Journal of Applied Analysis and Computation
Volume 15, Number 2, April 2025, 786–809
Website:http://www.jaac-online.com
DOI:10.11948/20240161

THE EXISTENCE OF THE GLOBAL SOLUTION OF THE SEMI-LINEAR SCHRÖDINGER EQUATION WITH NUMERICAL COMPUTATION ON THE DIRICHLET BOUNDARY

Pius W. M. Chin[1,†]

Abstract This paper concerns the analysis of the initial boundary value problem for the semi-linear Schrödinger equation. In the paper, we design a reliable scheme coupling the nonstandard finite difference method in time with the Galerkin combined with the compactness method in the space variables to analyze the problem. The analysis begins by showing that, given initial solutions in specified space, the global solution of the Schrödinger equation exists uniquely. We further show using the a priori estimates obtained from the existence process, that the numerical solution from the designed scheme is stable and converges optimally in specified norms. Furthermore, we show that the scheme replicates or preserves the qualitative properties of the exact solution. Numerical experiments are conducted using a carefully chosen example to justify our theoretical proposition.

Keywords Schrödinger equation, semi-linear equation, nonstandard finite difference method, Galerkin method, optimal rate of convergence.

MSC(2010) 35J47, 35J70, 65N15, 65N30.

1. Introduction

The semi-linear time-dependent evolution equation such as the Schrödinger equation because of its importance in the scientific arena, has attracted a lot of interest to mathematical scholars. This equation is one of the cornerstones of quantum physics which describes what a system of quantum objects such as atoms and subatomic particles will do in the future, based on its current state. The equation also models many physical phenomena such as optics, seismology and bimolecular dynamics to mention a few. See, [10,23,27,37,40–42,45] and [50] for more details. Of recent, the growing interest has been focussed on the solution of the equation both theoretically and numerically. Since we intend in our work to study the global solution by extending it from the local one, then we consider the simplest model of the two

[†]The corresponding author.
[1]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa 0204, Ga-rankuwa, Pretoria, South Africa
Email: pius.chin@smu.ac.za(P. W. M. Chin)

dimensional equation stated as follows:

$$i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = 0, \text{ on, } \Omega \times (0,T), \tag{1.1}$$

$$u(x,t) = 0 \text{ on } \partial\Omega \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \text{ for } x \in \Omega \ \ t = 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded set contained in $\mathbb{R}^2$ and $\partial\Omega$ is a Dirichlet boundary.

There are so many methods in the literature used in the studying of the above problem in $\mathbb{R}^2$ and not in the domain Ω. However, we are aware of some of the results in Ω with homogeneous boundary conditions presented by Y. Tsutsumi [48] and M. Tsutsumi [47] that proved well-posedness for the homogeneous problem in an exterior domain with sufficiently small and smooth initial data. Some other studies have been carried out with problems over Ω in two dimensional space as in [46]. It should also be noted that studies over the domain Ω could lead to results asserting the blowing up of the solution under certain conditions. Other studies have been carried out in one-dimension with non homogeneous boundary conditions and we have seen in Bu [12] that such result proved the well-posedness of smooth solutions with arbitrary large data and a nonlinear term of positive-energy. We also understand the fact that Carroll and Bu [14] for one-dimensional problems, proved similar results like Bu [12] with nonlinear cubic term using inverse scattering techniques. For many more theoretical methods of the problems see [19, 25, 30]. It is important to note at this point that, the theoretical analysis of the time-dependent Schrödinger equation can be quite tedious and can only be possible with limited methods like those listed above. For this reason, numerical methods are the most commonly used in the analysis of the solution of such problems. The most common of these methods are the centered finite difference methods for the spatial derivative and other regularly used methods are the Crank-Nicolson methods which are of second order in both space and time and are numerically stable. Other commonly used methods for the temporal part include the fourth order Runge-Kutta method which as the name implies has fourth order accuracy. There are still many more methods which do not use finite difference methods in the spatial part such as the split operator Fourier method see [15, 33]. These methods depend or rely mostly on their Fourier transforms and for that reason, the spatial accuracy is determined by the algorithms used to perform the transforms. In summary, a growing interest of the numerical techniques, are based on the centered based finite difference techniques described above. Many more methods are the finite element methods, the spectral or the more specialized coupling type of the Schrödinger equation, see [9, 13, 22, 24, 26, 28, 31, 39, 49] for more details.

To the best of this author's knowledge, none of the above mentioned studies and many more, have utilized the a priori estimates obtained from the existence of global solution of the problem, to obtain the optimal rate of convergence of the solution of the Schrödinger equation. For this reason, we exploit this gap and design a reliable numerical scheme consisting of the nonstandard finite difference method in the time variable and the Galerkin combined with the compactness method in the space variables. We then show using the Galerkin combined with the compactness method that the global solution of the problem exists uniquely. We further proceed with the a priori estimates obtained from the existence of the global solution of the problem to show that the numerical solution obtained from this scheme is stable. With the stable scheme, we proceed to show that the numerical solution converges optimally in both the H^1 and the L^2-norms. Furthermore, we show that the numerical solution replicates or preserves the qualitative properties of the exact solution of the problem. Numerical experiments are further conducted with a chosen example, to show that indeed the numerical solution from the scheme validate the theory proposed in the work.The nonstandard finite difference method mentioned above was initiated by Mickens some decade back, see [34] and for some major contributions to the foundation of the method, we refer to [4, 5]. For an overview of the technique, see [35]. As regard the comparison of the standard and nonstandard finite difference methods see [34].

The paper after this section is organized as follows: In section 2 we will briefly state the notations and preliminaries to be used in the paper. This will be followed by section 3 which is devoted to show analytically that the global solution of the problem exists uniquely. In section 4 we design the main numerical scheme NSFD-GM and show that the scheme converges optimally with it's numerical solution replicating the qualitative properties of the exact solution. We proceed in section 5 to conduct a numerical experiments with a chosen example to show that the numerical solution indeed validate the theory of optimal convergence shown in section 4. We conclude our findings and future remarks in section 6.

2. Notations and preliminaries

This section will be concerned with presenting some notations, definitions and preliminary results that will be used in this paper. Some of these results might be common to some of results already used in some papers for examples [16–19] to mention just a few. We may therefore spare duplications by stating just the very useful ones needed in this paper and for details of these, we refer to relevant references . We now present among others, the function spaces where the analysis of the problem is done. We begin with the space $\mathcal{D}(\Omega)$ defined as the linear space of functions which are infinitely differentiable with compact support on Ω i.e

$$\mathcal{D}(\Omega) := \{v|_\Omega : v \in supp(v) \subset \Omega\}.$$

The above space is followed by the space of distributions denoted by $\mathcal{D}'(\Omega)$ and often known as the dual of $\mathcal{D}(\Omega)$. The duality pairing between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$ is often denoted by $\langle\cdot,\cdot\rangle$ and it is remark that, if a function v is a locally integrable, then v can be identified with a distributions by

$$\langle v, \rho\rangle := \int_\Omega v(x)\rho(x)dx, \quad \forall \rho \in \mathcal{D}(\Omega). \tag{2.1}$$

For more on these spaces see [1, 21, 32]. $L^p(\Omega)$ spaces will also be needed in the paper for $1 < p < +\infty$ and this is briefly defined by

$$L^p(\Omega) := \left\{ v : \left(\int_\Omega |v(x)|^p dx \right)^{1/p} < \infty \right\} \tag{2.2}$$

and (2.2) is known as a Banach space with its norm defined by

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx \right)^{1/p}. \tag{2.3}$$

The next very important function space is the Sobolev space denoted and defined for $m \in \mathbb{N}$ and $p \in \mathbb{R}$ with $1 < p \leq \infty$ by

$$W^{m,p}(\Omega) := \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \ \text{for all multi index} \ |\alpha| \leq m\}, \quad (2.4)$$

and this is also a Banach space with the norm

$$\|v\|_{m,p,\Omega} = \left(\sum_{|\alpha| \leq m} \|D^\alpha v\|_{L^p(\Omega)} \right)^{1/p}, \quad p < \infty \quad (2.5)$$

or

$$\|v\|_{m,\infty,\Omega} = \sup_{|\alpha| \leq m} \left(\sup_{x \in \Omega} ess |D^\alpha v(x)| \right), \quad p = \infty. \quad (2.6)$$

In the case when $p = 2$, the above Sobolev space $W^{m,2}(\Omega)$ is often denoted by $H^m(\Omega)$ which is a Hilbert space. If there is no ambiguity, we will be allowed to drop the superscript $p = 2$ when referring to its norm and semi-norm. We would like to continue but this time, with a more general Sobolev space denoted by $H^m[(0,T); X]$ where X is a Hilbert space. This space in view of Lions and Magenes [32] is defined as the space of square integrable functions taking values from $[0,T]$ to X. The norm of the above space is given by

$$\|v\|_{H^m[(0,T);X]} := \left(\sum_{|\alpha| \leq m} \int_0^T \|D^\alpha v(x)\|_X^2 dt \right)^{1/2}. \quad (2.7)$$

In view of (2.7), X will either be L^p or $W^{m,p}$ space and in our paper in particular, $X = L^2, L^4, H_0^1$ and H_0^m. Since we shall be dealing with differential equation involving complex functions, then it better to state the composition of such function and for more on the properties of such functions we refer to text relevant to such materials. We high-light that a complex function $v(x)$ with $x \in \mathbb{R}$ is one of the form $v(x) = Re(v) + iIm(v)$ where $Re(v)$ and $Im(v)$ are real and $i = \sqrt{-1}$ is a complex number. We shall be using some important inequalities like the Hölder, Gronwall's, Young's, Poincaré and Gagliador-Nirenburg inequalities to mention a few, will be referred to some standard text books such as [1, 20, 21, 32] and [43] when required.

We will finally conclude this section by introducing the space where the discrete problem will be defined. This will be a space of finite dimensional space $\mathcal{V}_h$ defined by

$$\mathcal{V}_h := \left\{ v_h \in C^0(\bar{\Omega}) : v_h|_{\partial\Omega} = 0, v_h|_{\mathcal{J}} \in P_1, \ \forall \mathcal{J} \in \mathcal{J}_h \right\} \quad (2.8)$$

where P_1 is the space of polynomial of degree less than or equal to 1, $\mathcal{J}_h$ a regular family of discretization of Ω consisting of compatible triangles $\mathcal{J}$ of sizes $h_{\mathcal{J}} < h$ see [20] for more details. In view of (2.8), we observe that for each mesh size $\mathcal{J}_h$, we associate the finite element space $\mathcal{V}$ of continuous piece-wise linear function that is 1 and zero at every other nodes of $\mathcal{V}$, That is, if $\{P_j\}_{j=1}^n$ are the interior nodes of $\mathcal{J}_h$, then any function in $\mathcal{V}_h$ is uniquely determined by its values at the point P_j and it should also be noted that $\mathcal{V}_h \subset H_0^1(\Omega)$.

3. The semi-linear Schrödinger equation

This section will be devoted to show using the Galerkin method and the compactness method that the solution of the semi-linear Schrödinger equation (1.1)-(1.3) exists uniquely in the space

$$L^\infty\left[(0,T);L^2(\Omega)\right]\cap L^2\left[(0,T);H_0^1\cap H^2(\Omega)\right]\cap L^4\left[(0,T);L^4(\Omega)\right]$$

for given initial data given in (1.3). The first known result of the above type on a bounded domain was due to Brezis and Gallouet [11]. The weak solution of the problem can be proceed by multiplying (1.1)-(1.3) by a test function v and integrating it over Ω and incorporating the Dirichlet boundary condition (1.2) with the use of Green's theorem to arrive at a the following weak problem: find $u \in L^\infty\left[(0,T);L^2(\Omega)\right]\cap L^2\left[(0,T);H_0^1\cap H^2(\Omega)\right]\cap L^4\left[(0,T);L^4(\Omega)\right]$ such that for all $u_0 \in H_0^1\cap H^2(\Omega)$ we have

$$i\left\langle \frac{\partial u}{\partial t}, v\right\rangle + \langle \nabla u, \nabla v\rangle + \left\langle |u|^2 u, v\right\rangle = 0, \tag{3.1}$$

$$\langle u(x,0), v\rangle = \langle u_0, v\rangle\,, \tag{3.2}$$

for all $v \in H_0^1(\Omega)$.

The above variational or weak problem (3.1)-(3.2) leads to the introduction of the Galerkin frame-work which will be used to solve the problem as follows: We introduce the L^2 orthonormal basis given by $\{e_1, e_2, e_2, \cdots, e_m\} \subset H_0^1\cap H^2(\Omega)$ where $m\in\mathbb{N}$. We use the basis functions together with the test function v spanned by $v \in span\{e_1, e_2, e_2, \cdots, e_m\}$ to approximate the solution u defined by

$$u_m = \sum_{i=1}^{m}\gamma_i(t)e_i. \tag{3.3}$$

Although an explicit basis can not be computed, we know a priori that one exists because $H_0^1(\Omega)$ is a separable Hilbert space. Substituting equation (3.3) into the semi-linear Schrödinger equation, (1.1)-(1.3) satisfies the following ordinary differential equation

$$i\frac{\partial u_m}{\partial t} - \Delta u_m + P_m\left(|u_m|^2 u_m\right) = 0, \ \text{ on } \ \Omega\times(0,T), \tag{3.4}$$

$$u_m(x,t) = 0 \ \text{ on } \ \partial\Omega\times(0,T), \tag{3.5}$$

$$u(x,0) = P_m u_0 \ \text{ on } \Omega. \tag{3.6}$$

It should be observed at this point that the ordinary differential equation (3.4)-(3.6) is also satisfied with the discrete solution $\{u_m\}$ taking values in the finite dimensional subspace $V_m \in H_0^1(\Omega)$ defined by the equation (2.8). We should also observe that the operator P_m stated in (3.4) denotes the orthogonal projection,

$$P_m : H^{-1}(\Omega) \longrightarrow \mathcal{V}_m \subset H^{-1}(\Omega), \tag{3.7}$$

meaning that the operator is extended from $L^2(\Omega)$ onto $H^{-1}(\Omega)$ and defined on the $H^{-1}(\Omega)$ by

$$P_m\left(\sum_{k\in m}\gamma_m^k(t)u_k\right) = \sum_{k=1}^{m}\gamma_m^k(t)u_k. \tag{3.8}$$

The above connection between the semi-linear Schrödinger equation (1.1)-(1.3) and the system of ordinary differential equation (3.4)-(3.6), validate the fact that the solution of these equations are equivalent as seen classically in Temam 1997 [44] and Evans 1998 [21]. To this end we use the above frame-work to show that the global solution of the semi-linear Schrödinger equation exists uniquely. The results will be stated in the following Theorem 3.1:

Theorem 3.1. *Given the initial solution $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u|_{\partial\Omega} = 0$. Then there exists a unique global solution of the semi-linear Schrödinger equation (1.1)-(1.3)*

$$u \in L^\infty\left[(0,T);L^2(\Omega)\right] \cap L^2\left[(0,T);H_0^1(\Omega)\cap H^2(\Omega)\right] \cap L^4\left[(0,T);L^4(\Omega)\right]$$
$$\text{and } \frac{\partial u}{\partial t} \in L^2\left[(0,T);H^{-1}(\Omega)\right]$$

such that the variational problem (3.1)-(3.2) with the estimate

$$\begin{aligned}&\|u(\cdot,t)\|^2 + \left\|\frac{\partial u(\cdot,t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^4_{L^4(\Omega)}\\ \leq& C\left(\|u_0\|^2_{H^2(\Omega)} + \|u_0\|^6_{H^2(\Omega)}\right)\end{aligned} \tag{3.9}$$

are satisfied.

The proof of the above theorem above will be executed in three main subsections which will be 3.1, 3.2 and 3.3. Subsection 3.1 will address the uniform approximation estimates, followed by 3.2 which address the compactness and passage to the limit. Finally subsection 3.3 will be focused on the uniqueness of the solution of the equation

3.1. Uniform approximate estimates of the solution of the equation

For us to show under this subsection the uniform approximation estimates of the semi-linear Schrödinger equation, we will first consider here and after that all constants independent of m will be denoted by C In view of equation (3.4)-(3.6), the variational equation is stated by

$$i\left\langle \frac{\partial u_m}{\partial t}, v\right\rangle + \langle \nabla u_m, \nabla v\rangle + P_m\left\langle |u_m|^2 u_m, v\right\rangle = 0, \tag{3.10}$$
$$\langle u_m(x,0), v\rangle = \langle u_0, v\rangle. \tag{3.11}$$

Setting $v = \bar{u_m}(t)$ in equation (3.10) we have

$$\int_\Omega \left[i\frac{\partial u_m}{\partial t}\bar{u_m} + |\nabla u_m|^2 + |u_m|^4\right] dx = 0$$

from where we obtain

$$i\frac{1}{2}\|u_m\|^2_{L^2} + \|\nabla u_m\|^2_{L^2} + \int_\Omega |u_m|^4 dx = 0. \tag{3.12}$$

Integrating both sides of (3.12) over the time interval t and using the initial value inequality $\|u_m(\cdot,t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{L^2(\Omega)}$ we have

$$\|u_m(\cdot,t)\|^2_{L^2(\Omega} + 2\|u_m\|^4_{L^4(\Omega)} \le \|u_0\|^2_{L^2(\Omega)} \tag{3.13}$$

after subtracting the complex conjugate of (3.12) from itself. Hence, using Gagliadoe-Nirenburg and the Gronwall's Lemma on inequalities (3.13) for any $t \in [0,T]$ yield

$$\|u_m(\cdot,t)\|^2_{L^2(\Omega} + \frac{1}{2}\|u_m\|^4_{L^4(\Omega)} \le C_1\|u_0\|^2_{L^2(\Omega)}. \tag{3.14}$$

From the above inequality (3.14), the estimate of $\|u_m(\cdot,t)\|^2_{L^2(\Omega)}$ can be obtained .

As for the estimate for the derivative $\frac{\partial \bar{u_m}}{\partial t}$, we differentiate equation (3.10) and set $v = \frac{\partial \bar{u_m}}{\partial t}$ to have

$$i\left\langle \frac{\partial^2 u_m}{\partial t^2}, \frac{\partial \bar{u}_m}{\partial t}\right\rangle + \left\langle \frac{\partial}{\partial t}(\nabla u_m), \nabla\frac{\partial \bar{u}_m}{\partial t}\right\rangle + \left\langle \frac{\partial}{\partial t}|u_m|^2 u_m, \frac{\partial \bar{u}_m}{\partial t}\right\rangle = 0$$

and integrating it as was the case in (3.12) yield

$$\int_\Omega \left[i\frac{\partial^2 u_m}{\partial t^2}\frac{\partial \bar{u}_m}{\partial t} + \left|\frac{\partial}{\partial t}(\nabla u_m)\right|^2 + \frac{\partial}{\partial t}(|u_m|^2 u_m)\frac{\partial \bar{u}_m}{\partial t}\right] dx = 0. \tag{3.15}$$

Subtracting the complex conjugate of (3.15) from itself we have

$$\begin{aligned}
&\int_0^t \int_\Omega i\left[\frac{\partial}{\partial t}\left|\frac{\partial u_m}{\partial t}\right|^2\right] dxdt \\
&+ \int_0^t \int_\Omega iIm\left[\frac{\partial}{\partial t}(|u_m|^2 u_m)\frac{\partial \bar{u}_m}{\partial t} + \frac{\partial}{\partial t}(|u_m|^2 u_m)\frac{\partial u_m}{\partial t}\right] dxdt \\
=& - Re\int_0^t \int_\Omega \left[\frac{\partial}{\partial t}(|u_m|^2 u_m)\frac{\partial \bar{u}_m}{\partial t} - \frac{\partial}{\partial t}(|u_m|^2 u_m)\frac{\partial u_m}{\partial t}\right] dxdt.
\end{aligned} \tag{3.16}$$

Since it can easily be calculated that

$$\frac{\partial}{\partial t}\left(|u_m|^2 u_m\right)\frac{\partial \bar{u}_m}{\partial t} = 2|u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2 + (u_m)^2\left(\frac{\partial u_m}{\partial t}\right)^2$$

then using this into equation (3.16) leads to the calculation of the following two identities:

$$\frac{\partial}{\partial t}\left(|u_m|^2 u_m\right)\frac{\partial \bar{u}_m}{\partial t} + \frac{\partial}{\partial t}\left(|u_m|^2 \bar{u}_m\right)\frac{\partial u_m}{\partial t} = 2|u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2 + \left(\frac{\partial}{\partial t}(|u_m|^2)\right)^2 \tag{3.17}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(|u_m|^2 u_m\right)\frac{\partial \bar{u}_m}{\partial t} - \frac{\partial}{\partial t}\left(|u_m|^2 \bar{u}_m\right)\frac{\partial u_m}{\partial t} \\
=&(u_m)^2\left(\frac{\partial \bar{u}_m}{\partial t}\right) - (\bar{u}_m)^2\left(\frac{\partial u_m}{\partial t}\right)^2.
\end{aligned} \tag{3.18}$$

Using these identities (3.17) and (3.18) back into equation (3.16) yield

$$\begin{aligned}&\left\|\frac{\partial u_m}{\partial t}\right\|_{L^2}^2+2Im\int_0^t\int_\Omega|u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2dxdt+Im\int_0^t\int_\Omega\left(\frac{\partial}{\partial t}|u_m|^2\right)^2dxdt\\ \le&2|Re|\int_0^t\int_\Omega|u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2dxdt+\left\|\frac{\partial u_m(\cdot,0)}{\partial t}\right\|_{L^2(\Omega)}^2. \end{aligned}\tag{3.19}$$

Using the fact that $2(Imz-Rez)\geq 2|Rez|>0$ then the above inequality (3.19) becomes

$$\left\|\frac{\partial u_m(\cdot,t)}{\partial t}\right\|_{L^2(\Omega)}^2+\int_0^t\int_\Omega|u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2dxdt\le C_2\left\|\frac{\partial u_m(\cdot,0)}{\partial t}\right\|_{L^2(\Omega)}^2. \tag{3.20}$$

We now estimate the term $\left\|\frac{\partial u_m(\cdot,0)}{\partial t}\right\|_{L^2(\Omega)}^2$. on the right hand side of inequality (3.20) by considering equation (3.10) as follows:

$$i\left\langle\frac{\partial u_m(\cdot,t)}{\partial t},v\right\rangle_{L^2(\Omega)}=\langle\Delta u_m,v\rangle_{L^2(\Omega)}-\left\langle|u_m|^2u_m,v\right\rangle_{L^2(\Omega)}. \tag{3.21}$$

Setting $v=\frac{\partial\bar u_m(\cdot,0)}{\partial t}$ for $t=0$ in equation (3.21) we have

$$\int_\Omega i\left|\frac{\partial u_m(\cdot,0)}{\partial t}\right|^2dx=\int_\Omega\Delta u_m(\cdot,o)\frac{\partial\bar u_m(\cdot,0)}{\partial t}dx-\int_\Omega|u_m|^2u_m\frac{\partial\bar u_m(\cdot,0)}{\partial t}dx$$

and hence we have

$$\begin{aligned}\left\|\frac{\partial u_m(\cdot,t)}{\partial t}\right\|_{L^2(\Omega)}^2&\le C_3\|\Delta u_m(\cdot,0)\|^2+C_4\int_\Omega|u_m(\cdot,0)|^6\\ &\le C_3\|\Delta u_m(\cdot,0)\|^2+C_4\|u_m(\cdot,0)\|_{L^6(\Omega)}^6.\end{aligned}\tag{3.22}$$

Since

$$\|u_m(\cdot,t)\|_{L^6(\Omega)}^6\le\beta^6\left(\|\nabla u_m(\cdot,t)\|_{L^2}^4\|u_m(\cdot,t)\|_{L^2(\Omega)}^2\right)\quad\text{for any}\quad t\in[0,T] \tag{3.23}$$

then thanks to [29] we have

$$\begin{aligned}\|u_m(\cdot,t)\|_{L^6(\Omega)}^6&\le C_5\left(\|\nabla u_m(\cdot,t)\|_{L^2}^4\|u_m(\cdot,t)\|_{L^2(\Omega)}^2\right)\\ &\le C_6\|u_m(\cdot,0)\|_{H_0^1(\Omega)}^6\\ &\le C_6\|u_0\|_{H^2(\Omega)}^6(\Omega).\end{aligned}\tag{3.24}$$

Using the inequality (3.22), it is well-known that

$$\|\Delta u_m(\cdot,0)\|\le C_7\|u_0\|_{H^2(\Omega)}^2. \tag{3.25}$$

Thus, using (3.24) and (3.25) back into inequality (3.21) yield

$$\left\|\frac{\partial u_m(\cdot,0)}{\partial t}\right\|_{L^2(\Omega)}^2\le C_8\|u_0\|_{H^2(\Omega)}^2+C_9\|u_0\|_{H^2}^6. \tag{3.26}$$

Using the inequality (3.26) back into (3.20) yield

$$\left\|\frac{\partial u_m(\cdot,t)}{\partial t}\right\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega |u_m|^2\left|\frac{\partial u_m}{\partial t}\right|^2 dxdt \le C_{10}\left(\|u_0\|^2_{H^2(\Omega)} + \|u_0\|^6_{H^2(\Omega)}\right). \tag{3.27}$$

From the above inequality (3.27) we can obtain the estimate $\left\|\frac{\partial u_m(\cdot,t)}{\partial t}\right\|^2_{L^2(\Omega)}$.

The only estimate that remains to be evaluated now is ∇u_m. We obtain this estimate by multiplying equation (3.10) $\frac{\partial \bar{u}_m}{\partial t}$ to have

$$\int_0^t\int_\Omega\left[\left(i\frac{\partial u_m}{\partial t}\right)\frac{\partial \bar{u}_m}{\partial t}\right]dxdt + \int_0^t\int_\Omega\frac{\partial}{\partial t}|\nabla u_m|^2dxdt + \int_0^t\int_\Omega\frac{\partial}{\partial t}|u_m|^4dxdt = 0$$

after integrating over the time interval and using the Green theorem with consideration only taken on the real part. This further leads to the following equation

$$\int_0^t\frac{\partial}{\partial t}\left[\|\nabla u_m\|^2_{L^2(\Omega)} + \frac{1}{4}\|u_m\|^4_{L^4(\Omega)}\right]dt = 0. \tag{3.28}$$

By the use of Sobolev embedding theorem $H^1(\Omega)\hookrightarrow L^p(\Omega), p\le 6$ we obtain for all $t\in[0,T]$

$$\|\nabla u_m\|^2_{L^{L^2}(\Omega)} + \frac{1}{4}\|u_m\|^4_{L^4(\Omega)} \le C'\|\nabla u_m(\cdot,0)\|^2_{L^2(\Omega)}. \tag{3.29}$$

Combining all these inequalities (3.14), (3.27) and (3.29) above conclude the fact that the sequence of solutions $\{u_m\},\ m\in\mathbb{N}$ is uniformly bounded in the space.

$$L^\infty\left[(0,T);L^2(\Omega)\right]\cap L^2\left[(0,T);H^1_0\cap H^2(\Omega)\right]\cap L^4\left[(0,T);L^4(\Omega)\right]$$
$$\text{and}\ \ \frac{\partial u}{\partial t}\in L^2\left[(0,T);H^{-1}(\Omega)\right].$$

3.2. Compactness and passage to the limit of the solution of the problem

In view of the above estimates for uniform boundedness of the solution obtained in section 3.1, we reserve this subsection to show that the sequences of solutions $\{u_m\}$ of the semi-linear Schrödinger equation converges strongly to the solution u. We proceed to show this by first of all recall that, we have obtained the following approximate solution u_m defined on the interval $[0,T]$:

$$u_m \text{ is uniformly bounded in } L^\infty\left[(0,T);L^2(\Omega)\right],$$
$$u_m \text{ is uniformly bounded in } L^2\left[(0,T);H^1_0\cap H^2(\Omega)\right],$$
$$u_m \text{ is uniformly bounded in } L^4\left[(0,T);L^4(\Omega)\right],$$
$$\frac{\partial u_m}{\partial t} \text{ is uniformly bounded in } L^2\left[(0,T);L^2.(\Omega)\right].$$

In view of the following embedding

$$H^1_0(\Omega)\hookrightarrow L^2(\Omega),$$

by Banach-Alaoglu's Theorem found in [36], there exists a subsequence of u_m still denoted by u_m such that

$$\begin{aligned}
&u_m \longrightarrow u \text{ weakly star in } L^\infty\left[(0,T);L^2(\Omega)\right],\\
&u_m c u \text{ weakly in } L^2\left[(0,T);H_0^1\cap H^2(\Omega)\right],\\
&u_m \longrightarrow u \text{ weakly in } L^4\left[(0,T);\ L^4(\Omega)\right],\\
&\frac{\partial u_m}{\partial t} \longrightarrow \frac{\partial u}{\partial t} \text{ weakly in } L^2\left[(0,T);L^2.(\Omega)\right],
\end{aligned}$$

and in view of the following Theorem 3.2 found in [36] $u_m \longrightarrow u$ strongly in $L^2\left[(0,T);L^2(\Omega)\right]$.

Theorem 3.2. *Suppose that $X \hookrightarrow Y \hookrightarrow Z$ are Banach spaces where X, Z are reflexive and X is compactly embedding in Y. Let $1 < p < \infty$. If the functions $w_N : (0,T) \longrightarrow X$ are such that $\{w_N\}$ is uniformly bounded in $L^2\left[(0,T);X\right]$ and $\left\{\frac{\partial u}{\partial t}\right\}$ is uniformly bounded in $L^p\left[(0,T);Z\right]$, then there is a subsequence that converges strongly in $L^2\left[(0,T);Y\right]$.*

With the above strong convergence of u_m in $L^2\left[(0,T);L^2(\Omega)\right]$ shown, we only need to show below that this solution satisfies equation (3.11). To this end, we introduce another test function say θ which is continuously differentiable on $[0,T]$ with values $\theta(0) = 1$ and $\theta(T) = 0$. With these claims in place, we proceed in view of the variational formulation (3.10) with the test function θ to have

$$i\left\langle \frac{\partial u_m}{\partial t}, v\right\rangle \theta(t) + \langle \nabla u_m, \nabla v\rangle \theta(t) + \left\langle |u_m|^2 u_m, v\right\rangle \theta(t) = 0. \tag{3.30}$$

Integrating equation (3.30) by part over the interval $[0,T]$ yield

$$\begin{aligned}
&-\int_0^T i\left\langle \frac{\partial u_m}{\partial t}, \theta(t)\right\rangle v dt + \int_0^T \langle \nabla u_m, \nabla v\theta(t)\rangle dt + \int_0^T \left\langle |u_m|^2 u_m, v\theta(t)\right\rangle dt\\
&= \langle u(0), v\rangle \theta(t).
\end{aligned} \tag{3.31}$$

In view of the Theorem 3.2, $u_m(t)$ is uniformly bounded, which by passing to the limit, we have in view of (3.31) yield

$$\begin{aligned}
&-\int_0^T i\left\langle \frac{\partial u}{\partial t}, \theta(t)\right\rangle v dt + \int_0^T \langle \nabla u, \nabla v\theta(t)\rangle dt + \int_0^T \left\langle |u|^2 u, v\theta(t)\right\rangle dt\\
&= \langle u(0), v\rangle \theta(0),
\end{aligned} \tag{3.32}$$

which in particular holds for $\theta(t)\mathcal{D}'(0,T)$, meaning therefore that u from equation (3.32) is satisfied in the distributional sense. Comparing equations (3.30) and (3.32) yield

$$\langle u(0) - u_0, v\rangle = 0, \ \ \forall v \in H_0^1(\Omega)$$

which is the equation (3.11) as required.

3.3. Uniqueness of the solution of the equation

We set this subsection aside to show the uniqueness of the solution of the semilinear Schrödinger equation (1.1)-(1.3). This is shown by letting $u = u_1 - u_2$.

Since the solution u satisfies equation (1.1) and (1.2) where $u|_{\partial\Omega} = 0$, then $u(0) = u_1(0) - u_2(0) = 0$. In view of this, we proceed using equation (1.1) to obtain

$$i\frac{\partial u}{\partial t} - \Delta u + |u_1|^2 u_1 - |u_2|^2 u_2 = 0. \tag{3.33}$$

In view of equation (3.33) and the factorization of $|u_1|^2 - |u_2|^2 = (|u_1| - |u_2|)(|u_1| + |u_2|)$ we have

$$i\left\langle \frac{\partial u}{\partial t}, \bar{u} \right\rangle + \|\nabla u\|^2_{L^2(\Omega)} \leq |2|^2_{L^4(\Omega)} \left[|u_1|^2 + |u_1||u_2| + |u_2|^2 \right] \tag{3.34}$$

after multiplying through out by $\bar{u}$. Using the Gagliardor-Nirenberg and the Young's inequalities on inequality (3.34) in view of (3.13) yield

$$\|u(t)\|^2_{L^2(\Omega)} \leq \|u(0)\|^2_{L^2(\Omega)} = 0, \ \ \forall \ \ t \geq 0, \tag{3.35}$$

after applying the Gronwall inequality. Hence, from (3.35) the uniqueness of the solution of the problem is achieved.

4. The Schrödinger NSFD-GM numerical scheme

We set aside this section for the the numerical analysis of the semi-linear Schrödinger equation. This analysis is done on the design reliable scheme NSFD-GM mentioned earlier. The scheme will consist of coupling the nonstandard finite difference method on the time and the Galerkin combined with compactness method on the space variables. With the scheme, we intend to show that the numerical solution obtained from it is stable and converges optimally in the L^2 as well as the H^1-norms. Furthermore, we show that the scheme replicates the decaying properties of the exact solution. The above stated objectives will be achieved by first stating the discrete version of the variational form of the semi-linear Schrödinger equation (4.1)-(4.2) which states, find $u_h : [0, T] \longrightarrow \mathcal{V}_h$, the discrete solution such that

$$i\left\langle \frac{\partial u_h}{\partial t}, v_h \right\rangle + \langle \nabla u_h, \nabla v_h \rangle + \left\langle |u_h|^2 u_h, v_h \right\rangle = 0, \tag{4.1}$$

$$\langle u_h(x,0), v_h \rangle = \langle P_h u_0, v_h \rangle, \tag{4.2}$$

where P_h is the orthogonal projection onto $\mathcal{V}_h$.

The above discrete version leads to the following frame-work that is geared toward asserting the analysis of the numerical solution of equation (4.1)-(4.2) and also by using the fact that, the subspace $\mathcal{V}_h \subset H^1_0(\Omega)$ as seen in [20]. Besides, the above assumption, we will also assume that the projector P_h with respect to the Dirichlet inner product $\nabla u, \nabla v$ satisfies the inequality

$$\|P_h v - v\| \leq Ch^2 \|v\|_{H^2}, \ \ \text{for} \ \ v \in H^1_0 \cap H^2 \tag{4.3}$$

where $\|\cdot\|$ is the usual norm in L^2 and H^2 is a standard Sobolev space with some constant C. It is also well-known in view of [51] that if u is sufficiently smooth on a closed time interval $[0, T]$ and the discrete initial data are suitably chosen, then

$$|u(t) - u_h(t)| \leq C_1(u, C_2, C_3)H^2, \quad \forall t \in [0, T] \tag{4.4}$$

where C_2 is the bound on U and ∇u with C_3 the constant in (4.3).

We now continue to address the afore-mentioned objective after putting in place the desired frame-work. To this end, we set the time step size $t_n = n\Delta t$ for $n = 0, 1, 2, \cdots, \mathbb{N}$ over the time interval $[0, T]$. With this time step, we find the NSFD-GM approximate solution $\{U_h^n\}$ such that $U_h^n \approx u_h^n$ at each discrete time t_n in the finite dimensional space $\mathcal{V}_h$ for sufficiently smooth functions. The above approximation, permits us to define the NSFD-GM scheme as that which consists of finding a fully discrete solution of the following semi-linear Schrödinger equation $U_h^n \in \mathcal{V}_h$ for $v_h \in \mathcal{V}_h$ such that for all $\mathcal{V}_h \subset H_0^1(\Omega)$ we have

$$\langle \delta_n U_h^n(t), v_h \rangle + \langle \nabla U_h^n, \nabla v_h \rangle + \left\langle |U_h^n|^2 U_h^n, v_h \right\rangle = 0, \tag{4.5}$$
$$\langle U_h^n, v_h \rangle = \langle P_h u_0, v_h \rangle, \tag{4.6}$$

are satisfied, where

$$\delta_n U_h^n = \frac{U_h^n - U_h^{n-1}}{\phi(\Delta t)} \text{ and } \phi(\Delta t) = \frac{e^{\lambda \Delta t} - 1}{\lambda}. \tag{4.7}$$

The above new frame-work needs the following comments:

(a) that the special and complicated function $\phi(\Delta t)$ is in such a way that

$$0 < \phi(\Delta t) < 1 \text{ for } n = 1, 2, 3, \cdots, N. \tag{4.8}$$

(b) That if the nonlinear function $|U_h^n|^2 U_h$ is made very small that its effect is negligible, or even zero then the scheme (4.5) will coincide to the exact scheme.

$$\left\langle \frac{U_h^{n+1} - U_h^n}{\phi(\Delta t)} \right\rangle + \langle \nabla U_h^n, \nabla v_h \rangle = 0 \tag{4.9}$$

which according to Michens [34], replicates the decaying to zero property.

4.1. The stability of the NSFD-GM scheme for the equation

This subsection is reserved to show that the numerical scheme of the Schrodinger equation (4.5)-(4.6) is stable. In other words, we want to show that the numerical solution $U_h^n(t)$ of the Schrödinger scheme NSFD-GM (4.5)-(4.6) is uniformly bounded as stated in the following Theorem 4.1

Theorem 4.1. *Assume that the solution of the semi-linear Schrödinger equation u in equation (4.1)-(4.2) is regular. Then given $U_h^0 \in \mathcal{V}_h$, the solution $U_h^n(t)$ of the NSFD-GM Schrödinger scheme (4.5)-(4.6) remain bounded in the following sense:*

$$|U_h^0|^2 \leq |U_h^0|^2, \tag{4.10}$$
$$\sum_{n=1}^{N} |U^n - U_h^{n-1}|^2 \leq |U_h^0|^2. \tag{4.11}$$

Proof. We proceed with the proof of the above theorem by setting $v_h = \bar{U_h^n}(t)$ in equation (4.5) to obtain

$$i \left\langle U_h^n(t) - U_n^{n-1}(t), U_h^n(t) \right\rangle + \phi(\Delta t) \|\nabla U_h^n(t)\|_{L^2}^2 + \phi(\Delta t) \left| U_h^n \right|^2 U_h^n = 0.$$

In view of (3.12), we have from the above equation

$$i\left\langle U_h^n(t)-U_n^{n-1}(t),U_h^n(t)\right\rangle+\phi(\Delta t)\|\nabla U_h^n(t)\|_{L^2}^2+\phi(\Delta t)\left\|U_h^n\right\|_{L^4(\Omega)}^4\leq 0. \quad (4.12)$$

It is well-known in view of the first term of the left hand side of the inequality (4.12) that

$$\left\langle U_h^n(t)-U_h^{n-1}(t),U_h^n(t)\right\rangle=\frac{1}{2}|U_h^n|^2-\frac{1}{2}|U_h^{n-1}|^2+\frac{1}{2}|U_h^n-U_h^{n-1}|^2$$

re-introducing this identity back into the inequality ((4.12) and considering only the real part of the problem yield

$$|U_h^n|^2-|U_h^{n-1}|^2+|U_h^n-U_h^{n-1}|^2+2\phi(\Delta t)\left\|\nabla U_h^n\right\|_{L^2}^2+2\phi(\Delta t)\left\|U_h^n\right\|_{L^4}^4\leq 0. \quad (4.13)$$

Summing the above inequality (4.13) for $n=1,2,\cdots,\mathbb{N}$ we have

$$|U_h^n|^2+\sum_{n=1}^{\mathbb{N}}|U_h^n-U_h^{n-1}|^2+2\phi(\Delta t)\sum_{n=1}^{\mathbb{N}}\left\|\nabla U_h^n\right\|_{L^2}^2+2\phi(\Delta t)\sum_{n=1}^{\mathbb{N}}\left\|U_h^n\right\|_{L^4}^4\leq|U_h^0|^2. \quad (4.14)$$

Hence, in view of (3.14) and (3.29), we can immediately read the results (4.10) and (4.11) from inequality (4.14) as required. □

4.2. Optimal convergence of NSFD-GM scheme

We show under this subsection that the numerical solution obtained from the NSFD-GM Schrödinger scheme converges optimally in both the L^2 and H^1-norms. Furthermore, we show that these numerical solutions replicates the decaying properties of the exact solution. The two objectives under this subsection will be achieved by first statin without proof the following results from Shen [38].

Lemma 4.1. *Let Δt, γ and a_k, b_k, d_k, γ_k for the integer $k\geq 0$ be non-negative numbers such that*

$$a_J+\sum_{k=0}^{J}b_k\Delta t\leq\sum_{k=0}^{J}d_k a_J\Delta t+\sum_{k=0}^{J}\gamma_k\Delta t+\gamma,\ \ \forall\ J\geq 0. \quad (4.15)$$

Suppose that

$$d_k\Delta t<1\ \text{ and set }\ \sigma_k=(1-d_k\Delta t)^{-1},\ \forall\ k\geq 0. \quad (4.16)$$

Then we have

$$a_J+\sum_{k=0}^{J}b_k\Delta t\leq\exp\left(\sum_{k=0}^{J}d_k\Delta t\right)\left(\sum_{k=0}^{J}\gamma_k\Delta t+\gamma\right)\ \forall\ J\geq 0. \quad (4.17)$$

With the NSFD-GM Schrödinger scheme frame-work in place, and the availability of the above Lemma 4.1 thanks to Shen [38], we can state and prove the error estimate in the next Theorem 4.2.

Theorem 4.2. *Assume that Φ_k be a non-negative number and the solution of the continuous and discrete semi-linear Schrödinger equation (4.1)-(4.2) and (4.5)-(4.6) respectively exists uniquely satisfying*

$$\Phi_k\phi(\Delta t) < 1 \ \ \textit{and} \ \ \sigma_k = (1 - \Phi_k\phi(\Delta t))^{-1}, \ \forall\, k \geq 0.$$

Then we have

$$\begin{aligned} &\|u(t_J) - U_h(t_J)\| + \phi(\Delta t)\sum_{k=0}^{J} |\nabla(u(t_J) - U_h(t_J))|^2 \\ \leq &C(t_J)(\phi(\Delta t))^2, \ \ \forall J \geq 0. \end{aligned} \tag{4.18}$$

Proof. We use the implicit nonstandard finite difference method in the time to the above theorem stated as

$$i\frac{U_{n+1} - U_n}{\phi(\Delta t)} = \Delta U_{n+1} - |U_{n+1}|^2 U_{n+1}. \tag{4.19}$$

Followed by the nonstandard Taylor's integral Theorem stated by

$$\begin{aligned} i\frac{u(t_{n+1} - u(t_n))}{\phi(\Delta t} =& \frac{\partial u(t_{n+1})}{\partial t} - \frac{1}{2}\int_{t_n}^{t_{n+1}} \frac{\partial^2 u(t)}{\partial t^2}(t_{n+1} - t)dt \\ =& \Delta u(t_{n+1}) + |u(t_{n+1})|^2 u(t_{n+1}) \\ &- \frac{1}{2}\int_{t_n}^{t_{n+1}} \frac{\partial^2 u(t)}{\partial t^2}(t_{n+1} - t)dt. \end{aligned} \tag{4.20}$$

Subtracting equation (4.20) from (4.19) and noting that $\Theta_n = u(t_n - U_n)$ yield

$$\begin{aligned} &\frac{1}{\phi(\Delta t)} i\left[\Theta_{n+1} - \Theta_n, \bar{\Theta}_{n+1}\right] \\ =&\left\langle \left(|u^{n+1}|^2 u^{n+1} - |U_{n+1}|^2 U_{n+1}\right), \bar{\Theta}_{n+1}\right\rangle - \|\nabla\Theta_{n+1}\|^2_{L^2(\Omega)} \\ &+ \frac{1}{2}\int_{t_n}^{t_{n+1}} \left\langle \frac{\partial^2 u(t)}{\partial t^2}, \bar{\Theta}_{n+1}\right\rangle (t_{n+1} - t)dt \end{aligned} \tag{4.21}$$

after setting $u^{n+1} = u(t_{n+1})$ and multiplying equation (4.19) by the complex conjugate $\bar{\Theta}_{n+1}$.

Estimating the first term of the right hand side of (4.21) yield

$$\begin{aligned} &\int_\Omega \left|\left\langle\left(|u^{n+1}|^2 u^{n+1} - |U_{n+1}|^2 U_{n+1}, \bar{\Theta}_{n+1}\right)\right\rangle\right| dx \\ \leq &\int_\Omega |\Theta_{n+1}|^2_{L^4(\Omega)} \left[|u^{n+1}|^2 + |u^{n+1}|\,|U_{n+1}| + |U_{n+1}|^2\right] dx \end{aligned} \tag{4.22}$$

since $|u^{n+1}|^2 - |U_{n+1}|^2 = \left(|u^{n+1}| - |U_{n+1}|\right)\left(|u^{n+1}| + |U_{n+1}|\right)$. Using Gagliardo-Nirenberg and Young's inequalities with the fact that $H^1 \subset L^\infty$ on the right hand side of inequality (4.22) yield

$$\int_\Omega |\theta_{n+1}|^2_{L^4(\Omega)} \left[|u^{n+1}|^2 + |u^{n+1}|\,|U_{n+1}| + |U_{n+1}|^2\right] dx$$

$$\leq \int_{\Omega} |\nabla\Theta_{n+1}|\,|\Theta_{n+1}|\left(|u^{n+1}|^2 + |u^{n+1}|\,|U_{n+1}| + |U_{n+1}|^2\right)$$
$$\leq \frac{\epsilon}{2}\|\nabla\Theta_{n+1}\|_{L^2}^2 + \frac{1}{2\epsilon}\|\Theta_{n+1}\|_{L^2(\Omega)}^2\left(|u^{n+1}|^2 + |u^{n+1}|\,|U_{n+1}| + |U_{n+1}|^2\right). \tag{4.23}$$

Re-introducing (4.23) into inequality (4.21) yield

$$\frac{1}{\phi(\Delta t)} i\left[\Theta_{n+1} - \Theta_n, \bar{\Theta}_{n+1}\right]$$
$$\leq (\frac{\epsilon}{2} - 1)\|\nabla\Theta_{n+1}\|_{L^2}^2 + \frac{1}{2\epsilon}\|\Theta_{n+1}\|_{L^2(\Omega)}^2\left(|u^{n+1}|^2 + |u^{n+1}|\,|U_{n+1}| + |U_{n+1}|^2\right)^2. \tag{4.24}$$

Going back to inequality (4.21), we estimate the third term of the right hand side to yield

$$\left|\frac{1}{2\phi(\Delta t}\int_{t_n}^{t_{n+1}}\left\langle\frac{\partial^2 u(t)}{\partial t^2}, \Theta_{n+1}\right\rangle(t - t_{n+1})dt\right|$$
$$\leq \frac{C}{2\phi(\Delta t)}|\nabla\Theta_{n+1}|\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right||t - t_{n+1}|dt \tag{4.25}$$

since by Poincare inequality, $|\Theta_{n+1}|_{H_0^1} \leq C|\nabla\Theta_{n+1}|_{L^2}$. Applying Holder's inequality on the right hand side of (4.25) yields

$$\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2}, \Theta_{n+1}\right)(t - t_{n+1})dt\right|$$
$$\leq \frac{C}{2\phi(\Delta t)}|\nabla\Theta_{n+1}|\left(\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u}{\partial t^2}\right|_{H^{-1}}\right)^{1/2}\left(\int_{t_n}^{t_{n+1}}|t - t_{n+1}|^2 dt\right)^{1/2}. \tag{4.26}$$

But, we have in view of $t_n < t < t_{n+1}$ that there exists a function $\phi(t_n) < \phi(t) < \phi(t_{n+1})$ such that

$$|\phi(t) - \phi(t_n)| = \phi(\Delta t) = |t - t_n|\Delta t.$$

Thus

$$\left(\int_{t_n}^{t_{n+1}}|t - t_{n+1}|\right)^{1/2} \leq \phi(\Delta t)(t - t_{n+1})^{1/2} \leq (\phi(\Delta t))^{1/2}$$

and substituting this into inequality (4.26) we have

$$\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2}, \Theta_{n+1}\right)(t - t_{n+1})dt\right|$$
$$\leq C(\phi(\Delta t))^{1/2}\left(\int_{t_n}^{t_{n+1}}\left|\frac{\partial^2 u(t)}{\partial t^2}\right|_{L^2(\Omega)}^2 dt\right)^{1/2}|\nabla\Theta_{n+1}|_{L^2(\Omega)}. \tag{4.27}$$

Using Young's inequality for arbitrary $\epsilon > 0$ in inequality (4.27) we have

$$\left|\frac{1}{2\phi(\Delta t)}\int_{t_n}^{t_{n+1}}\left(\frac{\partial^2 u(t)}{\partial t^2}, \Theta_{n+1}\right)(t - t_{n+1})dt\right|$$

$$\leq \frac{\epsilon}{2} \left\| \nabla \Theta_{n+1} \right\|_{L^2(\Omega)}^2 + \frac{C}{2\epsilon} \phi(\Delta t) \int_{t_n}^{t_{n+1}} \left| \frac{\partial^2 u(t)}{\partial t^2} \right|^2 dt. \tag{4.28}$$

Re-introducing inequality (4.28) back into inequality (4.24) we have

$$\begin{aligned} &\frac{1}{\phi(\Delta t)} i \left[\Theta_{n+1} - \Theta_n, \bar{\Theta}_{n+1} \right] \\ \leq &(\frac{\epsilon}{2} - 1) \left\| \nabla \Theta_{n+1} \right\|_{L^2}^2 + \frac{1}{2\epsilon} \left\| \Theta_{n+1} \right\|_{L^2(\Omega)}^2 \left(|u^{n+1}|^2 + |u^{n+1}| \, |U_{n+1}| + |U_{n+1}|^2 \right)^2 \\ &+ \frac{C}{2\epsilon} \phi(\Delta t) \int_{t_n}^{t_{n+1}} \left| \frac{\partial^2 u(t)}{\partial t^2} \right|^2 dt. \end{aligned} \tag{4.29}$$

Choosing $\epsilon > 0$ such that $\epsilon - 1 = 1$ then have in view of (4.29)

$$\begin{aligned} &\frac{1}{\phi(\Delta t)} i \left[\Theta_{n+1} - \Theta_n, \bar{\Theta}_{n+1} \right] + \left\| \nabla \Theta_{n+1} \right\|_{L^2}^2 \\ \leq &\frac{1}{4} \left\| \Theta_{n+1} \right\|_{L^2(\Omega)}^2 \left(|u^{n+1}|^2 + |u^{n+1}| \, |U_{n+1}| + |U_{n+1}|^2 \right)^2 \\ &+ \frac{C}{4} \phi(\Delta t) \int_{t_n}^{t_{n+1}} \left| \frac{\partial^2 u(t)}{\partial t^2} \right|^2 dt \end{aligned}$$

and hence

$$\begin{aligned} &i \left[\Theta_{n+1} - \Theta_n, \bar{\Theta}_{n+1} \right] + \phi(\Delta t) \left\| \nabla \Theta_{n+1} \right\|_{L^2}^2 \\ \leq &\frac{\phi(\Delta t)}{4} \left\| \Theta_{n+1} \right\|_{L^2(\Omega)}^2 \Psi_{n+1} + C \frac{\phi(\Delta t)}{4} \Phi_{n+1} \end{aligned} \tag{4.30}$$

where

$$\Psi_{n+1} = |u^{n+1}|^2 + |u^{n+1}| \, |U_{n+1}| + |U_{n+1}|^2$$

and

$$\Phi_{n+1} = \int_{t_n}^{t_{n+1}} \left| \frac{\partial^2 u}{\partial t^2} \right|_{H^{-1}}^2 dt.$$

It is well-known that the identity $\langle \Theta_{n+1} - \Theta_n, \Theta_{n+1} \rangle$ in the first term of the left hand side of inequality (4.30) is given by

$$(\Theta_{n+1} - \Theta_n, \Theta_{n+1}) = \frac{1}{2} \left[|\Theta_{n+1}|_{L^2}^2 - |\Theta_n|_{L^2}^2 + |\Theta_{n+1} + \Theta_n|_{L^2}^2 \right]$$

and introducing this back into inequality 4.30) yield

$$\begin{aligned} &|\Theta_{n+1}|_{L^2}^2 - |\Theta_n|_{L^2}^2 + |\Theta_{n+1} - \Theta_n|_{L^2}^2 + \phi(\Delta t) \|\nabla \Theta_{n+1}\|_{L^2}^2 \\ \leq &C \frac{1}{2} \phi(\Delta t) \|\Theta_{n+1}\|_{L^2}^2 \Psi_{n+1} + C \frac{1}{2} \phi(\Delta t)^2 \Phi_{n+1}. \end{aligned} \tag{4.31}$$

Re-arranging the terms in inequality (4.31) and setting $a_k = \|\Theta_{n+1}\|_{L^2(\Omega)}^2$ and $b_k = \|\nabla \Theta_{n+1}\|_{L^2(\Omega)}^2$ and summing partially from $k = 0, 1, 2, \cdots, n-1$ and also using the fact that $a_0 = \Theta_0 = u_0 - U_0 = 0$ we have

$$a_n + \sum_{k=0}^{n} b_k \phi(\Delta t) \leq \sum_{k=0}^{n} a_k \phi(\Delta t) \Psi_{n+1} + \sum_{k=0}^{n} \phi(\Delta t)^2 \Phi_{n+1}. \tag{4.32}$$

Applying Lemma 4.1 in inequality (4.32) yield

$$a_n + \sum_{k=0}^{n} b_k \phi(\Delta t) \leq \exp\left(\sum_{k=0}^{n} \sigma_k \phi(\Delta t) \Psi_{n+1}\right) \left(\sum_{k=0}^{n} \Phi_{n+1} (\phi(\Delta t))^2\right), \quad (4.33)$$

provided $\Psi_{n+1}\phi(\Delta t) < 1$ and $\sigma_k = (1 - \Psi_{n+1}\phi(\Delta t))^{-1} \quad \forall k \geq 0$. Since a_n, b_k, Ψ_{n+1} and Φ_{n+1} are all positive series, then in view of Lemma 4.1

$$a_n + \sum_{k=0}^{n} b_k \phi(\Delta t) \leq C(\phi(\Delta t))^2 \quad (4.34)$$

and the proof of the theorem is completed. □

The error estimate shown above leads to the following optimal rate of convergence in specified norms:

Theorem 4.3. *In view of the assumptions of Theorem 4.2, the numerical solution of the semi-linear Schrödinger equations (4.5)-(4.6) using the NSFD-GM method converges optimally with the following estimates*

$$\|u(t) - U_h(t)\|_{L^2} \leq C(t)\left(h^2 + \phi(\Delta t)\right) \quad (4.35)$$

where the constant $C(t)$ depends on t. Furthermore, the discrete solution $U_h(t)$ preserves all the qualitative properties of the exact solution of the equation under study.

Proof. To prove the above theorem, we use the following error decomposition equation

$$\begin{aligned} \|u(t) - U_h(t)\|_{L^2} &= \|u(t) - P_h u(t) + P_h u(t) - U_h(t)\|_{L^2} \\ &\leq \|\xi_n\|_{L^2} + \|\eta_n\|_{L^2} \end{aligned} \quad (4.36)$$

where $\|\xi_n\|_{L^2} = \|u(t) - P_h u(t)\|_{L^2}$ represents the error inherent in the Galerkin approximation of the linearized semi-linear Schrödinger equation and $\|\eta_n\|_{L^2} = \|P_h u(t) - U_h(t)\|_{L^2}$ the error caused by the nonlinearity in the problem. Hence, in view of the inequality (4.4) and Theorem 4.2 we have from inequality (4.36) that

$$\begin{aligned} \|u(t_n) - U_h(t_n)\|_{L^2} &\leq C(t_{n+1})h^2 + \sup_{t\in[t_n,t_{n+1}]} \|P_h u(t_{n+1}) - U_h(t_{n+1})\|_{L^2} \\ &\leq C(t_{n+1})h^2 + C(t_{n+1})(\phi(\Delta t))^2, \ \ \forall t \in [t_n, t_{n+1}]. \end{aligned} \quad (4.37)$$

In view of the inequality (4.37) we can conclude without any difficulties the validity of inequality (4.35).

As for the replication of the decaying properties of the exact solution, we proceed by first high-lighting the fact that in view of Mickens 1994 [34], the above scheme was designed for

$$\phi(\Delta t) = \frac{e^{\lambda \Delta t} - 1}{\lambda} \approx \Delta t + O((\Delta t)^2). \quad (4.38)$$

Based on the above approximation of $\phi(\Delta t)$ jn (4.38), we observe that as $\Delta \longrightarrow 0$, the function $\phi(\Delta t) \approx \Delta t$. n view of this, we deduced that the numerical scheme (4.5)-(4.6) converges point-wise in $\mathcal{V}_h \subset H_0^1(\Omega)$ to the solution u as $\Delta t \longrightarrow 0$ by the compactness Theorem. We justify this as follows: If we choose the source term of our scheme (4.5) to be $U_h^0 \in H_0^1(\Omega)$ and $\mathbf{F} \in L^2\left[(0,T); L^2(\Omega)\right]$, then we have

$$\langle \delta_n U_h^n(t), v_h \rangle + \langle \nabla U_h^n, \nabla v_h \rangle + \left\langle |U_h^n|^2 U_h^n, v_h \right\rangle = \mathbf{F}, \tag{4.39}$$

when only the real part is considered. If we, in addition let the support of $\mathbf{F}$ be very small that the test function $v_h = 0$ far inside the support say $\Omega_1 \subset \Omega$ and $\mathbf{F}$ is regular, then integrating equation (4.39) over Ω will culminate to

$$\int_\Omega \mathbf{F} v_v dx = \mathbf{F}(a) \text{ measure over the } supp(v_h), \quad a \in \Omega_1 \subset \Omega.$$

Thus, the uniform convergence of the solution over Ω is equivalent to the point-wise convergence of the scheme (4.39). For more on such analysis see [1]. Hence, $U_h^n(a)$ is the NSFD-GM solution converges to u and thereby replicating all the qualities of the solution u in (4.9). This completes the second part of the proof and hence complete the proof of the Theorem. □

5. Numerical experiments

We set aside this section to present the numerical experiments to validate the theoretical proposition of the optimal convergence of the numerical solution of the Schrödinger equation obtained above in section 4. In view of this, we consider the equation (1.1)-(1.3) over a three-dimensional domain $\Omega_t = [0,1] \times [0,1] \times (0,T)$ where Ω_t consists of a two-dimensional domain Ω which will be discretized into regular mesh $\mathcal{J}_h$ with a mesh size of h and $(0,T)$ the time domain discretized with a mesh size of Δt. The experiments will be conducted using the software Matlab 7-100(R2014a). The right hand side of the Schrödinger equation (1.1)-(1.3) will be denoted and calculated by $f(x,t)$ using the carefully chosen example

$$u(x_1, x_2, t) = 5e^{it}(1+2t^2)(1-x_1)(1-x_2)\sin(x_1)\sin(x_2) \tag{5.1}$$

respectively. In other words, equation (1.1) will be

$$i\frac{\partial u}{\partial t} - \Delta u + |u|^2 u = f(x,t) \tag{5.2}$$

where $f(x,t)$ is obtained by introducing the chosen exact solution (5.1) on the left hand side of equation (1.1). Using this on the NSFD-GM scheme designed in equation (4.5) we compute the approximate or numerical solution of the scheme (4.5). This computation process is done with the following initial solution

$$u(x_1, x_2, 0) = 5(1-x_1)(1-x_2)\sin(x_1)\sin(x_2). \tag{5.3}$$

The above numerical experiments is runned with the following specifications: a uniform triangular partition with $M+1$ nodes in each direction, where $h = \frac{1}{M}$, a time discretization of $\Delta t = 0.01$ and $T = 1.0$ and the value of $\lambda = 4$ on a complicated denominator $\phi(\Delta t)$. With all these specifications in place, we proceed using the Newton's iterative method to compute and display the following figures:

With the display figures in place, we present numerical results in Table 1 and 2 using the following number of nodes: $M = 40, 80, 120, 160, 200, 240$ and 280.

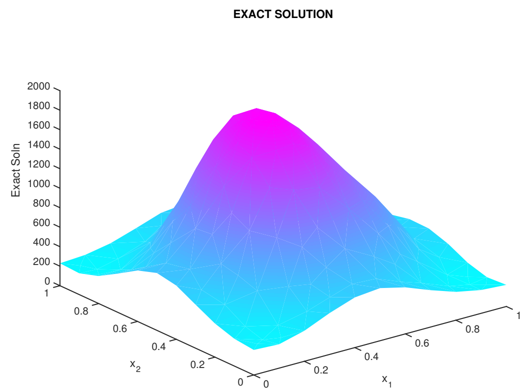

Figure 1. The Exact Computed Solution

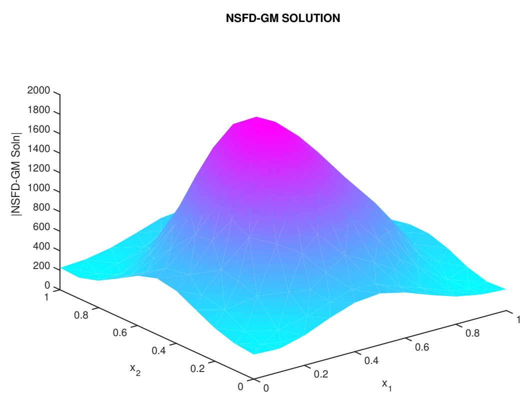

Figure 2. Approximate solution for NSFD-GM Scheme

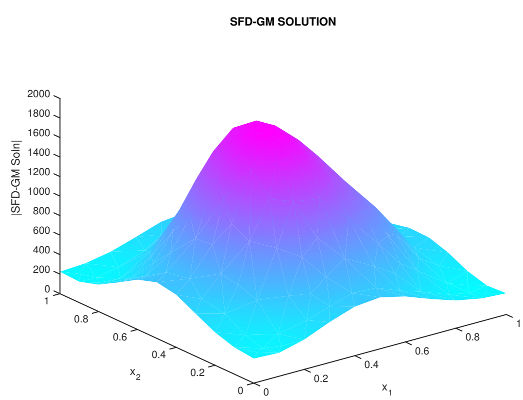

Figure 3. Approximate solution for SFD-GM Scheme

Table 1. NSFD-GM Error in both L^2 and H^1-norms.

M	L^2-error	Rate L^2	H^1-error	Rate H^1
40	4.2248e-03		1.0743e-01	
80	2.0124e-03	1.07	7.0877e-02	0.61
120	1.2676e-03	1.14	5.4578e-02	0.64
160	8.7965e-04	1.27	4.3688e-02	0.78
200	6.2802e-04	1.51	3.6627e-02	0.79
240	4.5232e-04	1.80	3.1426e-02	0.84
280	3.3386e-04	1.97	2.7439e-02	0.88

Table 2. SFD-GM Error in both L^2 and H^1-norms.

M	L^2-error	Rate L^2	H^1-error	Rate H^1
40	3.9874e-03		1.0425e-01	
80	1.6194e-03	1.39	6.9258e-02	0.59
120	9.0697e-04	1.43	5.3645e-02	0.63
160	5.8237e-04	1.54	4.3860e-02	0.70
200	4.0570e-04	1.62	3.7350e-02	0.72
240	2.9867e-04	1.68	3.2458e-02	0.77
280	2.2838e-04	1.74	2.8692e-02	0.80

Observations 1. The interpretation of the above results are based on the rate of convergence of both the NSFD-GM and SFD-GM schemes. Our expectations to this regard were that the rate of convergence of the L^2-norm will be 2 and that of the H^1-norm will be 1 using both the NSFD-GM and SFD-GM schemes respectively. It therefore appears from our numerical computational data that the rates of convergence in the L^2-norm in both cases for NSFD-GM and SFD-GM schemes were approximately 2 with that of the NSFD-GM scheme tilting more closer to 2 than that of the SFD-GM scheme.

The same trend was viewed for the rate of convergence of the H^1-norm in both NSFD-GM and SFD-GM schemes. Both rates were approximately 1 with that of the NSFD-GM scheme getting more closer to 1 than the SFD-GM scheme. In addition to the above trend from the tables, we observes diagrammatically by using Fig 1, 2 and 3 that the results from the two schemes NSFD-GM and SFD-GM are closer to each other. These results can best be explained by viewing their computed errors in both Tables 1 and 2. The errors in both tables manifest themselves a lot clearer as their closeness indicate how similar Fig 2 and 3 are more close to Fig 1 which illustrates the exact solution of the problem under investigation.

These differences in the performance of these schemes did not surprise us. This is because, where the schemes have been designed previously, the NSFD-GM scheme has always shown some age over the SFD-GM scheme. This may be due to some qualities of efficiency, accuracy and viability that comes from it's preserving of the qualitative properties of the exact solution. Based on these extra differences, we could favor as a fair alternative the NSFD-GM scheme over the more traditional SFD-GM scheme. All these computational analysis are what makes the study very interesting.

6. Conclusion and future remarks

In this article, we designed and analyzed the scheme consisting of the nonstandard finite difference method in the time and and Galerkin combined with the compactness method in the space variables of the semi-linear Schrödinger equation. We showed firstly that, when given initial solutions in a specified space, the global solution of the above mentioned equation exists uniquely. We proceeded to show secondly that, using the a priori estimates obtained from the existence process, the numerical solution from the designed scheme is stable and converges optimally in both the L^2 as well as the H^1-norms. We thirdly showed that the numerical solutions from the scheme replicates or preserves the qualitative properties of the exact solution of the problem. Furthermore, numerical experiments were conducted to validate the above proposed theory after using a carefully chosen examples. The results speak for themselves.

We will like to consider in future, real life problems or dispersive partial differential equations with meaning in real life such as the KdV, the Fisher's equation with coefficients of diffusion term much more smaller than that of the reaction term and also the Kawahara equation to mention a few. We will try to design numerical schemes from these problems using the same techniques and try to compare them with those that emanate from fractional differential equation. These types of studies could be very interesting computationally and successful type of such studies for nonstandard finite difference scheme have been done in one dimension. For more on these studies, see [2, 3, 6, 7]. Similar techniques involving semi or quasi-linear problems over non-smooth geometry could also be considered.

Acknowledgments

The research contained in this article has been supported by Sefako Makgatho Health Sciences University, Medunsa 0204, Ga-rankuwa, Pretoria, South Africa.

References

[1] A. R. Adams, *Sobolev Space*, Academic Press, New York, 1975.

[2] A. A. Aderogba and A. R. Appadu, *Classical and multisymplectic schemes for linearized KdV equation: Numerical results and dispersion analysis*, Fluids, 2021, 6, 214. https://doi.org/10.3390/fluids/6060214.

[3] A. M. Agbavon, A. R. Appadu and M. Khumalo, *On the numerical solution of Fisher's equation with coefficient of diffusion term much smaller than coefficient of reaction term*, Adv. Differ. Equ., 2019, 146(2019). https://doi.org/10.1186/s13662-019-2080-x.

[4] R. Anguelov and J. M.-S. Lubuma, *Contributions to the mathematics of the nonstandard finite difference method and applications*, Numerical Methods for Partial Differential Equations, 2001, 17(5), 518–543.

[5] R. Anguelov and J. M.-S. Lubuma, *Nonstandard finite difference method by nonlocal approximation*, Mathematics and Computers in Simulation, 2003, 61(3), 465–475.

[6] A. R. Appadu and A. S. Kelil, *Comparison of modified ADM and classical finite difference method for some third-order and fifth-order KdV equations*, Demonstratio Mathematica, 2021, 54, 377–409. https://doi.org/10.1515/dema-2021-0039.

[7] A. R. Appadu and A. S. Kelil, *Some finite difference methods for solving linear fractional KdV equation*, Front. Appl. Math. Stat., 2023, 9, 1261270. https://doi.org/10.3389/fams.2023.1261270.

[8] J. P. Aubin, *Un théoréme de compacité*, C. R. Acad. Sc. Paris, 1963, 256, 5012–5014.

[9] H. P. Bhatt and A. Q. M. Khaliq, *Higher order exponential time differencing scheme for system of coupled nonlinear Schrödinger equations*, Appl. Math. Comput., 2014, 228, 271–291.

[10] A. Biswas, *Soliton solutions of the perturbed resonant nonlinear Schrödinger's equation with full nonlinearity by semi-inverse variational principle*, Quant. Phys. Lett., 2012, 1(2), 79–86.

[11] H. Brezis and T. Gallouet, *Nonlinear Schrodinger evolution equations*, Nonlinear Analysis, 1980, 4, 677–681.

[12] C. Bu, *Nonlinear Schrödinger equation on the semi-infinite line*, Chinese Annals of Math., 2000, 21A, 1–12.

[13] J. Cai, Y. Wang and H. Liang, *Local energy-preserving and momentum-preserving algorithms for coupled nonlinear Schrödinger system*, J. Comput. Phys., 2013, 239, 30–50.

[14] B. Carroll and C. Bu, *Solution of the forced nonlinear Schrödinger equation (NLS) using PDE techniques*, Appl. Anal., 1991, 41, 33–51.

[15] Y. Chen, H. Zhu and S. Song, *Multi-symplectic splitting method for the coupled nonlinear Schrödinger system*, Comput. Phys. Comm., 2010, 181(7), 1231–1241.

[16] P. W. M. Chin, *The Galerkin reliable scheme for the numerical analysis of the Burgers'-Fisher equation*, Progress in Computational Fluid Dynamics, 2021, 21(4), 234–247.

[17] P. W. M. Chin, *The analysis of an efficient numerical scheme for the Allen-Cahn equations using the Galerkin method*, Communications in Nonlinear Science and Numerical Simulation, 2022, 105, 106061.

[18] P. W. M. Chin, *The analysis of the solution of the Burgers-Huxley equation using the Galerkin method*, Numerical Methods of Partial Differential Equations, 2023, 39(4), 2787–2807.

[19] P. W. M. Chin, *The effect on the solution of the Fitzhugh-Nagumo equation by the external parameter α using the Galerkin method*, Journal of Applied Analysis and Computation, 2023, 13(4), 1983–2005.

[20] P. Q. Ciarlet, *The Finite Element Method for Elliptic Problems*, Elsevier, North Holland, 1978.

[21] L. C. Evan, *Partial Differential Equations, Graduate, Studies in Mathematics*, volume 19, American Mathematical Society, Rhode Island, 1998.

[22] D. C. R. Gomes, M. A. Rincon, M. D. G. da Silva and G. O. Antunes, *Numerical and mathematical analysis of a nonlinear Schrödinger problem with moving ends*, Journal of Applied Analysis and Computation, 2024, 14(2), 886–910. DOI: 10.11948/20230189.

[23] H. Hashimoto and H. Ono *Nonlinear modulations of gravity waves*, J. Phys. Soc. Jpn., 1972, 33, 805–811.

[24] X. Hu and L. Zhang, *Conservative compact difference schemes for the coupled nonlinear Schrödinger system*, Numer. Methods Partial Differential Equations, 2014, 30(3), 749–772.

[25] L. Huang and J. Su, *Multiple solutions for nonhomogeneous quasilinear Schrödinger-Poisson system*, Journal of Applied Analysis and Computation, 2023, 13(8), 1597–1612.

[26] M. S. Ismail and T. R. Taha, *A linearly implicit conservative scheme for the coupled nonlinear Schrödinger equation*, Math. Comput. Simulation, 2007, 74(4–5), 302–311.

[27] P. L. Kelley, *Self-focusing of optical beams*, Physical, Review Letter, 1965, 15, 1005–1008.

[28] L. Kong, J. Hong, L. Ji and P. Zhu, *Compact and efficient conservative scheme for coupled nonlinear Schrödinger equation*, Numer. Methods Partial Differential Equations, 2015, 31(6), 1814–1843.

[29] O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer-Verlag, New York, 1985, 322p.

[30] A. Li, C. Wei and L. Zhao, *Nontrivial generalized solution of Schrödinger-poisson system* $\mathbb{R}^3$ *with zero mass and periodic potential*, Journal of Applied Analysis and Computations, 2023, 13(6), 3491–3503. DOI: 10.11948/20230122.

[31] L. Liu, H. Chen and J. Yang, *Decay properties and asymptotic behavior of positive solutions for the nonlinear fractional Schrödinger-Poisson system*, Journal of Applied Analysis and Computational, 2023, 13(6), 3136–3157. DOI: 10.11948/20220378.

[32] J.-L. Louis, E. Magenes and P. Kenneth, *Non-Homogeneous Boundary Value Problems and Applications*, Volume 1, Springer Berlin, 1972.

[33] Y. Ma, L. Kong, J. Hong and Y. Cao, *Higher-order compact splitting multi-symplectic method for the coupled nonlinear Schrödinger equations*, Comput. Math. Appl., 2011, 61(2), 319–333.

[34] R. E. Mickens, *Nonstandard Finite Difference Models of Differential Equations*, World Scientific Publishing, River Edge NJ, USA, 1994.

[35] K. C. Partidar, *On the use of nonstandard finite difference methods*, Journal of Difference Equations and Applications, 2005, 11(8), 735–758.

[36] W. Rudin, *Functional Analysis.* McGraw-Hill, New York, 1991.

[37] K. Schimizu and Y. H. Ichikawa, *Automodulation of ion oscillation modes in plasma*, J. Phys. Soc. Jpn., 1972, 33, 789–792.

[38] J. Shen, *Long time stability and convergence for the fully discrete nonlinear Galerkin methods*, Appl. Anal., 1990, 38, 201–229.

[39] W. J. Sonnier and C. I. Christov, *Strong coupling of Schrödinger equations: Condservative scheme approach*, Math. Comput. Simulation, 2005, 69(5–6), 514–525.

[40] K. Stewartson and J. T. Stuart, *A nonlinear instability theory for a wave system in plane poiseuille flow*, J. of Fluid Mechanic, 1971, 48(3), 529–545.

[41] C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation, Applied Mathematical Sciences*, 139, Springer-Verlag, New York, 1999.

[42] V. I. Talanov, *Self-focusing of wave beams in nonlinear media*, Soviet Physics, JETP Letters, 1965, 2, 138–141.

[43] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Amsterdam, North-Holland, 1984.

[44] R. Temam, *Infinite Dimensional Dynamical System in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[45] H. Triki, T. Hayat, O. M. Aldossary and A. Biswas, *Bright and dark solitons for the resonant nonlinear Schrödinger equation with time-dependent coefficients*, Optics. and Laser Tecnology, 2012, 44(7), 2223–2231.

[46] M. Tsutsumi, *On smooth solutions to the initial-boundary value problem for the nonlinear Schrödinger equations in two space dimension*, Nonlinear Anal., 1989, 13, 1051–1056.

[47] M. Tsutsumi, *On global solutions to the initial-boundary value problem for the nonlinear Schrödinger equations in exterior domain*, Comm. Partial Differential Equations, 1991, 16, 885–907.

[48] Y. Tsutsumi, *Global solutions of the nonlinear Schrödinger equations in exterior domain*, Comm. Partial Differential Equations, 1983, 8, 1337–1374.

[49] T. Wang, *Maximum norm error band of a linearised difference scheme for a coupled nonlinear Schrödinger equations*, J. Comput. Appl. Math., 2011, 235(14), 4237–4250.

[50] A.-M. Wazwaz, *Exact solutions for the fourth order nonlinear Schrödinger equations with cubic and power law nonlinearities*, Math. Comput. Modelling, 2006, 43(7–8), 802–808.

[51] M. F. Wheeler, *A priori L^2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 1973, 10, 723–759.