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LIE SYMMETRY AND EXACT SOLUTIONS FOR THE POROUS MEDIUM EQUATION

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Abstract This paper aims to study a (2+1)-dimensional Biological population model with the porous medium by Lie symmetry method. By using commutation tables, the one-dimensional optimal subalgebras for the porous medium equation is given. Group invariant solutions of this model are constructed by the reduction equations. Further, the dynamic behavior of the model graphically is presented.

Keywords Porous medium equation, Lie symmetry method, optimal system, exact solutions.

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1. Introduction

Nonlinear partial differential equations(NPDEs) are studied by many scholars in various fields such as plasma physics, chemical physics, applied mathematics, mechanical systems, ocean waves, optics, quantum mechanics, biological mathematics and so on [2,5,14,16,24,37,38]. Because the solutions of NPDEs can describe different complicated physical phenomena, there are a variety of mathematical methods to construct the exact solutions, such as the bi-factor method [11], the inverse scattering method [39], Lagrange characteristic method [9], extended transformed rational function method [40], the first integral method [22], the modified extended tanh-function method [1], the modified simple equation method [12, p11], the extended F-expansion method [21] and so on. Recently, Silem et al [35] studied the vc-nNLS equation by the Hirota method. The authors [20, p11] studied the Mixed Integer Linear Programming models with strong relaxations for the shallow water waves. Lie symmetry analysis [28] plays a significant role in obtaining exact solutions, linearization and conservation laws of nonlinear PDEs. A number of the literatures have referred to the method [4,6,13,17–19,29,30].

The dispersal or emigration is a key factor in the regulation of population of the species. Gurtin and MacCamy [8] gave a special transformation and confirmed existence and uniqueness for the one-dimensional initial-value problem as well as the solution for an initial point source, which could be applied to the above equation.

$$\frac{d}{dt}\int_{\Gamma} udV + \int_{\partial\Gamma} u\vec{\nu}\cdot\hat{n}dV = \int_{\Gamma} gdV,$$

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where Γ represents any regular subregion, u is the population density, $\vec{\nu}$ is the diffusion velocity and $\hat{n}$ is the outward unit normal to the $\partial\Gamma$ of Γ, g stands for the population supply due to births and deaths. Denote $\vec{\nu} = -F(u) \vartriangle u$ and $g = g(u)$ [7], the degenerate parabolic equations are given by

$$u_t = F(u)_{xx} + F(u)_{yy} + g(u), \quad t \geq 0, x, y \in \mathbb{R}. \tag{1.1}$$

When $g(u) = \alpha u$ and α is an constant, it satisfies Malthusian Law [8]. When $g(u) = \alpha_1 u - \alpha_2 u^2$ and α_1, α_2 are constants, it satisfies Verhulst model [8]. When $g(u) = \alpha u^k$, $\alpha > 0, 0 < k \leq 1$, it is a porous media model [3, 27]. Different graphical representations generated by (1.1) show the specific spread. It is very helpful in demonstrating the enlargement of viruses, parasites and diseases, finding the greatest harvest for farmers, working and controlling the delicate species and many other fields [10, 25].

To consider a walk through a rectangular mesh, in which individuals may either stay at their present location or may move in a direction of the lowest population density, a model leads to the normal biological population model

$$u_t = u_{xx}^2 + u_{yy}^2 + g(u), \tag{1.2}$$

which means $F(u) = u^2$ in Eq. (1.1). In [23], Lu investigated the Hölder estimates of solutions of Eq. (1.2). Shakeri and Dehghan [33] used the variational iteration method and Adomian decomposition method to study numerical solution of a more general form of g as $g(u) = hu^a(1 - ru^b)$. Liew et al [41, p11] considered numerical modeling of the biological population problems by using an improved element-free Galerkin method. Shagolshem et al [32, p11] constructed exact solutions for biological population model with Malthusian law by using Lie point symmetry method, furthermore, the conservation laws were analysed. Arora et al [34] considered invariant solutions of a Verhulst biological population model by using Lie symmetry analysis and conservation laws for this model by the multiplier method. However, Lie symmetry analysis of Eq. (1.2) with porous media law is still open.

Some authors tackled the time fractional-order biological population model

$$\partial_t^\theta u = u_{xx}^2 + u_{yy}^2 + g(u). \quad 0 < \theta < 1. \tag{1.3}$$

For example, Srivastava et al [36, p11] found the analytical solution of two-dimensional time fractional-order biological population model. Zhang et al [42] firstly studied exact solutions of Eq. (1.3) by Lie symmetry analysis and the F-expansion method. Khater [15] considered the nonlinear fractional biology population model

$$\partial_t^\theta u = \partial_{xx}^{2\theta} u^2 + \partial_{yy}^{2\theta} u^2 + c(u^2 - s), \quad 0 < \theta \leq 1, \tag{1.4}$$

in which θ, c and s are random constants, the exact solutions are constructed by using the generalized Khater (GK) technique and utilizing Atangana's conformable fractional derivative operator. Various forms of solutions of the biological population model with a novel beta$-$time derivative operators were obtained via the extended Sinh$-$Gordon equation expansion method and the Expa function method by Nisar et al [26, p11]. Sarwar [31, p11] studied the fractional-order biological population models with Malthusian, Verhulst, and porous media laws by the optimal homotopy asymptotic method.

Motivated by the above nonlinear population system, in this paper, we perform Lie symmetry analysis method for the (2+1)-dimensional Biological population model with porous media law

$$u_t - (u^2)_{xx} - (u^2)_{yy} + \alpha\sqrt{u} = 0. \tag{1.5}$$

In Section 2, Lie symmetry analysis and the one-dimensional optimal system of infinitesimal generators by commutator table are considered. Section 3 constructs several exact solutions of Eq. (1.5) by the reduction equations based on the optimal subalgebras. In Section 4, physical analysis of some exact solutions are discussed. Finally we conclude the results in Section 5.

2. Lie point symmetry and optimal system

In this Section, Lie point symmetries [43, p11] can be analyzed and an optimal system is derived. Consider the Lie group of point transformations

$$\begin{aligned}
\overline{t} &= t + \epsilon\tau(t,x,y,u) + O(\varepsilon^2),\\
\overline{x} &= x + \epsilon\zeta(t,x,y,u) + O(\varepsilon^2),\\
\overline{y} &= y + \epsilon\chi(t,x,y,u) + O(\varepsilon^2),\\
\overline{u} &= u + \epsilon\psi(t,x,y,u) + O(\varepsilon^2),
\end{aligned} \tag{2.1}$$

in which ε is a parameter, the functions τ, ζ, η, ψ are the infinitesimals generators. Then the vector field associated with Lie algebra of Eq. (1.5) is

$$\Re = \tau(t,x,y,u)\partial t + \zeta(t,x,y,u)\partial x + \chi(t,x,y,u)\partial y + \psi\partial u.$$

By applying the second prolongation $Pr^2\Re$ to Eq. (1.5), and solving the determined equations, we obtain

$$\begin{aligned}
\tau &= c_1 t + c_2,\\
\zeta &= -c_3 y + \frac{3}{2}c_1 x + c_5,\\
\chi &= \frac{3}{2}c_1 y + c_3 x + c_4,\\
\psi &= 2c_1 u,
\end{aligned} \tag{2.2}$$

in which $c_1, \cdots, c_5$ are arbitrary constants. Then $\Re$ can be rewritten as

$$\Re = (-c_3 y + \frac{3}{2}c_1 x + c_5)\partial x + (\frac{3}{2}c_1 y + c_3 x + c_4)\partial y + (c_1 t + c_2)\partial t + 2c_1 u\partial u.$$

Furthermore, corresponding to the vector field $\Re_i$,

$$\begin{aligned}
\Re_1 &= t\partial t + \frac{3}{2}x\partial x + \frac{3}{2}y\partial y + 2u\partial u,\\
\Re_2 &= \partial t,\\
\Re_3 &= -y\partial x + x\partial y,\\
\Re_4 &= \partial y,
\end{aligned}$$

$$\Re_5 = \partial x. \tag{2.3}$$

We get the symmetry groups $G_i : (t, x, y, u) \to (\bar{t}, \overline{x}, \overline{y}, \overline{u})$:

$$\begin{aligned}
&G_1 : (t, x, y, u) \to (te^{\epsilon}, xe^{\frac{3}{2}\epsilon}, ye^{\frac{3}{2}\epsilon}, ue^{2\epsilon}),\\
&G_2 : (t, x, y, u) \to (t + \epsilon, x, y, u),\\
&G_3 : (t, x, y, u) \to (t, x\cos\epsilon - y\sin\epsilon, x\sin\epsilon + y\cos\epsilon, u),\\
&G_4 : (t, x, y, u) \to (t, x, y + \epsilon, u),\\
&G_5 : (t, x, y, u) \to (t, x + \epsilon, y, u).
\end{aligned} \tag{2.4}$$

Theorem 2.1. *If $u = f(t, x, y)$ satisfies Eq.* (1.5)*, the new solutions $u_i, (i = 1, \cdots, 5)$ can be given by*

$$\begin{aligned}
&u_1 = e^{2\epsilon} f(te^{-\epsilon}, xe^{-\frac{3}{2}\epsilon}, ye^{-\frac{3}{2}\epsilon}),\\
&u_2 = f(t - \epsilon, x, y),\\
&u_3 = f(t, x\cos\epsilon + y\sin\epsilon, y\cos\epsilon - x\sin\epsilon),\\
&u_4 = f(t, x, y - \epsilon),\\
&u_5 = f(t, x - \epsilon, y).
\end{aligned} \tag{2.5}$$

For (2.3), by the definition of Lie brackets $[\Re_i, \Re_j] = \Re_i\Re_j - \Re_j\Re_i$, the following Table can be obtained.

Table 1. Commutator Table of Lie algebra for Eq. (1.5).

$*$	$\Re_1$	$\Re_2$	$\Re_3$	$\Re_4$	$\Re_5$
$\Re_1$	0	$-\Re_2$	0	$-\frac{3}{2}\Re_4$	$-\frac{3}{2}\Re_5$
$\Re_2$	$\Re_2$	0	0	0	0
$\Re_3$	0	0	0	$\Re_5$	$-\Re_4$
$\Re_4$	$\frac{3}{2}\Re_4$	0	$-\Re_5$	0	0
$\Re_5$	$\frac{3}{2}\Re_5$	0	$\Re_4$	0	0

Generators $\Re_1, \cdots, \Re_5$ are linearly independent so that any infinitesimal of Eq. (1.5) can be expressed by

$$\Re = l_1\Re_1 + l_2\Re_2 + l_3\Re_3 + l_4\Re_4 + l_5\Re_5.$$

Next, for constructing the one-dimensional optimal system, $l = (l_1, l_2, l_3, l_4, l_5)$, for $i, = 1, \cdots, 5$, we have

$$E_i = c_{ij}^k l_j \partial_{l_k}$$

in which c_{ij}^k can be derived by $[\Re_i, \Re_j] = c_{ij}^k \Re_k$. Then E_1, E_2, E_3, E_4, E_5 are given by

$$\begin{aligned}
&E_1 = c_{12}^2 l_2\partial_{l_2} + c_{14}^4 l_4\partial_{l_4} + c_{15}^5 l_5\partial_{l_5} = -l_2\partial_{l_2} - \frac{3}{2} l_4\partial_{l_4} - \frac{3}{2} l_5\partial_{l_5},\\
&E_2 = c_{21}^2 l_1\partial_{l_2} = l_1\partial_{l_2},\\
&E_3 = c_{34}^5 l_4\partial_{l_5} + c_{35}^4 l_5\partial_{l_4} = l_4\partial_{l_5} - l_5\partial_{l_4},\\
&E_4 = c_{41}^4 l_1\partial_{l_4} + c_{43}^5 l_3\partial_{l_5} = \frac{3}{2} l_1\partial_{l_4} - l_3\partial_{l_5},
\end{aligned}$$

$$E_5 = c_{51}^5 l_1 \partial_{l_5} + c_{53}^4 l_3 \partial_{l_4} = \frac{3}{2} l_1 \partial_{l_5} + l_3 \partial_{l_4}. \tag{2.6}$$

With the parameters a_j and $\bar{l} \mid_{a_j=0} = l, j = 1, \cdots, 5$, Lie equations can be expressed as

$$\begin{aligned}
&\frac{d\overline{l_1}}{da_1} = 0, \frac{d\overline{l_2}}{da_1} = -\overline{l_2}, \frac{d\overline{l_3}}{da_1} = 0, \frac{d\overline{l_4}}{da_1} = -\frac{3}{2}\overline{l_4}, \frac{d\overline{l_5}}{da_1} = -\frac{3}{2}\overline{l_5},\\
&\frac{d\overline{l_1}}{da_2} = 0, \frac{d\overline{l_2}}{da_2} = \overline{l_1}, \frac{d\overline{l_3}}{da_2} = 0, \frac{d\overline{l_4}}{da_2} = 0, \frac{d\overline{l_5}}{da_2} = 0,\\
&\frac{d\overline{l_1}}{da_3} = 0, \frac{d\overline{l_2}}{da_3} = 0, \frac{d\overline{l_3}}{da_3} = 0, \frac{d\overline{l_4}}{da_3} = -\overline{l_5}, \frac{d\overline{l_5}}{da_3} = \overline{l_4},\\
&\frac{d\overline{l_1}}{da_4} = 0, \frac{d\overline{l_2}}{da_4} = 0, \frac{d\overline{l_3}}{da_4} = 0, \frac{d\overline{l_4}}{da_4} = \frac{3}{2}\overline{l_1}, \frac{d\overline{l_5}}{da_2} = -\overline{l_3},\\
&\frac{d\overline{l_1}}{da_5} = 0, \frac{d\overline{l_2}}{da_5} = 0, \frac{d\overline{l_3}}{da_5} = 0, \frac{d\overline{l_4}}{da_5} = \overline{l_3}, \frac{d\overline{l_5}}{da_2} = \frac{3}{2}\overline{l_1}.
\end{aligned} \tag{2.7}$$

By solving Eqs. (2.7), we obtain the linear transformations

$$\begin{aligned}
&T_1 : (\overline{l_1}, \overline{l_2}, \overline{l_3}, \overline{l_4}, \overline{l_5}) = \left(l_1, e^{-a_1} l_2, l_3, e^{-\frac{3}{2}a_1} l_4, e^{-\frac{3}{2}a_1} l_5\right),\\
&T_2 : (\overline{l_1}, \overline{l_2}, \overline{l_3}, \overline{l_4}, \overline{l_5}) = (l_1, a_2 l_1 + l_2, l_3, l_4, l_5),\\
&T_3 : (\overline{l_1}, \overline{l_2}, \overline{l_3}, \overline{l_4}, \overline{l_5}) = (l_1, l_2, l_3, -l_5 \sin a_3 + l_4 \cos a_3, l_4 \sin a_3 + l_5 \cos a_3),\\
&T_4 : (\overline{l_1}, \overline{l_2}, \overline{l_3}, \overline{l_4}, \overline{l_5}) = \left(l_1, l_2, l_3, \frac{3}{2} l_1 a_4 + l_4, -l_3 a_4 + l_5\right),\\
&T_5 : (\overline{l_1}, \overline{l_2}, \overline{l_3}, \overline{l_4}, \overline{l_5}) = \left(l_1, l_2, l_3, l_3 a_5 + l_4, \frac{3}{2} l_1 a_5 + l_5\right).
\end{aligned} \tag{2.8}$$

Simplify the vector l through the transformation $T_1 - T_5$ in (2.8).

Case 1. $l_1 \neq 0$. Let $a_2 = -\frac{l_2}{l_1}, a_4 = -\frac{2}{3}\frac{l_4}{l_1}, a_5 = -\frac{2}{3}\frac{l_5}{l_1}$ in T_2, T_4 and T_5, the simplified vector is

$$(l_1, 0, l_3, 0, 0),$$

we get the representatives as follows

$$\Re_1, \Re_1 \pm \Re_3.$$

Case 2. $l_1 = 0, l_3 \neq 0$. The vector reduces to

$$(0, l_2, l_3, l_4, l_5).$$

Let $a_4 = \frac{l_5}{l_3}, a_5 = -\frac{l_4}{l_3}$ in T_4 and T_5, we let $\overline{l_4} = 0, \overline{l_5} = 0$. Thus we can get the vector

$$(0, l_2, l_3, 0, 0).$$

The representatives can be given by

$$\Re_3, \Re_3 \pm \Re_2.$$

Case 3. $l_1 = 0, l_3 = 0$ and $l_4 \neq 0$. The vector is

$$(0, l_2, 0, l_4, l_5).$$

Let $a_3 = -\arctan \frac{l_5}{l_4}$ in T_3 and get $\overline{l_5} = 0$. The simplified vector is

$$(0, l_2, 0, l_4, 0),$$

which means

$$\Re_4, \Re_4 \pm \Re_2.$$

Case 4. $l_1 = l_3 = l_4 = 0$. Then the vector is

$$(0, l_2, 0, 0, l_5).$$

The representatives should be

$$\Re_2, \Re_5, \Re_2 \pm \Re_5.$$

Theorem 2.2. $\Re_1, \Re_2, \Re_3, \Re_4, \Re_5$ *generate the one-dimensional optimal system* S : *generated by*

$$\Re_1, \Re_1 \pm \Re_3, \Re_3, \Re_3 \pm \Re_2, \Re_4, \Re_4 \pm \Re_2, \Re_2, \Re_5, \Re_2 \pm \Re_5.$$

3. Symmetry reductions and exact solutions

In this section, symmetry reductions and exact solutions of Eq. (1.5) will be discussed.

3.1. $\Re_1 = t\partial t + \frac{3}{2}x\partial x + \frac{3}{2}y\partial y + 2u\partial u$

The corresponding characteristic equation for $\Re_1$ is

$$\frac{dx}{\frac{3}{2}x} = \frac{dy}{\frac{3}{2}y} = \frac{dt}{t} = \frac{du}{2u},$$

which generates

$$u(t, x, y) = t^2\phi(\xi, \eta),$$

where $\xi = \frac{x}{t^{\frac{3}{2}}}$ and $\eta = \frac{y}{t^{\frac{3}{2}}}$ are the invariants. Then Eq. (1.5) reduces to

$$\frac{-3}{2}\phi_\xi \xi + \frac{-3}{2}\phi_\eta \eta + 2\phi - 2\phi_\xi^2 - 2\phi\phi_{\xi\xi} - 2\phi_\eta^2 - 2\phi\phi_{\eta\eta} + \alpha\sqrt{\phi} = 0. \tag{3.1}$$

The symmetry group of Eq. (3.1) is spanned by

$$\psi_\phi = 0,\ \zeta_\xi = -C_1\eta,\ \zeta_\eta = C_1\xi,$$

where C_1 is a constant. Then we can get the characteristic equation

$$\frac{d\xi}{-C_1\eta} = \frac{d\eta}{C_1\xi} = \frac{d\phi}{0},$$

which means Eq. (3.1) has a solution given by

$$\phi(\xi, \eta) = \rho(\tau),$$

where $\tau = \xi^2 + \eta^2$. Then the reduction equation is

$$8\tau\rho\rho'' + (8\rho + 8\tau\rho' + 3\tau)\rho' - (\alpha + 2\sqrt{\rho})\sqrt{\rho} = 0, \tag{3.2}$$

which can be rewritten as

$$(8\tau\rho\rho' + 3\tau\rho)' = (\alpha\sqrt{\rho} + 5\rho). \tag{3.3}$$

Integrate (3.3) once, we obtain

$$8\rho\rho' + 3\rho = \frac{1}{\tau}\int_0^\tau (\alpha\sqrt{\rho} + 5\rho)d\omega. \tag{3.4}$$

Then we can get an implicit solution

$$\rho(\tau) = \rho(0) - 3\tau + \int_0^\tau \frac{\int_0^\chi (\alpha\sqrt{\rho} + 5\rho)d\omega}{8\chi\rho(\chi)}d\chi, \tag{3.5}$$

and a special solution

$$\rho = \frac{\alpha^2}{4}.$$

Then we get an exact solution of Eq. (1.5)

$$u(t, x, y) = \frac{\alpha^2 t^2}{4}. \tag{3.6}$$

3.2. $\Re_3 = -y\partial x + x\partial y$

The invariance are $t, u, r = x^2 + y^2$, which means the invariant solution is

$$u(t, x, y) = \phi(t, r).$$

Then Eq. (1.5) can be transformed to

$$-4\phi_r^2 r - 4\phi\phi_{rr} r - 4\phi\phi_r + \alpha\sqrt{\phi} + \phi_t = 0. \tag{3.7}$$

The infinitesimals generators are given by

$$\psi_\phi = 2C_1\phi, \zeta_t = C_1 t + C_2, \zeta_r = 3C_1 r.$$

The characteristic equations is

$$\frac{dr}{3C_1 r} = \frac{dt}{C_1 t + C_2} = \frac{d\phi}{2C_1\phi}.$$

Let $C_1 = 1, C_2 = 0$, thus the solution of (3.7) is

$$\phi(t, r) = \rho(\tau)t^2,$$

in which $\tau = \frac{r}{t^3}$ can be obtained. The reduced equation of Eq. (3.7) is

$$-4\tau\rho'^2 - 4\tau\rho\rho'' - 4\rho\rho' - 3\tau\rho' + \alpha\sqrt{\rho} + 2\rho = 0, \tag{3.8}$$

which can be rewritten as

$$(4\tau\rho\rho' + 3\tau\rho)' = (\alpha\sqrt{\rho} + 5\rho). \tag{3.9}$$

Integrate (3.9) once,

$$4\rho\rho' + 3\rho = \frac{1}{\tau}\int_0^\tau (\alpha\sqrt{\rho} + 5\rho)d\omega, \tag{3.10}$$

then we can get an implicit solution

$$\rho(\tau) = \rho(0) - 3\tau + \int_0^\tau \frac{\int_0^\chi (\alpha\sqrt{\rho} + 5\rho)d\omega}{4\chi\rho(\chi)}d\chi, \tag{3.11}$$

and a special solution

$$\rho = \frac{\alpha^2}{4}.$$

Then we get the same exact solutions as (3.6).

3.3. $\Re_4 = \partial y$

The invariant solution of Eq. (1.5) is

$$u(x, y, t) = \phi(x, t).$$

Then Eq. (1.5) can be written as

$$\phi_t - 2\phi_x^2 - 2\phi\phi_{xx} + \alpha\sqrt{\phi} = 0. \tag{3.12}$$

The infinitesimal generators of Eq. (3.12) are

$$\psi_\phi = 2C_1\phi, \zeta_t = C_1 t + C_2, \zeta_x = \frac{3}{2}C_1 x + C_3,$$

where $C_i, i = 1, 2, 3$ are arbitrary constants. Then we have

$$\frac{dx}{\frac{3}{2}C_1 x + C_3} = \frac{dt}{C_1 t + C_2} = \frac{d\phi}{2C_1\phi}.$$

By making $C_3 = 1, C_1 = C_2 = 0$, ϕ is given as

$$\phi(x, t) = \rho(t).$$

Eq. (3.12) reduces to

$$\rho' + \alpha\sqrt{\rho} = 0.$$

The solution is

$$\rho = \left(\frac{c - \alpha t}{2}\right)^2.$$

Then we obtain invariant solution of Eq. (1.5)

$$u(t, x, y) = \frac{c^2}{4} - \frac{\alpha c t}{2} + \frac{\alpha^2 t^2}{4}. \tag{3.13}$$

Considering $C_1 = 0, C_2 = C_3 = 1$, thus we can obtain

$$\phi(x, t) = \rho(W),$$

where $W = x - t$. Eq. (3.12) reduces to

$$-2\rho\rho'' - (1 + 2\rho')\rho' + \alpha\sqrt{\rho} = 0. \tag{3.14}$$

The implicit solution of Eq. (3.14) is

$$\rho^2 + \rho = c_0 - \int_0^W \alpha\sqrt{\rho}d\omega. \tag{3.15}$$

If $\alpha = 0$, we have

$$\rho = 2c_1 \left[LambertW\Big(\frac{1}{2c_1 e} e^{-\frac{W+c_2}{4c_1}}\Big) + 1 \right]$$

and

$$\rho = \frac{1}{2}W + c.$$

Hence we get the invariant solutions

$$u(t, x, y) = \rho = 2c_1 \left[LambertW\Big(\frac{1}{2c_1 e} e^{-\frac{x-t+c_2}{4c_1}}\Big) + 1 \right] \tag{3.16}$$

and

$$u(x, y, t) = \frac{1}{2}(x - t) + c.$$

When $C_1 = 1, C_2 = C_3 = 0$, we have

$$\phi(x, t) = \rho(W)t^2,$$

where $W = \frac{x}{t^{\frac{3}{2}}}$. Eq. (3.12) can be reduced to

$$-2\rho\rho'' + \Big(-\frac{3}{2}W - 2\rho'\Big)\rho' + \alpha\sqrt{\rho} + 2\rho = 0, \tag{3.17}$$

which can be rewritten as

$$\Big(2\rho\rho' + \frac{3}{2}W\rho\Big)' = \Big(\alpha\sqrt{\rho} + \frac{7}{2}\rho\Big). \tag{3.18}$$

Integrate (3.18) once,

$$2\rho\rho' + \frac{3}{2}W\rho = \int_0^W \Big(\alpha\sqrt{\rho} + \frac{7}{2}\rho\Big)d\omega, \tag{3.19}$$

then we can get an implicit solution

$$\rho(\tau) = \rho(0) - \frac{3}{8}W^2 + \int_0^W \frac{\int_0^\chi \Big(\alpha\sqrt{\rho} + \frac{7}{2}\rho\Big)d\omega}{2\rho(\chi)}d\chi, \tag{3.20}$$

and a special solution

$$\rho = \frac{\alpha^2}{4}.$$

Then an exact solution of Eq. (1.5) is the same as (3.6).

3.4. $\Re_5 = \partial x$

The invariant solution of Eq. (1.5) is

$$u(x,y,t) = \phi(y,t).$$

We can get

$$\phi_t - 2\phi_y^2 - 2\phi\phi_{yy} + \alpha\sqrt{\phi} = 0. \tag{3.21}$$

Furthermore, Eq. (3.21) yields

$$\psi_\phi = 2C_1\phi, \zeta_t = C_1 t + C_2, \zeta_y = \frac{3}{2}C_1 t + C_3,$$

where C_1, C_2, C_3 are the arbitrary constants. So that the characteristic equations is

$$\frac{dy}{\frac{3}{2}C_1 y + C_3} = \frac{dt}{C_1 t + C_2} = \frac{d\phi}{2C_1\phi}.$$

If $C_1 = 0, C_2 = C_3 = 1$, ϕ is given by

$$\phi(y,t) = \rho(W),$$

where $W = y - t$. The reduced equation is

$$-2\rho\rho'' - (1 + 2\rho')\rho' + \alpha\sqrt{\rho} = 0. \tag{3.22}$$

Similar to Eq. (3.14), we can get the implicit solution of Eq. (3.22) is

$$\rho^2 + \rho = c_0 - \int_0^W \alpha\sqrt{\rho}d\omega. \tag{3.23}$$

If $\alpha = 0$, the invariant solutions of Eq. (1.5) are given by

$$u(t,x,y) = \rho = 2c_1 \left[LambertW\Big(\frac{1}{2c_1 e} e^{-\frac{y-t+c_2}{4c_1}}\Big) + 1 \right] \tag{3.24}$$

and

$$u(x,y,t) = \frac{1}{2}(y-t) + c.$$

When $C_1 = 1, C_2 = C_3 = 0$, ϕ can be given by

$$\phi(y,t) = \rho(W)t^2, \quad \text{where} \quad W = \frac{y}{t^{\frac{3}{2}}}.$$

Then we get the equation as follow

$$-2\rho\rho'' + \Big(-\frac{3}{2}W - 2\rho'\Big)\rho' + \alpha\sqrt{\rho} + 2\rho = 0, \tag{3.25}$$

which is similar to Eq. (3.17).

3.5. $\Re_4 + \Re_2 = \partial y + \partial t$

The characteristic equation for $\Re_2 + \Re_4$ is

$$\frac{dy}{1} = \frac{dt}{1}.$$

Then corresponding invariant solution is

$$u(t, x, y) = \phi(x, z)$$

where $z = y - t$. Substituting u into Eq. (1.5),

$$-\phi_z - 2\phi_z^2 - 2\phi\phi_{zz} - 2\phi_x^2 - 2\phi\phi_{xx} + \alpha\sqrt{\phi} = 0. \tag{3.26}$$

Correspondingly we have

$$\eta_\phi = 0, \zeta_z = C_2, \zeta_x = C_1.$$

Therefore, the characteristic equation is

$$\frac{dx}{C_1} = \frac{dz}{C_2} = \frac{d\phi}{0}.$$

Choosing $C_1 = 1, C_2 = -1$, ϕ could be given as

$$\phi(x, z) = \rho(\omega),$$

where $\omega = z + x = y + x - t$. Then

$$-4\rho\rho'' - \rho' - 4\rho'^2 + \alpha\sqrt{\rho} = 0 \tag{3.27}$$

is obtained. One special solution is given by

$$\rho = 4c_1 \left[\mathrm{LambertW} \left[\frac{1}{4c_1} e^{-\frac{\omega + c_2}{16c_1} - 1} \right] + 1 \right].$$

So the invariant solutions of Eq. (1.5) can be given by

$$u(t, x, y) = 4c_1 \left[\mathrm{LambertW} \left[\frac{1}{4c_1} e^{-\frac{y + x - t + c_2}{16c_1} - 1} \right] + 1 \right]. \tag{3.28}$$

3.6. $\Re_5 + \Re_2 = \partial x + \partial t$

The process is similar to that when $\Re_4 + \Re_2$. First, we can get

$$\frac{dx}{1} = \frac{dt}{1}.$$

Then the invariant solution of Eq. (1.5) is

$$u(t, x, y) = \phi(y, z)$$

where $z = x - t$. Substituting u into Eq. (1.5),

$$-\phi_z - 2\phi_z^2 - 2\phi\phi_{zz} - 2\phi_y^2 - 2\phi\phi_{yy} + \alpha\sqrt{\phi} = 0. \tag{3.29}$$

Correspondingly we have

$$\eta_\phi = 0, \zeta_y = C_2, \zeta_z = C_1.$$

Therefore, the characteristic equation is

$$\frac{dy}{C_2} = \frac{dz}{C_1} = \frac{d\phi}{0}.$$

Choosing $C_2 = 1, C_1 = -1$, ϕ could be given as

$$\phi(y, z) = \rho(\omega),$$

where $\omega = z + y = y + x - t$. Then

$$-4\rho\rho'' - \rho' - 4\rho'^2 + \alpha\sqrt{\rho} = 0 \tag{3.30}$$

is obtained. Hence the invariant solutions are the same as Eq. (3.28).

3.7. $\Re_2 + \Re_3 = \partial t - y\partial x + x\partial y$

The similarity variables are $\psi = x\sin t + y\cos t$ and $\varsigma = x\cos t - y\sin t$. The group invariant solution of Eq. (1.5) is $u = \phi(\psi, \varsigma)$. Then Eq. (1.5) can be rewritten as

$$-2\phi_\psi^2 - 2\phi_\varsigma^2 - 2\phi\phi_{\psi\psi} - 2\phi\phi_{\varsigma\varsigma} - \psi\phi_\varsigma + \phi_\psi\varsigma + \alpha\sqrt{\phi} = 0. \tag{3.31}$$

Correspondingly we have

$$\eta_\phi = 0, \zeta_\psi = -C_1\varsigma, \zeta_\varsigma = C_1\psi.$$

Therefore, the characteristic equation is

$$\frac{d\psi}{-C_1\varsigma} = \frac{d\psi}{C_1\psi} = \frac{d\phi}{0}.$$

Then ϕ could be given as

$$\phi(\psi, \varsigma) = \rho(\omega),$$

where $\omega = \psi^2 + \varsigma^2 = x^2 + y^2$. Obviously, we can get

$$-8\omega\rho'^2 - 8\rho\rho' - 8\omega\rho\rho'' + \alpha\sqrt{\rho} = 0 \tag{3.32}$$

is obtained. Thus one special solution of Eq. (1.5) is

$$u(x, y, t) = \sqrt{2c_1\ln(x^2 + y^2) + 2c_2}.$$

4. Results and discussion

It's better to use graphical analysis to express mathematical expressions and understand the dynamical behavior physically. In this section, we provide the solutions with the physical presentations. The solutions include arbitrary constants and functions. So we can take the appropriate values. The solution (3.24) in the form of

LambertW function. In Figure 1(c), the population density u rises over t but decreases with increasing y. This phenomenon occurs only when the population dcline or migrate out a region. For the solution (3.28) in the form of LambertW function, when we take a fixed time, the population density can be visually represented in Figure 2. The population density u is increasing over time, decreasing with both x and y. One of the key factor to this phenomenon is an expand in the birth rate.

Zhang et al [41, p11] applied an improved element-free Galerkin method for numerical modeling of the biological population problems and our model is a special case studied in this article. The results of this paper can provide theoretical knowledge for numerical simulation in [41, p11]. Compared with [34], the exact solutions of Eq. (1.5) show some different phenomenon from a Verhulst biological population model.

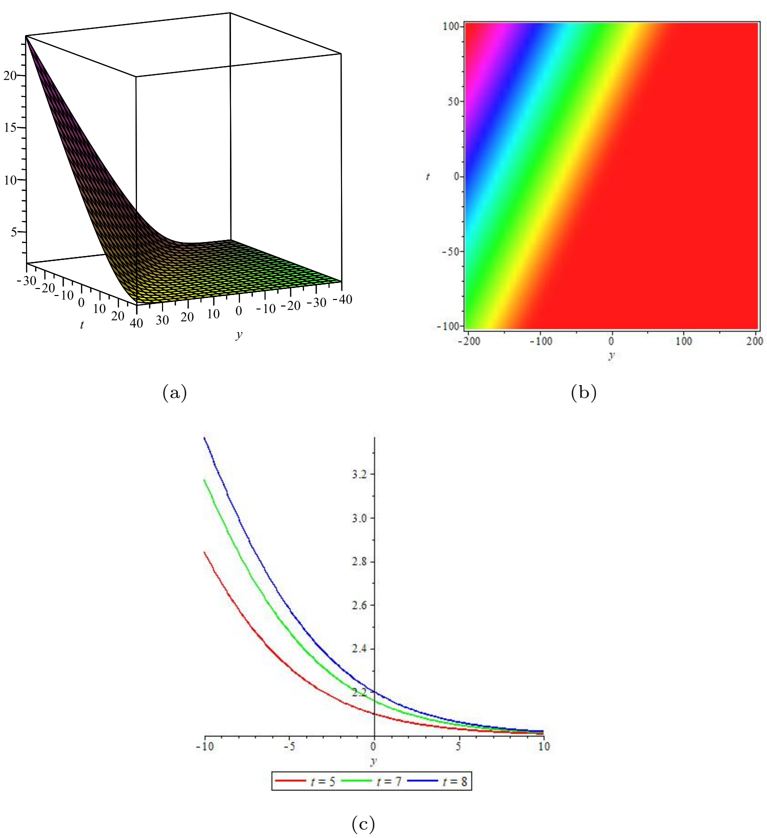
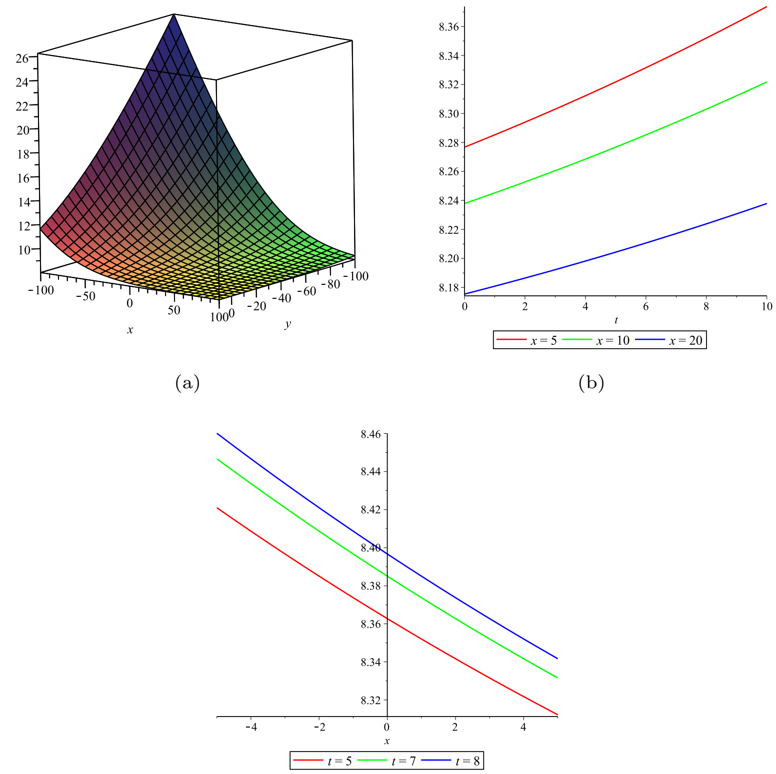


Figure 1. The solution (3.24) at $c_1 = 1, c_2 = 10$: (a). 3D profile; (b). the density of the solution; (c). 2D profile of the solution with respect to y at $t = 5$, $t = 7$, $t = 8$.

5. Conclusion

This paper constructs group invariant solutions of the $(2 + 1)$ dimensional Biological population model with porous media law by exploring Lie symmetry analysis



(a) (b) (c)

Figure 2. The solution (3.28) at $c_1 = 2, c_2 = 3$: (a). 3D profile with $t = 1$; (b). 2D sketch of (3.28) for t at $x = 5, x = 10, x = 20$ and $y = 0$; (c). 2D profile of (3.28) with respect to x at $t = 5$, $t = 7$, $t = 8$ and $y = 1$.

method. Lie point symmetries of Eq. (1.5) are analysed and the optimal system with the help of commutator table is obtained. Furthermore, we find group invariant solutions of this model according to the corresponding reduced nonlinear ordinary differential equations, which are related to the population density and affect the population control. Finally we present the discussion and dynamical analysis by the graphical representations. In the future, numerical simulations and machine learning for the biological population model will overcome the paper's drawbacks and advance the population dynamics study.

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