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CENTERS AND LIMIT CYCLE BIFURCATIONS IN A FAMILY OF PIECEWISE SMOOTH SEPTIC Z_2-EQUIVARIANT SYSTEMS

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Abstract In this paper, we investigate the center-focus problem and the number of limit cycles bifurcating from three foci for a family of piecewise smooth planar septic Z_2-equivariant systems, which include $(\pm 1, 0)$ and infinity as their singularities. We achieve a comprehensive classification of the conditions under which $(\pm 1, 0)$ act as centers. Moreover, we rigorously prove that, under small Z_2-equivariant perturbations, the perturbed system possesses at least 15 limit cycles, comprising 14 with small amplitude and 1 large amplitude with the scheme $1 \supset (7 \cup 7)$.

Keywords Center, limit cycle, bifurcation, Z_2-equivariant switching systems.

MSC(2010) 34C07, 37C23.

1. Introduction

For the past few years, the center and bifurcation problems for planar differential systems have been extensively studied. Theoretically, the center and bifurcation problems are closely related to the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by D. Hilbert in 1900 [19]. A simplified form of this problem was also proposed by S. Smale [36] as one of the 18 most challenging mathematical problems for 21st century. In practical applications, many complex dynamical behaviors are triggered by the bifurcation of limit cycles.

In the study of dynamical systems, center problems, which are closely related to the Hilbert's 16th problem, are far from being completely solved. A complete study on the bi-center problem for Z_2-equivariant cubic vector fields was given in [40, 41], and the bi-center problem for some Z_2-equivariant quintic systems was studied in [35]. In 2017, the bi-isochronous center problem for cubic systems in Z_2-equivariant vector fields with real coefficients was considered in [15]. In 2020, the isochronous center problem for the Z_2-equivariant cubic vector fields with complex coefficients was completely solved [24]. The Z_2-equivariant cubic vector fields with

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weak saddles or resonant saddles were studied in [25–27]. Weak centers and local bifurcation of critical periods in a Z_2-equivariant vector field of degree 5 were studied in [39]. For degenerate singular point, a normal form method was given in [42] and bifurcation of limit cycles and center problem for $p : q$ homogeneous weight systems were studied in [29].

A great many problems appearing frequently in science, particularly in mechanics, electrical engineering and automatic control, are described by dynamical systems whose vector fields (i.e., the right-hand sides of the equations) are not continuous or not differentiable. These systems are indistinctly called discontinuous or non-smooth systems and discussed in the classical books [16, 23]. In recent years, there has been considerable interest in studying bifurcations and chaos in non-smooth systems because these systems are widely encountered in applications. Examples include the squealing noise in car brakes [2, 20], the absence of a thermal equilibrium in gases modeled by scattering billiards [21, 22], relay feedback systems in control theory [1, 6], switching circuits in power electronics [3], impact and dry frictions in mechanical engineering [7, 13, 14], etc. Due to various forms of non-smoothness, non-smooth systems can exhibit not only all kinds of bifurcations belonging to smooth systems, but also complicated nonstandard bifurcation phenomena that are exclusive to non-smooth ones, such as grazing [4, 8], sliding effects [7], border collision [33] etc.

A non-smooth system is called a switching system if such a system is divided by one or more curves which may not be continuous on these curves. In recent decades increasing attention has been paid to the following switching system

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \begin{cases} (F^+(x, y, \mu), G^+(x, y, \mu)), \text{ for } y > 0, \\ (F^-(x, y, \mu), G^-(x, y, \mu)), \text{ for } y < 0, \end{cases} \tag{1.1}$$

where $F^{\pm}(x, y, \mu)$ and $G^{\pm}(x, y, \mu)$ are analytic functions in x and y. It is seen that system (1.1) actually includes two systems: the first equation is called the upper system, defined for $y > 0$, and the second is called the lower system, defined for $y < 0$. Note that $y = 0$ (i.e., the x-axis) is a switching line.

Analogous to the study of smooth dynamical systems, we are interested in the following two fundamental problems in the analysis of Hopf bifurcation in switching systems (1.1)

• The center-focus problem, determining if a singular point on the line $y = 0$ is either a center, an attractor or a repeller.

• The cyclicity problem, finding the maximal number of limit cycles around the singular point under the variation of the parameters inside the systems.

Discontinuous planar differential equations have richer dynamical behavior than smooth dynamical systems. For the center problem, it is well-known that a singular point is a center in planar smooth systems if and only if there exists a local first integral around the singular point. However, the situation is quite complicated in switching systems. The origin of system (1.1) can be a center even if it is not a center of either the upper system or the lower system. On the other hand, if the origin is a center for both the upper system and the lower system of (1.1), system (1.1) may not have a center at the origin. Published literatures show that switching systems may exhibit more limit cycles than continuous ones.

A good deal of work has been done investigating whether some classical bifurcation methods for treating smooth systems, such as the Hopf, homoclinic and subharmonic bifurcation methods, can be generalized to non-smooth cases, see, for example, [5, 10, 11, 17, 30, 31, 37] and the references therein. Both of the center problem and cyclicity problem of switching systems can be investigated via the computation of Lyapunov constants [10, 11, 17]. Gasull and Torregrosa [17] proposed a new method to compute the return map near the critical point based on a suitable decomposition of certain one-forms associated with the expression of (1.1) in polar coordinates. The studies on center-focus problem for switching systems were started in [32, 34]. Up to now, some center conditions have been obtained for some switching Kukles systems [17], switching Liénard systems [12, 30] and switching Bautin systems [11]. It is well known that planar linear systems can not possess limit cycles. However, compared to smooth systems, piecewise linear switching systems may have 2 or 3 limit cycles. Han and Zhang [18] proved that 2 limit cycles can appear near a focus of either FF (focus-focus), FP (focus-parabolic) or PP (parabolic-parabolic) type. The number of small amplitude limit cycles bifurcating from a focus of quadratic switching Bautin systems was investigated in [10,11,17,31]. Particularly, examples with linear lower system possessing 5 small amplitude limit cycles were constructed in [17]. By using the perturbation method, it was shown in [10] that the cyclicity of discontinuous quadratic systems is at least 9. Recently, Tian and Yu [38] constructed an example of switching systems to show the existence of 10 small amplitude limit cycles bifurcating from a center, which is a new lower bound of the maximal number of small amplitude limit cycles obtained in quadratic switching systems near a singular point. Nilpotent center conditions in cubic switching polynomial Liénard systems by higher-order analysis were studied in [9].

In this paper, we deal with the center problem and bifurcation of limit cycles for a class of planar septic Z_2-equivalent systems with 4 switching lines expressed as follows:

$$\begin{cases} \dfrac{dx}{dt} = -x^4y + A_1x^3y^2 + (3 + A_2 + 2A_3)x^2y^3 \\ \qquad -(A_1 - 2A_4)xy^4 - A_2y^5 - y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + A_3)x^3y^2 + (2A_1 - A_4)x^2y^3 \\ \qquad +(2A_2 + A_3)xy^4 + A_4y^5 + x(x^2 + y^2)^3, \end{cases} \quad (x > 0,\ y > 0),$$

$$\begin{cases} \dfrac{dx}{dt} = -x^4y + B_1x^3y^2 + (3 + B_2 + 2B_3)x^2y^3 \\ \qquad -(B_1 - 2B_4)xy^4 - B_2y^5 - y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + B_3)x^3y^2 + (2B_1 - B_4)x^2y^3 \\ \qquad +(2B_2 + B_3)xy^4 + B_4y^5 + x(x^2 + y^2)^3, \end{cases} \quad (x < 0,\ y > 0),$$

$$\begin{cases} \dfrac{dx}{dt} = -x^4y + A_1x^3y^2 + (3 + A_2 + 2A_3)x^2y^3 \\ \qquad -(A_1 - 2A_4)xy^4 - A_2y^5 - y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + A_3)x^3y^2 + (2A_1 - A_4)x^2y^3 \\ \qquad +(2A_2 + A_3)xy^4 + A_4y^5 + x(x^2 + y^2)^3, \end{cases} \quad (x < 0,\ y < 0),$$

$$\begin{cases} \dfrac{dx}{dt} = -x^4y + B_1x^3y^2 + (3 + B_2 + 2B_3)x^2y^3 \\ \qquad -(B_1 - 2B_4)xy^4 - B_2y^5 - y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + B_3)x^3y^2 + (2B_1 - B_4)x^2y^3 \\ \qquad +(2B_2 + B_3)xy^4 + B_4y^5 + x(x^2 + y^2)^3, \end{cases} \quad (x > 0,\ y < 0). \tag{1.2}$$

It is easy to verify that system (1.2) is unchanged under a real planar counterclockwise rotation through π, so it lies in a Z_2-equivariant vector field. System (1.2) admits three singular points: $(\pm 1, 0)$ and the infinity, the former two have the same topological structure.

The remaining sections are depicted as follows. A method of computing Lyapunov constants for switching systems is given in Section 2 as preliminary. Section 3 is devoted to looking for center conditions. The bifurcation of limit cycles generated from the equilibria is considered in Section 4.

2. Preliminary

Lyapunov constants are effective in distinguishing weak foci from centers and in determining the cyclicity of linear center type equilibria. The vanishing of all Lyapunov constants is a necessary and sufficient condition for a singular point to become a center. We will give some properties and technical results about the Poincaré return map and Lyapunov constants associated to switching systems, as offered by [28].

If the functions $F^{\pm}(x, y, \mu)$ and $G^{\pm}(x, y, \mu)$ in system (1.1) are analytic functions in x and y in the neighborhood of the origin, systems (1.1) can be expanded as the following system

$$\begin{cases} \dfrac{dx}{dt} = \delta^+ x - y + \displaystyle\sum_{k=2}^{\infty} X_k^+(x, y), \\ \dfrac{dy}{dt} = x + \delta^+ y + \displaystyle\sum_{k=2}^{\infty} Y_k^+(x, y), \end{cases} \quad \text{on } y \geq 0 \text{ and,}$$

$$\begin{cases} \dfrac{dx}{dt} = \delta^- x - y + \displaystyle\sum_{k=2}^{\infty} X_k^-(x, y), \\ \dfrac{dy}{dt} = x - \delta^- y + \displaystyle\sum_{k=2}^{\infty} Y_k^-(x, y), \end{cases} \quad \text{on } y \leq 0, \tag{2.1}$$

where $X_k^{\pm}(x, y), Y_k^{\pm}(x, y)$ are homogeneous polynomials in x, y of degree k, According to Lemma 2.1 of [17], a Poincare map can be defined by using the upper and lower systems of (2.1). At first, the lower system of (2.1) can be changed to

$$\begin{cases} \dfrac{dx}{dt} = -\delta^- x - y - \displaystyle\sum_{k=2}^{\infty} X_k^-(x, -y), \\ \dfrac{dy}{dt} = x - \delta^- y + \displaystyle\sum_{k=2}^{\infty} Y_k^-(x, -y), \end{cases} \quad \text{on } y \geq 0, \tag{2.2}$$

by the transformation $(x, y, t) \to (x, -y, -t)$. Then, under the transformation

$$x = r\cos\theta,\ y = r\sin\theta, \tag{2.3}$$

the upper system (2.1) and system (2.2) become

$$\begin{aligned} \frac{dr}{dt} &= r\left[\delta^{\pm} + \sum_{k=1}^{\infty} \varphi_{k+2}^{\pm}(\theta) r^k\right], \\ \frac{d\theta}{dt} &= 1 + \sum_{k=1}^{\infty} \psi_{k+2}^{\pm}(\theta) r^k, \end{aligned} \tag{2.4}$$

where $\varphi_k(\theta), \psi_k(\theta)$ are polynomials of $\cos\theta$ and $\sin\theta$, given by

$$\begin{aligned} \varphi_k^{\pm}(\theta) &= \cos\theta X_{k-1}^{\pm}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}^{\pm}(\cos\theta, \sin\theta), \\ \psi_k^{\pm}(\theta) &= \cos\theta Y_{k-1}^{\pm}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}^{\pm}(\cos\theta, \sin\theta). \end{aligned} \tag{2.5}$$

We see from (2.4) that

$$\frac{dr}{d\theta} = r\frac{\delta^{\pm} + \sum\limits_{k=1}^{\infty} \varphi_{k+2}^{\pm}(\theta) r^k}{1 + \sum\limits_{k=1}^{\infty} \psi_{k+2}^{\pm}(\theta) r^k}. \tag{2.6}$$

To study the solutions of this equation, we shall consider a general differential equation

$$\frac{dr}{d\theta} = r\sum_{k=0}^{\infty} R_k^{\pm}(\theta) r^k, \tag{2.7}$$

where $\theta \in (0, \pi)$. Suppose system (2.7) have the following solution of convergent power series

$$r_1 = \tilde{r}_1(\theta, h) = \sum_{k=1}^{\infty} u_k(\theta) h^k \tag{2.8}$$

and

$$r_2 = \tilde{r}_2(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k, \tag{2.9}$$

respectively, satisfying the initial condition $r_1|_{\theta=0} = r_2|_{\theta=0} = h$, where h is sufficiently small and

$$u_1(0) = v_1(0) = 0,\ u_k(0) = v_k(0) = 0,\ k = 2, 3, \cdots. \tag{2.10}$$

We can then define the following successive functions

$$\Delta_1(h) = \tilde{r}_1(\pi, h) - h \tag{2.11}$$

and

$$\Delta_2(h) = \tilde{r}_2(\pi, h) - h \tag{2.12}$$

for the upper system and lower system of (2.2), respectively. Therefore, the successive function for the switching system (2.2) can be defined as

$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h). \tag{2.13}$$

Definition 2.1. (See [28]) Define

$$\Delta(h) = \sum_{k=1}^{\infty} [u_k(\pi) - v_k(\pi)]\, h^k = \sum_{k=1}^{n} V_k h^k, \tag{2.14}$$

where V_k is called the kth order Lyapunov constant of the switching system (2.2).

Definition 2.2. (See [28]) If the functions on the right-hand side of system (2.2) satisfy the following conditions:

$$X^+(x,y) = -X^-(x,-y), Y^+(x,y) = Y^-(x,-y), \tag{2.15}$$

then system (2.2) is said to be symmetric with the x-axis; If the functions on the right-hand side of system (2.2) satisfy the following conditions:

$$\begin{aligned} X^+(x,y) = X^+(-x,y), Y^+(x,y) = -Y^+(-x,y),\\ X^-(x,y) = X^-(-x,y), Y^-(x,y) = -Y^-(-x,y), \end{aligned} \tag{2.16}$$

then system (2.2) is said to be symmetric with the y-axis.

With the above definitions, we have the following result.

Theorem 2.1. (See [28]) *If system* (2.2) *is symmetric with the x-axis or the y-axis, then the origin is a center.*

Theorem 2.2. (See [28]) *If the upper half plane and lower half plane of system* (2.2) *have analytic first integrals $H_1(x,y)$ and $H_2(x,y)$, respectively, then the origin of system* (2.2) *is a center if and only if for $x_1 > 0$ small there exists $x_2 < 0$ such that* $H_j(x_1,0) = H_j(x_2,0), j = 1,2$.

3. Center conditions

This section is dedicated to finding the center conditions of system (1.2) with the aid of symbolic computation.

Since system (1.2) is a Z_2-equivariant system with two symmetric critical points $(\pm 1, 0)$, without loss of generality, it suffices to discuss the case of $(1,0)$.

It is well-known that the Hopf bifurcation is always considered in a small neighborhood of the critical point. Therefore, when we consider the Hopf bifurcation in the neighborhood of $(\pm 1, 0)$, it can be treated as the Hopf bifurcation in a switching system with a switching line. Namely, the Lyapunov constants at $(1,0)$ are the same as

$$\begin{cases} \dfrac{dx}{dt} = -x^4 y + A_1 x^3 y^2 + (3 + A_2 + 2A_3) x^2 y^3 \\ \qquad -(A_1 - 2A_4) x y^4 - A_2 y^5 - y(x^2+y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + A_3) x^3 y^2 + (2A_1 - A_4) x^2 y^3 \\ \qquad +(2A_2 + A_3) x y^4 + A_4 y^5 + x(x^2+y^2)^3, \end{cases} \quad (y > 0),$$

$$\begin{cases} \dfrac{dx}{dt} = -x^4y + B_1x^3y^2 + (3 + B_2 + 2B_3)x^2y^3 \\ \qquad -(B_1 - 2B_4)xy^4 - B_2y^5 - y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = -x^5 - (5 + B_3)x^3y^2 + (2B_1 - B_4)x^2y^3 \\ \qquad +(2B_2 + B_3)xy^4 + B_4y^5 + x(x^2 + y^2)^3, \end{cases} \quad (y < 0). \tag{3.1}$$

With the translation substitution $(x, y, t) \to (x + 1, y, 2t)$, system (1.2) is transformed to the form

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}A_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}A_1xy^2 + \dfrac{1}{2}(A_2 + 2A_3)y^3 \\ \qquad -12x^3y - (3 - A_2 - 2A_3)xy^3 - \dfrac{1}{2}(A_1 - 2A_4)y^4 - 8x^4y \\ \qquad +\dfrac{1}{2}A_1x^3y^2 - \dfrac{1}{2}(15 - A_2 - 2A_3)x^2y^3 - \dfrac{1}{2}(A_1 - 2A_4)xy^4 \\ \qquad -\dfrac{1}{2}(3 + A_2)y^5 - 3xy(x^2 + y^2)^2 - \dfrac{1}{2}y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2 + A_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}A_3xy^2 \qquad\qquad (y > 0), \\ \qquad +\dfrac{1}{2}(2A_1 - A_4)y^3 + 15x^4 + \dfrac{3}{2}(5 - A_3)x^2y^2 + (2A_1 - A_4)xy^3 \\ \qquad +\dfrac{1}{2}(3 + 2A_2 + A_3)y^4 + 10x^5 + \dfrac{1}{2}(25 - A_3)x^3y^2 \\ \qquad +\dfrac{1}{2}(2A_1 - A_4)x^2y^3 + \dfrac{1}{2}(9 + 2A_2 + A_3)xy^4 \\ \qquad +\dfrac{1}{2}(7x^2 + y^2)(x^2 + y^2)^2 + \dfrac{1}{2}x(x^2 + y^2)^3, \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}B_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}B_1xy^2 + \dfrac{1}{2}(B_2 + 2B_3)y^3 \\ \qquad -12x^3y - (3 - B_2 - 2B_3)xy^3 - \dfrac{1}{2}(B_1 - 2B_4)y^4 - 8x^4y \\ \qquad +\dfrac{1}{2}B_1x^3y^2 - \dfrac{1}{2}(15 - B_2 - 2B_3)x^2y^3 - \dfrac{1}{2}(B_1 - 2B_4)xy^4 \\ \qquad -\dfrac{1}{2}(3 + B_2)y^5 - 3xy(x^2 + y^2)^2 - \dfrac{1}{2}y(x^2 + y^2)^3, \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2 + B_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}B_3xy^2 \qquad\qquad (y < 0). \\ \qquad +\dfrac{1}{2}(2B_1 - B_4)y^3 + 15x^4 + \dfrac{3}{2}(5 - B_3)x^2y^2 + (2B_1 - B_4)xy^3 \\ \qquad +\dfrac{1}{2}(3 + 2B_2 + B_3)y^4 + 10x^5 + \dfrac{1}{2}(25 - B_3)x^3y^2 \\ \qquad +\dfrac{1}{2}(2B_1 - B_4)x^2y^3 + \dfrac{1}{2}(9 + 2B_2 + B_3)xy^4 \\ \qquad +\dfrac{1}{2}(7x^2 + y^2)(x^2 + y^2)^2 + \dfrac{1}{2}x(x^2 + y^2)^3, \end{cases} \tag{3.2}$$

Executing the Mathematica program for computing the Lyapunov constants results in

Theorem 3.1. *The first 7 Lyapunov constants at the origin of system* (3.2) *are given as follows:*

$$
\begin{aligned}
V_1 &= -\frac{2}{3}(A_3 - B_3),\\
V_2 &= \frac{1}{16}(2A_1 - A_1A_3 - 3A_4 + 2B_1 - A_3B_1 - 3B_4)\pi,\\
V_3 &= \frac{2}{45}(4A_1^2 + 9A_2 - 2A_1^2A_3 - 3A_2A_3 - 6A_1A_4 + 4A_1B_1\\
&\quad - 2A_1A_3B_1 - 6A_4B_1 - 9B_2 + 3A_3B_2).
\end{aligned} \tag{3.3}
$$

Case (I). For $A_3 \neq 3$,

$$
\begin{aligned}
V_4 = &-\frac{1}{2304(A_3-3)}(A_1 + B_1)(864 + 90A_1^2 + 540A_2 - 252A_3 - 55A_1^2A_3\\
&-360A_2A_3 - 120A_3^2 + 5A_1^2A_3^2 + 60A_2A_3^2 + 36A_3^3 + 45A_1A_4\\
&-75A_1A_3A_4 - 270A_4^2 + 30A_1B_1 - 25A_1A_3B_1 + 5A_1A_3^2B_1\\
&-45A_4B_1 + 15A_3A_4B_1)\pi,\\
V_5 = &-\frac{4}{141754}(A_1 + B_1)(270A_1 + 10A_1^3 - 144A_1A_3 - 5A_1^3A_3 + 18A_1A_3^2\\
&-324A_4 - 15A_1^2A_4 + 108A_3A_4 - 54B_1 + 20A_1^2B_1 - 36A_3B_1\\
&-10A_1^2A_3B_1 + 18A_3^2B_1 - 30A_1A_4B_1 + 10A_1B_1^2\\
&-5A_1A_3B_1^2 - 15A_4B_1^2),
\end{aligned} \tag{3.4}
$$

Subcase ($\mathrm{I_a}$). For $A_3 \neq \frac{1}{36}[108 + 5(A_1 + B_1)^2]$,

$$
\begin{aligned}
V_6 &= \frac{1}{737280[108 - 36A_3 + 5(A_1 + B_1)^2]^2}(A_3 - 3)(1 + A_3)(A_1 + B_1)f_6\pi,\\
V_7 &= -\frac{1}{1134[108 - 36A_3 + 5(A_1 + B_1)^2]^3[48 + 35(A_1 + B_1)^2]}(A_3 - 3)\\
&\quad \times (1 + A_3)(A_1 - B_1)^3(A_1 + B_1)^3[144 + 5(A_1 + B_1)^2]f_7.
\end{aligned} \tag{3.5}
$$

Subcase ($\mathrm{I_b}$). For $A_3 = \frac{1}{36}[108 + 5(A_1 + B_1)^2]$,

$$
\begin{aligned}
V_6 &= -\frac{1}{7464960A_1}g_6\pi,\\
V_7 &= \frac{64}{14467005(5A_1^2 - 12)}A_1^3(9072 - 24A_1^2 + 35A_1^4)(9A_1 + 5A_1^3 + 27A_4).
\end{aligned} \tag{3.6}
$$

Case (II)**.** For $A_3 = 3$,

$$
\begin{aligned}
V_4 &= -\frac{5}{256}(A_1 + 3A_4)(A_2 - B_2)\pi,\\
V_5 &= V_6 = V_7 = 0,
\end{aligned} \tag{3.7}
$$

In the above expression of V_k, it is assumed that: $V_1 = V_2 = \cdots = V_{k-1} = 0, k = 2, 3, 4, 5, 6, 7$, and

$$\begin{aligned}
f_6 &= 4478976 + 5857920A_1^2 + 160800A_1^4 - 875A_1^6 - 2985984A_3 - 2315520A_1^2A_3 \\
&\quad -100800A_1^4A_3 + 497664A_3^2 + 362880A_1^2A_3^2 + 3006720A_1B_1 + 1248000A_1^3B_1 \\
&\quad +26250A_1^5B_1 - 4631040A_1A_3B_1 - 403200A_1^3A_3B_1 + 725760A_1A_3^2B_1 \\
&\quad +5857920B_1^2 + 2174400A_1^2B_1^2 + 112875A_1^4B_1^2 - 2315520A_3B_1^2 \\
&\quad -604800A_1^2A_3B_1^2 + 362880A_3^2B_1^2 + 1248000A_1B_1^3 + 171500A_1^3B_1^3 \\
&\quad -403200A_1A_3B_1^3 + 160800B_1^4 + 112875A_1^2B_1^4 - 100800A_3B_1^4 \\
&\quad +26250A_1B_1^5 - 875B_1^6, \\
f_7 &= 145152 + 96A_1^2 + 35A_1^4 + 192A_1B_1 + 140A_1^3B_1 + 96B_1^2 + 210A_1^2B_1^2 \\
&\quad +140A_1B_1^3 + 35B_1^4, \\
g_6 &= 61236A_1^2 + 59805A_1^4 + 43350A_1^6 - 875A_1^8 + 367416A_1A_4 + 51030A_1^3A_4 \\
&\quad -85050A_1^5A_4 + 551124A_4^2 - 229635A_1^2A_4^2.
\end{aligned}$$

By using the expressions of Lyapunov constants obtained in the above theorem, we can get the center conditions.

Theorem 3.2. *The first 7 Lyapunov constants at the origin of system* (3.2) *vanish if and only if one of the following conditions holds:*

$$B_1 = -A_1, B_2 = A_2, B_3 = A_3, B_4 = -A_4; \tag{3.8}$$
$$A_2 = B_2 = A_3 = B_3 = -1, A_4 = A_1, B_4 = B_1; \tag{3.9}$$
$$A_3 = B_3 = 3, A_4 = -\frac{1}{3}A_1, B_4 = -\frac{1}{4}B_1. \tag{3.10}$$

Proof. By linearly solving $V_7 = 0$ for A_4 given in (3.6), we have

$$A_4 = -\frac{1}{27}A_1(9 + 5A_1), \tag{3.11}$$

under which, V_6 is rewritten as

$$V_6 = \frac{1}{186624}A_1^3(36 + 5A_1^2)(12 + 35A_1^2)\pi \neq 0. \tag{3.12}$$

Therefore, the origin of system (3.2) is impossible to be a center in case ($\mathrm{I_b}$).

Using the expressions of Lyapunov constants in subcase ($\mathrm{I_a}$) and solving the nonlinear system $\{V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = 0\}$, we obtain conditions (3.8) and (3.9). Using the expressions of Lyapunov constants in case (II) and solving the nonlinear system $\{V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = 0\}$, we obtain condition (3.10) and

$$B_1 = -A_1, B_2 = A_2, B_3 = A_3 = 3, B_4 = -A_4, \tag{3.13}$$

which is a special case of condition (3.8). □

Theorem 3.3. *For system* (3.2), *all the Lyapunov constants at the origin vanish if and only if the first 7 Lyapunov constants vanish, i.e., one of the three conditions in Theorem 3.2 holds. Relevantly, the three conditions in Theorem 3.2 are the center conditions of the origin.*

Proof. When condition (3.8) is satisfied, system (3.2) becomes

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}A_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}A_1xy^2 + \dfrac{1}{2}(A_2+2A_3)y^3 \\ \qquad -12x^3y - (3-A_2-2A_3)xy^3 - \dfrac{1}{2}(A_1-2A_4)y^4 - 8x^4y \\ \qquad +\dfrac{1}{2}A_1x^3y^2 - \dfrac{1}{2}(15-A_2-2A_3)x^2y^3 - \dfrac{1}{2}(A_1-2A_4)xy^4 \\ \qquad -\dfrac{1}{2}(3+A_2)y^5 - 3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2+A_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}A_3xy^2 & (y>0), \\ \qquad +\dfrac{1}{2}(2A_1-A_4)y^3 + 15x^4 + \dfrac{3}{2}(5-A_3)x^2y^2 \\ \qquad +(2A_1-A_4)xy^3 + \dfrac{1}{2}(3+2A_2+A_3)y^4 + 10x^5 \\ \qquad +\dfrac{1}{2}(25-A_3)x^3y^2 + \dfrac{1}{2}(2A_1-A_4)x^2y^3 + \dfrac{1}{2}(9+2A_2+A_3)xy^4 \\ \qquad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy - \dfrac{1}{2}A_1y^2 - \dfrac{21}{2}x^2y - \dfrac{3}{2}A_1xy^2 + \dfrac{1}{2}(A_2+2A_3)y^3 \\ \qquad -12x^3y - (3-A_2-2A_3)xy^3 + \dfrac{1}{2}(A_1-2A_4)y^4 - 8x^4y \\ \qquad -\dfrac{1}{2}A_1x^3y^2 - \dfrac{1}{2}(15-A_2-2A_3)x^2y^3 \\ \qquad +\dfrac{1}{2}(A_1-2A_4)xy^4 - \dfrac{1}{2}(3+A_2)y^5 \\ \qquad -3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, & (y<0), \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2+A_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}A_3xy^2 - \dfrac{1}{2}(2A_1-A_4)y^3 \\ \qquad +15x^4 + \dfrac{3}{2}(5-A_3)x^2y^2 - (2A_1-A_4)xy^3 + \dfrac{1}{2}(3+2A_2+A_3)y^4 \\ \qquad +10x^5 + \dfrac{1}{2}(25-A_3)x^3y^2 - \dfrac{1}{2}(2A_1-A_4)x^2y^3 \\ \qquad +\dfrac{1}{2}(9+2A_2+A_3)xy^4 + \dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases} \tag{3.14}$$

whose vector field is symmetric with respect to the x-axis.

Under condition (3.9), system (3.2) writes

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}A_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}A_1xy^2 - \dfrac{3}{2}y^3 - 12x^3y \\ \qquad +\dfrac{3}{2}A_1x^2y^2 - 6xy^3 + \dfrac{1}{2}A_1y^4 \\ \qquad -\dfrac{1}{2}y(16x^2 - A_1xy + 2y^2)(x^2+y^2) \\ \qquad -3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, & (y>0), \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}y^2 + \dfrac{25}{2}x^3 + \dfrac{3}{2}xy^2 + \dfrac{1}{2}A_1y^3 + 15x^4 + 9x^2y^2 \\ \qquad +A_1xy^3 + \dfrac{1}{2}(20x^3 + 6xy^2 + A_1y^3)(x^2+y^2) \\ \qquad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}B_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}B_1xy^2 - \dfrac{3}{2}y^3 - 12x^3y \\ \qquad +\dfrac{3}{2}B_1x^2y^2 - 6xy^3 + \dfrac{1}{2}B_1y^4 \\ \qquad -\dfrac{1}{2}y(16x^2 - B_1xy + 2y^2)(x^2+y^2) \\ \qquad -3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, & (y<0). \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{1}{2}y^2 + \dfrac{25}{2}x^3 + \dfrac{3}{2}xy^2 + \dfrac{1}{2}B_1y^3 + 15x^4 + 9x^2y^2 \\ \qquad +B_1xy^3 + \dfrac{1}{2}(20x^3 + 6xy^2 + B_1y^3)(x^2+y^2) \\ \qquad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases} \tag{3.15}$$

By means of complex transformation

$$z = x+iy, w = x-iy, T = it, i = \sqrt{-1}, \tag{3.16}$$

system (3.15) is transferred into

$$\begin{cases} \dfrac{dz}{dT} = \dfrac{1}{8}(1+z)^2(1+w)[8z + (6+iA_1)z^2 \\ \qquad +2(6-iA_1)zw + (2+iA_1)w^2 \\ \qquad +8z^2w + 8zw^2 + 4z^2w^2], \\ & (\mathrm{Im}(z) = \mathrm{Im}(w) > 0), \\ \dfrac{dw}{dT} = -\dfrac{1}{8}(1+w)^2(1+z)(8w + (6-iA_1)w^2 \\ \qquad +2(6+iA_1)wz + (2-iA_1)z^2 \\ \qquad +8w^2z + 8wz^2 + 4w^2z^2), \end{cases}$$

$$\begin{cases} \dfrac{dz}{dT} = \dfrac{1}{8}(1+z)^2(1+w)[8z+(6+iB_1)z^2 \\ \qquad +2(6-iB_1)zw+(2+iB_1)w^2 \\ \qquad +8z^2w+8zw^2+4z^2w^2], \\ \dfrac{dw}{dT} = -\dfrac{1}{8}(1+w)^2(1+z)[8w+(6-iB_1)w^2 \\ \qquad +2(6+iB_1)wz+(2-iB_1)z^2 \\ \qquad +8w^2z+8wz^2+4w^2z^2], \end{cases} \quad (\mathrm{Im}(z)=\mathrm{Im}(w)<0). \tag{3.17}$$

The upper system and lower one in system (3.17) have the following first integrals

$$\begin{aligned} &H_1^+(z,w) \\ =&\frac{(2-iA_1)z(2+z)+2(2+iA_1+8z+4z^2)w+(2+iA_1+8z+4z^2)w^2+2i(2i-A_1)(1+z)(1+w)\ln(1+w)}{8(1+z)(1+w)} \\ &+\frac{1}{4}i(2i+A_1)\ln(1+z), \end{aligned}$$

$$\begin{aligned} &H_1^-(z,w) \\ =&\frac{(2-iB_1)z(2+z)+2(2+iB_1+8z+4z^2)w+(2+iB_1+8z+4z^2)w^2+2i(2i-B_1)(1+z)(1+w)\ln(1+w)}{8(1+z)(1+w)} \\ &+\frac{1}{4}i(2i+B_1)\ln(1+z), \end{aligned} \tag{3.18}$$

respectively. Furthermore,

$$H_1^+(z,z)=H_1^-(z,z)=\frac{1}{2}[2z+z^2-2\ln(1+z)]. \tag{3.19}$$

When condition (3.10) holds, system (3.2) takes the form

$$\begin{cases} \dfrac{dx}{dt} = -y-5xy+\dfrac{1}{2}A_1y^2-\dfrac{21}{2}x^2y+\dfrac{3}{2}A_1xy^2+\dfrac{1}{2}(6+A_2)y^3 \\ \qquad -12x^3y+\dfrac{3}{2}A_1x^2y^2+(3+A_2)xy^3-\dfrac{5}{6}A_1y^4-8x^4y \\ \qquad +\dfrac{1}{2}A_1x^3y^2-\dfrac{1}{2}(9-A_2)x^2y^3-\dfrac{5}{6}A_1xy^4-\dfrac{1}{2}(3+A_2)y^5 \\ \qquad -3xy(x^2+y^2)^2-\dfrac{1}{2}y(x^2+y^2)^3, \\ \dfrac{dy}{dt} = x+\dfrac{11}{2}x^2-\dfrac{5}{2}y^2+\dfrac{25}{2}x^3-\dfrac{9}{2}xy^2+\dfrac{7}{6}A_1y^3+15x^4 \\ \qquad +3x^2y^2+\dfrac{7}{3}A_1xy^3+(3+A_2)y^4+10x^5 \\ \qquad +11x^3y^2+\dfrac{7}{6}A_1x^2y^3+(6+A_2)xy^4-\dfrac{1}{6}A_1y^5 \\ \qquad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2+\dfrac{1}{2}x(x^2+y^2)^3, \end{cases} \quad (y>0),$$

$$\begin{cases} \dfrac{dx}{dt} = -y - 5xy + \dfrac{1}{2}B_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}B_1xy^2 + \dfrac{1}{2}(6+B_2)y^3 \\ \quad -12x^3y + \dfrac{3}{2}B_1x^2y^2 + (3+B_2)xy^3 - \dfrac{5}{6}B_1y^4 - 8x^4y \\ \quad +\dfrac{1}{2}B_1x^3y^2 - \dfrac{1}{2}(9-B_2)x^2y^3 - \dfrac{5}{6}B_1xy^4 - \dfrac{1}{2}(3+B_2)y^5 \\ \quad -3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, \\ \dfrac{dy}{dt} = x + \dfrac{11}{2}x^2 - \dfrac{5}{2}y^2 + \dfrac{25}{2}x^3 - \dfrac{9}{2}xy^2 + \dfrac{7}{6}B_1y^3 + 15x^4 \\ \quad +3x^2y^2 + \dfrac{7}{3}B_1xy^3 + (3+B_2)y^4 + 10x^5 \\ \quad +11x^3y^2 + \dfrac{7}{6}B_1x^2y^3 + (6+B_2)xy^4 - \dfrac{1}{6}B_1y^5 \\ \quad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases} \quad (y<0). \quad (3.20)$$

With the complex substitution (3.16), system (3.20) can be written in the form:

$$\begin{cases} \dfrac{dz}{dT} = \dfrac{1}{48}(1+z)^2[48z + 6(10+iA_1)z^2 + 12(6-iA_1)zw \\ \quad +6(6+iA_1)w^2 + (15+4iA_1+3A_2)z^3 \\ \quad +3(21-2iA_1-3A_2)z^2w + 9(13+A_2)zw^2 \\ \quad +(21+2iA_1-3A_2)w^3 + 72z^2w^2 \\ \quad +48zw^3 + 24z^2w^3], \\ \dfrac{dw}{dT} = -\dfrac{1}{48}(1+w)^2[48w + 6(10-iA_1)w^2 \\ \quad +12(6+iA_1)wz + 6(6-iA_1)z^2 \\ \quad +(15-4iA_1+3A_2)w^3 + 3(21+2iA_1-3A_2)w^2z \\ \quad +9(13+A_2)wz^2 + (21-2iA_1-3A_2)z^3 \\ \quad +72w^2z^2 + 48wz^3 + 24w^2z^3], \end{cases} \quad (\mathrm{Im}(z) = \mathrm{Im}(w) > 0),$$

(3.21)

$$\begin{cases} \dfrac{dz}{dT} = \dfrac{1}{48}(1+z)^2[48z + 6(10+iB_1)z^2 + 12(6-iB_1)zw \\ \quad +6(6+iB_1)w^2 + (15+4iB_1+3B_2)z^3 \\ \quad +3(21-2iB_1-3B_2)z^2w + 9(13+B_2)zw^2 \\ \quad +(21+2iB_1-3B_2)w^3 + 72z^2w^2 \\ \quad +48zw^3 + 24z^2w^3], \\ \dfrac{dw}{dT} = -\dfrac{1}{48}(1+w)^2[48w + 6(10-iB_1)w^2 \\ \quad +12(6+iB_1)wz + 6(6-iB_1)z^2 \\ \quad +(15-4iB_1+3B_2)w^3 + 3(21+2iB_1-3B_2)w^2z \\ \quad +9(13+B_2)wz^2 + (21-2iB_1-3B_2)z^3 \\ \quad +72w^2z^2 + 48wz^3 + 24w^2z^3], \end{cases} \quad (\mathrm{Im}(z) = \mathrm{Im}(w) < 0),$$

whose upper system and lower system have the following first integrals

$$\begin{aligned}
&H_2^+(z,w)\\
=&-\frac{1}{192(1+z)^4(1+w)^4}[48+192(z+w)+288(z+w)^2+8(18+iA_1)z^3\\
&+24(30-iA_1)z^2w+24(30+iA_1)zw^2+8(18-iA_1)w^3+(15+4iA_1+3A_2)z^4\\
&+4(69-2iA_1-3A_2)z^3w+18(45+A_2)z^2w^2+4(69+2iA_1-3A_2)zw^3\\
&+(15-4iA_1+3A_2)w^4+288(z+w)z^2w^2+96z^3w^3],\\
&H_2^-(z,w)\\
=&-\frac{1}{192(1+z)^4(1+w)^4}[48+192(z+w)+288(z+w)^2+8(18+iB_1)z^3\\
&+24(30-iB_1)z^2w+24(30+iB_1)zw^2+8(18-iB_1)w^3+(15+4iB_1+3B_2)z^4\\
&+4(69-2iB_1-3B_2)z^3w+18(45+B_2)z^2w^2+4(69+2iB_1-3B_2)zw^3\\
&+(15-4iB_1+3B_2)w^4+288(z+w)z^2w^2+96z^3w^3],
\end{aligned} \tag{3.22}$$

respectively. Moreover,

$$H_2^+(z,z)=H_2^-(z,z)=-\frac{1+4z+2z^2}{4(1+z)^4}. \tag{3.23}$$

□

Naturally, we have the following result.

Corollary 3.1. *The equilibria* $(\pm1,0)$ *of system* (1.2) *become centers if and only if one of the three conditions in Theorem 3.2 is satisfied.*

4. Bifurcation of limit cycles

In this section, we will employ the computation method of Lyapunov constants as presented in Section 2 to demonstrate that system (1.2) can exhibit 14 small amplitude limit cycles bifurcating from $(\pm1,0)$ and one large amplitude limit cycle bifurcating from the infinity.

Theorem 4.1. *For system* (3.2)*, the origin is a 7th order weak focus if and only if either*

$$\begin{aligned}
&B_3=A_3,\\
&B_4=\frac{(A_1+B_1)(2-A_3)-3A_4}{3},\\
&B_2=\frac{2A_1^2(A_3-2)+2A_1(3A_4-2B_1+A_3B_1)+3A_2(A_3-3)+6A_4B_1}{3(A_3-3)},\\
&A_2=-\frac{864+5A_1^2(A_3-2)(A_3-9)-5A_1[3(5A_3-3)A_4-(A_3-2)(A_3-3)B_1]-3A_3(84+40A_3-12A_3^2-5A_4B_1)-45A_4(6A_4+B_1)}{60(A_3-3)^2},\\
&A_4=\frac{5A_1^2(2-A_3)(A_1+2B_1)+A_1[18(A_3-3)(A_3-5)-5(A_3-2)B_1^2]+18(A_3-3)(1+A_3)B_1}{3[108-36A_3+5(A_1+B_1)^2]},\\
&f_6=0,
\end{aligned}$$

$$\{36A_3 - [108 + 5(A_1 + B_1)^2]\}(A_1 - B_1)(A_1 + B_1)(1 + A_3)(A_3 - 3)(3A_3 - 13) \neq 0, \tag{4.1}$$

or

$$\begin{aligned}
&B_1 = A_1,\\
&B_3 = A_3 = \frac{27 + 5A_1^2}{9},\\
&B_4 = -\frac{18A_1 + 10A_1^3 + 27A_4}{27},\\
&B_2 = \frac{A_1^2(243 - 60A_1^2 + 125A_1^4) + 27A_4(54A_1 + 50A_1^3 + 81A_4)}{150A_1^4},\\
&A_2 = \frac{A_1^2(81 - 140A_1^2 - 25A_1^4) + 9A_4(54A_1 + 10A_1^3 + 81A_4)}{50A_1^4},\\
&g_6 = 0,\\
&A_1(5A_1^2 - 12)(9A_1 + 5A_1^3 + 27A_4) \neq 0,
\end{aligned} \tag{4.2}$$

holds.

Proof. In subcase ($\mathrm{I_a}$), it is easy to verify that the equation $f_7 = 0$ has no real solution. Then, setting $\{V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = 0, V_7 \neq 0\}$ gives the desired condition (4.1).

In subcase ($\mathrm{I_b}$), the fifth Lyapunov constant is simplified as

$$V_5 = \frac{1}{51030}(A_1 - B_1)(A_1 + B_1)^3[144 + 5(A_1 + B_1)^2].$$

We solve $V_5 = 0$ to get $A_1 + B_1 = 0$ or $A_1 - B_1 = 0$. However, the latter is impossible. Otherwise, we get $A_3 = 3$, which is contradictory with case (I). Then, solving $\{V_1 = V_2 = V_3 = V_4 = V_6 = 0, V_7 \neq 0\}$ yields condition (4.2). □

Theorem 4.2. *Suppose that the origin is a 7th order weak focus, then 7 small amplitude limit cycles can bifurcate from the perturbed system of* (3.2)*:*

$$\begin{cases}
\dfrac{dx}{dt} = \delta^+ x - y - 5xy + \dfrac{1}{2}A_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}A_1xy^2\\
\qquad +\dfrac{1}{2}(A_2 + 2A_3)y^3 - 12x^3y - (3 - A_2 - 2A_3)xy^3\\
\qquad -\dfrac{1}{2}(A_1 - 2A_4)y^4 - 8x^4y + \dfrac{1}{2}A_1x^3y^2\\
\qquad -\dfrac{1}{2}(15 - A_2 - 2A_3)x^2y^3 - \dfrac{1}{2}(A_1 - 2A_4)xy^4 - \dfrac{1}{2}(3 + A_2)y^5\\
\qquad -3xy(x^2 + y^2)^2 - \dfrac{1}{2}y(x^2 + y^2)^3,\\
\dfrac{dy}{dt} = x + \delta^+ y + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2 + A_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}A_3xy^2\\
\qquad +\dfrac{1}{2}(2A_1 - A_4)y^3 + 15x^4 + \dfrac{3}{2}(5 - A_3)x^2y^2\\
\qquad +(2A_1 - A_4)xy^3 + \dfrac{1}{2}(3 + 2A_2 + A_3)y^4 + 10x^5\\
\qquad +\dfrac{1}{2}(25 - A_3)x^3y^2 + \dfrac{1}{2}(2A_1 - A_4)x^2y^3 + \dfrac{1}{2}(9 + 2A_2 + A_3)xy^4\\
\qquad +\dfrac{1}{2}(7x^2 + y^2)(x^2 + y^2)^2 + \dfrac{1}{2}x(x^2 + y^2)^3,
\end{cases} \quad (y > 0),$$

$$\begin{cases} \dfrac{dx}{dt} = \delta^- x - y - 5xy + \dfrac{1}{2}B_1y^2 - \dfrac{21}{2}x^2y + \dfrac{3}{2}B_1xy^2 \\ \quad +\dfrac{1}{2}(B_2+2B_3)y^3 - 12x^3y - (3 - B_2 - 2B_3)xy^3 \\ \quad -\dfrac{1}{2}(B_1 - 2B_4)y^4 - 8x^4y + \dfrac{1}{2}B_1x^3y^2 \\ \quad -\dfrac{1}{2}(15 - B_2 - 2B_3)x^2y^3 - \dfrac{1}{2}(B_1 - 2B_4)xy^4 - \dfrac{1}{2}(3+B_2)y^5 \\ \quad -3xy(x^2+y^2)^2 - \dfrac{1}{2}y(x^2+y^2)^3, \\ \dfrac{dy}{dt} = x + \delta^- y + \dfrac{11}{2}x^2 - \dfrac{1}{2}(2+B_3)y^2 + \dfrac{25}{2}x^3 - \dfrac{3}{2}B_3xy^2 \\ \quad +\dfrac{1}{2}(2B_1 - B_4)y^3 + 15x^4 + \dfrac{3}{2}(5 - B_3)x^2y^2 \\ \quad +(2B_1 - B_4)xy^3 + \dfrac{1}{2}(3 + 2B_2 + B_3)y^4 + 10x^5 \\ \quad +\dfrac{1}{2}(25 - B_3)x^3y^2 + \dfrac{1}{2}(2B_1 - B_4)x^2y^3 + \dfrac{1}{2}(9 + 2B_2 + B_3)xy^4 \\ \quad +\dfrac{1}{2}(7x^2+y^2)(x^2+y^2)^2 + \dfrac{1}{2}x(x^2+y^2)^3, \end{cases} \quad (y<0), \tag{4.3}$$

where $0 < \delta^{\pm} \ll 1$.

Proof. When the origin of system (3.2) is a 7th order weak focus, i.e., condition (4.1) or (4.2) holds, after computing the determinant of the Jacobian matrix, we arrive at

$$\begin{aligned} &\det\left(\frac{\partial(V_1,V_2,V_3,V_4,V_5,V_6)}{\partial(B_3,B_4,B_2,A_2,A_4,A_3)}\right)\Big|_{(4.1)} \\ &= \left(\frac{\partial V_1}{\partial B_3}\cdot\frac{\partial V_2}{\partial B_4}\cdot\frac{\partial V_3}{\partial B_2}\cdot\frac{\partial V_4}{\partial A_2}\cdot\frac{\partial V_5}{\partial A_4}\cdot\frac{\partial V_6}{\partial A_3}\right)\Big|_{(4.1)} \\ &= \frac{1}{1003290624000[108 - 36A_3 + 5(A_1+B_1)^2]^2}(A_3-3)^2(A_1+B_1)^3\pi^3 f \\ &\neq 0, \end{aligned} \tag{4.4}$$

or

$$\begin{aligned} &\det\left(\frac{\partial(V_1,V_2,V_3,V_4,V_5,V_6)}{\partial(B_3,B_4,B_2,A_2,B_1,A_4)}\right)\Big|_{(4.2)} \\ &= \left(\frac{\partial V_1}{\partial B_3}\cdot\frac{\partial V_2}{\partial B_4}\cdot\frac{\partial V_3}{\partial B_2}\cdot\frac{\partial V_4}{\partial A_2}\cdot\frac{\partial V_5}{\partial B_1}\cdot\frac{\partial V_6}{\partial A_4}\right)\Big|_{(4.2)} \\ &= \frac{1}{13060694016}A_1^7(5A_1^2-12)(36+5A_1^2)(9A_1+5A_1^3+27A_4)\pi^3 \\ &\neq 0, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} f = &\ 483729408 + 890196480A_1^2 + 30326400A_1^4 - 141000A_1^6 - 4375A_1^8 \\ &- 967458816A_3 - 974177280A_1^2A_3 - 85795200A_1^4A_3 - 1749000A_1^6A_3 \\ &+ 4375A_1^8A_3 + 644972544A_3^2 + 522547200A_1^2A_3^2 + 39139200A_1^4A_3^2 \end{aligned}$$

$$
\begin{aligned}
&+756000A_1^6A_3^2-179159040A_3^3-138101760A_1^2A_3^3-5443200A_1^4A_3^3\\
&+17915904A_3^4+13063680A_1^2A_3^4-100776960A_1B_1+208396800A_1^3B_1\\
&+8982000A_1^5B_1+122500A_1^7B_1-1321297920A_1A_3B_1\\
&-343180800A_1^3A_3B_1-15786000A_1^5A_3B_1-122500A_1^7A_3B_1\\
&+1045094400A_1A_3^2B_1+156556800A_1^3A_3^2B_1+4536000A_1^5A_3^2B_1\\
&-276203520A_1A_3^3B_1-21772800A_1^3A_3^3B_1+26127360A_1A_3^4B_1\\
&+890196480B_1^2+356140800A_1^2B_1^2+37197000A_1^4B_1^2+822500A_1^6B_1^2\\
&-974177280A_3B_1^2-514771200A_1^2A_3B_1^2-47403000A_1^4A_3B_1^2\\
&-822500A_1^6A_3B_1^2+522547200A_3^2B_1^2+234835200A_1^2A_3^2B_1^2\\
&+11340000A_1^4A_3^2B_1^2-138101760A_3^3B_1^2-32659200A_1^2A_3^3B_1^2\\
&+13063680A_3^4B_1^2+208396800A_1B_1^3+56148000A_1^3B_1^3+2117500A_1^5B_1^3\\
&-343180800A_1A_3B_1^3-66732000A_1^3A_3B_1^3-2117500A_1^5A_3B_1^3\\
&+156556800A_1A_3^2B_1^3+15120000A_1^3A_3^2B_1^3-21772800A_1A_3^3B_1^3\\
&+30326400B_1^4+37197000A_1^2B_1^4+2843750A_1^4B_1^4-85795200A_3B_1^4\\
&-47403000A_1^2A_3B_1^4-2843750A_1^4A_3B_1^4+39139200A_3^2B_1^4\\
&+11340000A_1^2A_3^2B_1^4-5443200A_3^3B_1^4+8982000A_1B_1^5+2117500A_1^3B_1^5\\
&-15786000A_1A_3B_1^5-2117500A_1^3A_3B_1^5+4536000A_1A_3^2B_1^5\\
&-141000B_1^6+822500A_1^2B_1^6-1749000A_3B_1^6-822500A_1^2A_3B_1^6\\
&+756000A_3^2B_1^6+122500A_1B_1^7-122500A_1A_3B_1^7-4375B_1^8+4375A_3B_1^8,
\end{aligned}
$$

due to

$$
\begin{aligned}
&\text{Resultant}[f,f_6,B_1]\\
=&20201064598082918022646712795034104987910144000000000000000000000000000(12+35A_1^2)^2\\
&\times(27+5A_1^2-9A_3)^4(A_3-3)^6(1+A_3)^{10}(3A_3-13)^4\\
\neq&0,
\end{aligned}
$$

which means f and f_6 do not have common solutions. Therefore, for $0<\delta^{\pm}\ll 1$, 7 small amplitude limit cycles can bifurcate from the origin of the perturbed system (4.3). □

By employing the aforementioned theorem, we are able to establish the following result.

Theorem 4.3. *For system* (1.2)*, 15 limit cycles can appear near* $(\pm 1,0)$ *and the infinity under small perturbation. The distribution is that, 7 small amplitude limit cycles around* $(-1,0)$*, one large amplitude limit cycle around the infinity, and 7 small amplitude limit cycles around* $(1,0)$*.*

Proof. The first Lyapunov constant at the infinity of system (1.2) can be computed as

$$V_1^{\infty}=-\frac{1}{8}[(A_1+B_1)+3(A_4+B_4)].$$

It follows from Theorem 4.2 that 7 small amplitude limit cycles can bifurcate from each of $(\pm 1,0)$ of system (1.2) because of its equivariance. At the same time, we have

$$V_1^{\infty}\big|_{(4.1)} = \frac{1}{8}(A_3 - 3)(A_1 + B_1)\pi \neq 0,$$

or

$$V_1^{\infty}\big|_{(4.2)} = \frac{5}{36}A_1^3\pi \neq 0,$$

which shows that when $(\pm 1, 0)$ are 7th order weak foci, the infinity is a first order weak focus. □

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