ for all $t\in\mathbb{R}$ and $x\in\bar{\Omega}$, and $D_i > 0$, where $i = S, E, I, V, R$.

To determine $L(t,\tau,x)$, we need to integrate along characteristics. Let $\xi \geq 0$ and consider solutions of (2.1) along the characteristic line $t = a+\xi$. Let $P(\xi,a,x) = L(a+\xi,a,x)$. Then, for $a\in[0,\tau]$, we obtain

$$\begin{cases} \dfrac{\partial P(\xi,a,x)}{\partial a} =\dfrac{\partial L(a+\xi,a,x)}{\partial a} \\ \qquad\qquad = \left[\dfrac{\partial L(t,a,x)}{\partial t} + \dfrac{\partial L(t,a,x)}{\partial a}\right]_{t=a+\xi} \\ \qquad\qquad = D_E(1-q_E)\Delta L(a+\xi,a,x) \\ \qquad\qquad\quad - (r(a+\xi,a,x) + d(a+\xi,x) + q(a))L(a+\xi,a,x) \\ \qquad\qquad = D_E(1-q_E)\Delta P(\xi,a,x) - (r_E(a+\xi,x) \\ \qquad\qquad\quad + d(a+\xi,x) + q_E)P(\xi,a,x), \\ P(\xi,0,x) = L(\xi,0,x) \\ \qquad\qquad = \beta_1(\xi,x)\dfrac{S(\xi,x)I(\xi,x)}{S(\xi,x)+I(\xi,x)}\beta_2(\xi,x)\dfrac{V(\xi,x)I(\xi,x)}{V(\xi,x)+I(\xi,x)}. \end{cases} \tag{2.4}$$

For the last equation above, we can treat $\xi$ as a parameter and integrate with respect to it, which gives us

$$P(\xi,a,x) = \int_\Omega \Gamma(\xi+a,\xi,x,y)\Big[\beta_1(\xi,y)\frac{S(\xi,y)I(\xi,y)}{S(\xi,y)+I(\xi,y)} + \beta_2(\xi,y)\frac{V(\xi,y)I(\xi,y)}{V(\xi,y)+I(\xi,y)}\Big]\,dy,$$

where $\Gamma(t,s,x,y)$ with $t > s \geq 0$ and $x,y\in\Omega$ are the fundamental solutions associated with the partial differential operator $\partial_t - D_E(1-q_E)\Delta - d(t,\cdot) - r_E(t,\cdot) - q_E$ subject to the Neumann boundary condition [11]. Since $L(t,a,x) = P(t-a,a,x), \forall t\geq a$, we get

$$L(t,a,x) = \int_\Omega \Gamma(t,t-a,x,y)\Big[\beta_1(t-a,y)\frac{S(t-a,y)I(t-a,y)}{S(t-a,y)+I(t-a,y)} + \beta_2(t-a,y)\frac{V(t-a,y)I(t-a,y)}{V(t-a,y)+I(t-a,y)}\Big]\,dy.$$

Taking $a = \tau$, we have

$$L(t,\tau,x) = \int_\Omega \Gamma(t,t-\tau,x,y)\Big[\beta_1(t-\tau,y)\frac{S(t-\tau,y)I(t-\tau,y)}{S(t-\tau,y)+I(t-\tau,y)} + \beta_2(t-\tau,y)\frac{V(t-\tau,y)I(t-\tau,y)}{V(t-\tau,y)+I(t-\tau,y)}\Big]\,dy. \tag{2.5}$$

For convenience, let $(u_1, u_2, u_3, u_4, u_5, u_6) = (S, E, I, V, Q, R)$, $(D_1, D_2, D_3, D_4, D_6) = (D_S, D_E(1-q_E), D_I(1-q_I), D_V, D_R)$. From (2.5), we can rewrite model (2.3) as a time-delayed and nonlocal almost periodic reaction-diffusion system with no flux boundary condition:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} =& D_1\Delta u_1(t,x) + \Lambda(t,x) - \beta_1(t,x)\dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)} \\ & - \alpha(t,x)u_1(t,x) - d(t,x)u_1(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_2(t,x)}{\partial t} =& D_2\Delta u_2(t,x) + \beta_1(t,x)\dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)} \\ & + \beta_2(t,x)\dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)} \\ & - (r_E(t,x)+q_E+d(t,x))u_2(t,x) - L(t,\tau,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_3(t,x)}{\partial t} =& D_3\Delta u_3(t,x) + L(t,\tau,x) \\ & - (r_I(t,x)+q_I+d(t,x))u_3(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_4(t,x)}{\partial t} =& D_4\Delta u_4(t,x) - \beta_2(t,x)\dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)} + \alpha(t,x)u_1(t,x) \\ & - d(t,x)u_4(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_5(t,x)}{\partial t} =& q_E u_2(t,x) + q_I u_3(t,x) - d(t,x)u_5(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_6(t,x)}{\partial t} =& D_6\Delta u_6(t,x) + r_E(t,x)u_2(t,x) + r_I(t,x)u_3(t,x) \\ & - d(t,x)u_6(t,x),\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_i(t,x)}{\partial \mathbf{n}} =& 0,\ x\in\partial\Omega,\ t>0,\ i=1,...,6, \end{cases} \tag{2.6}$$

where $L(t,\tau,x)$ is given as (2.5).

## 3. Non-negative and uniform boundness of the solution of (2.6)

Let $Y_0 := C(\bar{\Omega}, \mathbb{R}^6)$ be the Banach space with supremum norm $\|\cdot\|_{Y_0}$, and let $Y_0^+ := C(\bar{\Omega}, \mathbb{R}_+^6)$. Define $Y := C([-\tau, 0], Y_0)$ and $Y^+ := C([-\tau, 0], Y_0^+)$. The norm of $Y$ is defined by $\|\varphi\|_Y = \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{Y_0}$, $\forall \varphi \in Y$.

**Lemma 3.1.** *For any $\phi = (\phi_1, \phi_2, \cdots, \phi_6) \in Y^+$, system (2.6) admits a unique non-negative mild solution*

$$u(t,x;\phi) = (u_1(t,x;\phi), u_2(t,x;\phi), u_3(t,x;\phi), u_4(t,x;\phi), u_5(t,x;\phi), u_6(t,x;\phi))$$

*on $[0,\infty)$ with initial value $\phi$. Moreover, $u(t,x;\phi)$ is a classical solution when $t > \tau$.*

**Proof.** Since the equation about the isolated person $u_5$ does not have a diffusion coefficient, we first consider the following equation with diffusion coefficients:

$$\begin{cases}\dfrac{\partial u_1(t,x)}{\partial t} =& D_1\Delta u_1(t,x)+\Lambda(t,x)-\beta_1(t,x)\dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)}\\ & -\alpha(t,x)u_1(t,x)-d(t,x)u_1(t,x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial u_2(t,x)}{\partial t} =& D_2\Delta u_2(t,x)+\beta_1(t,x)\dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)}\\ & +\beta_2(t,x)\dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)}\\ & -(r_E(t,x)+q_E+d(t,x))u_2(t,x)-L(t,\tau,x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial u_3(t,x)}{\partial t} =& D_3\Delta u_3(t,x)+L(t,\tau,x)\\ & -(r_I(t,x)+q_I+d(t,x))u_3(t,x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial u_4(t,x)}{\partial t} =& D_4\Delta u_4(t,x)-\beta_2(t,x)\dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)}+\alpha(t,x)u_1(t,x)\\ & -d(t,x)u_4(t,x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial u_6(t,x)}{\partial t} =& D_6\Delta u_6(t,x)+r_E(t,x)u_2(t,x)+r_I(t,x)u_3(t,x)\\ & -d(t,x)u_6(t,x),\ x\in\Omega,\ t>0,\\ \dfrac{\partial u_i(t,x)}{\partial \mathbf{n}} =& 0,\ x\in\partial\Omega,\ t>0,\ i=1,2,3,4,6.\end{cases} \tag{3.1}$$

Define $\mathcal{F}=(F_1,F_2,F_3,F_4,F_6):[0,+\infty)\times C([-\tau,0],C(\bar{\Omega},\mathbb{R}^5_+))\to C(\bar{\Omega},\mathbb{R}^5)$ by

$$\begin{aligned}F_1(t,\phi) &= \Lambda(t,\cdot)-\beta_1(t,\cdot)\frac{\phi_1(0,\cdot)\phi_3(0,\cdot)}{\phi_1(0,\cdot)+\phi_3(0,\cdot)},\\ F_2(t,\phi) &= \beta_1(t,\cdot)\frac{\phi_1(0,\cdot)\phi_3(0,\cdot)}{\phi_1(0,\cdot)+\phi_3(0,\cdot)}+\beta_2(t,\cdot)\frac{\phi_4(0,\cdot)\phi_3(0,\cdot)}{\phi_4(0,\cdot)+\phi_3(0,\cdot)}\\ &\quad -\int_\Omega \bar{\Gamma}(t,t-\tau,\cdot,y)\Big[\bar{\beta}_1(t-\tau,y)\frac{\phi_1(-\tau,y)\phi_3(-\tau,y)}{\phi_1(-\tau,y)+\phi_3(-\tau,y)}\\ &\quad +\bar{\beta}_2(t-\tau,y)\frac{\phi_4(-\tau,y)\phi_3(-\tau,y)}{\phi_4(-\tau,y)+\phi_3(-\tau,y)}\Big]\,dy,\\ F_3(t,\phi) &= \int_\Omega \bar{\Gamma}(t,t-\tau,\cdot,y)\Big[\bar{\beta}_1(t-\tau,y)\frac{\phi_1(-\tau,y)\phi_3(-\tau,y)}{\phi_1(-\tau,y)+\phi_3(-\tau,y)}\\ &\quad +\bar{\beta}_2(t-\tau,y)\frac{\phi_4(-\tau,y)\phi_3(-\tau,y)}{\phi_4(-\tau,y)+\phi_3(-\tau,y)}\Big]\,dy,\\ F_4(t,\phi) &= -\beta_2(t,\cdot)\frac{\phi_4(0,\cdot)\phi_3(0,\cdot)}{\phi_4(0,\cdot)+\phi_3(0,\cdot)}+\alpha(t,\cdot)\phi_1(0,\cdot),\\ F_6(t,\phi) &= r_E(t,\cdot)\phi_2(0,\cdot)+r_I(t,\cdot)\phi_3(0,\cdot).\end{aligned}$$

Then (3.1) becomes

$$\begin{cases}\dfrac{\partial u(t,x)}{\partial t}=\mathcal{A}(t)u(t,x)+\mathcal{F}(t,u_t), \quad t>0,\ x\in\Omega,\\ u(\sigma,x)=\phi(\sigma,x), \quad \sigma\in[-\tau,0],\ x\in\Omega,\end{cases}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x), u_4(t,x), u_6(t,x))$,

$$\mathcal{A}(t) = \begin{pmatrix} A_1(t) & & & & \\ & A_2(t) & & & \\ & & A_3(t) & & \\ & & & A_4(t) & \\ & & & & A_6(t) \end{pmatrix}.$$

$A_1(t)$ is defined by

$$D(A_1(t)) = \left\{ \varphi \in C^2(\bar{\Omega}) \Big| \frac{\partial \varphi(x)}{\partial \mathbf{n}} = 0, x \in \partial\Omega \right\},$$
$$A_1(t)\varphi = D_1 \Delta \varphi - (\alpha(t,\cdot) + d(t,\cdot))\varphi, \quad \forall \varphi \in D(A_1(t)).$$

$A_2(t)$ is defined by

$$D(A_2(t)) = \left\{ \varphi \in C^2(\bar{\Omega}) \Big| \frac{\partial \varphi(x)}{\partial \mathbf{n}} = 0, x \in \partial\Omega \right\},$$
$$A_2(t)\varphi = D_2 \Delta \varphi - (r_E(t,\cdot) + q_E + d(t,\cdot))\varphi, \quad \forall \varphi \in D(A_2(t)).$$

$A_3(t)$ is defined by

$$D(A_3(t)) = \left\{ \varphi \in C^2(\bar{\Omega}) \Big| \frac{\partial \varphi(x)}{\partial \mathbf{n}} = 0, x \in \partial\Omega \right\},$$
$$A_3(t)\varphi = D_3 \Delta \varphi - (r_I(t,\cdot) + q_I + d(t,\cdot))\varphi, \quad \forall \varphi \in D(A_3(t)).$$

$A_4(t)$ is defined by

$$D(A_4(t)) = \left\{ \varphi \in C^2(\bar{\Omega}) \Big| \frac{\partial \varphi(x)}{\partial \mathbf{n}} = 0, x \in \partial\Omega \right\},$$
$$A_4(t)\varphi = D_4 \Delta \varphi - d(t,\cdot)\varphi, \quad \forall \varphi \in D(A_4(t)).$$

$A_6(t)$ is defined by

$$D(A_6(t)) = \left\{ \varphi \in C^2(\bar{\Omega}) \Big| \frac{\partial \varphi(x)}{\partial \mathbf{n}} = 0, x \in \partial\Omega \right\},$$
$$A_6(t)\varphi = D_6 \Delta \varphi - d(t,\cdot)\varphi, \quad \forall \varphi \in D(A_6(t)).$$

Then, similar to the proof process of Theorem 2.2 in [44], we can obtain that there exists a unique nonnegative mild solution to (3.1) with initial value $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_6)$ and it is a classical solution when $t > \tau$.

The next step is to consider the fifth equation of (2.6). Since there exists a unique non-negative solution for (3.1), substituting it into the fifth equation. The fifth equation can be regarded as an ordinary differential equation, and it is easy to obtain that its solution is positive and unique with initial value $\phi_5$. Thus, for any $\phi \in Y^+$, system (2.6) admits a unique non-negative mild solution

$$u(t,x;\phi) = (u_1(t,x;\phi), u_2(t,x;\phi), u_3(t,x;\phi), u_4(t,x;\phi), u_5(t,x;\phi), u_6(t,x;\phi))$$

on $[0,\infty)$ with initial value $\phi$ and it is a classical solution when $t > \tau$. This completes the proof. $\square$

**Lemma 3.2.** *Let*

$$u(t,x;\phi)=(u_1(t,x;\phi),u_2(t,x;\phi),u_3(t,x;\phi),u_4(t,x;\phi),u_5(t,x;\phi),u_6(t,x;\phi))$$

*be the solution of* (2.6) *with initial datum* $\phi\in Y^+$. *Then the solution* $u(t,x;\phi)$ *is uniformly bounded on* $\bar{\Omega}$.

**Proof.** By model (2.6) we get

$$\begin{aligned}&\sum_{i=1}^{6}\frac{\partial u_i(t,x)}{\partial t}-\sum_{i=1}^{4}D_i\Delta u_i(t,x)-D_6\Delta u_6(t,x)\\ =&\Lambda(t,x)-d(t,x)\sum_{i=1}^{6}u_i(t,x).\end{aligned}$$

Integrating on both sides yields

$$\begin{aligned}&\int_\Omega\left(\sum_{i=1}^{6}\frac{\partial u_i(t,x)}{\partial t}-\sum_{i=1}^{4}D_i\Delta u_i(t,x)-D_6\Delta u_6(t,x)\right)dx\\ =&\int_\Omega\left[\Lambda(t,x)-d(t,x)\sum_{i=1}^{6}u_i(t,x)\right]dx.\end{aligned}$$

By Green's formula and Neumann boundary $\frac{\partial u_i}{\partial \mathbf{n}}=0, i=1,2,3,4,6, x\in\partial\Omega, t>0$, we get

$$D_i\int_\Omega\Delta u_i\,dx=D_i\int_{\partial\Omega}\frac{\partial u_i}{\partial \mathbf{n}}\,ds=0,\ \ i=1,2,3,4,6.$$

Hence, for $t>0$

$$\begin{aligned}\int_\Omega\left(\sum_{i=1}^{6}\frac{\partial u_i(t,x)}{\partial t}\right)dx&=\int_\Omega[\Lambda(t,x)-d(t,x)\sum_{i=1}^{6}u_i(t,x)]\,dx\\ &\le\int_\Omega[\Lambda^*-d_*\sum_{i=1}^{6}u_i(t,x)]\,dx\\ &=\Lambda^*\left|\Omega\right|-d_*\int_\Omega\sum_{i=1}^{6}u_i(t,x)\,dx.\end{aligned}\tag{3.2}$$

Denote $\int_\Omega\sum_{i=1}^{6}u_i(t,x)\,dx=F(t)$, then (3.2) becomes

$$\frac{dF(t)}{dt}\le\Lambda^*\left|\Omega\right|-d_*F(t),\quad t>0.$$

By which we get

$$0\le F(t)\le\frac{\Lambda^*}{d_*}\left|\Omega\right|+F(0)e^{-d_*t},\quad t>0,$$

where

$$F(0)=\int_\Omega\sum_{i=1}^{6}u_i(0,x)\,dx\le 6\|\phi\|\left|\Omega\right|.$$

This indicates that $F(t) = \int_\Omega (\sum_{i=1}^6 u_i)\,dx$ is bounded. Let $Z = \frac{\Lambda^*}{d_*}|\Omega| + F(0)$, then we get

$$F(t) = \int_\Omega \Big(\sum_{i=1}^6 u_i\Big)\,dx \le Z.$$

By Theorem 3.1 in [1], there is a positive constant $Z^*$ depending on $Z$ such that

$$\Big\|\sum_{i=1}^6 u_i\Big\|_{L^\infty(\Omega)} \le Z^*.$$

Hence, we get that $u_i(t,x), i = 1,\cdots,6$ are uniformly bounded on $\bar{\Omega}$. This completes the proof. □

# 4. Threshold dynamics

We use the upper Lyapunov exponent $\lambda^*$ as the threshold to determine whether the disease is prevalent, so we first give the definition and some properties of $\lambda^*$ in the following subsection.

## 4.1. The upper Lyapunov exponent

Define $X_0 = C(\bar{\Omega},\mathbb{R})$ as a Banach space equipped with the supremum norm $\|\cdot\|_\infty$. Let

$$X_0^+ = C(\bar{\Omega},\mathbb{R}^+) = \{\varphi \in X_0 : \varphi(x) \ge 0, \forall x \in \bar{\Omega}\}.$$

We denote the interior of $X_0^+$ by $\mathrm{Int}(X_0^+)$, which is non-empty. For a constant $\tau > 0$, define $X := C([-\tau,0],X_0)$ equipped with the norm $\|\phi\| = \max_{\theta\in[-\tau,0]}\|\phi(\theta)\|_\infty$, $\forall\phi \in X$. For a function $\gamma \in C([-\tau,\rho],X_0)$ $(\rho > 0)$, define $\alpha_t(\theta) = \alpha(t+\theta), \forall\theta \in [-\tau,0], t \in [0,\rho)$. Let $\tilde{}$ denote the inclusion $\mathbb{R} \to X$ by $\alpha \to \tilde{\alpha}$.

Linearizing the the equation of infectious variable of (2.6) at a disease-free almost periodic solution, we obtain:

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} = D_3\Delta u_3(t,x) - k(t,x)u_3(t,x) \\ \qquad\qquad + \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)u_3(t-\tau,y)\,dy,\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} = 0,\ x \in \partial\Omega,\ t > 0, \end{cases} \tag{4.1}$$

where $k(t,x) = r_I(t,x) + q_I + d(t,x)$, $\beta(t-\tau,y) = \beta_1(t-\tau,y) + \beta_2(t-\tau,y)$.

Since $d(t,x), r_E(t,x)$ are uniformly almost periodic in $t$, it can be obtained that $\Gamma(t,s,x,y)$ is uniformly almost periodic in $t$ and $s$ [11]. Define the hull of $k$ as $H(k) = cls\{k(t+s,x) : s \in \mathbb{R}\}$ and the closure is taken under the compact open topology. Similarly, we define the hull of $\Gamma$ and $\beta$ as $H(\Gamma)$ and $H(\beta)$, respectively. Let $\pi = (k,\Gamma,\beta)$, $H(\pi)$ be the hull of $\pi$. Taking $\vartheta = (\bar{k},\bar{\Gamma},\bar{\beta}) \in H(\pi)$, the translation map $\sigma : \mathbb{R}\times H(\pi) \to H(\pi), (s,\vartheta) \mapsto \vartheta\cdot s$ given by $(\vartheta\cdot s)(t,x,y) = (\bar{k}(t+s,x),\bar{\Gamma}(t+s,t-\tau+s,x,y),\bar{\beta}(t+s,x))$, $t \in \mathbb{R}, x,y \in \bar{\Omega}$ defines a compact, almost periodic minimal and distal flow [34]. Consider

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} =& D_3\Delta u_3(t,x) - \bar{k}(t,x)u_3(t,x) \\ & + \displaystyle\int_\Omega \bar{\Gamma}(t,t-\tau,x,y)\bar{\beta}(t-\tau,y)u_3(t-\tau,y)\,dy,\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} =& 0,\ x\in\partial\Omega,\ t>0. \end{cases} \tag{4.2}$$

From the proof of the theorem in [44], we can conclude that equation (4.2) has a unique mild solution $w(t,x;\phi,\vartheta)$ with initial datum $\phi\in X^+ := C([-\tau,0],X_0^+)$. Furthermore, $w(t,x;\phi,\vartheta)$ is a classical solution for $t>\tau$ . Define $w_t(\phi,\vartheta)(\theta,x) := w(t+\theta,x;\phi,\vartheta), \forall t\geq 0, \theta\in[-\tau,0], x\in\bar{\Omega}$. Then, we can define a continuous linear skew-product semiflow:

$$\begin{aligned} &\Pi : \mathbb{R}^+\times X^+\times H(\pi)\mapsto X^+\times H(\pi), \\ &(t,\phi,\vartheta)\mapsto (w_t(\phi,\vartheta),\vartheta\cdot t). \end{aligned} \tag{4.3}$$

It follows from Corollary 1 of [29] that the skew-product semiflow (4.3) is monotone. Let $\Phi(t,\vartheta)\phi = w_t(\phi,\vartheta), \forall\phi\in X$. For any $\vartheta\in H(\pi)$, we define the Lyapunov exponent $\lambda_\vartheta$ as

$$\lambda_\vartheta = \lim_{t\to\infty}\sup\frac{\ln\|\Phi(t,\vartheta)\|}{t}. \tag{4.4}$$

The number $\lambda^* = \sup_{\vartheta\in H(\pi)}\lambda_\vartheta$ is called the upper Lyapunov exponent of (4.1) or (4.3).

The following Lemmas and their proofs are similar to those in [32], so their proofs are omitted.

**Lemma 4.1.** *There exist two almost periodic functions*

$$a\in C(\mathbb{R},\mathbb{R}), w_1\in \mathrm{Int}(C(\mathbb{R},X_0^+))$$

*such that* $w(t,x) = e^{\int_0^t a(s)\,ds w_1(t,x)}$ *is a solution of equation* (4.1)*. Furthermore,*

$$\lambda^* = \lim_{t\to\infty}\frac{\int_0^t a(s)\,ds}{t}.$$

**Lemma 4.2.** *Choose* $\phi\in\mathrm{Int}(X^+)$*, and let* $w(t,x;\phi)$ *be the solution of equation* (4.1) *with initial value* $w_0=\phi$*. Then*

$$\lambda^* = \lim_{t\to\infty}\frac{\ln w(t,x_0;\phi)}{t},\quad \forall x_0\in\bar{\Omega}.$$

For the convenience of later research, we give the following equation which is the perturbation equation of (4.1) with a positive parameter $\varepsilon<1$,

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} =& D_3\Delta u_3(t,x) - k(t,x)u_3(t,x) + (1-\varepsilon) \\ & \times \displaystyle\int_\Omega \Gamma(t,t-\tau,x,y)\beta(t-\tau,y)u_3(t-\tau,y)\,dy,\ x\in\Omega,\ t>0, \\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} =& 0,\ x\in\partial\Omega,\ t>0. \end{cases} \tag{4.5}$$

**Lemma 4.3.** *Let* $\lambda_\varepsilon^*$ *be the upper Lyapunov exponent saaociated with equation* (4.5)*, then*

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon^* = \lambda^*.$$

## 4.2. Global properties of disease-free and endemic solutions

In this subsection, we will give the global attractiveness of the disease-free solution and the uniform persistence of the endemic solution of system (2.6).

Letting $u_2 = u_3 = u_5 = u_6 = 0$ in (2.6), we get the following equation for $u_1(t,x)$ and $u_4(t,x)$:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = & D_1 \Delta u_1(t,x) + \Lambda(t,x) \\ & -(\alpha(t,x)+d(t,x))u_1(t,x),\ x \in \Omega,\ t>0, \\ \dfrac{\partial u_4(t,x)}{\partial t} = & D_4 \Delta u_4(t,x) + \alpha(t,x)u_1(t,x) \\ & - d(t,x)u_4(t,x),\ x \in \Omega,\ t>0, \\ \dfrac{\partial u_1(t,x)}{\partial \mathbf{n}} = & \dfrac{\partial u_4(t,x)}{\partial \mathbf{n}} = 0,\ x \in \partial\Omega,\ t>0. \end{cases} \tag{4.6}$$

To proceed further, we give the following Lemma:

**Lemma 4.4.** *Suppose assumptions* (A1) *and* (A2) *are satisfied, then system* (4.6) *admits a unique positive almost periodic solution* $(u_{1*}(t,\cdot), u_{4*}(t,\cdot))$ *which is globally attractive in* $X_0^+ \times X_0^+$.

**Proof.** Firstly, we consider the following equation:

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = & D_1 \Delta u_1(t,x) + \Lambda(t,x) - (\alpha(t,x)+d(t,x))u_1(t,x),\ x \in \Omega,\ t>0, \\ \dfrac{\partial u_1(t,x)}{\partial \mathbf{n}} = & 0,\ x \in \partial\Omega,\ t>0. \end{cases} \tag{4.7}$$

Based on Assumption $(A1)$, we get that $\Lambda(t,x)$, $\alpha(t,x)+d(t,x)$ are Hölder continuous on $\mathbb{R} \times \bar{\Omega}$ and uniformly almost periodic in $t$. Combining this with Assumption $(A2)$ and applying Lemma 3.2 from [32], we can conclude that system (4.7) has a unique positive almost periodic solution $u_{1*}$, which is globally attractive in $X_0^+$.

Then, we consider the following equation for $u_4(t,x)$:

$$\begin{cases} \dfrac{\partial u_4(t,x)}{\partial t} = & D_4 \Delta u_4(t,x) + \alpha(t,x)u_{1*}(t,x) - d(t,x)u_4(t,x),\ x \in \Omega,\ t>0, \\ \dfrac{\partial u_1(t,x)}{\partial \mathbf{n}} = & 0,\ x \in \partial\Omega,\ t>0. \end{cases} \tag{4.8}$$

Similarly, by applying Lemma 3.2 of [32], we can also show that system (4.8) has a unique positive almost periodic solution $u_{4*}$, which is globally attractive in $X_0^+$. Consequently, the system (4.6) has a unique positive almost periodic solution $(u_{1*}(t,\cdot), u_{4*}(t,\cdot))$ that is globally attractive in $X_0^+ \times X_0^+$.This completes the proof. $\square$

**Remark 4.1.** Lemma 4.4 follows that system (2.6) admits a unique positive disease-free almost periodic solution $E^0 = (u_1^*, 0, 0, u_4^*, 0, 0)$. Linearizing system (2.6) at $E^0$, we can obtain that infectious variable $u_3$ satisfies equation (4.1).

Now we show that $\lambda^*$ is a threshold value for the global extinction and uniform persistence of the disease.

**Theorem 4.1.** *Assume that* (A1) *and* (A2) *hold. Let* $u(t,x;\phi)$ *be the solution of system* (2.6) *with initial value* $\phi \in Y^+$. *If* $\lambda^* < 0$, *then* $\lim_{t\to\infty} \|u(t,\cdot;\phi) - E^0(t,\cdot)\| = 0$.

**Proof.** Let $A = (\Lambda, \beta_1, \beta_2, \alpha, d, r_E, r_I)$ and $H(A)$ be the hull of $A$. For any $\varsigma = (\bar{\Lambda}, \bar{\beta}_1, \bar{\beta}_2, \bar{\alpha}, \bar{d}, \bar{r}_E, \bar{r}_I) \in H(A)$, the translation map $\mathbb{R} \times H(A) \to H(A), (s, \varsigma) \mapsto \varsigma \cdot s$ given by $(\varsigma \cdot s)(t,x,y) = (\bar{\Lambda}(t+s,x), \bar{\beta}_1(t+s,x), \bar{\beta}_2(t+s,x), \bar{\alpha}(t+s,x), \bar{d}(t+s,x), \bar{r}_E(t+s,x), \bar{r}_I(t+s,x))(t \in \mathbb{R}, x, y \in \bar{\Omega})$ defines a compact, almost periodic minimal and distal flow. Consider

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1 \Delta u_1(t,x) + \bar{\Lambda}(t,x) - \bar{\beta}_1(t,x) \dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)} \\ \qquad - \bar{\alpha}(t,x)u_1(t,x) - \bar{d}(t,x)u_1(t,x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2 \Delta u_2(t,x) + \bar{\beta}_1(t,x) \dfrac{u_1(t,x)u_3(t,x)}{u_1(t,x)+u_3(t,x)} \\ \qquad + \bar{\beta}_2(t,x) \dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)} \\ \qquad - (\bar{r}_E(t,x) + q_E + \bar{d}(t,x))u_2(t,x) \\ \qquad - \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y) \Big[ \bar{\beta}_1(t-\tau, y) \dfrac{u_1(t-\tau,y)u_3(t-\tau,y)}{u_1(t-\tau,y)+u_3(t-\tau,y)} \\ \qquad + \bar{\beta}_2(t-\tau,y) \dfrac{u_4(t-\tau,y)u_3(t-\tau,y)}{u_4(t-\tau,y)+u_3(t-\tau,y)} \Big] dy, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_3(t,x)}{\partial t} = D_3 \Delta u_3(t,x) - (\bar{r}_I(t,x) + q_I + \bar{d}(t,x))u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y) \Big[ \bar{\beta}_1(t-\tau, y) \dfrac{u_1(t-\tau,y)u_3(t-\tau,y)}{u_1(t-\tau,y)+u_3(t-\tau,y)} \\ \qquad + \bar{\beta}_2(t-\tau,y) \dfrac{u_4(t-\tau,y)u_3(t-\tau,y)}{u_4(t-\tau,y)+u_3(t-\tau,y)} \Big] dy, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_4(t,x)}{\partial t} = D_4 \Delta u_4(t,x) - \bar{\beta}_2(t,x) \dfrac{u_4(t,x)u_3(t,x)}{u_4(t,x)+u_3(t,x)} + \bar{\alpha}(t,x)u_1(t,x) \\ \qquad - \bar{d}(t,x)u_4(t,x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_5(t,x)}{\partial t} = q_E u_2(t,x) + q_I u_3(t,x) - \bar{d}(t,x)u_5(t,x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_6(t,x)}{\partial t} = D_6 \Delta u_6(t,x) + \bar{r}_E(t,x)u_2(t,x) \\ \qquad + \bar{r}_I(t,x)u_3(t,x) - \bar{d}(t,x)u_6(t,x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_i(t,x)}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, \ t > 0, \ i = 1, ..., 6. \end{cases} \tag{4.9}$$

By Lemma 3.1, system (4.9) admits a unique solution

$$u(t,x;\phi,\varsigma) = (u_1(t,x;\phi,\varsigma), u_2(t,x;\phi,\varsigma), u_3(t,x;\phi,\varsigma), u_4(t,x;\phi,\varsigma), \\ u_5(t,x;\phi,\varsigma), u_6(t,x;\phi,\varsigma)),$$

with initial datum $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \in Y^+$. Define $u_t(\phi,\varsigma)(\theta,x) = u(t+\theta, x; \phi, \varsigma), \forall t \geq 0, \theta \in [-\tau, 0], x \in \bar{\Omega}$. The solution of (4.9) induces a skew-product semi-flow:

$$\Pi^Y : \mathbb{R}^+ \times Y^+ \times H(A) \mapsto Y^+ \times H(A),$$

$$(t, \phi, \varsigma) \mapsto (u_t(\phi, \varsigma), \varsigma \cdot t).$$

The following proof is similar to the proof of Theorem 4.2 in [32]. For the sake of completeness, we give detailed proof below. From the third equation of (4.9), we can get $u_3(t, x; \phi, \varsigma)$ satisfies

$$\begin{cases} \dfrac{\partial u_3(t,x)}{\partial t} \leq D_3 \Delta u_3(t,x) - (\bar{r}_I(t,x) + \bar{q}_I(t,x) + \bar{d}(t,x)) u_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y)(\bar{\beta}_1(t-\tau, y) \\ \qquad + \bar{\beta}_2(t-\tau, y)) u_3(t-\tau, y)\, dy, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases} \tag{4.10}$$

Consider the comparison system of (4.10)

$$\begin{cases} \dfrac{\partial \tilde{u}_3(t,x)}{\partial t} = D_3 \Delta \tilde{u}_3(t,x) - (\bar{r}_I(t,x) + \bar{q}_I(t,x) + \bar{d}(t,x)) \tilde{u}_3(t,x) \\ \qquad + \displaystyle\int_\Omega \bar{\Gamma}(t, t-\tau, x, y)(\bar{\beta}_1(t-\tau, y) \\ \qquad + \bar{\beta}_2(t-\tau, y)) \tilde{u}_3(t-\tau, y)\, dy, \ x \in \Omega, \ t > 0, \\ \dfrac{\partial \tilde{u}_3(t,x)}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases} \tag{4.11}$$

Due to $\lambda^* < 0$, it follows from Lemma 4.1 that there exist two almost periodic function $a(\varsigma \cdot t)$ and $w_1(t, x; \varsigma)$ such that $w(t, x; \varsigma) = e^{\int_0^t a(\varsigma \cdot t)\, ds} w_1(t, x; \varsigma)$ is a solution of (4.11), we know that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t a(\varsigma \cdot s)\, ds = \lambda^* < 0.$$

Thus, by the comparison principle we get $u_3(t, x; \phi, \varsigma) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

Note that $u_2(t, x; \phi, \varsigma)$ satisfies

$$\begin{cases} \dfrac{\partial u_2(t,x)}{\partial t} \leq D_2 \Delta u_2(t,x) + (\bar{\beta}_1(t,x) + \bar{\beta}_2(t,x)) u_3(t,x) \\ \qquad - \bar{d}(t,x) u_2(t,x), \ x \in \Omega, \ t > 0, \\ \dfrac{\partial u_2(t,x)}{\partial \mathbf{n}} = 0, \ x \in \partial\Omega, \ t > 0. \end{cases}$$

Let $\tilde{u}_2(t)$ be the solution of the following equation with initial datum $\tilde{u}_2(0) = \max_{x \in \bar{\Omega}} \phi_2(x)$,

$$\frac{du_2(t)}{dt} = B_1(t) - d_* u_2(t),$$

where $B_1(t) = \max_{x \in \bar{\Omega}} \{(\bar{\beta}_1(t,x) + \bar{\beta}_2(t,x)) u_3(t, x; \phi, \varsigma)\}$. It then follows from the comparison principle that $u_2(t, x; \phi, \varsigma) \leq \tilde{u}_2(t)$, $\forall x \in \bar{\Omega}$. By the Theorem 2.6 of [37], we further obtain $\tilde{u}_2(t) \to 0$ as $t \to \infty$. Hence, $u_2(t, x; \phi, \varsigma) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

Then, let $\tilde{u}_5(t)$ be the solution of the following equation with initial datum $\tilde{u}_5(0) = \max_{x\in\bar{\Omega}} \phi_5(x)$,

$$\frac{du_5(t)}{dt} = B_2(t) - d_* u_5(t),$$

where $B_2(t) = \max_{x\in\bar{\Omega}}\{q_E u_2(t,x;\phi,\varsigma) + q_I u_3(t,x;\phi,\varsigma)\}$. It then follows from the comparison principle that $u_5(t,x;\phi,\varsigma) \le \tilde{u}_5(t)$, $\forall x \in \bar{\Omega}$. By the Theorem 2.6 of [37], we further obtain $\tilde{u}_5(t) \to 0$ as $t \to \infty$. Hence, $u_5(t,x;\phi,\varsigma) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$. Similarly, it's easy to get $u_6(t,x;\phi,\varsigma)$ satisfies

$$\begin{cases} \dfrac{\partial u_6(t,x)}{\partial t} \le D_6\Delta u_6(t,x) + \bar{r}_E(t,x)u_2(t,x) + \bar{r}_I(t,x)u_3(t,x) \\ \qquad\qquad - d_* u_2(t,x),\ x \in \Omega,\ t > 0, \\ \dfrac{\partial u_6(t,x)}{\partial \mathbf{n}} = 0,\ x \in \partial\Omega,\ t > 0. \end{cases}$$

Then, let $\tilde{u}_6(t)$ be the solution of the following equation with initial datum $\tilde{u}_6(0) = \max_{x\in\bar{\Omega}} \phi_6(x)$,

$$\frac{du_6(t)}{dt} = B_3(t) - d_* u_6(t),$$

where $B_3(t) = \max_{x\in\bar{\Omega}}\{\bar{r}_E(t,x)u_2(t,x;\phi,\varsigma) + \bar{r}_I(t,x)u_3(t,x;\phi,\varsigma)\}$. It then follows from the comparison principle that $u_6(t,x;\phi,\varsigma) \le \tilde{u}_6(t)$, $\forall x \in \bar{\Omega}$. By the Theorem 2.6 of [37], we further obtain $\tilde{u}_6(t) \to 0$ as $t \to \infty$. Hence, $u_6(t,x;\phi,\varsigma) \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

Let $\omega(\phi,\varsigma)$ denote the omega limit set of $(\phi,\varsigma)$ for $\Pi^Y(t,\phi,\varsigma)$, which means

$$\omega(\phi,\varsigma) = \{(\phi^*,\varsigma^*) \in Y \times H(A) : \lim_{t_n\to\infty}(u_{t_n}(\phi,\varsigma),\varsigma_{t_n}) = (\phi^*,\varsigma^*)\}.$$

Since $\lim_{t\to\infty} u_i(t,\cdot,\phi,\varsigma) = 0 (i = 2,3,5,6)$, we have

$$\omega(\phi,\varsigma) \subset \{(\omega_1,\hat{0},\hat{0},\omega_4,\hat{0},\hat{0},\varsigma) : \omega_1,\omega_4 \in X^+, \varsigma \in H(A)\}.$$

From the first and the fourth equations of (4.9), we get that

$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} \ge D_1\Delta u_1(t,x) + \bar{\Lambda}(t,x) - \bar{\beta}_1(t,x)u_1(t,x) \\ \qquad\qquad - \bar{\alpha}(t,x)u_1(t,x) - \bar{d}(t,x)u_1(t,x),\ x \in \Omega,\ t > 0, \\ \dfrac{\partial u_4(t,x)}{\partial t} \ge D_4\Delta u_4(t,x) - \bar{\beta}_2(t,x)u_4(t,x) + \bar{\alpha}(t,x)u_1(t,x) \\ \qquad\qquad - \bar{d}(t,x)u_4(t,x),\ x \in \Omega,\ t > 0, \\ \dfrac{\partial u_1(t,x)}{\partial \mathbf{n}} = \dfrac{\partial u_4(t,x)}{\partial \mathbf{n}} = 0,\ t > 0,\ x \in \partial\Omega, \end{cases}$$

which implies that $\lim_{t\to\infty} u_1(t,x;\phi,\varsigma) \ge \frac{\Lambda_*}{M_1}$, $\lim_{t\to\infty} u_4(t,x;\phi,\varsigma) \ge \frac{\alpha_*\Lambda_*}{M_1M_2}$, where

$$M_1 = \sup_{(t,x)\in\mathbb{R}\times\bar{\Omega}} (\beta_1(t,x) + \alpha(t,x) + d(t,x)),$$
$$M_2 = \sup_{(t,x)\in\mathbb{R}\times\bar{\Omega}} (\beta_2(t,x) + d(t,x)).$$

This implies that for any $(\omega_1,\hat{0},\hat{0},\omega_4,\hat{0},\hat{0},\varsigma) \in \omega(\phi,\varsigma)$, we have $\omega_1 \in \mathrm{Int}(X^+)$ and $\omega_4 \in \mathrm{Int}(X^+)$.

By Lemma 4.4, we know that equation (4.6) has a unique solution

$$(u_1^*(t,x;\varsigma), u_4^*(t,x;\varsigma))$$

which is globally attractive and uniformly almost periodic in $t \in \mathbb{R}$. We can define $u_1^*(\varsigma)(\theta,x) = u_1^*(\theta,x;\varsigma)$ and $u_4^*(\varsigma)(\theta,x) = u_4^*(\theta,x;\varsigma)$ for all $\theta \in [-\tau,0]$ and $x \in \bar{\Omega}$. Since every trajectory of the omega limit set has a backward extension, we can conclude that $\omega_1 = u_1^*(\varsigma)$ and $\omega_4 = u_4^*(\varsigma)$. Thus

$$\begin{aligned}&\lim_{t\to\infty} \|(u_1(t,\cdot;\phi,\varsigma), u_2(t,\cdot;\phi,\varsigma), u_3(t,\cdot;\phi,\varsigma), u_4(t,\cdot;\phi,\varsigma), u_6(t,\cdot;\phi,\varsigma), u_1(t,\cdot;\phi,\varsigma))\\ &-(u_1^*(t,\cdot;,\varsigma),0,0,u_4^*(t,\cdot;,\varsigma),0,0)\| = 0.\end{aligned}$$

This completes the proof. □

**Theorem 4.2.** *Assume the assumptions* (A1) *and* (A2) *hold. Let $u(t,x;\phi)$ be the solution of* (2.6) *with initial value $\phi \in Y^+$. If $\lambda^* > 0$, there exists an $\varepsilon > 0$ such that for any $\phi \in Y^+$ with $\phi_3(0,\cdot) \neq 0$, we have*

$$\liminf_{t\to\infty} u_3(t,x;\phi) \geq \varepsilon,$$

*uniformly on $x \in \bar{\Omega}$.*

**Proof.** Define

$$\begin{aligned}&S_0 = \{\phi = (\phi_1,\phi_2,\cdots,\phi_6) \in Y^+ : \phi_3(0,\cdot) \not\equiv 0\};\\ &\partial S_0 = Y_0^+ \setminus S_0 = \{\phi \in Y^+ : \phi_3(0,\cdot) \equiv 0\},\end{aligned}$$

and

$$P = Y^+ \times H(A), P_0 = S_0 \times H(A), \partial P_0 = \partial U_0 \times H(A),$$

where $H(A)$ is defined in the proof of Theorem 4.1. Let

$$\begin{aligned}u(t,x;\phi,\varsigma) =&(u_1(t,x;\phi,\varsigma), u_2(t,x;\phi,\varsigma), u_3(t,x;\phi,\varsigma),\\ &u_4(t,x;\phi,\varsigma), u_5(t,x;\phi,\varsigma), u_6(t,x;\phi,\varsigma))\end{aligned}$$

be the unique solution of (4.9) with initial value $\phi \in Y^+$, and let

$$\begin{aligned}&\Pi_t^P : P \to P,\\ &(\phi,\varsigma) \mapsto (u_t(\phi,\varsigma), \varsigma \cdot t),\end{aligned}$$

where $u_t(\phi,\varsigma)(x,\theta) = u(t+\theta,x;\phi,\varsigma)$, $\forall t \geq 0, \theta \in [-\tau,0], x \in \Omega$. It's easy to get that $\Pi_t^P(P_0) \subset P_0$ and $\Pi_t^P(\partial P_0) \subset \partial P_0, \forall t \geq 0$. From Lemma 3.2 we know that $\Pi_t^P$ is continuous and point dissipative. Moreover, from Theorem 2.1.8 of [38], we get $\Pi_t^P$ is compact for any $t > \tau$. It then follows from Theorem 3.4.8 of [14] that $\Pi_t^P : P \to P$ admits a global attractor $\mathcal{A}$.

Let $\mathcal{M} = \{(\phi,\varsigma) \in \partial P_0 : \Pi_t^P(\phi,\varsigma) \in \partial P_0, \forall t \geq 0\}$. Let $\omega(\phi,\varsigma)$ represent the $\omega$-limit set for $\Pi_t^P$ and define $M = \{(u_1^*(\varsigma),\hat{0},\hat{0},u_4^*(\varsigma),\hat{0},\hat{0}) : \varsigma \in H(A)\}$, where $u_1^*$ and $u_4^*$ are given in Theorem 4.1. By the similar argument of the proof of Theorem 4.1, we can get that for $t \geq 0$, $x \in \Omega, (\phi,\varsigma) \in \partial P_0$,

$$u_3(t,x;\phi,\varsigma) = 0,$$

$$\lim_{t\to\infty} u_i(t,\cdot;\phi,\varsigma)=0,\ \ i=2,5,6,$$

and

$$\lim_{t\to\infty}\|u_1(t,\cdot;\phi,\varsigma)-u_1^*(t,\cdot;\varsigma)\|=0,$$
$$\lim_{t\to\infty}\|u_4(t,\cdot;\phi,\varsigma)-u_4^*(t,\cdot;\varsigma)\|=0.$$

Therefore, we have $\cup_{(\phi,\varsigma)\in\partial P_0}\omega(\phi,\varsigma)=M$. It follows that

$$\omega(\mathcal{M})=\cup_{(\phi,\varsigma)\in\mathcal{M}}\omega(\phi,\varsigma)=M.$$

Moreover, $M$ is a compact and isolated invariant set, and no subset of $M$ forms a cycle for $\Pi^P$ in $\partial P_0$. Consider the following perturbation system with a positive parameter $\eta$:

$$\begin{cases}\dfrac{\partial u_3(t,x)}{\partial t} =D_3\Delta u_3(t,x)-(\bar r_I(t,x)+q_I+\bar d(t,x))u_3(t,x)\\ \qquad +\displaystyle\int_\Omega\bar\Gamma(t,t-\tau,x,y)\Big[\bar\beta_1(t-\tau,y)\frac{u_1^*(t,x;\varsigma)-\eta}{u_1^*(t,x;\varsigma)+2\eta}u_3(t-\tau,y)\\ \qquad +\bar\beta_2(t-\tau,y)\dfrac{u_4^*(t,x;\varsigma)-\eta}{u_4^*(t,x;\varsigma)+2\eta}u_3(t-\tau,y)\Big]\,dy,\ \ x\in\Omega,\ \ t>0,\\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} =0,\ \ x\in\partial\Omega,\ \ t>0.\end{cases}\tag{4.12}$$

Let $\lambda_\eta^*$ be the upper Lyapunov exponent associated with (4.12). Since $\lambda^*>0$ and the continuity of upper Lyapunov exponent (as shown in Lemma 4.3), there exist a sufficiently small number $\eta>0$ such that $\lambda_\eta^*>0$. Moreover, we have the following claim.

**Claim.** *$M$ is a uniform weak repeller for $\Pi_t^P$, that is,*

$$\limsup_{t\to\infty} d(\Pi_t^P(\phi,\varsigma),M)\geq\eta,\quad \forall(\phi,\varsigma)\in P_0.$$

Suppose, by contradiction, there exists some $(\phi^0,\varsigma^0)\in P_0$ such that

$$\limsup_{t\to\infty} d(\Pi_t^P(\phi^0,\varsigma^0),M)<\eta.$$

Then there exist $t_1>0$ such that $d(\Pi_t^P(\phi^0,\varsigma^0),M)<\eta$, $\forall t\geq t_1$, which implies that

$$|u_1(t,x;\phi^0,\varsigma^0)-u_1^*(t,x;\varsigma^0)|\leq\eta,$$
$$|u_4(t,x;\phi^0,\varsigma^0)-u_4^*(t,x;\varsigma^0)|\leq\eta,$$

and $u_i(t,x;\phi^0,\varsigma^0)\leq\eta,\forall t\geq t_1,x\in\bar\Omega,i=2,3,5,6$. It then follows that $u_3(t,x;\phi^0,\varsigma^0)$ satisfies

$$\begin{cases}\dfrac{\partial u_3(t,x)}{\partial t} \geq D_3\Delta u_3(t,x)-(\bar r_I(t,x)+q_I+\bar d(t,x))u_3(t,x)\\ \qquad +\displaystyle\int_\Omega\bar\Gamma(t,t-\tau,x,y)\Big[\bar\beta_1(t-\tau,y)\frac{u_1^*(t,x;\varsigma^0)-\eta}{u_1^*(t,x;\varsigma^0)+2\eta}u_3(t-\tau,y)\\ \qquad +\bar\beta_2(t-\tau,y)\dfrac{u_4^*(t,x;\varsigma^0)-\eta}{u_4^*(t,x;\varsigma^0)+2\eta}u_3(t-\tau,y)\Big]\,dy,\ \ x\in\Omega,\ \ t>t_1,\\ \dfrac{\partial u_3(t,x)}{\partial \mathbf{n}} =0,\ \ x\in\partial\Omega,\ \ t>t_1.\end{cases}\tag{4.13}$$

By Lemma 4.1, there exist two almost periodic functions $a(\varsigma^0 \cdot t)$ and $w(t,x;\varsigma^0)$ such that $w(t,x;\varsigma^0) = e^{\int_0^t a(\varsigma^0\cdot s)\,ds}\tilde{w}(t,x;\varsigma^0)$ is a solution of (4.12) and

$$\lim_{t\to\infty} \frac{1}{t} a(\varsigma^0 \cdot s)\,ds = \lambda^*_\eta > 0.$$

Recall $(\phi^0, \varsigma^0) \in P_0$, a similar argument to that in [44, Lemma 4.2 (i)] can show that $u_3(t,x;\phi^0,\varsigma^0) > 0$ for $t > 0$. Thus, there exist $t_2 > 0$ and a sufficiently small number $\delta > 0$ such that $u_3(t_2+\theta,x;\phi^0,\varsigma^0) \geq \delta\omega(t_2+\theta,x;\varsigma^0), \forall\theta \in [-\tau,0], x \in \bar{\Omega}$. It then follows from the comparison principle that

$$u_3(t,x;\phi^0,\varsigma^0) \geq \delta\omega(t,x;\varsigma^0) = \delta e^{\int_0^t a(\varsigma^0\cdot s)\,ds}\tilde{w}(t,x;\varsigma^0),\ \forall t \geq t_3,\ x \in \Omega,$$

where $t_3 = \max\{t_1,t_2\}$. Note that $\tilde{w}(t,x;\varsigma^0)$ is almost periodic and

$$\lim_{t\to\infty} e^{\int_0^t a(\varsigma^0\cdot s)\,ds} = \lim_{t\to\infty}\left(e^{\frac{\int_0^t a(\varsigma^0\cdot s\,ds)}{t}}\right)^t = \infty,$$

then we have $\lim_{t\to\infty} u_3(t,x;\phi^0,\varsigma^0) = \infty$, a contradiction.

Since $M$ is an isolated invariant set for $\Pi_t^P$ in $\partial P_0$, the previous claim implies that $M$ is also an isolated invariant set for $\Pi_t^P$ in $P$. Moreover, the claim establishes that $W^s(M) \cap P_0 = \emptyset$, where $W^s(M)$ denotes the stable set of $M$ for $\Pi_t^P$. Specifically, $W^s(M)$ is defined as

$$W^s(M) = \{(\phi,\varsigma) \in P : \omega(\phi,\varsigma) \neq \emptyset, \omega(\phi,\varsigma) \subset M\}.$$

Based on the continuous-time version of [47], we can conclude that the skew-product semiflow $\Pi_t^P : P \to P$ is uniformly persistent with respect to $(P_0, \partial P_0)$. Moreover, since $\Pi_t^P$ is compact for any $t > \tau$, it is also asymptotically smooth. Applying Theorem 3.7 and Remark 3.10 of [28], we can further deduce that $\Pi_t^P$ : $P_0 \to P_0$ possesses a global attractor $\mathcal{A}_0$.

Since $\mathcal{A}_0 \in P_0$ and $\Pi_t^P(\mathcal{A}_0) = \mathcal{A}_0$, it follows that $\mathcal{A}_0 \in \mathrm{Int}(P)$. Define a function $A : P \to [0,+\infty)$ by

$$A(\phi,\varsigma) = \min_{x\in\bar{\Omega}}\{\phi_3(0,x)\}, \quad \forall(\phi,\varsigma) \in P.$$

We can observe that $A$ is a continuous function and $A(\phi,\varsigma) > 0$ for all $(\phi,\varsigma) \in \mathcal{A}_0$. The compactness of $\mathcal{A}_0$ ensures that $\inf_{(\phi,\varsigma)\in\mathcal{A}_0} A(\phi,\varsigma) = \min_{(\phi,\varsigma)\in\mathcal{A}_0} A(\phi,\varsigma) > 0$. Therefore, there exists a positive number $\varepsilon$ such that

$$\liminf_{t\to\infty} u_3(t,\cdot;\phi,\varsigma) \geq \varepsilon, \quad \forall(\phi,\varsigma) \in P_0.$$

This completes the proof. □

# 5. Numerical simulation

In this section, we will give two examples to verify our conclusions. let $\Omega = (0,1), D_1 = 0.3\cdot 10^{-3}, D_2 = 0.2\cdot 10^{-2}, D_3 = 0.3\cdot 10^{-2}, D_4 = 0.5\cdot 10^{-3}, D_6 = 0.6\cdot 10^{-3}$ and some parameters in the model are taken as follows:

$$\Lambda = 1 + 0.5\cos(0.5\pi t), \qquad \alpha = 0.2(1 + 0.5\cos(0.5\pi t)), \quad d = 0.04(3 + \cos\pi t),$$

$$\beta_1 = 0.1(1 + 0.2\cos(0.5\pi t)), \beta_2 = 0.3(2 + 0.1\cos(0.5\pi t)), q_E = 0.1.$$

**Example 5.1.** Take the initial value as

$$u^1(0,x) = (u_1^1(0,x), u_2^1(0,x), u_3^1(0,x), u_4^1(0,x), u_5^1(0,x), u_6^1(0,x)),$$

where

$$u_1^1(0,x) = 2 \cdot (2 + \sin \pi x),\ u_2^1(0,x) = 4 \cdot (2 + \sin \pi x),$$
$$u_3^1(0,x) = 4 \cdot (2 + \sin \pi x),\ u_4^1(0,x) = 10 \cdot (2 + \sin \pi x),$$
$$u_5^1(0,x) = 5 \cdot (2 + \sin \pi x),\ u_6^1(0,x) = 3 \cdot (2 + \sin \pi x),$$

let $r_E = 2(1 + 0.2\cos \pi t)$, $r_I = 1.5 \sin \pi t$, $q_I = 0.01$, then $\lambda^* < 0$. Therefore, system (2.6) is globally asymptotically stable, and all its solutions converge to the disease-free equilibrium $E^0$, as shown in Figure 1.

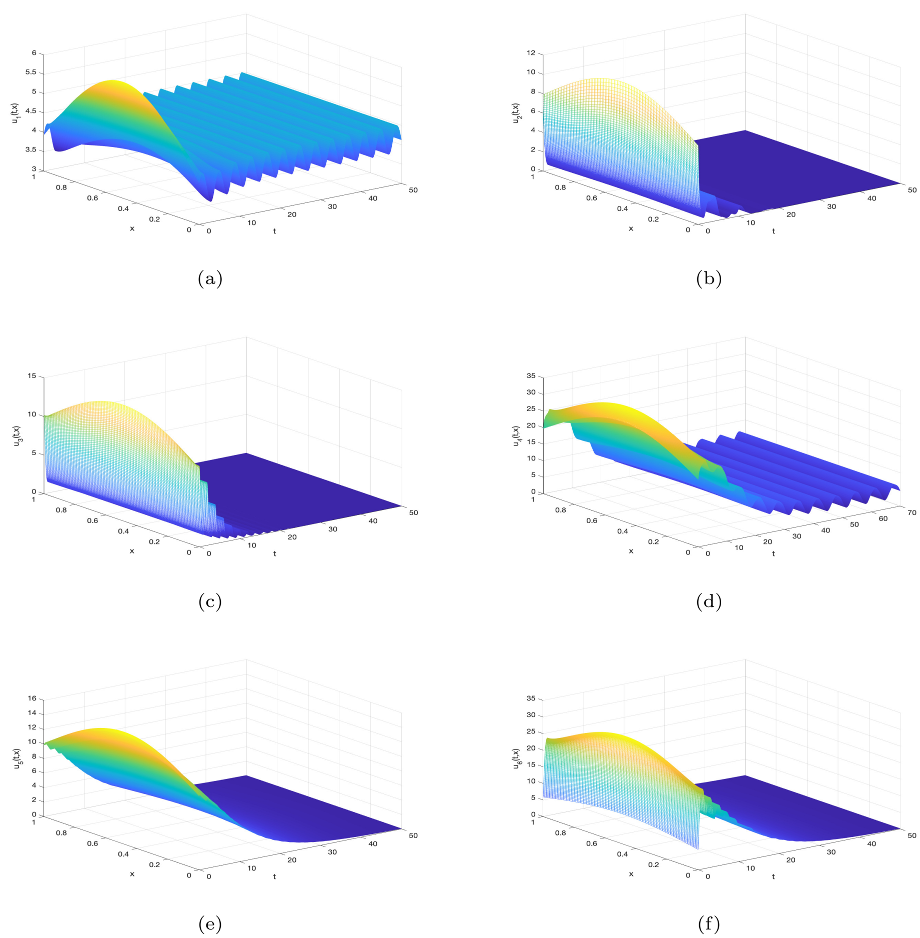


**Figure 1.** Almost periodic solution curves for the model (2.6) with initial values as $u^1(0,x)$ and $\lambda^* < 0$.

**Example 5.2.** Take the initial values as

$$u^2(0,x) = (u_1^2(0,x), u_2^2(0,x), u_3^2(0,x), u_4^2(0,x), u_5^2(0,x), u_6^2(0,x)),$$

and $u^1(0,x)$ respectively, where

$$u_1^2(0,x) = 8 \cdot (2 + \sin \pi x),\ u_2^2(0,x) = 8 \cdot (2 + \sin \pi x),$$
$$u_3^2(0,x) = 4 \cdot (2 + \sin \pi x),\ u_4^2(0,x) = 5 \cdot (2 + \sin \pi x),$$
$$u_5^2(0,x) = 6 \cdot (2 + \sin \pi x),\ u_6^2(0,x) = 4 \cdot (2 + \sin \pi x),$$

let $r_E = 0.02$, $r_I = 1.5\sin(0.5\pi t)$,$q_I = 0.6$, then $\lambda^* > 0$. Therefore, the disease persists, as shown in Figures 2 and 3. Comparing Figures 2 and 3, it can be seen that when $\lambda^* > 0$, the persistence of the disease does not depend on the initial value.

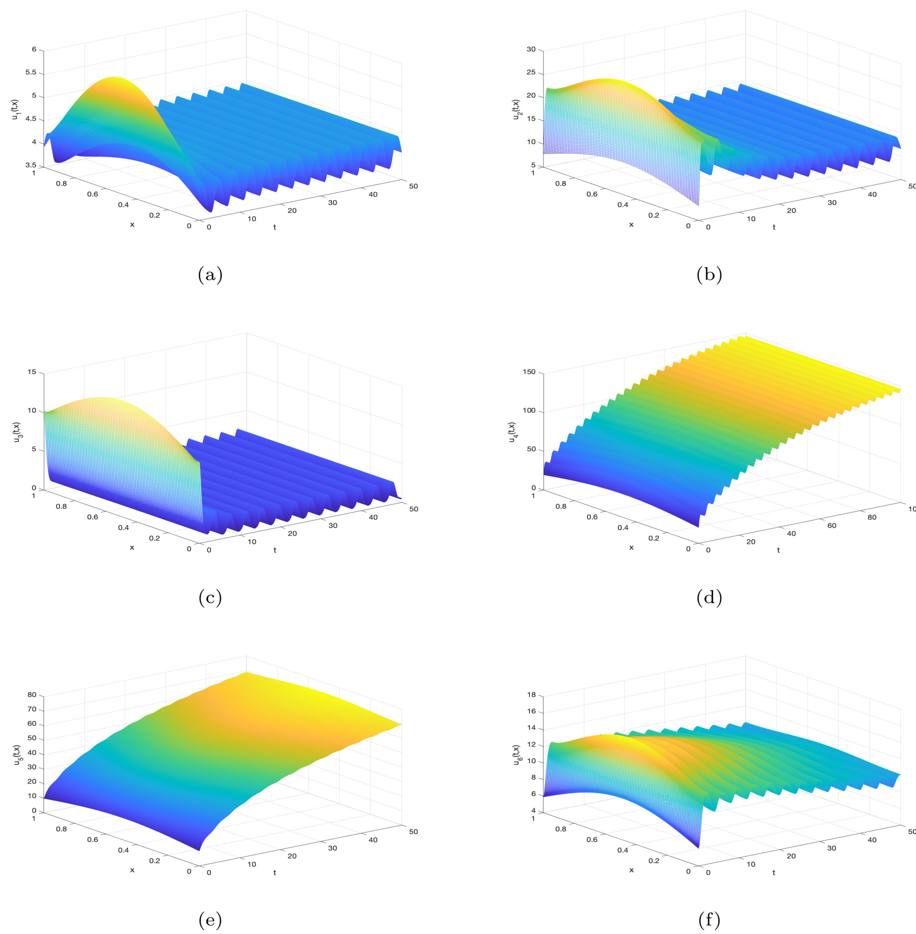


**Figure 2.** Almost periodic solution curves for the model (2.6) with initial values as $u^1(0,x)$ and $\lambda^* > 0$.

**Ethics statement.** The authors declare that the study does not include animal and human experiments that violate ethics.

**Declaration of competing interest.** The authors declare that they have no

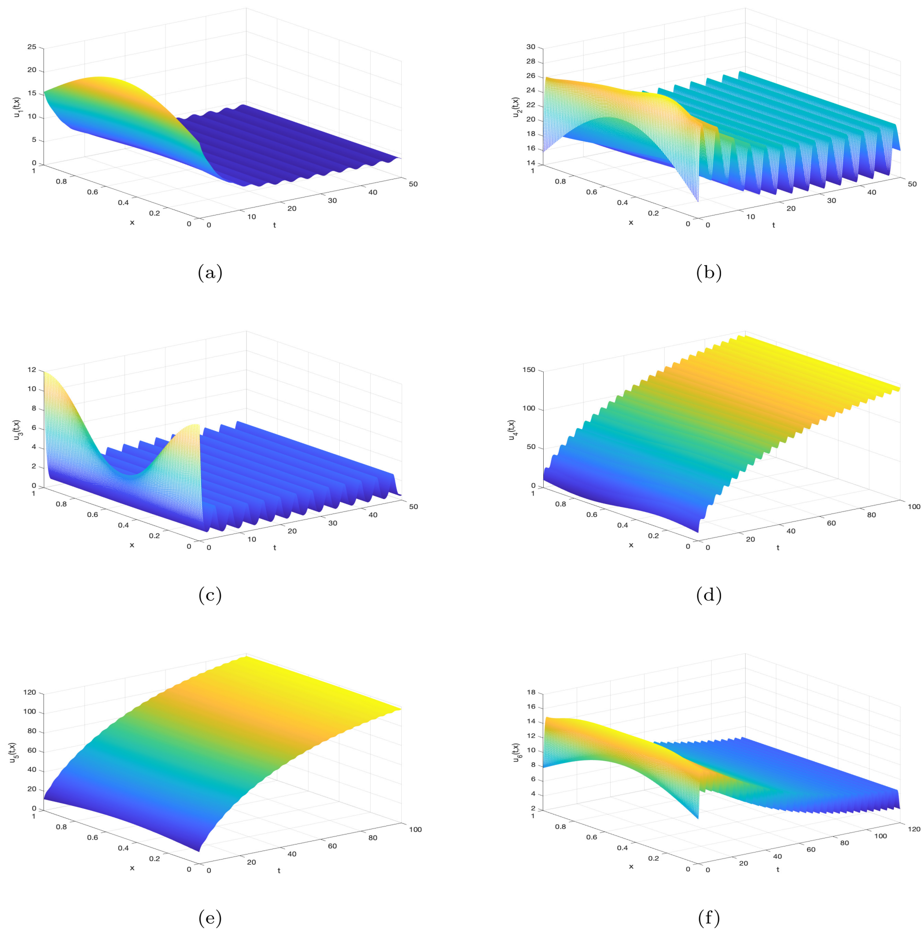


**Figure 3.** Almost periodic solution curves for the model (2.6) with initial values as $u^2(0, x)$ and $\lambda^* > 0$.

known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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