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SIMPSON'S THREE-EIGHTHS APPROACH FOR COMPUTING SOLUTIONS OF ABSOLUTE VALUE EQUATIONS NUMERICALLY

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Abstract In this study, we propose a two-step iterative procedure for solving absolute value equations. The method includes Simpson's Three-Eighths formula with five points as a corrector step and generalized Newton's approach as a predictor step. For solving large systems, this method is very effective because it is very simple. Moreover, we show the convergence analysis under certain conditions using different theorems. We conducted numerical experiments to examine the efficiency of the presented technique.

Keywords Absolute value equations, Simpson's three-eighths formula, convergence analysis, numerical outcomes.

MSC(2010) 65F10, 65H10.

1. Introduction

Consider the absolute value equation (AVE) of the form:

$$Ax - |x| = b, \tag{1.1}$$

where $A \in R^{n\times n}$, $|x| = (|x_1|, |x_2|, ..., |x_n|)^T$, and $b \in R^n$. Another generalized form of Eq. (1.1) is

$$Ax + B|x| = b, \tag{1.2}$$

where $B \in R^{n\times n}$ was first presented by Rohn in [25]. When $B = -I$, where I represents the identity matrix, equation (1.2) is transformed into equation (1.1). Many engineering and scientific computing applications use equation (1.1), including linear complementarity problems (LCPs), linear programming, and network price [18,19]. Numerical algorithms for AVEs are primarily examined with mathematical theories, the framework of solutions, and the accurate output of high-quality preconditioners and highly efficient numerical procedures AVEs. Numerous numerical techniques have been investigated in recent years to solve AVE, such as Salkuyeh [28] proposed

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the Picard-HSS iterative method for calculating AVE (1.1). A generalized Newton (GN) algorithm for solving AVE (1.1) was developed by Mangasarian [17], who also demonstrated that this method converges linearly when $||A^{-1}|| < \frac{1}{4}$. In their study, Cacceta et al. [3] investigated a smoothing Newton algorithm for equation (1.1) and determined that this method is convergent when $||A^{-1}|| < 1$. Saheya et al. [27] examined smoothing type techniques to calculate equation (1.1) and presented convergence outcomes for the proposed algorithms. Abdallah et al. [1] Presented equation (1.1) as an LCP and determined it Utilizing a smoothing approach. In [24], Prokopyev analyzed unique features of AVE and their relationship to LCP. For instance, [16] presented a preconditioned iterative AOR technique for solving equation (1.1) and explained the conditions for the technique to converge. Wu with Li in [29] examined a novel iterative technique for determining the AVE using the shift splitting (SS) technique. Haghani studied the generalized version of Traub's approach in [11]. Hu and Huang [12] gave several convexity and existence results for the solution of the AVE system and reconstructed the AVE system as a standard LCP without any assumptions. Fakharzadeh and Shams in [7], investigated the convergence properties of the mixed-type splitting approach to solve equation (1.1). Dong et al. in [5] developed a new SOR-like approach for computing AVE. Feng with Liu [8] introduced an improved generalized Newton technique. Iqbal et al. [13] presented the Levenberg-Marquardt iterative procedure to solve AVE (1.1). Gul et al. [9] developed a two-step iterative approach for computing AVE (1.1). Noor et al. [20, 21] suggested minimization algorithms to solve AVE.

In this paper, Simpson's Three-Eighths approach, along with the generalized Newton approach, are discussed for solving equation (1.1). This new algorithm is simple and very effective. The following is a summary of the points discussed in this article. In Sec. 2, we introduce the proposed technique, different definitions and notations utilized in this article. Sec.3 discusses the convergence for solving equation (1.1). We present the numerical outcomes and our conclusions in Sections 4 and 5, respectively. We use the following notations. Let $sign(x)$ be a vector with elements $-1, 0, 1$ based on the related elements of x. The generalized Jacobian $\partial|x|$ of $|x|$ based on a subgradient [23, 30] of the elements of $|x|$ is the diagonal matrix D is defined as

$$D(x) = \partial|x| = diag(sign(x)), \tag{1.3}$$

svd(A) represents the n singular values of A. ψ is the maximum eigenvalue of $A^T A$ in absolute and $||A|| = (\psi)^{\frac{1}{2}}$ representss the 2-norm of A. The norm $||x||$ will represent the 2-norm $\sqrt{(x^T x)}$ of the vector x. Note that $|x| = D(x)x$

2. Proposed method

Suppose

$$\Psi(x) = Ax - |x| - b. \tag{2.1}$$

The generalized Jacobian of $\Psi(x)$ at x is:

$$\partial\Psi(x) = A - D(x). \tag{2.2}$$

Consider the predictor step as:

$$\lambda^k = \left(A - D\left(x^k\right)\right)^{-1} b. \tag{2.3}$$

Let m be the solution to AVE (1.1). To construct the corrector step, we proceed as follows:

$$\int_u^m \Psi'(t)dt = \Psi(m) - \Psi(u) = -\Psi(u). \tag{2.4}$$

Now, we use the five-point Simpson's Three-Eighths formula, we get

$$\begin{aligned}\int_u^m \Psi'\Big(t\Big)dt =& \frac{m-u}{90}\Bigg[7\Psi'(u) + 32\Psi'\Big(\frac{3u+m}{4}\Big) + 12\Psi'\Big(\frac{u+m}{2}\Big) \\ &+ 32\Psi'\Big(\frac{u+3m}{4}\Big) + 7\Psi'\Big(m\Big)\Bigg].\end{aligned} \tag{2.5}$$

From Equations (2.4) and (2.5), we get

$$\begin{aligned}-\Psi\Big(u\Big) =& \frac{m-u}{90}\Bigg[7\Psi'\Big(u\Big) + 32\Psi'\Big(\frac{3u+m}{4}\Big) + 12\Psi'\Big(\frac{m+u}{2}\Big) \\ &+ 32\Psi'\Big(\frac{u+3m}{4}\Big) + 7\Psi'\Big(m\Big)\Bigg].\end{aligned} \tag{2.6}$$

Thus

$$\begin{aligned}m =& u - 90\Bigg[7\Psi'\Big(u\Big) + 32\Psi'\Big(\frac{3u+m}{4}\Big) + 12\Psi'\Big(\frac{m+u}{2}\Big) \\ &+ 32\Psi'\Big(\frac{u+3m}{4}\Big) + 7\Psi'\Big(m\Big)\Bigg]^{-1}\Psi(u).\end{aligned} \tag{2.7}$$

From equation (2.7) the algorithm for STE approach can be writte as:

Algorithm 2.1

1: Select $x^{(0)} \in R^n$.

2: For k compute $\lambda^k = \Big(A - D\Big(x^k\Big)\Big)^{-1} b.$

3: Using Step 2, compute

$x^{k+1} = x^k - 90\Big[7\Psi'\Big(x^k\Big) + 32\Psi'\Big(\frac{3x^k+\lambda^k}{4}\Big) + 12\Psi'\Big(\frac{\lambda^k+x^k}{2}\Big) + 32\Psi'\Big(\frac{x^k+3\lambda^k}{4}\Big) + 7\Psi'\Big(\lambda^k\Big)\Big]^{-1}\Psi\Big(x^k\Big).$

4: If $x^{k+1} = x^k$, then end. Otherwise, apply $k = k+1$ and continue from step 2.

3. Convergence

In this section, we prove the convergence of STE method. The predictor step is well defined (see Lemma 2 [17]) as

$$\lambda^k = \Big(A - D\Big(x^k\Big)\Big)^{-1} b. \tag{3.1}$$

Now, we want to prove that

$$7\Psi'\left(x^k\right)+32\Psi'\left(\frac{3x^k+\lambda^k}{4}\right)+12\Psi'\left(\frac{\lambda^k+x^k}{2}\right)+32\Psi'\left(\frac{x^k+3\lambda^k}{4}\right)+7\Psi'\left(\lambda^k\right) \tag{3.2}$$

is nonsingular, we first consider

$$\alpha^k=\left(\frac{3x^k+\lambda^k}{4}\right),\quad \beta^k=\left(\frac{\lambda^k+x^k}{2}\right),\quad \gamma^k=\left(\frac{x^k+3\lambda^k}{4}\right). \tag{3.3}$$

Now

$$\begin{aligned}
&7\Psi'\left(x^k\right)+32\Psi'\left(\frac{3x^k+\lambda^k}{4}\right)+12\Psi'\left(\frac{\lambda^k+x^k}{2}\right)\\
&+32\Psi'\left(\frac{x^k+3\lambda^k}{4}\right)+7\Psi'\left(\lambda^k\right)\\
=&\,7A-7D\left(x^k\right)+32A-32D\left(\alpha^k\right)+12A-12D\left(\beta^k\right)+32A-32D\left(\gamma^k\right)\\
&+7A-7D\left(\lambda^k\right)\\
=&\,90A-7D\left(x^k\right)-32D\left(\alpha^k\right)-12D\left(\beta^k\right)-32D\left(\gamma^k\right)-7D\left(\lambda^k\right),
\end{aligned}$$

which is nonsingular.

Lemma 3.1. *If svd$(A)>1$, then* $\left(90A-7D\left(x^k\right)-32D\left(\alpha^k\right)-12D\left(\beta^k\right)-32D\left(\gamma^k\right)-7D\left(\lambda^k\right)\right)^{-1}$ *exists for any D defined in equation (1.3).*

Proof. If $90A-7D\left(x^k\right)-32D\left(\alpha^k\right)-12D\left(\beta^k\right)-32D\left(\gamma^k\right)-7D\left(\lambda^k\right)$ is singular, then $\left(90A-7D\left(x^k\right)-32D\left(\alpha^k\right)-12D\left(\beta^k\right)-32D\left(\gamma^k\right)-7D\left(\lambda^k\right)x=0\right.$ for some $x\neq 0$. As the svd$(A)>1$, thus

$$\begin{aligned}
x^Tx<&x^TA^TAx\\
=&\frac{1}{8100}x^T\left(7D\left(x^k\right)+32D\left(\alpha^k\right)+12D\left(\beta^k\right)+32D\left(\gamma^k\right)+7D\left(\lambda^k\right)\right)\\
&\times\left(\left(7D\left(x^k\right)+32D\left(\alpha^k\right)+12D\left(\beta^k\right)+32D\left(\gamma^k\right)+7D\left(\lambda^k\right)\right)x\right.\\
=&\frac{1}{8100}x^T\left(49D\left(x^k\right)D\left(x^k\right)+49D\left(\lambda^k\right)D\left(\lambda^k\right)+1024D\left(\alpha^k\right)D\left(\alpha^k\right)\right.\\
&+144D\left(\beta^k\right)D\left(\beta^k\right)+1024D\left(\gamma^k\right)\left(\gamma^k\right)+448D\left(x^k\right)D\left(\alpha^k\right)\\
&+168D\left(x^k\right)D\left(\beta^k\right)+448D\left(x^k\right)D\left(\gamma^k\right)+768D\left(\alpha^k\right)D\left(\beta^k\right)
\end{aligned}$$

$$
\begin{aligned}
&+2048D\Big(\gamma^k\Big)D\Big(\alpha^k\Big)+448D\Big(\alpha^k\Big)D\Big(\lambda^k\Big)+168D\Big(z^k\Big)D\Big(\beta^k\Big)\\
&+768D\Big(\beta^k\Big)D\Big(\gamma^k\Big)+448D\Big(\lambda^k\Big)\Big)D\Big(\gamma^k\Big)\Big)x\\
\le&\frac{1}{8100}8100x^Tx\\
=&x^Tx,
\end{aligned}
$$

which is a contradiction, hence $90A-7D\Big(x^k\Big)-32D\Big(\alpha^k\Big)-12D\Big(\beta^k\Big)-32D\Big(\gamma^k\Big)-7D\Big(\lambda^k\Big)$ is non-singular. □

Lemma 3.2. *If $svd(A)>1$, then the sequence of STE approach is bounded and well-defined. Therefore, an accumulation point $\bar{x}$ exists such that*

$$
\begin{aligned}
\widetilde{x}=&\widetilde{x}-90\Big(7\Psi'\Big(x^k\Big)+32\Psi'\Big(\alpha^k\Big)+12\Psi'\Big(\beta^k\Big)\\
&+32\Psi'\Big(\gamma^k\Big)+7\Psi'\Big(\lambda^k\Big)\Big)^{-1}\Psi\Big(\widetilde{x}\Big),
\end{aligned}
\tag{3.4}
$$

or it is equivalent to

$$
\begin{aligned}
&\Big(7\Psi'\Big(x^k\Big)+32\Psi'\Big(\alpha^k\Big)+12\Psi'\Big(\beta^k\Big)+32\Psi'\Big(\gamma^k\Big)-7\Psi'\Big(\lambda^k\Big)\Big)\widetilde{x}\\
=&\Big(7\Psi'\Big(x^k\Big)+32\Psi'\Big(\alpha^k\Big)+12\Psi'\Big(\beta^k\Big)+32\Psi'\Big(\gamma^k\Big)+7\Psi'\Big(\lambda^k\Big)\Big)\widetilde{x}-90\Psi\Big(\widetilde{x}\Big).
\end{aligned}
\tag{3.5}
$$

Hence, there exists an accumulation point $\widetilde{x}$ with

$$
\Big(A-\widetilde{D}(\widetilde{x})\Big)\widetilde{x}=b, \tag{3.6}
$$

for some diagonal matrix $\widetilde{D}$ with diagonal elements 0 or ±1 depends on wether the corresponding component of $\widetilde{x}$ is positive, zero, or negative as defined in equation (1.3).

Proof. The proof of this Lemma is analogous to proposition 3 of [17]. Thus it is omitted. □

Theorem 3.1. *If $\left\|\Big(7\Psi'\Big(x^k\Big)+32\Psi'\Big(\alpha^k\Big)+12\Psi'\Big(\beta^k\Big)+32\Psi'\Big(\gamma^k\Big)+7\Psi'\times\Big(\lambda^k\Big)\Big)^{-1}\right\|<\frac{1}{270}$, then the STE approach converges to a solution m of equation (1.1).*

Proof. Consider

$$
\begin{aligned}
x^{k+1}-m=&x^k-90\Big(7\Psi'\Big(x^k\Big)+32\Psi'\Big(\alpha^k\Big)+12\Psi'\Big(\beta^k\Big)+32\Psi'\Big(\gamma^k\Big)\\
&+7\Psi'\Big(\lambda^k\Big)\Big)^{-1}\Psi\Big(x^k\Big)-m.
\end{aligned}
\tag{3.7}
$$

Fot simplicity, let

$$P = 7\Psi'\left(x^k\right) + 32\Psi'\left(\alpha^k\right) + 12\Psi'\left(\beta^k\right) + 32\Psi'\left(\gamma^k\right) + 7\Psi'\left(\lambda^k\right). \tag{3.8}$$

Then, equations (3.7) converts into

$$\begin{aligned} x^{k+1} - m &= x^k - m - 90P^{-1}\Psi\left(x^k\right), \\ P\left(x^{k+1} - m\right) &= P\left(x^k - m\right) - 90\Psi\left(x^k\right). \end{aligned} \tag{3.9}$$

We know that, m is the solution of equation (1.1), thus

$$\Psi(m) = Am - |m| - b = 0. \tag{3.10}$$

From equations (3.9) and (3.10), we obtain

$$\begin{aligned} &P\left(x^{k+1} - m\right) \\ &= P\left(x^k - m\right) - 90\Psi\left(x^k\right) + 90\Psi\left(m\right) \\ &= P\left(x^k - m\right) - 90\left(\Psi\left(x^k\right) - \Psi\left(m\right)\right) \\ &= P\left(x^k - m\right) - 90\left(Ax^k - |x^k| - Am + |m|\right) \\ &= \left(P - 90A\right)\left(x^k - m\right) - 90\left(|m| - |x^k|\right) \\ &= -\left(E\right)\left(x^k - m\right) + 90\left(|x^k| - |m|\right) \end{aligned}$$

where

$$E = 7D\left(x^k\right) + 32D\left(\alpha^k\right) + 12D\left(\beta^k\right) + 32D\left(\gamma^k\right) + 7D\left(\lambda^k\right).$$

Now

$$x^{k+1} - m = \left(P\right)^{-1}\left[90\left(|x^k| - |m|\right) - \left(E\right)\left(x^k - m\right)\right], \tag{3.11}$$

$$x^{k+1} - m = \left(P\right)^{-1}\left[90\left(|x^k| - |m|\right) - E\left(x^k - m\right)\right], \tag{3.12}$$

$$\left\|x^{k+1} - m\right\| \leq \left\|\left(P\right)^{-1}\right\|\left[180\left\|x^k - m\right\| + \left\|\left(E\right)\right\|\left\|x^k - m\right\|\right]. \tag{3.13}$$

In equation (3.13), we utilized Lipschitz continuity of the absolute value (see [17]); that is,

$$\left\||x^k| - |m|\right\| \leq 2\left\|x^k - m\right\|.$$

Since $D\left(x^k\right)$, $D\left(\lambda^k\right)$, $D\left(\alpha^k\right)$, $D\left(\beta^k\right)$ and $D\left(\gamma^k\right)$ are diagonal matrices whose diagonal elements are 0 or ± 1, thus

$$\begin{aligned}\left\|E\right\| &= \left\|7D\left(x^k\right) + 32D\left(\alpha^k\right) + 12D\left(\beta^k\right) + 32D\left(\gamma^k\right) + 7D\left(\lambda^k\right)\right\| \\ &\leq \left\|7D\left(x^k\right)\right\| + \left\|32D\left(\alpha^k\right)\right\| + \left\|12D\left(\beta^k\right)\right\| \\ &\quad + \left\|32D\left(\gamma^k\right)\right\| + \left\|7D\left(\lambda^k\right)\right\| \\ &\leq 90. \end{aligned} \tag{3.14}$$

From equation (3.13) and (3.14), we obtain

$$\left\|x^{k+1} - m\right\| \leq 270\left\|\left(P\right)^{-1}\right\|\left\|x^k - m\right\| < \left\|x^k - m\right\|. \tag{3.15}$$

In equation (3.15), we have utalized the condition that $\left\|\left(7\Psi'\left(x^k\right) + 32\Psi'\left(\alpha^k\right) + 12\Psi'\left(\beta^k\right) + 32\Psi'\left(\gamma^k\right) + 7\Psi'\left(\lambda^k\right)\right)^{-1}\right\| < \frac{1}{270}$. Thus, x^k linearly converges to the solution of AVE (1.1). □

Lemma 3.3. *Let* $\left\|A^{-1}\right\| < \frac{1}{271}$ *and* $D\left(x^k\right)$, $D\left(\alpha^k\right)$, $D\left(\beta^k\right)$, $D\left(\gamma^k\right)$ *and* $D\left(\lambda^k\right)$ *be nonzero. Then STE method is well defined and converges to the unique solution of AVE (1.1) for any initial vector* x^0.

Proof. Since $\left\|A^{-1}\right\| < \frac{1}{271}$, therefore, AVE (1.1) is uniquely solvable for any b see ([18], Proposition 4). Since A^{-1} exists, therefore, by Lemma 2.3.2 [22], we have

$$\left\|\left(7\Psi'\left(x^k\right) + 32\Psi'\left(\alpha^k\right) + 12\Psi'\left(\beta^k\right) + 32\Psi'\left(\gamma^k\right) + 7\Psi'\left(\lambda^k\right)\right)^{-1}\right\|$$

$$= \left\|\left(90A - 7D\left(x^k\right) - 32D\left(\alpha^k\right) - 12D\left(\beta^k\right) - 32D\left(\gamma^k\right) - 7D\left(\lambda^k\right)\right)^{-1}\right\|$$

$$\leq \frac{\left\|\left(90A\right)^{-1}\right\|\left\|-7D\left(x^k\right) - 32D\left(\alpha^k\right) - 12D\left(\beta^k\right) - 32D\left(\gamma^k\right) - 7D\left(\lambda^k\right)\right\|}{1 - \left\|\left(90A\right)^{-1}\right\|\left\|\left(-7D\left(x^k\right) - 32D\left(\alpha^k\right) - 12D\left(\beta^k\right) - 32D\left(\gamma^k\right) - 7D\left(\lambda^k\right)\right\|}$$

$$\leq \frac{\frac{1}{90}\left\|\left(A\right)^{-1}\right\|90}{1 - \frac{1}{90}\left\|\left(A\right)^{-1}\right\|90}$$

$$< \frac{\frac{1}{271}}{1-\frac{1}{271}}$$
$$= \frac{1}{270}.$$

Based on Theorem 3.1, we conclude that the STE approach converges linearly to the unique solution of equation (1.1). □

4. Numerical outcomes

Here, we conduct numerical tests to demonstrate the efficiency of Simpson's Three-Eighths method. Moreover, the iteration, residual, and CPU time, are represented, by **Iters**, **RES**, and **CPU**, respectively. We utilized Intel (R) Core (TM) i5-8145U, 2.30 GHz CPU, and 8 GB of RAM for all numerical experiments. All numerical experiments are initiated with the null vector, and the analysis is terminated when the current iteration is concluded.

$$RES := \frac{\|Ax^k - |x^k| - b\|_2}{\|b\|_2} \leq 10^{-6}.$$

Example 4.1. Let A be a matrix in the form of

$$A = (a_{ij}) \begin{cases} 1000 + i, & \text{for } j = i, \\ 1, \quad \text{for} \begin{cases} j = i+1, i = 1, 2, ..., n-1, \\ j = i-1, i = 2, ..., n, \end{cases} \\ 0, & \text{Otherwise.} \end{cases}$$

Calculate $Au^\star - |u^\star| = b \in R^n$, with $u^\star = (x_1, x_2, x_3, ...x_n)^T \in R^n$ such that $x_i = (-1)^i$.

In Example 4.1, the starting vector and the stopping criterion are taken from [14]. Furthermore, we compare the suggested technique to the new iteration procedure (NA) in [14], and the SORLoapt method given in [4] and with Picard method in [26]. Table 1 summarizes the outcomes of the investigation. According to Table 1 , the developed method determines the AVE solution more rapidly than existing algorithms in terms of Iters and CPU.

Example 4.2. [6] Let

$$A = \text{Tridiag}(-1, 4, -1) = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 4 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 4 \end{pmatrix} \in R^{n\times n}, \; u^\star = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \\ \\ -1 \\ 1 \end{pmatrix} \in R^n,$$

where $Au^\star - |u^\star| = b \in R^n$.

Table 1. Numerical results for Example 4.1 with NA, SORLaopt, Technique I, and STE method.

Techniques	n	1000	2000	3000	4000	5000
NA	Iters	17	18	18	18	18
	CPU	1.9831	10.5160	28.6587	63.6419	117.3205
	RES	7.38e-09	2.60e-09	3.19e-09	3.68e-09	4.11e-09
SORLaopt	Iters	15	15	15	15	15
	CPU	1.4542	9.1963	25.5616	56.0278	102.4061
	RES	1.99e-09	3.62e-09	7.58e-09	3.68e-09	9.88e-09
Picard	Iters	5	5	5	5	5
	CPU	0.6201	1.3475	4.4926	13.3852	44.5911
	RES	1.34e-11	1.68e-11	2.38e-11	3.73e-11	3.13e-11
STE	Iters	2	2	2	2	2
	CPU	0.2638	0.6417	1.4300	2.8050	4.7831
	RES	2.04e-07	2.04e-07	2.04e-07	2.04e-07	2.04e-07

In this example, we compare the presented approach with the SOR-like approach [4] (written as SORLaopt) and the SS iterative approach proposed in [31] (represented by SSA) and with GGS approach [6].

In Table 2, we present the results of the study. Table 2 indicates that all experimented techniques quickly analyze equation (1.1). The proposed method offers superior Iters and CPU values in comparison to existing techniques.

Table 2. Numerical results for Example 4.2 with SORLaopt and SSA techniques, Technique I, and STE technique.

Techniques	n	1000	2000	3000	4000
GGS	Iters	21	22	22	22
	CPU	2.9658	7.7891	17.6613	31.6259
	RES	7.89e–07	4.90e–07	6.01e–07	6.94e–07
SORLaopt	Iters	18	18	18	18
	CPU	2.5147	6.1249	15.9104	27.1345
	RES	6.12e-07	6.13e-07	6.13e-07	6.14e-07
SSA	Iters	14	14	14	14
	CPU	1.7828	5.0954	13.3028	21.1644
	RES	8.91e-07	8.92e-07	8.93e-07	8.93e-07
STE	Iters	7	7	7	7
	CPU	0.6072	1.9383	3.7563	9.5864
	RES	5.47e-07	5.48e-07	5.48e-07	5.48e-07

Example 4.3. [10] We consider AVE (1.1) with

$$A = \text{Tridiag}(-I_n, Z_n, -I_n) = \begin{pmatrix} Z & -I & 0 & \cdots & 0 & 0 \\ -I & Z & -I & \cdots & 0 & 0 \\ 0 & -I & Z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Z & -I \\ 0 & 0 & 0 & \cdots & -I & Z \end{pmatrix} \in R^{n\times n},\ u^\star = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \\ -1 \\ 1 \end{pmatrix} \in R^n,$$

$$Z_n = \text{Tridiag}(-1, 8, -1) = \begin{pmatrix} 8 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 8 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 8 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 8 \end{pmatrix} \in R^{n \times n},$$

where $m = n^2$ and $b = Au^\star - |u^\star| \in R^n$.

In Example 4.3, we compare our proposed algorithm with the SOR-like method [10], MSOR approach [2], and NSOR method [5].

Table 3 shows the numerical outcomes of the four methods. According to the data presented in Table 3, every tested technique produced an accurate outcome in solving for equation (1.1). Compared with the existing methods, the Iters and CPU values in the suggested method are superior. Therefore, we can conclude that the proposed approach is both very effective and practicable in terms of Iters and CPU.

Table 3. Numerical results for Examlpe 4.3 with SOR-like, MSOR, NSOR and STE technique.

Techniques	n	1600	2500	3600	4900
	Iters	16	16	16	16
SOR-like	CPU	0.0433	0.0910	0.1945	0.3554
	RES	4.48e–08	5.68e–08	6.97e–08	8.19e–08
	Iters	12	12	12	12
MSOR	CPU	0.0351	0.0774	0.1585	0.2779
	RES	4.71e–07	5.99e–07	7.28e–07	8.56e–07
	Iters	9	9	10	10
NSOR	CPU	0.0252	0.0538	0.1276	0.2299
	RES	7.10e–07	9.16e–07	1.54e–07	1.82e–07
	Iters	6	6	6	6
STE	CPU	0.0131	1.0336	0.0955	0.1971
	RES	2.69e–07	2.76e–07	2.81e–07	2.84e–07

Example 4.4. [5]. Let

$$A = Tridiag(-1, 8, -1) = \begin{pmatrix} 8 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 8 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 8 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 8 \end{pmatrix} \in R^{n \times n}, \ u^\star = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ \vdots \\ \\ -1 \\ 1 \end{pmatrix} \in R^n,$$

and vector $b = Au^\star - |u^\star| \in R^n$.

Table 4 shows the numerical results of four techniques. Example 4.4 compares the proposed approach with SOR-like approach, MSOR method, and NSOR approach. As shown in Table 4, the four techniques have the potential to solve the problem efficiently and effectively. In Table 4, we report the Iters, the CPU, and the RES. Table 4 shows that the iterations and CPU of the suggested technique are better than the SOR-like, MNSOR, and NSOR methods.

Table 4. Numerical results for Examlpe 4.4 with SOR-like, MSOR, NSOR and STE technique.

Techniques	n	1000	2000	3000	4000	5000	6000
	Iters	12	13	13	13	13	13
SOR-like	CPU	0.0188	0.0501	0.0785	0.1030	0.1358	0.1581
	RES	9.45e–08	2.69e–08	3.29e–08	3.80e–08	4.25e–08	4.66e–08
	Iters	10	10	10	10	10	11
MSOR	CPU	0.0120	0.0464	0.0954	0.1711	0.2583	0.4067
	RES	4.14e–07	5.86e–07	7.18e–07	8.29e–07	9.27e–07	1.03e–07
	Iters	8	9	9	9	9	9
NSOR	CPU	0.0128	0.0347	0.0571	0.0761	0.0960	0.1153
	RES	8.69e–07	6.88e–08	8.43e–08	9.73e–08	1.08e–07	1.19e–07
	Iters	5	5	5	5	5	5
STE	CPU	0.0119	0.02942	0.0484	0.0731	0.1163	0.1920
	RES	1.97e–07	1.97e–07	1.97e–07	1.97e–07	1.97e–07	1.96e–07

Exmple 4.5. Let

$$A = I \otimes Q + H \otimes I \in \mathbb{R}^{q \times q},$$

where $I \in \mathbb{R}^{q \times q}$ is the identity matrix, and $\otimes$ represents the Kronecker product. Similarly, Q and H are $g \times g$ tridiagonal matrices given by:

$$\begin{cases} Q = \text{Tridiag}\Big[-(\dfrac{2+\bar{p}}{8}), 8, -(\dfrac{2-\bar{p}}{8})\Big], \\ H = \text{Tridiag}\Big[-(\dfrac{1+\bar{p}}{4}), 4, -(\dfrac{1-\bar{p}}{4})\Big], \\ \bar{p} = \dfrac{1}{n}; q = n^2. \end{cases}$$

The right-hand side vector $b = Au^\star - |u^\star| \in \mathbb{R}^q$, where $u^\star = ones(q, 1) \in \mathbb{R}^q$. The assumption and the ultimate limiting factor of this example is the starting point from [15]. In the section on the assumption, we evaluate the offered Techniques in light of those demonstrated in [4] (revealed by SPM), the special shift-splitting iteration technique [29], and the Technique described in [26] (revealed by Picard).

Table 5 contains the results of this investigation. Table 5 shows that for each value of q, the presented techniques analyze the solution $\bar{x}$. Based on the numerical resultss proposed in Table 5, we can determine that the strategies we have provided are more successful than the SPM, SSM, and Picard techniques under specific situations when seen from the perspective of the 'Iters' and 'CPU'.

In conclusion, our unique strategies are relevant to AVEs and are within their capabilities. Table 5 shows all the results of the proposed techniques.

Table 5. Numerical results for Example 4.5 with SPM, SSM, Picard, and STE techniques.

Techniques	n	256	1296	2401	4096
	Iters	12	12	12	12
SPM	CPU	1.6492	2.0258	4.1522	7.3811
	RES	3.77e-07	3.74e-07	3.73e-07	3.72e-07
	Iters	8	8	8	8
SSM	CPU	0.9853	1.1725	2.0863	4.3729
	RES	1.54e-07	1.55e-07	1.56e-07	1.56e-07
	Iters	6	6	6	6
Picard	CPU	0.7305	0.9477	1.3571	3.2084
	RES	2.13e-07	2.10e-07	2.09e-07	2.08e-07
	Iters	4	4	4	4
STE	CPU	0.0514	0.6840	1.6036	1.9221
	RES	2.95e-07	2.92e–07	2.91e–07	2.90e–07

5. Conclusion

In this article, we have used a five-point technique to solve AVE. The well-known generalized Newton approach is used as the predictor step in this innovative new technique, while the STE method for AVEs is utilized as the corrector step. The proposed technique converges in the third phase of the analysis. The new technique is extremely useful for finding solutions to complex systems. This concept can be developed further to answer generalized equations involving absolute value.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] L. Abdallah, M. Haddou and T. Migot, *Solving absolute value equation using complementarity and smoothing functions*, Journal of Computational and Applied Mathematics, 2018, 327, 196–207.

[2] Z. Z. Bai, *Modulus based matrix splitting iteration methods for linear complementarity problems*, Numerical Linear Algebra with Applications, 2010, 17(6), 917–933.

[3] L. Caccetta, B. Qu and G. Zhou, *A globally and quadratically convergent method for absolute value equations*, Computational Optimization and Applications, 2011, 48, 45–58.

[4] C. Chen, D. Yu and D. Han, *Optimal parameter for the SOR-like iteration method for solving the system of absolute value equations*, 2021. arXiv: 05781.

[5] X. Dong, X. H. Shao and H. L. Shen, *A new SOR-like method for solving absolute value equations*, Applied Numerical Mathematics, 2020, 156, 410–421.

[6] V. Edalatpour, D. Hezari and D. K. Salkuyeh, *A generalization of the Gauss–Seidel iteration method for solving absolute value equations*, Applied Mathematics and Computation, 2017, 293, 156–167.

[7] A. J. Fakharzadeh and N. N. Shams, *An efficient algorithm for solving absolute value equations*, J. Math. Extension, 2021, 15, 1–23.

[8] J. Feng and S. Liu, *An improved generalized Newton method for absolute value equations*, Springer Plus, 2016, 5, 1042. DOI: 10.1186/s40064-016-2720-5.

[9] N. Gul, H. Chen, J. Iqbal and R. Shah, *A new two-step iterative technique for efficiently solving absolute value equations*, Engineering Computations, 2024.

[10] P. Guo, S. L. Wu and C. X. Li, *On the SOR-like iteration method for solving absolute value equations*, Applied Mathematics Letters, 2019, 97, 107–113.

[11] F. K. Haghani, *On generalized Traub's method for absolute value equations*, J. Optim. Theory. Appl., 2015, 166, 619–625.

[12] S. L. Hu and Z. H. Huang, *A note on absolute value equations*, Optimization Letters, 2010, 4(3), 417–424.

[13] J. Iqbal, A. Iqbal and M. Arif, *Levenberg-Marquardt method for solving systems of absolute value equations*, J. Comput. Appl. Math., 2015, 282, 134–138.

[14] Y. Ke, *The new iteration algorithm for absolute value equation*, Applied Mathematics Letters, 2020, 99, 105990.

[15] Y. F. Ke and C. F. Ma, *SOR-like iteration method for solving absolute value equations*, Applied Mathematics and Computation, 2017, 311, 195–202.

[16] C. X. Li, *A preconditioned AOR iterative method for the absolute value equations*, Int. J. Comput. Methods, 2017, 14, 1–12.

[17] O. L. Mangasarian, *A generalized Newton method for absolute value equations*, Optim Lett., 2009, 3, 101–108.

[18] O. L. Mangasarian and R. R. Meyer, *Absolute value equation*, Linear. Algebra. Appl., 2006, 419, 359–367.

[19] M. Mansoori and A. Erfanian, *Dynamic model to solve the absolute value equations*, J. Comput. Appl. Math., 2018, 333, 28–35.

[20] M. A. Noor, J. Iqbal, S. Khattri and E. Al-Said, *A new iterative method for solving absolute value equations*, Inter. J. Phy. Sci., 2011, 6(7), 1793–1797.

[21] M. A. Noor, J. Iqbal, K. I. Noor and E. Al-Said, *On an iterative method for solving absolute value equations*, Optim. Lett., 2012, 6, 1027–1033.

[22] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Society for Industrial and Applied Mathematics, 2000.

[23] B. T. Polyak, *Introduction to Optimization*, Optimization Software, Inc., Publications Division, New York, 1987,

[24] O. Prokopyev, *On equivalent reformulations for absolute value equations*, Computational Optimization and Applications, 2009, 44, 363–372.

[25] J. Rohn, *A theorem of the alternatives for the equation* $Ax + B \mid x \mid = b$, Linear. Multilinear. Algebra., 2004, 52, 421–426.

[26] J. Rohn, V. Hooshyarbakhsh and R. Farhadsefat, *An iterative method for solving absolute value equations and sufficient conditions for unique solvability*, Optimization Letters, 2014, 8, 35–44.

[27] B. Saheya, C. H. Yu and J. S. Chen, *Numerical comparisons based on four smoothing functions for absolute value equation*, Journal of Applied Mathematics and Computing, 2018, 56, 131–149.

[28] D. K. Salkuyeh, *The Picard–HSS iteration method for absolute value equations*, Optimization Letters, 2014, 8, 2191–2202.

[29] S. L. Wu and C. X. Li, *A special shift splitting iteration method for absolute value equation*, AIMS. Math., 2020, 5, 5171–5183.

[30] L. Yong, S. Liu, Q. Feng and T. Zhou, *Hybrid differential evolution with biogeography-based optimization for absolute value equation*, Journal of Information and Computational Science, 2013, 10(8), 2417–2428.

[31] N. Zainali and T. Lotfi, *On developing a stable and quadratic convergent method for solving absolute value equation*, Journal of Computational and Applied Mathematics, 2018, 330, 742–747.