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NUMERICAL SIMULATION OF VARIABLE ORDER FRACTIONAL COUPLED FITZHUGH-NAGUMO REACTION-DIFFUSION PROBLEM AND IT'S ANALYSIS

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Abstract A novel scheme for numerical simulation of the variable order fractional partial differential equation (VOFPDE) has been presented in this article, which has been applied to find the approximate solution of the variable order time fractional coupled Fitzhugh-Nagumo reaction-diffusion equation. The solution of the considered model exists and is unique, and the aforementioned model will remain stable under the Ulam-Hyers test. It has been found that Vieta-Fibonacci wavelets are the appropriate basis function to solve the aforementioned problem numerically, and the operational matrices of the Vieta-Fibonacci wavelets have been derived for both integer as well as variable order fractional derivatives. Using these derived operational matrices and properties of Vieta-Fibonacci wavelets combined with the collocation method, the main problem is reduced to an algebraic system of equations, which has been solved easily. The salient feature of the article is the convergence analysis of the proposed method, which is discussed. The error analysis between the approximate solution of the particular cases of the concerned model using the proposed technique and their exact solutions has been presented through tables and figures.

Keywords Vieta-Fibonacci polynomials, collocation method, Vieta-Fibonacci wavelets, existence and uniqueness, Ulam-Hyers stability, convergence analysis.

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1. Introduction

A FPDE is a type of differential equation that involves fractional derivatives instead of integer derivatives. These equations describe complex phenomena where traditional integer-order differential equations fail. Several applications of FPDE have been discovered in the last few years, including those in physics, chemistry, engineering, and finance. For instance, they are used to model the diffusion of pollutants in

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groundwater, heat transfer in complex materials, and wave propagation in heterogeneous media. The modeling of fractional order systems allows for a more accurate representation of real-world phenomena and helps develop efficient algorithms for simulation and prediction. In addition, fractional partial differential equations enable us to understand and analyze complex systems that are inaccessible through traditional approaches. The study of FPDE is an exciting and rapidly growing field with numerous potential applications. We can better model and predict complex systems in various domains as we understand these equations. Finding numerical solutions to FDE has become increasingly important because of their wide range of applications. Several difficulties are encountered when solving fractional-order differential equations analytically, making it necessary to use a numerical method. Numerous numerical methods can be found in the literature viz., [6] have used operational matrix together with Tau method to investigate the approximate solution of FPDE, [8] have used Lucas polynomials with the collocation method and approximate two-dimensional and one-dimensional diffusion equation of fractional order, [18] have obtained the numerical solution of linear and nonlinear FPDE using homotopy perturbation method, [22] have solved the disease model using Genocchi wavelets method, [29] have studied Pine Wilt disease model of fractional order with the help of ADM and Laplace transform method, [17] have developed a numerical approach and found the AS of fractional BVP, [27] gave an analysis of dengue fever outbreaks using novel fractional operators. For the solution of FDE and integro-differential equations, Jacobi and block pulse operational matrices of fractional integral operators have been used in [37]. An iterative method was used by [30] to solve the epidemiological model, [4] has solved a two-dimensional, time-fractional, nonlinear drift reaction-diffusion equation using the shifted airfoil collocation method, [23] has investigated the numerical solution of a fractional model of host-parasitoid population dynamical system using Adam–Bashforth–Moulton and new Toufik–Atangana method, [21] has investigated the dynamics and numerical approximations for the fractional-order coronavirus disease system, [13] has explored the dynamics and chaotic behavior of a fractional predator-prey-pathogen model using the Atangana-Baleanu fractional operator, [36] has solved the fractional generalized nonlinear Schr¨odinger equation using the homotopy analysis transform method.

This article is primarily aimed at discovering the approximate solutions to the extended version of the variable order fractional Coupled Fitzhugh Nagumo (VOFFN) reaction-diffusion model. The mathematical form of VOFFN reaction-diffusion model is given as [16, 39]

$$\begin{aligned}
&\frac{\partial^{\alpha(\zeta,\varpi)}u(\zeta,\varpi)}{\partial\varpi^{\alpha(\zeta,\varpi)}}\\
&= D_u\frac{\partial^2 u(\zeta,\varpi)}{\partial\zeta^2} + u(\zeta,\varpi)(u(\zeta,\varpi)-a)(1-u(\zeta,\varpi)) - v(\zeta,\varpi) + f_1(\zeta,\varpi),\\
&\frac{\partial^{\beta(\zeta,\varpi)}v(\zeta,\varpi))}{\partial\varpi^{\beta(\zeta,\varpi)}} = D_v\frac{\partial^2 v(\zeta,\varpi)}{\partial\zeta^2} + \epsilon(u(\zeta,\varpi) - bv(\zeta,\varpi)) + f_2(\zeta,\varpi),
\end{aligned} \tag{1.1}$$

with conditions

$$\begin{aligned}
u(\zeta,0) &= u_0(\zeta), & v(\zeta,0) &= v_0(\zeta),\\
u(0,\varpi) &= u_1(\varpi), & u(1,\varpi) &= u_2(\varpi),
\end{aligned} \tag{1.2}$$

$$v(0,\varpi)=v_1(\varpi), \qquad v(1,\varpi)=v_2(\varpi),$$

where ζ, ϖ represents space and time, $D_u \geq 0$, $D_v \geq 0$, $a \geq 0$, $b \geq 0$ and $\epsilon \geq 0$. The coupled Fitzhugh-Nagumo reaction-diffusion model is a mathematical structure that describes the dynamics of interacting neuron populations in a spatial setting. This model comprises of two coupled partial differential equations, namely the Fitzhugh-Nagumo equations, which describe the spiking activity of individual neurons, and the reaction-diffusion equation, which describes the diffusion of ions and electrical charges across the neuronal membrane.

In this article, the new approximation approach is derived with Vieta-Fibonacci (VF) wavelets to find the numerical solution of the considered model (1). The wavelet method is a powerful and efficient technique for numerical solutions in PDEs. This is based on decomposing a function into a sum of wavelets, which are localized functions that capture the fine-scale features of the solution. This approach enables the wavelet method to capture both the local and global properties of the solution efficiently, making them ideal for problems with complex geometries, discontinuities, or singularities. In addition, the wavelet method can handle difficulties with adaptive resolutions by adjusting the wavelet coefficients to match the required level of accuracy. Moreover, wavelet methods have been widely used in different fields, including image and signal processing, data analysis, and quantum mechanics. They have been shown to provide accurate and stable numerical solutions of PDE, making them a popular choice for researchers and practitioners alike. Wavelet methods represent an innovative and versatile approach for solving partial differential equations, offering an efficient, accurate, and adaptable numerical solution. There are many wavelet methods available in literature such as in the article [38] an innovative numerical method is presented that can be used to solve both linear and nonlinear distributed fractional differential equations, [3] has developed a Haar wavelet placement technique for solving Volterra-Fredholm fractional integral-differential equations, [34] has generated the operational matrix for Gagenbauer wavelets and approximate Bagley-Torvik equation, [35] has used Taylor wavelet method for the approximation of delay differential equation. [7] numerical solution of fractional differential equations of variable order using Bernoulli wavelet method for anomalous infiltrations and diffusions. The CAS wavelet method was introduced by [28] to solve Fredholm integro-differential equations with nonlinearities, [12] has used Chebyshev wavelet for delay problem, [25] has approximated a problem arising in fluid dynamics, [26] has derived the operational matrix for Legendre polynomials to solve the singular ODE. In the literature the uses of wavelets can be found viz., Gegenbauer wavelet [24], Chebyshev wavelet [32], second kind Chebyshev wavelets [1], Lucas OM [31], variation iteration method [9, 10], Vieta-Lucas operational matrix [19] etc.

Here are the article outlines. Section 2 discusses some preliminary definitions and properties of fractional order derivatives, Vieta–Fibonacci polynomials, and their properties have been discussed. Section 3 introduces Vieta-Fibonacci Wavelets. In section 4, the authors have derived the VF wavelet operational matrix for integer order fractional derivative and VO fractional derivative operators. Section 5 contains the existence of the solution of the Fitzhugh-Nagumo reaction-diffusion system, which is unique and shows stability. Section 6 provides a detailed description of the proposed method to solve the VOFFN reaction-diffusion model. In section 7, the error bound and convergence analysis of the proposed method have been studied. The developed scheme has been applied to some numerical examples in section 8.

In the last section, a summary of the research is provided.

2. Basic of fractional calculus

In this section, we have presented some of the most important definitions and properties of fractional calculus that are relevant from the perspective of this article.

Definition 2.1. The Riemann-Liouville (R-L) fractional integral for $0 < \gamma(\zeta,\varpi) \leq 1$ of $u(\zeta,\varpi)$ is defined by [14]

$$I_{\zeta}^{\gamma(\zeta,\varpi)}u(\zeta,\varpi) = \frac{1}{\Gamma(\gamma(\zeta,\varpi))}\int_0^{\zeta}(\zeta - s)^{\gamma(\zeta,\varpi)-1}u(s,\varpi)ds. \tag{2.1}$$

Definition 2.2. The fractional derivative of variable-order $k-1 < \gamma(\zeta,\varpi) \leq k$ of the function $u(\zeta,\varpi)$ w.r to the variable ζ is given as [33]

$$\begin{aligned} &{}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}u(\zeta,\varpi)\\ &= \begin{cases} \dfrac{1}{\Gamma(k-\gamma(\zeta,\varpi))}\displaystyle\int_0^{\zeta}(\zeta - s)^{k-\gamma(\zeta,\varpi)-1}\ \dfrac{\partial^k u(s,\varpi)}{\partial s^k}ds, & k-1 < \gamma(\zeta,\varpi) < k,\\ \dfrac{\partial^k u(\zeta,\varpi)}{\partial \zeta^k}, \quad \gamma(\zeta,\varpi) = k. \end{cases} \end{aligned} \tag{2.2}$$

The variable order Caputo fractional derivative satisfies the property of linearity, i.e.

$${}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}(A\zeta_1(\zeta) + B\zeta_2(\zeta)) = A({}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}\zeta_1(\zeta)) + B({}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}\zeta_2(\zeta)).$$

Definition 2.3. According to this definition, a Mittag-Leffler function (which contains two positive parameters i and j) is defined as follows:

$$\mathbf{E}_{i,j}(\zeta) = \sum_{\lambda=0}^{\infty}\frac{\zeta^{\lambda}}{(i\lambda + j)}, \quad x \in \mathbb{R}. \tag{2.3}$$

The Caputo derivative and R-L integral of VO fractional order satisfy the following relations

$${}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}\zeta^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k+1-\gamma(\zeta,\varpi))}\zeta^{k-\gamma(\zeta,\varpi)}, & k \in \mathbb{N},\ \text{and}\ k \geq \lceil\gamma(\zeta,\varpi)\rceil\ \text{or}\\ & k \notin \mathbb{N},\ \text{and}\ k > \lceil\gamma(\zeta,\varpi)\rceil,\\ 0, \quad \text{else}, \end{cases}$$

$$I_{\zeta}^{\gamma(\zeta,\varpi)}\zeta^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\gamma(\zeta,\varpi))}\zeta^{\gamma(\zeta,\varpi)+k},$$

$${}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}\Big(I_{\zeta}^{\gamma(\zeta,\varpi)}\Big)u(\zeta,\varpi) = u(\zeta,\varpi),$$

$$I_{\zeta}^{n-\gamma(\zeta,\varpi)}\Big(\frac{d^n u(\zeta,\varpi)}{d\zeta^n}\Big) = {}_0^cD_{\zeta}^{\gamma(\zeta,\varpi)}u(\zeta,\varpi) - \sum_{i=\lceil\gamma(\zeta,\varpi)\rceil}^{n-1}\frac{u^i(0,\varpi)\zeta^{i-\gamma(\zeta,\varpi)}}{\Gamma(i+1-\gamma(\zeta,\varpi))}.$$

3. Vieta Fibonacci wavelets

3.1. Vieta Fibonacci and Shifted Vieta Fibonacci polynomials

In this section, we have discussed some properties of Vieta-Fibonacci polynomials as well as Shifted Vieta-Fibonacci polynomials.

Vieta-Fibonacci polynomials can be generated using the recurrence relation given below [2,15]

$$VF_n(\zeta) = \zeta VF_{n-1}(\zeta) - VF_{n-2}(\zeta), \quad n = 2, 3, \cdots \tag{3.1}$$

with conditions

$$VF_0(\zeta) = 0, \quad \text{and} \quad VF_1(\zeta) = 1. \tag{3.2}$$

$VF_k(\zeta)$ can be represented by the following formula as a power series

$$VF_k(\zeta) = \sum_{n=0}^{\lceil \frac{k-1}{2} \rceil} \frac{(-1)^n \Gamma(k-n)}{\Gamma(k-2n)\Gamma(n+1)} \zeta^{k-2n-1}, \quad k = 2, 3, \cdots, \tag{3.3}$$

where $\lceil \rceil$ is the ceiling function.

In series form, the shifted Vieta-Fibonacci polynomial appears as follows

$$VF_k^*(\zeta) = \sum_{n=0}^{k} \frac{(-1)^{k-n-1} 2^{2n} \Gamma(k+n+1)}{\Gamma(k-n)\Gamma(2n+2)} \zeta^n. \tag{3.4}$$

The $VF_n^*(\zeta)$ also possesses the property of orthogonality, which corresponds to the weight function $w(\zeta) = \sqrt{\zeta - \zeta^2}$ in the form

$$\Big\langle VF_n^*(\zeta), VF_m^*(\zeta) \Big\rangle = \int_0^1 VF_n^*(\zeta) VF_m^*(\zeta) w(\zeta) d\zeta = \begin{cases} \frac{\pi}{8}, & n = m \neq 0, \\ 0, & n \neq m. \end{cases} \tag{3.5}$$

Theorem 3.1. *Let $VF_n^*(\zeta)$ be the Vieta Fibonacci polynomials that are shifted into $[0,1]$, then we have*

$$\frac{d}{d\zeta}[VF_n^*(\zeta)] = \sum_{\substack{k=0 \\ (k+n)\ odd}}^{n-1} 4k \; VF_k^*(\zeta). \tag{3.6}$$

Proof. Proof is same as given in [26] □

3.2. Vieta Fibonacci wavelets

Vieta-Fibonacci wavelets are defined on the interval $[0,1]$ as [5]

$$\psi_{n,m}(\zeta) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{8}{\pi}} \; VF^*(2^k \zeta - n), & \zeta \in \Big[\frac{n}{2^k}, \frac{n+1}{2^k}\Big], \\ 0, & \text{otherwise} \end{cases} \tag{3.7}$$

where m varies from 1 to M and n varies from 0 to $2^k - 1$.

In addition, these wavelets are orthogonal to the weight function $\omega_n(\zeta)$ to the interval $[0,1]$

$$\omega_n(\zeta) = w(2^k\zeta - n) = \begin{cases} \sqrt{(2^k\zeta - n) - (2^k\zeta - n)^2}, & \zeta \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

Considering $f(\zeta)$ over the interval $[0,1]$, then linear combination of $f(\zeta)$ in terms of Vieta Fibonacci wavelets will be

$$f(\zeta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{n,m}(\zeta), \tag{3.9}$$

where $c_{nm} = \int_0^1 f(\zeta)\psi_{n,m}(\zeta)\omega_n(\zeta)d\zeta$.

Now, This infinite series is truncated as follows:

$$f(\zeta) = \sum_{n=0}^{2^k-1}\sum_{m=1}^{M} c_{nm}\psi_{n,m}(\zeta) \cong C^T\Psi(\zeta), \tag{3.10}$$

where C and $\Psi(\zeta)$ are column vector of order 2^kM and given by

$$C = [c_{01}, c_{02}\cdots c_{0M}|c_{11}, c_{12}\cdots c_{1M}|\cdots|c_{2^k-1\ 1}, c_{2^k-1\ 2}\cdots c_{2^k-1\ M}]^T, \tag{3.11}$$

and

$$\Psi(\zeta) = [\psi_{01}, \psi_{02}\cdots \psi_{0M}|\psi_{11}, \psi_{12}\cdots \psi_{1M}|\cdots|\psi_{2^k-1\ 1}, \psi_{2^k-1\ 2}\cdots \psi_{2^k-1\ M}]^T. \tag{3.12}$$

4. Operational matrix of the derivative

In this section, we derived the operational matrix of derivative of VF wavelets for both constant and variable order fractional derivative.

Theorem 4.1. *Let $\Psi(\zeta)$ be the VF wavelets defined in (3.12), then the derivative of $\Psi(\zeta)$ will be*

$$\frac{d\Psi(\zeta)}{d\zeta} = D\Psi(\zeta), \tag{4.1}$$

where D is the derivative matrix of order 2^kM and is given by

$$D = \begin{pmatrix} F & 0 & 0 & \cdots & 0 \\ 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & F \end{pmatrix}, \tag{4.2}$$

where F is a square matrix of order M, whose elements are calculated as

$$F_{r,s} = \begin{cases} 2^{k+2}s, \quad r = 2,\cdots,M, \quad s = 1,\cdots,(r-1) \text{ and } (r+s) \text{ odd} \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.3}$$

Proof. With the help of VF polynomial shifted into $[0,1]$, the rth element of the vector $\Psi(\zeta)$ is given by

$$\Psi_r(\zeta)=\psi_{n,m}=2^{\frac{k}{2}}\sqrt{\frac{8}{\pi}}\;VF^*_m(2^k\zeta-n)\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]},\tag{4.4}$$

where $r=nM+m,\ n=0,1,\cdots,(2^k-1),\ m=1,2,\cdots,M,\ \chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$ is the characteristic function.

Differentiating equation (4.4) w.r.to ζ, we get

$$\frac{d\Psi_r(\zeta)}{d\zeta}=2^{\frac{3k}{2}}\sqrt{\frac{8}{\pi}}VF^{*'}_m(2^k\zeta-n)\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}.\tag{4.5}$$

Since it is zero outside the interval $\left[\frac{n}{2^k},\frac{n+1}{2^k}\right]$, therefore $\frac{d\Psi_r(\zeta)}{d\zeta}=0$ for $r=1,M+1,2M+1,\cdots,(2^k-1)M+1$.

Now, substituting the value of $VF^{*'}_m(2^k\zeta-n)$ in equation (4.5) from Theorem 3.1, we obtain

$$\frac{d\Psi_r(\zeta)}{d\zeta}=2^{\frac{3k}{2}}\sqrt{\frac{8}{\pi}}\sum_{\substack{j=1\\(m+j)\text{ odd}}}^{m-1}4j\;VF^*_j(2^k\zeta-n)\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}.\tag{4.6}$$

Expanding the above equation in VF wavelets, we have

$$\frac{d\Psi_r(\zeta)}{d\zeta}=2^{\frac{3k}{2}}\sqrt{\frac{8}{\pi}}\sum_{\substack{j=1\\(m+j)\text{ odd}}}^{m-1}4j\;VF^*_j(2^k\zeta-n)\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}=2^k\sum_{\substack{s=1\\(r+s)\text{ odd}}}^{r-1}4s\;\Psi_{nM+s}(\zeta).\tag{4.7}$$

If we choose $F_{r,s}$ as

$$F_{r,s}=\begin{cases}2^{k+2}s, & r=2,\cdots,M,\quad s=1,\cdots,(r-1)\text{ and }(r+s)\text{ odd},\\ 0, & \text{otherwise.}\end{cases}$$

Then, the required result is obtained. □

4.1. Vieta-Fibonacci wavelet operational matrix of VOF derivative

The VOF derivative of order $q-1<\gamma(\zeta,\varpi)\le q$ of the VF wavelet vector defined in (3.12) is given by

$${}^{c}_{0}D^{\gamma(\zeta,\varpi)}_{\zeta}\Psi(\zeta)\simeq Q^{\gamma(\zeta,\varpi)}_{\zeta}\Psi(\zeta),\tag{4.8}$$

where $Q^{\gamma(\zeta,\varpi)}_{\zeta}$ is an square matrix of order $\hat{m}(\hat{m}=2^kM)$ for the VF wavelet. The explicit form of the matrix is derived by introducing another family of basis functions as

$$\theta_{nm}(\zeta)=\begin{cases}\zeta^{m-1}, & \zeta\in[\frac{n}{2^k},\frac{n+1}{2^k}],\\ 0, & \text{otherwise.}\end{cases}\tag{4.9}$$

A vector form representation of these $\hat{m}$ set monomials is given by

$$\Theta(\zeta) = [\theta_1(\zeta), \theta_2(\zeta), \cdots, \theta_{\hat{m}}(\zeta)], \tag{4.10}$$

where $\theta_j(\zeta) = \theta_{nm}(\zeta)$ and $i = Mn + m$.

VF wavelet and these monomials are related in the following way

$$\Theta(\zeta) \simeq P\Psi(\zeta). \tag{4.11}$$

The elements of P are obtained by using $(p_{i,j}) =< \omega_i(\zeta), \psi_j(\zeta) >$.

Lemma 4.1. *Let $\theta_{nm}(\zeta)$ be defined in equation (4.9) and $q-1 < \gamma(\zeta,\varpi) \leq q$, then we have*

$$\begin{aligned} &{}_0^cD_\zeta^{\gamma(\zeta,\varpi)}\theta_{nm}(\zeta) \\ &= \begin{cases} \dfrac{(m-1)!}{\Gamma(j-\gamma(\zeta,\varpi))}\zeta^{m-1-\gamma(\zeta,\varpi)}, & m = q+1, q+2, \cdots, M, \quad \zeta \in [\dfrac{n}{2^k}, \dfrac{n+1}{2^k}], \\ 0, \quad \textit{otherwise}. \end{cases} \end{aligned} \tag{4.12}$$

Proof. The proof is straightforward. □

Theorem 4.2. *Let $\Theta(\zeta)$ be a vector defined in (4.10), then the fractional derivative of variable order $q-1 < \gamma(\zeta,\varpi) \leq q$ is given by*

$$ {}_0^cD_\zeta^{\gamma(\zeta,\varpi)}\Theta(\zeta) = V_\zeta^{\gamma(\zeta,\varpi))}\Theta(\zeta), \tag{4.13}$$

where $V_\zeta^{\gamma(\zeta,\varpi)}$ is a $\hat{m} \times \hat{m}$ order matrix defined as

$$V_\zeta^{\gamma(\zeta,\varpi)} = \begin{pmatrix} D_\zeta^{\gamma(\zeta,\varpi)} & 0 & 0 & \cdots & 0 \\ 0 & D_\zeta^{\gamma(\zeta,\varpi))} & 0 & \cdots & 0 \\ 0 & 0 & D_\zeta^{\gamma(\zeta,\varpi)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & D_\zeta^{\gamma(\zeta,\varpi)} \end{pmatrix},$$

where $D_\zeta^{\gamma(\zeta,\varpi)}$ is $M \times M$ diagonal matrix defined by

$$\begin{aligned} D_\zeta^{\gamma(\zeta,\varpi)} =&\zeta^{-\gamma(\zeta,\varpi)} diag\Bigg[0,\ 0,\ \cdots, \frac{(q-1)!}{\Gamma(q-\gamma(\zeta,\varpi))}, \\ &\cdots, \frac{(M-1)!}{\Gamma(M-\gamma(\zeta,\varpi)-1)},\ \frac{M!}{\Gamma(M-\gamma(\zeta,\varpi))}\Bigg]. \end{aligned}$$

Proof. For proof, see Lemma. □

Theorem 4.3. *Let $\Psi(\zeta)$ be the VF wavelets defined in (3.12), then the VOF derivative of order $q-1 < \gamma(\zeta,\varpi) \leq q$ of $\Psi(\zeta)$ is given by*

$$ {}_0^cD_\zeta^{\gamma(\zeta,\varpi)}\Psi(\zeta) = Q_\zeta^{\gamma(\zeta,\varpi)}\Psi(\zeta) = (P^{-1}V_\zeta^{\gamma(\zeta,\varpi)}P)\Psi(\zeta). \tag{4.14}$$

Proof. Considering equation (4.14) and Theorem 4.2, we have

$$\Psi(\zeta) = P^{-1}\Theta(\zeta),$$

and

$${}_0^cD_\zeta^{\gamma(\zeta,\varpi)}\Psi(\zeta) = P^{-1}\ {}_0^cD_\zeta^{\gamma(\zeta,\varpi)}\Theta(\zeta) = P^{-1}V_\zeta^{\gamma(\zeta,\varpi)}\Theta(\zeta) = (P^{-1}V_\zeta^{\gamma(\zeta,\varpi)}P)\Psi(\zeta), \tag{4.15}$$

which completes the proof. □

5. Mathematical analysis of the proposed model

This section has two purposes: to provide the mathematical analysis of the present model. The first one is to prove that a solution exists and is unique, and the second one is to demonstrate the stability of the model.

5.1. Existence and uniqueness

Consider the variable order fractional coupled Fitzhugh-Nagumo model

$$\begin{aligned}\frac{\partial^{\alpha(\zeta,\varpi)}u(\zeta,\varpi)}{\partial\varpi^{\alpha(\zeta,\varpi)}} &= D_u\frac{\partial^2 u}{\partial\zeta^2} + u(u-a)(1-u) - v + f_1(\zeta,\varpi),\\ \frac{\partial^{\beta(\zeta,\varpi)}v(\zeta,\varpi)}{\partial\varpi^{\beta(\zeta,\varpi)}} &= D_v\frac{\partial^2 v}{\partial\zeta^2} + \epsilon(u-bv) + f_2(\zeta,\varpi).\end{aligned} \tag{5.1}$$

Now, introducing the R-L integral operator of fractional order in equation (5.1), we have

$$\begin{aligned}u(\zeta,\varpi) - u(\zeta,0) &= I_\varpi^{\alpha(\zeta,\varpi)}\Big(D_u u(\zeta,\varpi)\\ &\quad + u(u-a)(1-u) - v + f_1(\zeta,\varpi)\Big)\\ &= \frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^\varpi (\varpi - s)^{\alpha(\zeta,\varpi)-1}\Big(D_u u_{\zeta\zeta} + u(\zeta,s)\\ &\quad \times (u(\zeta,s)-a)(1-u(\zeta,s)) - v(\zeta,s) - f_1(\zeta,s)\Big)ds,\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}v(\zeta,\varpi) - v(\zeta,0) &= \frac{1}{\Gamma(\beta(\zeta,\varpi))}\int_0^\varpi (\varpi - s)^{\beta(\zeta,\varpi)-1}\Big(D_v v_{\zeta\zeta} + \epsilon(u(\zeta,s)\\ &\quad - bv(\zeta,s)) + f_2(\zeta,s))\Big)ds.\end{aligned} \tag{5.3}$$

Let

$$K_1(\varpi, u(\zeta,\varpi)) = D_u u_{\zeta\zeta} + u(u-a)(1-u) - v + f_1, \tag{5.4}$$

and

$$K_2(\varpi, v(\zeta,\varpi)) = D_v v_{\zeta\zeta} + \epsilon(u-bv) + f_2, \tag{5.5}$$

and for continuous functions $u(\zeta,\varpi),\ u_1(\zeta,\varpi),\ v(\zeta,\varpi)$ and $v_1(\zeta,\varpi) \in L^2\Big((0,1)\times(0,1)\Big)$, there exist some constants $\gamma_1 > 0$ and $\gamma_1' > 0$ such that

$$\begin{aligned}&||u_{\zeta\zeta} - (u_1)_{\zeta\zeta}|| \leq \gamma_1 ||u - u_1||,\\ &||v_{\zeta\zeta} - (v_1)_{\zeta\zeta}|| \leq \gamma_1' ||v - v_1||.\end{aligned} \tag{5.6}$$

Also, here $|D_u| \leq s_1$, $|a| \leq s_2$, $|D_v| \leq l_1$, $|b| \leq l_2$ and $|\epsilon| \leq l_3$. Now, we will show that $K_1(\varpi, u(\zeta,\varpi))$ and $K_2(\varpi, v(\zeta,\varpi))$ satisfy the Lipschitz condition. For this purpose, we have

$$\begin{aligned}&||K_1(\varpi,u) - K_1(\varpi,u_1)||\\ =&||D_u u_{\zeta\zeta} + u(u-a)(1-u) - v + f_1(\zeta,\varpi) - (D_u(u_1)_{\zeta\zeta}\\ &+ u_1(u_1-a)(1-u_1) - v + f_1(\zeta,\varpi))||\\ \leq&\lambda_1||u_{\zeta\zeta} - (u_1)_{\zeta\zeta}|| + \lambda_2||u-u_1|| + \lambda_3||u-u_1|| + \lambda_4||u-u_1||\\ \leq&\Big(s_1\gamma_1 + (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2\Big)||u-u_1||.\end{aligned}$$

Setting

$$M_1 = s_1\gamma_1 + (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2,$$

where u and u^* are bounded function such that $|u| \leq \lambda_1$ and $|u^*| \leq \lambda_2$. Thus, we have

$$||K_1(\varpi,u) - K_1(\varpi,u_1)|| \leq M_1||u-u_1||. \tag{5.7}$$

Similarly, for the function $v(\zeta,\varpi)$, we have

$$||K_2(\varpi,v) - K_2(\varpi,v_1)|| \leq M_2||v-v_1||, \tag{5.8}$$

where $M_2 = (l_1\gamma_1' + l_2 l_3)$. Here, the Lipschitz condition is satisfied by the kernel. In addition, if $0 \leq M_i < 1, \quad i = 1,2$, it is contraction.

Theorem 5.1. *Let us assume that $u(\zeta,\varpi)$ and $v(\zeta,\varpi)$ are bounded functions, then the operator $\Phi(u(\zeta,\varpi))$ and $\Phi(v(\zeta,\varpi))$ are defined by* [20]

$$\Phi(u(\zeta,\varpi)) = u(\zeta,0) + \frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi - s)^{\alpha(\zeta,\varpi)-1}K_1(\zeta,u)ds, \tag{5.9}$$

and

$$\Phi(v(\zeta,\varpi)) = v(\zeta,0) + \frac{1}{\Gamma(\beta(\zeta,\varpi))}\int_0^{\varpi}(\varpi - s)^{\beta(\zeta,\varpi)-1}K_2(\zeta,v)ds, \tag{5.10}$$

which satisfy the Lipschitz condition.

Proof. Assume that $u(\zeta,\varpi)$ and $w(\zeta,\varpi)$ are bounded functions such that $u(\zeta,0)=$

$w(\zeta,0)$, then

$$\begin{aligned}\Phi(u(\zeta,\varpi))-\Phi(w(\zeta,\varpi))=&\frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\alpha(\zeta,\varpi)-1}\Big(K_1(\zeta,u)\\&-K_1(\zeta,w)\Big)ds,\\ ||\Phi(u(\zeta,\varpi))-\Phi(w(\zeta,\varpi))||\le&\frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\alpha(\zeta,\varpi)-1}||K_1(\zeta,u)\\&-K_1(\zeta,w)||ds\\ \le&\frac{t_0^{\alpha(\zeta,\varpi)}M_1}{\Gamma(\alpha(\zeta,\varpi))\alpha(\zeta,\varpi)}||u-w||.\end{aligned}$$

Letting $m_1=\frac{t_0^{\alpha(\zeta,\varpi)}M_1}{\Gamma(\alpha(\zeta,\varpi))\alpha(\zeta,\varpi)}$, we get

$$||\Phi(u(\zeta,\varpi))-\Phi(w(\zeta,\varpi))||\le m_1||u-w||.$$

Similarly, by assuming $v(\zeta,\varpi)$ and $\phi(\zeta,\varpi)$ as bounded functions we can proof

$$||\Phi(v(\zeta,\varpi))-\Phi(\phi(\zeta,\varpi))||\le m_2||v-\phi||.$$

Hence, the proof is complete. □

Theorem 5.2. *Let us assume that $u(\zeta,\varpi)$ and $v(\zeta,\varpi)$ are bounded functions, then the operators are defined as*

$$\Phi(u)=D_u u_{\zeta\zeta}+u(u-a)(1-u)-v+f_1(\zeta,\varpi), \tag{5.11}$$

and

$$\Phi(v)=D_v v_{\zeta\zeta}+\epsilon(u-bv)+f_2(\zeta,\varpi), \tag{5.12}$$

which satisfy the conditions

$$\Big|\Big\langle\Phi(u)-\Phi(w),u-w\Big\rangle\Big|\le M_1||u-w||^2, \tag{5.13}$$

and

$$\Big|\Big\langle\Phi(v)-\Phi(\phi),v-\phi\Big\rangle\Big|\le M_2||v-\phi||^2, \tag{5.14}$$

respectively.

Proof. By considering the function $u(\zeta,\varpi)$ as bounded function, we have

$$\begin{aligned}&\Big|\Big\langle\Phi(u)-\Phi(w),u-w\Big\rangle\Big|\\=&\Big|\Big\langle D_u(u_{\zeta\zeta}-w_{\zeta\zeta})-(u^3-w^3)+(1+a)(u^2-w^2)\\&-a(u-w),u-w\Big\rangle\Big|\\ \le&|D_u||<(u-w)_{\zeta\zeta},u-w>|+|<u^3-w^3,u-w>|\\&+(1+a)|<u^2-w^2,u-w>|+a|<u-w,u-w>|\end{aligned}$$

$$\leq |D_u| ||(u-w)_{\zeta\zeta}|| ||u-w|| + ||u^3 - w^3|| ||u-w|| \\ + (1+a)||u^2 - w^2|| ||u-w|| + a||u-w||^2,$$

$$\Big|\Big\langle \Phi(u) - \Phi(w), u-w \Big\rangle\Big| \\ \leq \Big(s_1\gamma_1 + (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2\Big)||u-w||^2,$$

which implies that

$$\Big|\Big\langle \Phi(u) - \Phi(w), u-w \Big\rangle\Big| \leq M_1||u-w||^2.$$

Repeating this same process for the bounded function $v(\zeta,\varpi))$, we have

$$\Big|\Big\langle \Phi(v) - \Phi(\phi), v-\phi \Big\rangle\Big| \leq M_2||v-\phi||^2.$$

□

Theorem 5.3. *Suppose that* $u(\zeta,\varpi)$ *and* $v(\zeta,\varpi)$ *are bounded functions and* $0 < ||z|| < \infty$*, then*

$$\Phi(u) = D_u u_{\zeta\zeta} + u(u-a)(1-u) - v + f_1(\zeta,\varpi), \tag{5.15}$$

and

$$\Phi(v) = D_v v_{\zeta\zeta} + \epsilon(u - bv) + f_2(\zeta,\varpi), \tag{5.16}$$

satisfy the conditions

$$\Big|\Big\langle \Phi(u) - \Phi(w), z \Big\rangle\Big| \leq M_1||u-w||\ ||z||, \tag{5.17}$$

and

$$\Big|\Big\langle \Phi(v) - \Phi(\phi), z \Big\rangle\Big| \leq M_2||v-\phi||\ ||z||, \tag{5.18}$$

respectively.

Proof. Let $u(\zeta,\varpi)$ is a bounded function and $0 < ||z|| < \infty$, then

$$\Big|\Big\langle \Phi(u) - \Phi(w), z \Big\rangle\Big| \\ = \Big|\Big\langle D_u(u_{\zeta\zeta} - w_{\zeta\zeta}) - (u^3 - w^3) + (1+a)(u^2 - w^2) - a(u-w), z \Big\rangle\Big| \\ \leq |D_u| | < (u-w)_{\zeta\zeta}, z > | + | < u^3 - w^3, z > | \\ + (1+a)| < u^2 - w^2, z > | + a| < u-w, z > | \\ \leq |D_u| ||(u-w)_{\zeta\zeta}||\ ||z|| + ||u^3 - w^3||\ ||z|| \\ + (1+a)||u^2 - w^2||\ ||z|| + a||z||\ ||u-w||,$$

$$\Big|\Big\langle \Phi(u) - \Phi(w), z \Big\rangle\Big| \\ \leq \Big(s_1\gamma_1 + (\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2\Big)||u-w||\ ||z||,$$

which implies that

$$\left|\left\langle \Phi(u) - \Phi(w), z \right\rangle\right| \leq M_1 ||u - w|| \; ||z||.$$

Repeating this in the similar way for the bounded function $v(\zeta, \varpi)$, we have

$$\left|\left\langle \Phi(v) - \Phi(\phi), z \right\rangle\right| \leq M_2 ||v - \phi|| \; ||z||.$$

Hence, the proof is complete. □

Now, the iterated formulae for equations (5.2) and (5.3) are formulated as

$$u_{n+1}(\zeta, \varpi) = \frac{1}{\Gamma(\alpha(\zeta, \varpi))} \int_0^{\varpi} (\varpi - s)^{\alpha(\zeta,\varpi)-1} K_1(\varpi, u_n) ds, \tag{5.19}$$

$$v_{n+1}(\zeta, \varpi) = \frac{1}{\Gamma(\beta(\zeta, \varpi))} \int_0^{\varpi} (\varpi - s)^{\beta(\zeta,\varpi)-1} K_2(\varpi, v_n) ds, \tag{5.20}$$

$u(\zeta, 0) = u_0$ and $v(\zeta, 0) = v_0$.

The successive difference is presented in the following way

$$\begin{aligned} \xi_n(\zeta, \varpi) = u_n(\zeta, \varpi) - u_{n-1}(\zeta, \varpi) = \frac{1}{\Gamma(\alpha(\zeta, \varpi))} \int_0^{\varpi} &(\varpi - s)^{\alpha(\zeta,\varpi)-1} \\ &\times \Big(K_1(\varpi, u_{n-1}) - K_1(\varpi, u_{n-2})\Big) ds, \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} \chi_n(\zeta, \varpi) = v_n(\zeta, \varpi) - v_{n-1}(\zeta, \varpi) = \frac{1}{\Gamma(\beta(\zeta, \varpi))} \int_0^{\varpi} &(\varpi - s)^{\beta(\zeta,\varpi)-1} \\ &\times \Big(K_2(\varpi, v_{n-1}) - K_2(\varpi, v_{n-2}))\Big) ds. \end{aligned} \tag{5.22}$$

Note that

$$u_n(\zeta, \varpi) = \sum_{i=0}^{n} \xi_i(\zeta, \varpi), \tag{5.23}$$

$$v_n(\zeta, \varpi) = \sum_{j=0}^{n} \chi_j(\zeta, \varpi). \tag{5.24}$$

Applying norm on both sides of equation (5.21), we get

$$||\xi_n(\zeta, \varpi)|| = \left|\left| \frac{1}{\Gamma(\alpha(\zeta, \varpi))} \int_0^{\varpi} (\varpi - s)^{\alpha(\zeta,\varpi)-1} \Big(K_1(\varpi, u_{n-1}) - K_1(\varpi, u_{n-2})\Big) ds \right|\right|, \tag{5.25}$$

or

$$||\xi_n(\zeta, \varpi)|| \leq \frac{1}{\Gamma(\alpha(\zeta, \varpi))} \int_0^{\varpi} (\varpi - s)^{\alpha(\zeta,\varpi)-1} \left|\left| \Big(K_1(\varpi, u_{n-1}) - K_1(\varpi, u_{n-2})\Big) ds \right|\right|. \tag{5.26}$$

As the Lipschitz condition is satisfied by the kernel, therefore

$$||\xi_n(\zeta, \varpi)|| \leq \frac{M_1}{\Gamma(\alpha(\zeta, \varpi))} \int_0^{\varpi} (\varpi - s)^{\alpha(\zeta,\varpi)-1} ||u_{n-1} - u_{n-2}|| ds. \tag{5.27}$$

Proceeding for the bounded function $v(\zeta,\varpi)$, we get

$$||\chi_n(\zeta,\varpi)|| \leq \frac{M_2}{\Gamma(\beta(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\beta(\zeta,\varpi)-1}||v_{n-1}-v_{n-2}||ds. \tag{5.28}$$

Based on the above result, we prove the following theorem.

Theorem 5.4. *The Fitzhugh-Nagumo system defined in equation (1.1) has a solution if there exists t_0 such that*

$$\frac{M_1 t_0^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))} < 1, \tag{5.29}$$

$$\frac{M_2 t_0^{\beta(\zeta,\varpi)}}{\beta(\zeta,\varpi)\Gamma(\beta(\zeta,\varpi))} < 1. \tag{5.30}$$

Proof. Considering $u(\zeta,\varpi)$ as a bounded function and performing the recursive scheme, we have

$$||\xi_n(\zeta,\varpi)|| \leq \Big[\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big]^n u(\zeta,0), \tag{5.31}$$

and thus, the function

$$u_n(\zeta,\varpi) = \sum_{i=0}^{n}\xi_i(\zeta,\varpi),$$

exists and smooth.

Now, to show that equation (5.27) is a solution of (1.1), let us assume that $u(\zeta,\varpi) - u(\zeta,0) = u_n(\zeta,\varpi) - L_n(\zeta,\varpi)$, then we have

$$\begin{aligned}||L_n(\zeta,\varpi)|| &= \Big|\Big|\frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\alpha(\zeta,\varpi)-1}\Big(K_1(\varpi,u)-K_1(\varpi,u_{n-1})\Big)ds\Big|\Big| \\ &\leq \Big[\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big]||u-u_{n-1}||. \end{aligned} \tag{5.32}$$

Recursively, we get

$$||L_n(\zeta,\varpi)|| \leq \Big[\frac{\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big]^{n+1} M_1^{n+1}\delta, \tag{5.33}$$

at $\varpi = t_0$, the above equation becomes

$$||L_n(\zeta,\varpi)|| \leq \Big[\frac{t_0^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big]^{n+1} M_1^{n+1}\delta_1. \tag{5.34}$$

Now, $||L_n(\zeta,\varpi)|| \to 0$ as $n\to\infty$.

Similarly for the function $v(\zeta,\varpi)$,

$$||d_n(\zeta,\varpi)|| \leq \Big[\frac{t_0^{\beta(\zeta,\varpi)}}{\beta(\zeta,\varpi)\Gamma(\beta(\zeta,\varpi))}\Big]^{n+1} M_2^{n+1}\delta_2, \tag{5.35}$$

which tends to zero as $n\to\infty$.

Hence, the proof is complete. □

Next, we examine whether there is any unique solution to the fractional Fitzhugh-Nagumo system (1.1) of variable order. Let $u(\zeta,\varpi)$ and $r(\zeta,\varpi)$ are two solutions of the system (1.1). Thus we have

$$u(\zeta,\varpi)-r(\zeta,\varpi)=\frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\alpha(\zeta,\varpi)-1}\Big(K_1(\varpi,u)-K_1(\varpi,r)\Big)ds. \tag{5.36}$$

If we take the norm on both sides of equation (5.36), we get

$$\begin{aligned}&||u(\zeta,\varpi)-r(\zeta,\varpi)||\\ =&\Big|\Big|\frac{1}{\Gamma(\alpha(\zeta,\varpi))}\int_0^{\varpi}(\varpi-s)^{\alpha(\zeta,\varpi)-1}\Big(K_1(\varpi,u)-K_1(\varpi,r)\Big)ds\Big|\Big|,\end{aligned} \tag{5.37}$$

or

$$||u(\zeta,\varpi)-r(\zeta,\varpi)||\leq\Big(\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big)||u(\zeta,\varpi)-r(\zeta,\varpi)||, \tag{5.38}$$

which implies that

$$||u(\zeta,\varpi)-r(\zeta,\varpi)||\Big(1-\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big)\leq 0. \tag{5.39}$$

Similarly, for the function $v(\zeta,\varpi)$

$$||v(\zeta,\varpi)-r^*(\zeta,\varpi)||\Big(1-\frac{M_2\varpi^{\beta(\zeta,\varpi)}}{\beta(\zeta,\varpi)\Gamma(\beta(\zeta,\varpi))}\Big)\leq 0. \tag{5.40}$$

Theorem 5.5. *A fractional Fitzhugh-Nagumo system has a unique solution if the following conditions are met*

$$\begin{aligned}&\Big(1-\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big)>0,\\ &\Big(1-\frac{M_2\varpi^{\beta(\zeta,\varpi)}}{\beta(\zeta,\varpi)\Gamma(\beta(\zeta,\varpi))}\Big)>0.\end{aligned} \tag{5.41}$$

Proof. We have the following result if the condition of the previous Theorem is satisfied

$$||u(\zeta,\varpi)-r(\zeta,\varpi)||\Big(1-\frac{M_1\varpi^{\alpha(\zeta,\varpi)}}{\alpha(\zeta,\varpi)\Gamma(\alpha(\zeta,\varpi))}\Big)\leq 0. \tag{5.42}$$

Consequently, it implies that

$$||u(\zeta,\varpi)-r(\zeta,\varpi)||=0,$$

as a result $u(\zeta,\varpi)=r(\zeta,\varpi)$.

Similarly, for $v(\zeta,\varpi)$, we have $v(\zeta,\varpi)=r_1(\zeta,\varpi)$.

Therefore, system (1.1) has a unique solution. □

5.2. Ulam-Hyers stability

Our goal here is to demonstrate that the VOFFN system presented in equation (1.1) is Ulam-Hyers stable.

Definition 5.1. The equation (1.1) is Ulam-Hyers stable if the following relationship holds for $\delta_1,\ \delta_2 > 0$, for $u(\zeta,\varpi)$ and $v(\zeta,\varpi)$ [11],

$$|{}_0^cD_{\varpi}^{\alpha(\zeta,\varpi)}u - D_u u_{\zeta\zeta} + (1-u)(u-a)u + v(\zeta,\varpi) - f_1(\zeta,\varpi)| < \delta_1, \tag{5.43}$$

and

$$|{}_0^cD_{\varpi}^{\beta(\zeta,\varpi)}v - D_v v_{\zeta\zeta} - \epsilon(u-bv) - f_2(\zeta,\varpi)| < \delta_2, \tag{5.44}$$

there exist solutions u^* and v^*, such that

$$|u-u^*| < a_1\delta_1, \quad a_1 \in \mathbb{R}, \tag{5.45}$$

and

$$|v-v^*| < a_2\delta_2, \quad a_2 \in \mathbb{R}. \tag{5.46}$$

When u and v satisfy the following equations (5.43) and (5.44), then there exist functions $q_1(\zeta,\varpi)$ and $q_2(\zeta,\varpi)$ which are defined as follows:

$${}_0^cD_{\varpi}^{\alpha(\zeta,\varpi)}u - D_u u_{\zeta\zeta} - (1-u)(u-a)u + v - f_1 = q_1(\zeta,\varpi), \tag{5.47}$$

and

$${}_0^cD_{\varpi}^{\beta(\zeta,\varpi)}v - D_v v_{\zeta\zeta} - \epsilon(u-bv) - f_2 = q_2(\zeta,\varpi). \tag{5.48}$$

Using R-L fractional integral to both sides of the equation (5.47), we achieve

$$\begin{aligned} &u(\zeta,\varpi) - u(\zeta,0) + I_{\varpi}^{\alpha(\zeta,\varpi)}\Big(- D_u u_{\zeta\zeta} - u(u-a)(1-u) + v - f_1\Big) \\ =&I_{\varpi}^{\alpha(\zeta,\varpi)}q_1(\zeta,\varpi). \end{aligned} \tag{5.49}$$

Now,

$$\begin{aligned} &|u(\zeta,\varpi) - u(\zeta,0) + I_{\varpi}^{\alpha(\zeta,\varpi)}(-D_u u_{\zeta\zeta} - u(u-a)(1-u) + v - f_1(\zeta,\varpi))| \\ =&|I_{\zeta}^{\alpha(\zeta,\varpi)}q_1(\zeta,\varpi)| \\ \leq&|q_1|I_{\varpi}^{\alpha(\zeta,\varpi)}(1)| \\ =&|q_1|\frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}, \end{aligned}$$

$$\begin{aligned} &|u(\zeta,\varpi) - u(\zeta,0) + I_{\varpi}^{\alpha(\zeta,\varpi)}(-D_u u_{\zeta\zeta} - u(u-a)(1-u) + v - f_1(\zeta,\varpi)))| \\ \leq&\frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1. \end{aligned}$$

Similarly for $v(\zeta,\varpi)$, we have

$$|v(\zeta,\varpi) - v(\zeta,0) + I_{\varpi}^{\beta(\zeta,\varpi)}(-D_v v_{\zeta\zeta} - \epsilon(u-bv) - f_2(\zeta,\varpi))| \leq \frac{T^{\beta(\zeta,\varpi)}}{\Gamma(\beta(\zeta,\varpi)+1)}\delta_2.$$

Suppose that $u^*(\zeta,\varpi)$ and $v^*(\zeta,\varpi)$ be the solutions of the system (1.1) with $u(\zeta,0)=u^*(\zeta,0)=\zeta_0$ and $v(\zeta,0)=v^*(\zeta,0)=\zeta_1$, then

$$u^*(\zeta,\varpi)=u(\zeta,0)+I_{\varpi}^{\alpha(\zeta,\varpi)}\Big(D_u u_{\zeta\zeta}+(1-u)(u-a)u-v+f_1(\zeta,\varpi)\Big).$$

Now,

$$\begin{aligned}||u-u^*||=&||u-u(\zeta,0)-I_{\varpi}^{\alpha(\zeta,\varpi)}(D_u u^*_{\zeta\zeta}+-v+f_1(\zeta,\varpi))||\\
=&||u-u(\zeta,0)-I_{\varpi}^{\alpha(\zeta,\varpi)}(D_u u_{\zeta\zeta}+(1-u)(u-a)u-v+f_1(\zeta,\varpi))\\
&-I_{\varpi}^{\alpha(\zeta,\varpi)}(D_u u^*_{\zeta\zeta}+(1-u^*)(u^*-a)u^*-v^*\\
&+f_1(\zeta,\varpi))+I_{\varpi}^{\alpha(\zeta,\varpi)}(D_u u_{\zeta\zeta}+(1-u)(u-a)u-v+f_1(\zeta,\varpi))||,\\
||u-u^*||\leq&\frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1+\Big(s_1\gamma_1+(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)+(1+a)(\lambda_1+\lambda_2)+s_2\Big)\\
&\times(I_{\varpi}^{\alpha(\zeta,\varpi)}(||u-u^*||)).\end{aligned}\tag{5.50}$$

Similarly for the function $v(\zeta,\varpi)$, we have

$$\begin{aligned}||v-v^*||=&||v(\zeta,\varpi)-v(\zeta,0)-I_{\varpi}^{\beta(\zeta,\varpi)}\Big(D_v v^*_{\zeta\zeta}+\epsilon(u-bv^*)+f_2\Big)||\\
=&||v(\zeta,\varpi)-v(\zeta,0)-I_{\varpi}^{\beta(\zeta,\varpi)}\Big(D_v v_{\zeta\zeta}+\epsilon(u-bv)+f_2\Big)\\
&-I_{\varpi}^{\beta(\zeta,\varpi)}\Big(D_v v^*_{\zeta\zeta}+\epsilon(u-bv^*)+f_2\Big)\\
&+I_{\varpi}^{\beta(\zeta,\varpi)}\Big(D_v v_{\zeta\zeta}+\epsilon(u-bv)+f_2\Big)||,\\
||v-v^*||\leq&\frac{T^{\beta(\zeta,\varpi)}}{\Gamma(\beta(\zeta,\varpi)+1)}\delta_2+(l_1\gamma_1'+l_2l_3)(I_{\varpi}^{\beta(\zeta,\varpi)}||v-v^*||).\end{aligned}\tag{5.51}$$

Lemma 5.1. *Let $\gamma(\zeta,\varpi)>0$ and $z_1(\zeta,\varpi)$ be locally integrable, nonnegative and nondecreasing functions on interval (a,b). Also, let $u(\zeta,\varpi)$ be nonnegative and locally integrable on interval $[a,b)$ and $z_2(\zeta,\varpi)$ is bounded by some constant, then the inequality*

$$u\leq z_1+z_2\Big(I_{\varpi}^{\gamma(\zeta,\varpi)}u\Big),\tag{5.52}$$

implies

$$u\leq z_1\boldsymbol{E}_{\gamma(\zeta,\varpi)}\Big(z_2(\varpi-a)^{\gamma(\zeta,\varpi)}\Big),\tag{5.53}$$

where $\boldsymbol{E}_{\gamma(\zeta,\varpi)}(\varpi)=\sum_{i=0}^{\infty}\frac{\varpi^i}{\Gamma(\gamma(\zeta,\varpi)i+1)}$.

We obtain the following result by applying the Gronwall relation to equation (5.50) with $z_1(\zeta,\varpi)=\frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1$ and $z_2(\zeta,\varpi)=\Big(s_1\gamma_1+(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)+(1+a)(\lambda_1+\lambda_2)+s_2\Big)$

$$\begin{aligned}||u-u^*||\leq&\frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1\mathbf{E}_{\alpha(\zeta,\varpi)}\\
&\times\Big(\Big(s_1\gamma_1+(\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2)+(1+a)(\lambda_1+\lambda_2)+s_2\Big)\varpi^{\alpha(\zeta,\varpi)}\Big),\end{aligned}$$

$$\leq \frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1 \mathbf{E}_{\alpha(\zeta,\varpi)} \times \Big(\Big(s_1\gamma_1 + (\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2\Big)T^{\alpha(\zeta,\varpi)}\Big). \tag{5.54}$$

Applying the Gronwall relation to equation (5.51), we get

$$\begin{aligned}||v-v^*|| &\leq \frac{T^{\beta(\zeta,\varpi)}}{\Gamma(\beta(\zeta,\varpi)+1)}\delta_2 \mathbf{E}_{\beta(\zeta,\varpi)}\Big((l_1\gamma_1' + l_2 l_3)\varpi^{\beta(\zeta,\varpi)}\Big)\\ &\leq \frac{T^{\beta(\zeta,\varpi)}}{\Gamma(\beta(\zeta,\varpi)+1)}\delta_2 \mathbf{E}_{\beta(\zeta,\varpi)}\Big((l_1\gamma_1' + l_2 l_3)T^{\beta(\zeta,\varpi)}\Big).\end{aligned} \tag{5.55}$$

Equation (5.54) and (5.55), implies $||u-u^*|| \leq a_1\delta_1$ and $||v-v^*|| \leq a_2\delta_2$, with

$$a_1 = \frac{T^{\alpha(\zeta,\varpi)}}{\Gamma(\alpha(\zeta,\varpi)+1)}\delta_1 \mathbf{E}_{\alpha(\zeta,\varpi)} \times \Big(\Big(s_1\gamma_1 + (\lambda_1^2+\lambda_2^2+\lambda_1\lambda_2) + (1+a)(\lambda_1+\lambda_2) + s_2\Big)T^{\alpha(\zeta,\varpi)}\Big),$$

and

$$a_2 = \frac{T^{\beta(\zeta,\varpi)}}{\Gamma(\beta(\zeta,\varpi)+1)}\delta_2 \mathbf{E}_{\beta(\zeta,\varpi)}\Big((l_1\gamma_1' + l_2 l_3)T^{\beta(\zeta,\varpi)}\Big),$$

which completes the stability result.

6. Description of the proposed method

To investigate the numerical solutions of the system (1.1), the solution can be written in terms of VF wavelet as

$$\begin{aligned}u(\zeta,\varpi) &\simeq \sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}} u_{ij}\psi_i(\zeta)\psi_j(\varpi)\\ &\triangleq \Psi^T(\zeta)U\Psi(\varpi),\\ v(\zeta,\varpi) &\simeq \sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}} v_{ij}\psi_i(\zeta)\psi_j(\varpi)\\ &\triangleq \Psi^T(\zeta)V\Psi(\varpi),\end{aligned} \tag{6.1}$$

where U and V are unknown matrices of order $\hat{m}\times\hat{m}$ to be determined and $\Psi(\zeta)$ is vector defined in equation (3.12). Theorem 4.1 yields

$$\begin{aligned}\frac{\partial^2 u(\zeta,\varpi)}{\partial\zeta^2} &\simeq \Psi^T(\zeta)(D^{(2)})^T U\Psi(\varpi),\\ \frac{\partial^2 v(\zeta,\varpi)}{\partial\zeta^2} &\simeq \Psi^T(\zeta)(D^{(2)})^T V\Psi(\varpi).\end{aligned} \tag{6.2}$$

Also, from Theorem 4.3, we get

$$\frac{\partial^{\alpha(\zeta,\varpi)} u(\zeta,\varpi)}{\partial\varpi^{\alpha(\zeta,\varpi)}} \simeq \Psi^T(\zeta)U(P^{-1}V_{\varpi}^{\alpha(\zeta,\varpi)}P)\Psi(\varpi),$$

$$\frac{\partial^{\beta(\zeta,\varpi)}v(\zeta,\varpi)}{\partial\varpi^{\beta(\zeta,\varpi)}} \simeq \Psi^T(\zeta)V(P^{-1}V_{\varpi}^{\beta(\zeta,\varpi)}P)\Psi(\varpi). \tag{6.3}$$

The residual function can be obtained by inserting equation (6.1)-(6.3) into equation (1.1)

$$\begin{aligned}
R_1(\zeta,\varpi) \triangleq &- \Psi^T(\zeta)U\Big(P^{-1}V_{\varpi}^{\alpha(\zeta,\varpi)}P\Big)\Psi(\varpi)\\
&+ D_u\Psi^T(\zeta)(D^{(2)})^TU\Psi(\varpi) + \Psi^T(\zeta)U\Psi(\varpi) \qquad (6.4)\\
&\times\Big(\Psi^T(\zeta)U\Psi(\varpi) - a\Big)\Big(1 - \Psi^T(\zeta)U\Psi(\varpi)\Big) - \Psi^T(\zeta)V\Psi(\varpi) + f_1(\zeta,\varpi),\\
R_2(\zeta,\varpi) \triangleq &- \Psi^T(\zeta)V\Big(P^{-1}V_{\varpi}^{\beta(\zeta,\varpi)}P\Big)\Psi(\varpi) + D_v\Psi^T(\zeta)V\Psi(\varpi)\\
&+ \epsilon\Big(\Psi^T(\zeta)U\Psi(\varpi) - b\Psi^T(\zeta)V\Psi(\varpi)\Big) + f_2(\zeta,\varpi). \qquad (6.5)
\end{aligned}$$

Now, associated initial and boundary conditions can be approximated via VF wavelets as follows

$$\Psi(\zeta)^TU\Psi(0) - u_0(\zeta) \triangleq U_0(\zeta) \simeq 0, \qquad \Psi(\zeta)^TV\Psi(0) - v_0(\zeta) \triangleq V_0(\zeta) \simeq 0, \tag{6.6}$$

and

$$\begin{aligned}
&\Psi(0)^TU\Psi(\varpi) - u_1(\varpi) \triangleq U_1(\varpi) \simeq 0, \qquad \Psi(1)^TU\Psi(\varpi) - u_2(\varpi) \triangleq U_2(\varpi) \simeq 0,\\
&\Psi(0)^TV\Psi(\varpi) - v_1(\varpi) \triangleq V_1(\varpi) \simeq 0, \qquad \Psi(1)^TV\Psi(\varpi) - v_2(\varpi) \triangleq V_2(\varpi) \simeq 0.
\end{aligned} \tag{6.7}$$

At this stage collocating equation (6.4)-(6.7) at the certain collocation points i.e.

$$\begin{cases}
R_r(\zeta_i,\varpi_j) = 0, \quad r = 1,2, \quad 2 \le i \le \hat{m}-1, \quad 2 \le j \le \hat{m},\\
U_0(\zeta_i) = 0, \quad V_0(\zeta_i) = 0, \quad 1 \le i \le \hat{m},\\
U_1(\varpi_j) = 0, \quad U_2(\varpi_j) = 0, \quad 2 \le j \le \hat{m},\\
V_1(\varpi_j) = 0, \quad V_2(\varpi_j) = 0, \quad 2 \le j \le \hat{m}.
\end{cases} \tag{6.8}$$

This system (6.8) generates $2(\hat{m}\times\hat{m})$ equations. To derive a numerical solution to the original system, we will have to solve these algebraic equations to compute the unknown matrices.

7. Convergence analysis and error bound

Theorem 7.1. *Let $f(x)$ be a square-integrable function defined in $[0,1]$ with $|f''(x)| \le B$, can be written as the infinite sum of the VF wavelets, and the series converges uniformly to the function $f(x)$, that is*

$$f(x) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} c_{nm}\Psi_{nm}(x),$$

where

$$|c_{nm}| < \frac{3}{2}\sqrt{\frac{\pi}{8}}\frac{B}{(1+n)^2(m-1)^2(m-2)^2}, \qquad m > 2, \quad n \ge 0.$$

Proof.

$$c_{nm} = \int_0^1 f(x)\Psi_{nm}(x)\omega_n(x)dx = \int_{\frac{n}{2^k}}^{\frac{n+1}{2^k}} f(x)2^k\sqrt{\frac{8}{\pi}}VF_m^*(2^kx-n)w(2^kx-n)dx.$$

Now, let $2^kx - n = t$, then $dx = \frac{1}{2^k}dt$. Then

$$c_{nm} = \sqrt{\frac{8}{\pi}}\int_0^1 f\Big(\frac{t+n}{2^k}\Big)VF^*(t)w(t)dt.$$

Substituting $t = \frac{2+2\cos(\theta)}{4}$ in the above equation, we have

$$\begin{aligned}
c_{nm} =& \sqrt{\frac{1}{2\pi}}\int_0^\pi f\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin m\theta\sin\theta\ d\theta\\
=&\frac{1}{8}\sqrt{\frac{8}{\pi}}\int_0^\pi f\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\Big(\cos(m-1)\theta-\cos(m+1)\theta\Big)d\theta\\
=&\frac{1}{8}\sqrt{\frac{8}{\pi}}\frac{1}{2^{k+1}}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\\
&\times\Big(\frac{\sin(m-1)\theta\sin\theta}{(m-1)}-\frac{\sin(m+1)\theta\sin\theta}{(m+1)}\Big)d\theta\\
=&\frac{1}{8}\sqrt{\frac{8}{\pi}}\frac{1}{2^{k+1}}\left(\frac{1}{(m-1)}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin(m-1)\theta\sin\theta\ d\theta\right.\\
&\left.-\frac{1}{(m+1)}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin(m+1)\theta\sin\theta\ d\theta\right)\\
=&\frac{1}{8}\sqrt{\frac{8}{\pi}}\frac{1}{2^{k+1}}\Big(I_1-I_2\Big),
\end{aligned}$$

where $I_1 = \frac{1}{(m-1)}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin\theta\sin(m-1)\theta\ d\theta$ and $I_2 = \frac{1}{(m+1)} \times \int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin\theta\sin(m+1)\theta\ d\theta$.

Now, the task is to estimate the values of I_1 and I_2.

$$\begin{aligned}
I_1 =& \frac{1}{(m-1)}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin(m-1)\theta\sin\theta\ d\theta\\
=&\frac{1}{2(m-1)}\int_0^\pi f'\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\Big(\cos(m-2)\theta-\cos m\theta\Big)d\theta\\
=&\frac{2^{-k-1}}{2(m-1)}\left(\frac{1}{(m-2)}\int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin(m-2)\theta\sin\theta\ d\theta\right.\\
&\left.-\frac{1}{m}\int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\sin m\theta\sin\theta\ d\theta\right)\\
=&\frac{2^{-k-1}}{2(m-1)}\int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\left(\frac{\sin\theta\sin(m-2)\theta}{(m-2)}-\frac{\sin m\theta\sin\theta}{m}\right)d\theta.
\end{aligned}$$

In a similar way one can get

$$I_2 = \frac{2^{-k-1}}{2(m+1)}\int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\left(\frac{\sin\theta\sin m\theta}{m}-\frac{\sin\theta\sin(m+2)\theta}{(m+2)}\right)d\theta.$$

Thus, we have

$$c_{nm} = \frac{2^{-2k-2}}{\sqrt{8\pi}} \int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\Omega(\theta)\ d\theta,$$

where

$$\begin{aligned}\Omega(\theta) =&\Bigg[\frac{1}{2(-1+m)}\left(\frac{\sin\theta\sin(m-2)\theta}{(m-2)} - \frac{\sin m\theta\sin\theta}{m}\right)\\ &- \frac{1}{2(1+m)}\left(\frac{\sin m\theta\sin\theta}{m} - \frac{\sin\theta\sin(m+2)\theta}{(m+2)}\right)\Bigg],\end{aligned}$$

or

$$\begin{aligned}|c_{nm}| &= \left|\frac{2^{-2k-2}}{\sqrt{8\pi}} \int_0^\pi f''\Big(\frac{2n+\cos\theta+1}{2^{k+1}}\Big)\Omega(\theta)\ d\theta\right|,\\ |c_{nm}| &\leq \frac{2^{-2k-2}B}{\sqrt{8\pi}} \int_0^\pi |\Omega(\theta)| d\theta.\end{aligned}$$

After some mathematical calculations, we get

$$|c_{nm}| < \frac{3}{2}\sqrt{\frac{\pi}{8}}\frac{B}{(1+n)^2(m-1)^2(m-2)^2}, \quad m > 2,$$

which completes the proof. □

Theorem 7.2. *Let $f(x)$ be a continuous function defined in $[0,1]$ such that its second derivative is bounded by some constant B. Then, the error bound will be*

$$\begin{aligned}\sigma_{k,M} <&\frac{3B}{2}\sqrt{\frac{\pi}{8}}\Bigg(\sum_{n=0}^{\infty}\sum_{m=M+1}^{\infty}\left(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\right)^2\\ &+ \sum_{n=2^k}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\right)^2\Bigg)^{\frac{1}{2}},\end{aligned}$$

where

$$\sigma_{k,M}^2 = \int_0^1 \Big[f(x) - \sum_{n=0}^{2^k-1}\sum_{m=1}^{M} c_{nm}\psi_{n,m}(x)\Big]^2 \omega_n(x) dx.$$

Proof.

$$\begin{aligned}\sigma_{k,M}^2 &= \int_0^1 \Big[f(x) - \sum_{n=0}^{2^k-1}\sum_{m=1}^{M} c_{nm}\psi_{n,m}(x)\Big]^2 \omega_n(x) dx\\ &= \int_0^1 \Big[\sum_{n=0}^{\infty}\sum_{M=1}^{\infty} c_{cm}\psi_{n,m}(x) - \sum_{n=0}^{2^k-1}\sum_{m=1}^{M} c_{nm}\psi_{n,m}(x)\Big]^2 \omega_n(x) dx\\ &= \int_0^1 \left(\sum_{n=0}^{\infty}\sum_{m=M+1}^{\infty} c_{nm}\psi_{n,m}(x) + \sum_{n=2^k}^{\infty}\sum_{m=1}^{\infty} c_{nm}\psi_{n,m}(x)\right)^2 \omega_n(x) dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{m=M+1}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty} \sum_{m=1}^{\infty} c_{nm}^2 \\
&< \Big(\frac{9\pi B^2}{32}\Big)\Bigg(\sum_{n=0}^{\infty} \sum_{m=M+1}^{\infty} \bigg(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\bigg)^2 \\
&\quad + \sum_{n=2^k}^{\infty} \sum_{m=1}^{\infty} \bigg(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\bigg)^2\Bigg).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\sigma_{k,M}^2 <&\Big(\frac{9\pi B^2}{32}\Big)\Bigg(\sum_{n=0}^{\infty} \sum_{m=M+1}^{\infty} \bigg(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\bigg)^2 \\
&+ \sum_{n=2^k}^{\infty} \sum_{m=1}^{\infty} \bigg(\frac{1}{(1+n)^2(m-1)^2(m-2)^2}\bigg)^2\Bigg).
\end{aligned}
$$

By taking the square root of both sides, the proof can be obtained. □

8. Numerical applications

The purpose of this section is to illustrate some numerical experiments and apply the method that have been proposed to solve those. The results of these numerical experiments demonstrate the validity and applicability of our proposed method. During the numerical computations, Mathematica 11.3 software was used on Windows 10, 64 bit to operate all the numerical calculations. To illustrate the absolute error at (ζ_i, ϖ_j), the following notation is adopted in order to demonstrate it:

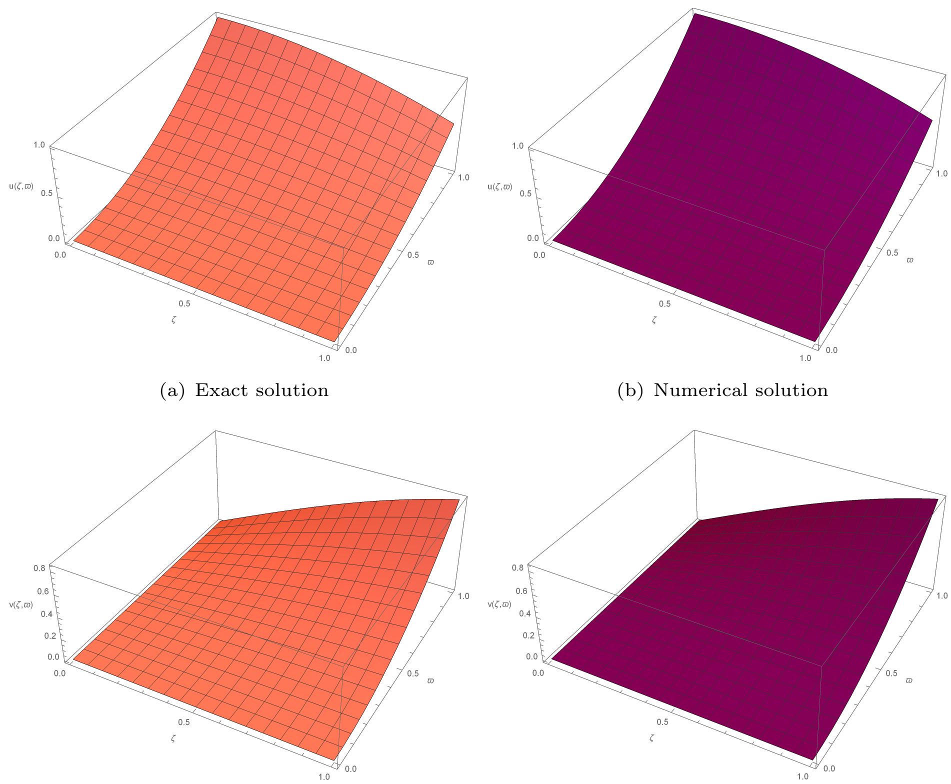
$$
e_u(\zeta_i, \varpi_j) = |u(\zeta_j, \varpi_j) - u_{\hat{m}}(\zeta_i, \varpi_j)|, \tag{8.1}
$$

where $u(\zeta_i, \varpi_j)$ and $u_{\hat{m}}(\zeta_i, \varpi_j)$ are exact and approximate solutions, respectively at (ζ_i, ϖ_j).

Example 8.1. Consider model (1.1) by assuming $D_u = D_v = 0.5$, $a = 0.25$, $b = 0.5$ and $\epsilon = 0.001$ with

$$
\begin{aligned}
f_1(\zeta, \varpi) &= \frac{2\varpi^{2-\alpha(\zeta,\varpi)}\cos(\zeta)}{\Gamma(3-\alpha(\zeta,\varpi))} + D_u \varpi^2 \cos(\zeta) \\
&\quad - \cos(\zeta)\varpi^2(\varpi^2\cos(\zeta) - a)(1 - \varpi^2\cos(\zeta)) + \sin(\zeta)\varpi^2, \\
f_2(\zeta, \varpi) &= \frac{2\varpi^{2-\beta(\zeta,\varpi)}\sin(\zeta)}{\Gamma(3-\beta(\zeta,\varpi))} + D_v \varpi^2 \sin(\zeta) - \epsilon(\cos(\zeta) - b\sin(\zeta))\varpi^2.
\end{aligned}
$$

The IC and BCs are obtained using the exact solutions $u(\zeta, \varpi) = \varpi^2\cos(\zeta)$ and $v(\zeta, \varpi) = \varpi^2\sin(\zeta)$.



(a) Exact solution (b) Numerical solution

(c) Exact solution (d) Numerical solution

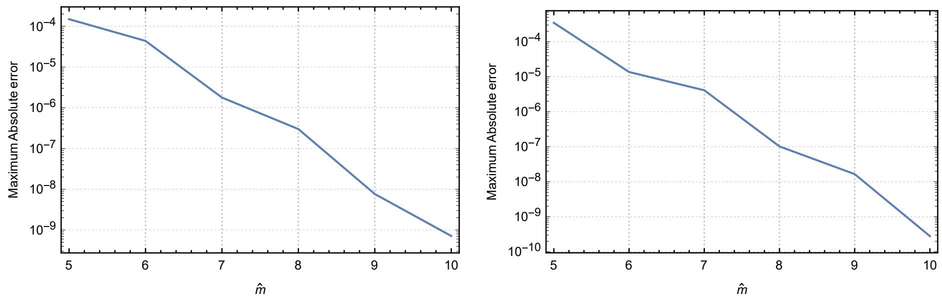
Figure 1. Behaviour of exact solution and their corresponding numerical solutions for Example 8.1 at $\hat{m} = 4$.

Table 1. Tabular presentation of absolute error of Example 8.1 for two different values of $\hat{m}$.

	$\hat{m} = 4$		$\hat{m} = 8$	
ζ	$e_u(\zeta, \varpi)$	$e_v(\zeta, \varpi)$	$e_u(\zeta, \varpi)$	$e_v(\zeta, \varpi)$
0.1	3.5993×10^{-4}	1.0668×10^{-4}	2.5985×10^{-8}	8.6582×10^{-9}
0.3	1.1373×10^{-3}	3.3706×10^{-4}	8.3329×10^{-8}	2.7798×10^{-8}
0.5	1.8997×10^{-3}	5.8260×10^{-4}	1.5829×10^{-7}	5.2981×10^{-8}
0.7	2.2062×10^{-3}	7.2479×10^{-4}	2.5998×10^{-7}	8.7720×10^{-8}
0.9	1.2675×10^{-3}	4.5465×10^{-4}	2.6505×10^{-7}	9.1621×10^{-8}

The main observations are as follows:

- The numerical results of Example 8.1 for the fractional order $\alpha(\zeta, \varpi) = (2 + \sin(\zeta\varpi))/4$ and $\beta(\zeta, \varpi) = 0.55 + 0.35 \sin(\zeta\varpi)$ at $\varpi = 0.5$ and for particular values of ζ are shown in Table 1. From Table 1, it is clear that increasing the number of approximations significantly reduces the absolute error, indicating the efficiency of the method in capturing the solution's details by increasing the number of approximation.



(a) Error in $u(\zeta, \varpi)$ (b) Error in $v(\zeta, \varpi)$

Figure 2. Comparison of maximum absolute error for Example 8.1 for different value of $\hat{m}$ at $\varpi = 0.5$.

- Behaviour of exact and numerical results can be seen in Fig 1, in which approximate results are obtained at $\hat{m} = 4$. It is clear from the figures that the exact solution and approximate solution graph are very similar for small values of approximation. The close alignment between the graphs of the exact solution and the approximate solution, particularly for small approximation values, demonstrates the robustness of the numerical method in replicating the true dynamics of the problem.

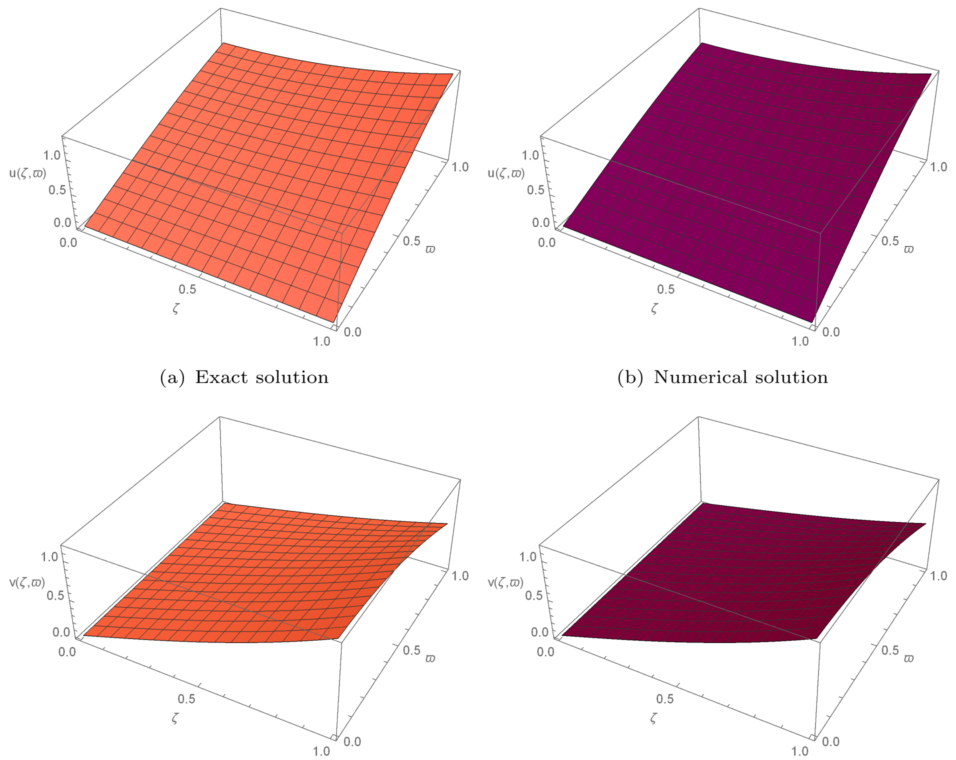
- A comparison of maximum absolute error for various values of $\hat{m}$ is indicated in Fig 2, which clearly shows that on increasing the number of approximations, the error decreases rapidly for both $u(\zeta, \varpi)$ and $v(\zeta, \varpi)$, signifying that the method not only converges but does so efficiently across different components of the solution. The results confirm the reliability of the proposed computational approach in accurately solving fractional-order problems, with the decreasing error providing strong evidence of its effectiveness.

Example 8.2. By assuming $D_u = D_v = 1$, $a = 0.35$, $b = 0.45$ and $\epsilon = 0.00125$ in the model (1.1), the problem is reduced to

$$\begin{aligned}
\frac{\partial^{\alpha(\zeta,\varpi)} u(\zeta, \varpi)}{\partial \varpi^{\alpha(\zeta,\varpi)}} &= \frac{\partial^2 u}{\partial \zeta^2} + u(u - 0.35)(1 - u) - v + f_1(\zeta, \varpi), \\
\frac{\partial^{\beta(\zeta,\varpi)} v(\zeta, \varpi)}{\partial \varpi^{\beta(\zeta,\varpi)}} &= \frac{\partial^2 v}{\partial \zeta^2} + 0.00125(u - 0.45v) + f_2(\zeta, \varpi),
\end{aligned}$$

whose exact solutions are given by $u(\zeta, \varpi) = \sin(\varpi)\cosh(\zeta)$, $v(\zeta, \varpi) = \cos(\varpi) \times \sinh(\zeta)$, considering

$$\begin{aligned}
f_1(\zeta, \varpi) =& \varpi^{1-\alpha(\zeta,\varpi)} \mathbf{E}_{2,2-\alpha(\zeta,\varpi)}(-\varpi^2)\cosh(\zeta) - \cosh(\zeta)\sin(\varpi) \\
& - \sin(\varpi)\cosh(\zeta)(\sin(\varpi)\cosh(\zeta) - 0.35)(1 - \sin(\varpi)\cosh(\zeta)) \\
& + \cos(\varpi)\sinh(\zeta), \\
f_2(\zeta, \varpi) =& - \varpi^{2-\beta(\zeta,\varpi)} \mathbf{E}_{2,3-\alpha(\zeta,\varpi)}(-\varpi^2)\sinh(\zeta) - \sinh(\zeta)\cos(\varpi) \\
& - 0.00125(\cosh(\zeta)\sin(\varpi) - 0.45\cos(\varpi)\sinh(\zeta)).
\end{aligned}$$



(a) Exact solution (b) Numerical solution

(c) Exact solution (d) Numerical solution

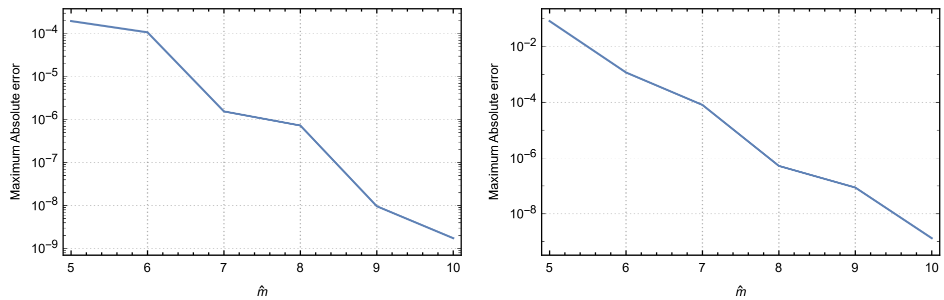
Figure 3. Behaviour of exact solution and their corresponding numerical solutions for Example 8.2 at $\hat{m} = 10$.

Table 2. Tabular presentation of absolute error of Example 8.2 for two different values of $\alpha(\zeta, \varpi)$ and $\beta(\zeta, \varpi)$.

	$\alpha(\zeta,\varpi)) = 0.45 + 0.25\sin(\zeta+\varpi), \beta(\zeta,\varpi) = 1$		$\alpha(\zeta,\varpi) = 1, \beta(\zeta,\varpi) = 0.95 - 0.20\cos(3\zeta\varpi)$	
ζ	$e_u(\zeta,\varpi)$	$e_v(\zeta,\varpi)$	$e_u(\zeta,\varpi)$	$e_v(\zeta,\varpi)$
0.1	1.7767×10^{-10}	1.8072×10^{-10}	1.5870×10^{-10}	1.6476×10^{-10}
0.3	5.5243×10^{-10}	5.4063×10^{-10}	5.0208×10^{-10}	4.9739×10^{-10}
0.5	9.8704×10^{-10}	8.9514×10^{-10}	9.2273×10^{-10}	8.3810×10^{-10}
0.7	1.5216×10^{-09}	1.2361×10^{-09}	1.4667×10^{-09}	1.1857×10^{-09}
0.9	1.7289×10^{-09}	1.2531×10^{-09}	1.7064×10^{-09}	1.2318×10^{-09}

The main observations are as follows:

- Table 2 is designed to show the absolute errors for Example 8.2 for $\alpha(\zeta,\varpi) = 0.45 + 0.25\sin(\varpi+\zeta)$, $\beta(\zeta,\varpi) = 1$ and $\alpha(\zeta,\varpi) = 1$, $\beta(\zeta,\varpi) = 0.95 - 0.20\cos(3\zeta\varpi)$ at $\varpi = 0.5$ for different values of ζ with $\hat{m} = 10$. From the Table, it can be observed that errors computed by the presented approach are very close to zero, and thus, the proposed method provides good numerical results.
- Graph of exact and approximate results for fractional order $\alpha(\zeta,\varpi) = 0.45 + 0.25\sin(\varpi+\zeta)$, and $\beta(\zeta,\varpi) = 1$ can be seen in Fig.3. The visual comparison



(a) Error in $u(\zeta, \varpi)$ (b) Error in $v(\zeta, \varpi)$

Figure 4. Comparison of maximum absolute error for Example 8.2 for different value of $\hat{m}$ at $\varpi = 0.5$.

between these results highlights the substantial agreement between the exact solution and the numerical approximation, demonstrating the method's capability to replicate the behavior of the system under study accurately. The close agreement of the graphs for exact and approximate solutions further confirms the effectiveness of the method.

- A comparison of maximum absolute error for various values of $\hat{m}$ for $\alpha(\zeta, \varpi) = 1$ and $\beta(\zeta, \varpi) = 0.95 - 0.20\cos(3\zeta\varpi)$ is indicated in Fig 4, which clearly shows that on increasing the value of $\hat{m}$, error is decreasing. This trend confirms the method's convergence and highlights its efficiency in reducing the error by increasing the number of approximations. The results in Fig. 4 show that the proposed method is highly effective in solving fractional-order differential equations.

9. Conclusion

This article has developed and thoroughly investigated a new computational scheme to find the approximate solution of variable-order partial differential equations where the derivatives are considered in the Caputo sense. The study delves into the uniqueness and existence of the solution for the Fitzhugh-Nagumo model, along with a detailed discussion on the Ulam-Hyers stability of the model. Vieta-Fibonacci wavelets are employed to find the numerical solution of the PDEs, and the operational matrices corresponding to both integer and variable-order differential operators are meticulously derived. These operational matrices play a crucial role in transforming the original problem into a system of algebraic equations. The error analysis associated with this method is rigorously examined, providing valuable insights into the accuracy and reliability of the proposed approach. The effectiveness and precision of the method are further validated by applying it to several numerical examples. These examples demonstrate the robustness of the proposed scheme in handling complex differential equations and underline its potential as a powerful tool in the numerical analysis of VOPDEs.

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