Journal of Applied Analysis and Computation
Volume 15, Number 2, April 2025, 958–972
Website:http://www.jaac-online.com
DOI:10.11948/20240227

THE SHARP BOUNDS OF HANKEL DETERMINANTS FOR THE FOUR-LEAF-TYPE BOUNDED TURNING FUNCTIONS*

Dong Guo[1], Huo Tang[2], Xi Luo[3] and Zong-Tao Li[4,†]

Abstract In the paper, a family of bounded turning functions involving a four-leaf-type domain is studied in the unit disk. The goal of the study is to explore the bounds of second and the third Hankel determinant for functions in the class. All of obtained bounds have been sharp.

Keywords Hankel determinant, bounded turning functions, four-leaf-type.

MSC(2010) 30C45, 30C50.

1. Introduction

Let $\mathcal{H}$ be the class of all functions $g(\zeta)$ which are holomorphic in unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and with the normalization $g(0) = g'(0) - 1 = 0$. Therefore, for $g(\zeta) \in \mathcal{H}$, one has

$$g(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n \quad (\zeta \in \mathbb{U}). \tag{1.1}$$

Let $\mathcal{S} \subset \mathcal{H}$ represent all functions that are univalent in $\mathbb{U}$.

For given parameters $j, n \in \mathbb{N} = \{1, 2, 3...\}$, the Hankel determinant $H_{j,n}(g)$ was defined by Pommerenke [8, 9] for a function $g \in \mathcal{S}$ having power series expansion

†The corresponding author.
[1]School of Mathematical Sciences, Yangzhou Polytechnic College, Jiangsu 225009, China
[2]College of Mathematics and Computer Science, Chifeng University, Inner Mongolia 024000, China
[3]School of Mathematics, Jiaying University, Guangdong 514015, China
[4]School of Public Basic Education, Guangzhou Civil Aviation College, Guangdong 510403, China
*The authors were supported by the Natural Science Foundation of Anhui Provincial Department of Education (No. KJ2020A0993, KJ2020ZD74) and the high-level talent research start-up project (No. DC2300000286).
Email: gd791217@163.com(D. Guo),
thth2009@163.com(H. Tang),
93030910@qq.com(X. Luo),
lizt2046@163.com(Z.-T. Li)

(1.1) as follows:

$$H_{j,n}(g) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+j-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+j} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n+j-1} & b_{n+j} & \cdots & b_{n+2j-2} \end{vmatrix}.$$

The bound of $|H_{j,n}(g)|$ has been evaluated for different subclasses of univalent functions. Exceptionally, for each of the sets $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{R}$ the sharp bound of the determinant $|H_{2,2}(g)| = |b_2b_4 - b_3^3|$ were obtained by Janteng et al. [3,4]. For more work on $|H_{2,2}(g)|$, see [2,7,13]. The determinant

$$H_{3,1}(g) = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_3b_5 - b_2^2b_5 \tag{1.2}$$

is known as third order Hankel determinant and the estimation of this determinant $|H_{3,1}(g)|$ is a challenging task. Recently, the sharp bounds of $|H_{3,1}(g)|$ for some subclasses of univalent functions were found by few authors [5,10,11,14–18,20–23].

In [1], Alshehry and his coauthors introduced a subfamily of starlike functions associated with a four-leaf function defined by

$$\mathcal{S}^*_{4l} = \left\{ g \in \mathcal{S} : \frac{\zeta g'(\zeta)}{g(\zeta)} \prec 1 + \frac{5}{6}\zeta + \frac{1}{6}\zeta^5, (\zeta \in \mathbb{U}) \right\}.$$

Similar to the definition of $\mathcal{S}^*_{4l}$, Sunthrayuth et al. [19] defined a new subclass of bounded turning functions defined by

$$\mathcal{B}T_{4l} = \left\{ g \in \mathcal{S} : g'(\zeta) \prec 1 + \frac{5}{6}\zeta + \frac{1}{6}\zeta^5, (\zeta \in \mathbb{U}) \right\}.$$

Alshehry et al. [1] obtained the sharp bounds of logarithmic coefficients, and the second-order Hankel determinant of of logarithmic coefficients. In 2022, Shi et al. [14] obtained the sharp bound of the third-order Hankel determinant and showned that $|H_{3,1}| \leq \frac{25}{324}$ for the class $\mathcal{S}^*_{4l}$. Sunthrayuth and his coauthors [19] studied the second-order Hankel determinant, Kruskal inequality and the bounds of the coefficient inequalities of the class $\mathcal{B}T_{4l}$.

In the paper, we study and obtain the sharp bounds of the second and third-order Hankel determinant of bounded turning class $\mathcal{B}T_{4l}$.

Let $\mathcal{P}$ be the family of functions p that are holomorphic in $\mathbb{U}$ with $Rep(\zeta) > 0$ and the power series form as follow:

$$p(\zeta) = p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + \ldots \quad (\zeta \in \mathbb{U}). \tag{1.3}$$

Lemma 1.1 ([6,12]). *If $p \in \mathcal{P}$ is of the form* (1.3), *then*

$$2p_2 = p_1^2 + y(4 - p_1^2), \tag{1.4}$$
$$4p_3 = p_1^3 + 2p_1y(4 - p_1^2) - p_1y^2(4 - p_1^2) + 2(4 - p_1^2)(1 - |y|^2)\delta, \tag{1.5}$$
$$\begin{aligned} 8p_4 = {} & p_1^4 + (4 - p_1^2)y[p_1^2(y^2 - 3y + 3) + 4y] - 4(4 - p_1^2)(1 - |y|^2) \\ & \times \left[p_1(y - 1)\delta + \overline{y}\delta^2 - (1 - |\delta|^2)\sigma\right], \end{aligned} \tag{1.6}$$

for some y, δ, σ with $|y| \leq 1$, $|\delta| \leq 1$ and $|\sigma| \leq 1$.

2. Main results

Theorem 2.1. *If $g \in \mathcal{B}T_{4l}$, then*

$$|H_{3,1}(g)| \leq \frac{25}{576}.$$

The bound is sharp.

Proof. Since $g \in \mathcal{B}T_{4l}$, from subordination definition there exists a Schwarz function $w(\zeta)$ with $w(0) = 0$ and $|w(\zeta)| < 1$, in such a way that

$$g'(\zeta) = 1 + \frac{5}{6}w(\zeta) + \frac{1}{6}w^5(\zeta).$$

Define a function

$$p(\zeta) = \frac{1+\omega(\zeta)}{1-\omega(\zeta)} = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + \cdots.$$

Clearly, we have $p(\zeta) \in \mathcal{P}$ and

$$\begin{aligned}\omega(\zeta) = \frac{p(\zeta)-1}{1+p(\zeta)} =& \frac{p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + \cdots}{2 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + \cdots} \\ =& \frac{1}{2}p_1\zeta + (\frac{1}{2}p_2 - \frac{1}{4}p_1^2)\zeta^2 + (\frac{1}{8}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{2}p_3)\zeta^3 \\ &+ (\frac{1}{2}p_4 - \frac{1}{2}p_1p_3 - \frac{1}{4}p_2^2 - \frac{1}{16}p_1^4 + \frac{3}{8}p_1^2p_2)\zeta^4 + \cdots.\end{aligned}$$

This gives

$$\begin{aligned}&1 + \frac{5}{6}w(\zeta) + \frac{1}{6}w^5(\zeta) \\ =&1 + \frac{5}{12}p_1\zeta + (\frac{5}{12}p_2 - \frac{5}{24}p_1^2)\zeta^2 + (\frac{5}{12}p_3 - \frac{5}{12}p_1p_2 + \frac{5}{48}p_1^3)\zeta^3 \\ &+ (\frac{5}{12}p_4 - \frac{5}{24}p_2^2 - \frac{5}{12}p_1p_3 + \frac{5}{16}p_1^2p_2 - \frac{5}{96}p_1^4)\zeta^4 + \cdots. \end{aligned} \tag{2.1}$$

From (1.1), we yield

$$g'(\zeta) = 1 + 2b_2\zeta + 3b_3\zeta^2 + 4b_4\zeta^3 + 5b_5\zeta^4 + \cdots. \tag{2.2}$$

Comparing (2.1) and (2.2), we may obtain

$$\begin{cases} b_2 = \dfrac{5p_1}{24}, \\ b_3 = \dfrac{1}{72}(-5p_1^2 + 10p_2), \\ b_4 = \dfrac{1}{192}(20p_3 + 5p_1^3 - 20p_1p_2), \\ b_5 = \dfrac{1}{480}(-20p_2^2 + 40p_4 + 30p_1^2p_2 - 5p_1^4 - 40p_1p_3). \end{cases} \tag{2.3}$$

From (2.3) and (1.2), we have

$$H_{3,1}(g) = \frac{1}{14929920}(-8400p_1^4p_2 + 14400p_1^3p_3 + 1175p_1^6 + 7800p_1^2p_2^2 + 172800p_2p_4$$

$$+241200p_1p_2p_3 - 140400p_1^2p_4 - 126400p_2^3 - 162000p_3^2). \qquad (2.4)$$

Let $p_1 = p \in [0,2]$, $l = 4 - p^2$ in (1.4), (1.5) and (1.6), we get

$$-8400p^4p_2 = -4200p^6 - 4200p^4ly,$$

$$14400p^3p_3 = 3600p^6 + 7200p^4ly - 3600p^4ly^2 + 7200p^3l\left(1-|y|^2\right)\delta,$$

$$7800p^2p_2^2 = 1950p^6 + 3900p^4ly + 1950p^2l^2y^2,$$

$$\begin{aligned}-140400p^2p_4 = &-17550p^4ly^3 + 70200p^2l\bar{y}\left(1-|y|^2\right)\delta^2 + 70200p^3ly\left(1-|y|^2\right)\delta \\ &+ 52650p^4ly^2 - 70200p^2l\left(1-|y|^2\right)\left(1-|\delta|^2\right)\sigma \\ &- 70200p^3l\left(1-|y|^2\right)\delta - 52650p^4ly - 17550p^6 - 70200p^2ly^2,\end{aligned}$$

$$\begin{aligned}241200pp_2p_3 = &-30150p^2l^2y^3 - 30150p^4ly^2 + 60300pyl^2\left(1-|y|^2\right)\delta + 60300p^2l^2y^2 \\ &+ 60300p^3l\left(1-|y|^2\right)\delta + 30150p^6 + 90450p^4ly,\end{aligned}$$

$$-126400p_2^3 = -15800l^3y^3 - 47400p^2l^2y^2 - 47400p^4ly - 15800p^6,$$

$$\begin{aligned}172800p_2p_4 = &10800p^6 + 10800p^4ly^3 - 32400p^4ly^2 + 43200p^4ly + 43200p^2ly^2 \\ &- 43200p^3ly(1-|y|^2)\delta + 43200p^3l(1-|y|^2)\delta - 43200p^2l(1-|y|^2)\bar{y}\delta^2 \\ &+ 43200p^2l(1-|y|^2)(1-|\delta|^2)\sigma + 10800p^2l^2y^4 - 32400p^2l^2y^3 \\ &+ 32400p^2l^2y^2 + 43200l^2y^3 + 43200pl^2y\delta(1-|y|^2) \\ &- 43200l^2(1-|y|^2)y\bar{y}\delta^2 + 43200l^2y(1-|y|^2)(1-|\delta|^2)\sigma \\ &- 43200pl^2\delta(1-|y|^2)y^2,\end{aligned}$$

$$\begin{aligned}-162000p_3^2 = &-10125p^6 - 40500p^4ly - 40500p^2l^2y^2 + 20250p^4ly^2 \\ &- 40500p^3l(1-|y|^2)\delta + 40500p^2l^2y^3 - 81000pl^2y\delta(1-|y|^2) \\ &- 10125p^2l^2y^4 + 40500pl^2y^2(1-|y|^2)\delta - 40500l^2\delta^2(1-|y|^2)^2.\end{aligned}$$

Substituting these expressions into (2.4), we obtain

$$\begin{aligned}H_{3,1}(g) = \frac{1}{14929920}[&43200l^2y^3 + 675p^2l^2y^4 + 6750p^4ly^2 + 6750p^2l^2y^2 \\ &-22050p^2l^2y^3 - 6750p^4ly^3 - 27000p^2ly^2 - 15800l^3y^3 \\ &+22500pl^2y(1-|y|^2)\delta + 27000p^3y(1-|y|^2)l\delta \\ &+27000p^2(1-|y|^2)l\bar{y}\delta^2 - 27000p^2(1-|y|^2)(1-|\delta|^2)l\sigma \\ &-2700pl^2y^2(1-|y|^2)\delta - 43200l^2(1-|y|^2)y\bar{y}\delta^2 \\ &+43200l^2y(1-|y|^2)(1-|\delta|^2)\sigma - 40500l^2\delta^2(1-|y|^2)^2].\end{aligned}$$

Thus, we achieve

$$H_{3,1}(g) = \frac{1}{14929920}\left(\eta_1(p,y) + \eta_2(p,y)\delta + \eta_3(p,y)\delta^2 + \eta_4(p,y,\delta)\sigma\right),$$

where

$$\begin{aligned}\eta_1(p,y) = &\left(4-p^2\right)y\left[\left(4-p^2\right)y\left(-20000y + 675p^2y^2 + 6750p^2 - 6250p^2y\right)\right. \\ &\left.+6750p^4y - 6750p^4y^2 - 27000p^2y\right],\end{aligned}$$

$$\eta_2(p,y) = py\left(4-p^2\right)\left(1-|y|^2\right)\left[(22500-2700y)\left(4-p^2\right)+27000p^2\right],$$
$$\eta_3(p,y) = \left(4-p^2\right)\left(1-|y|^2\right)\left[-\left(2700|y|^2+40500\right)\left(4-p^2\right)+27000\bar{y}p^2\right],$$
$$\eta_4(p,y,\delta) = \left(4-p^2\right)\left(1-|y|^2\right)\left(1-|\delta|^2\right)\left[43200y\left(4-p^2\right)-27000p^2\right].$$

Now, by setting $|y|=y$, $|\delta|=x$ and upon taking $|\sigma|\le 1$, we get

$$\begin{aligned}|H_{3,1}(g)| &\le \frac{1}{14929920}\left(|\eta_1(p,y)|+|\eta_2(p,y)|x+|\eta_3(p,y)|x^2+|\eta_4(p,y,\delta)|\right)\\ &\le \frac{1}{14929920}\varphi(p,y,x),\end{aligned} \tag{2.5}$$

where

$$\varphi(p,y,x) = \theta_1(p,y)+\theta_2(p,y)x+\theta_3(p,y)x^2+\theta_4(p,y)(1-x^2), \tag{2.6}$$

with

$$\begin{aligned}\theta_1(p,y) &= \left(4-p^2\right)y\left[\left(4-p^2\right)y\left(20000y+675p^2y^2+6750p^2+6250p^2y\right)\right.\\ &\quad \left.+6750p^4y+6750p^4y^2+27000p^2y\right],\\ \theta_2(p,y) &= py\left(4-p^2\right)\left(1-y^2\right)\left[(22500+2700y)\left(4-p^2\right)+27000p^2\right],\\ \theta_3(p,y) &= \left(4-p^2\right)\left(1-y^2\right)\left[\left(2700y^2+40500\right)\left(4-p^2\right)+27000yp^2\right]\end{aligned}$$

and

$$\theta_4(p,y) = \left(4-p^2\right)\left(1-y^2\right)\left[43200y\left(4-p^2\right)+27000p^2\right].$$

(1) For $p=2$,

$$\varphi(2,y,x)=0.$$

(2) For $p=0$,

$$\begin{aligned}\varphi(0,y,x) &= 691200y-371200y^3\\ &\quad +(648000-691200y-604800y^2+691200y^3-43200y^4)x^2\\ &= \varrho_1(y,x).\end{aligned}$$

The optimal points of ϱ_1 satisfy

$$\begin{cases}\dfrac{\partial\varrho_1}{\partial y} = 691200-1113600y^2+(-691200-1209600y+2073600y^2-172800y^3)x^2\\ \qquad = 0,\\ \dfrac{\partial\varrho_1}{\partial x} = (1296000-1382400y-1209600y^2+1382400y^3-86400y^4)x = 0.\end{cases}$$

Utilizing numerical computations, we yield

$$\begin{cases}y_1=-0.7878,\\ x_1=0,\end{cases}\quad \begin{cases}y_2=0.7878,\\ x_2=0,\end{cases}\quad \begin{cases}y_3=-1,\\ x_3=0.3909,\end{cases}\quad \begin{cases}y_4=-1,\\ x_4=-0.3909.\end{cases}$$

Thus the function ϱ_1 has no optimal point in $(0,1)\times(0,1)$.
(3) For $y=0$,

$$\begin{aligned}\varphi(p,0,x) &= 40500(4-p^2)^2x^2+27000p^2(4-p^2)(1-x^2)\\ &= 108000p^2-27000p^4+(648000-432000p^2+67500p^4)x^2\\ &= \varrho_2(p,x).\end{aligned}$$

Consider

$$\begin{cases}\dfrac{\partial\varrho_2}{\partial p}=216000p-108000p^3+(-864000p+270000p^3)x^2=0,\\ \dfrac{\partial\varrho_2}{\partial x}=(1296000-864000p^2+135000p^4)x=0.\end{cases}$$

Utilizing numerical computations, we yield

$$\begin{cases}p_1=0,\\ x_1=0,\end{cases}\quad \begin{cases}p_2=1.4142,\\ x_2=0,\end{cases}\quad \begin{cases}p_3=-1.4142,\\ x_3=0,\end{cases}\quad \begin{cases}p_4=2,\\ x_4=1,\end{cases}\quad \begin{cases}p_5=2,\\ x_5=-1,\end{cases}$$

$$\begin{cases}p_6=-2,\\ x_6=1,\end{cases}\quad \begin{cases}p_7=-2,\\ x_7=-1.\end{cases}$$

Therefore, ϱ_2 has no optimal point in $(0,2)\times(0,1)$.
(4) For $y=1$

$$\varphi(p,1,x)=175p^6-62400p^4+166800p^2+320000=\varrho_3(p)\le\varrho_3(1.1594)=431890.$$

(5) For $x=0$

$$\begin{aligned}\varphi(p,y,0)=&108000p^2-27000p^4+(691200-345600p^2+43200p^4)y\\ &+(-27000p^4+108000p^2)y^2\\ &+(-500p^6-46200p^4+285600p^2-371200)y^3\\ &+(675p^6-5400p^4+10800p^2)y^4\\ =&\varrho_4(p,y).\end{aligned}$$

Consider

$$\begin{cases}\dfrac{\partial\varrho_4}{\partial p}=216000p-108000p^3+(-691200p+172800p^3)y\\ \qquad +(-108000p^3+216000p)y^2+(-3000p^5-184800p^3+571200p)y^3\\ \qquad +(4050p^5-21600p^3+21600p)y^4\\ \quad =0,\\ \dfrac{\partial\varrho_4}{\partial y}=691200-345600p^2+43200p^4+(-54000p^4+216000p^2)y+(-1500p^6\\ \qquad -138600p^4+856800p^2-1113600)y^2+(2700p^6-21600p^4+43200p^2)y^3\\ \quad =0.\end{cases}$$

Using numerical computations, we obtain

$$\begin{cases} p_1 = 0, \\ y_1 = -0.7878, \end{cases} \quad \begin{cases} p_2 = 0, \\ y_2 = 0.7878, \end{cases} \quad \begin{cases} p_3 = 2, \\ y_3 = -1.4656, \end{cases} \quad \begin{cases} p_4 = -2, \\ y_4 = -1.4656, \end{cases}$$

$$\begin{cases} p_5 = 1.5304, \\ y_5 = -1.2699, \end{cases} \quad \begin{cases} p_6 = -1.5304, \\ y_6 = -1.2699, \end{cases} \quad \begin{cases} p_7 = 5.7826, \\ y_7 = 2.5389, \end{cases} \quad \begin{cases} p_8 = -5.7826, \\ y_8 = 2.5389. \end{cases}$$

Thus, ϱ_4 has no critical point in $(0,2) \times (0,1)$.

(6) For $x = 1$

$$\begin{aligned} \varphi(p,y,1) =& 40500p^4 - 324000p^2 + 648000 + (-4500p^5 - 27000p^4 - 72000p^3 \\ & + 108000p^2 + 360000p)y + (2700p^5 - 91800p^4 - 21600p^3 + 518400p^2 \\ & + 43200p - 604800)y^2 + (-500p^6 + 4500p^5 + 24000p^4 + 72000p^3 \\ & - 168000p^2 - 360000p + 320000)y^3 + (675p^6 - 2700p^5 - 8100p^4 \\ & + 21600p^3 + 32400p^2 - 43200p - 43200)y^4 \\ =& \varrho_5(p,y). \end{aligned}$$

Taking the partial derivative with respect to p, and y respectively, we have

$$\begin{aligned} \frac{\partial \varrho_5}{\partial p} =& 162000p^3 - 648000p + (-22500p^4 - 108000p^3 - 216000p^2 + 216000p \\ & + 360000)y + (13500p^4 - 367200p^3 - 64800p^2 + 1036800p + 43200)y^2 \\ & + (-3000p^5 + 22500p^4 + 96000p^3 + 216000p^2 - 336000p - 360000)y^3 \\ & + (4050p^5 - 13500p^4 - 32400p^3 + 64800p^2 + 64800p - 43200)y^4 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \varrho_5}{\partial y} =& -4500p^5 - 27000p^4 - 72000p^3 + 108000p^2 + 360000p + (5400p^5 - 183600p^4 \\ & - 43200p^3 + 1036800p^2 + 86400p - 1209600)y + (-1500p^6 + 13500p^5 \\ & + 72000p^4 + 216000p^3 - 504000p^2 - 1080000p + 960000)y^2 + (2700p^6 \\ & - 10800p^5 - 32400p^4 + 86400p^3 + 129600p^2 - 172800p - 172800)y^3. \end{aligned}$$

A computation show that the following system of equations has no solution:

$$\frac{\partial \varrho_5}{\partial p} = \frac{\partial \varrho_5}{\partial y} = 0$$

in $(0,2) \times (0,1)$.

(7) For $y = x = 0$,

$$\varphi(p,0,0) = 108000p^2 - 27000p^4 = \varrho_6(p) \leq= \varrho_6(\sqrt{2}) = 108000.$$

(8) For $y=0, x=1$

$$\varphi(p,0,1)=648000-324000p^2+40500p^4\leq 648000.$$

(9) For $p=y=0$,

$$\varphi(0,0,x)=648000x^2\leq 648000.$$

(10) For $y=x=1$, or $y=1, x=0$,

$$\begin{aligned}\varphi(p,1,1)&=\varphi(p,1,0)\\&=175p^6-62400p^4+166800p^2+32000\\&=\varrho_3(p)\\&\leq\varrho_3(1.1594)\\&=431890.\end{aligned}$$

(11) For $p=0, y=1$

$$\varphi(0,1,x)=320000.$$

(12) For $p=2, y=1$, or $p=2, y=0$, or $p=2, x=0$, or $p=2, x=1$,

$$\varphi(2,1,x)=\varphi(2,0,x)=\varphi(2,y,0)=\varphi(2,y,1)=0.$$

(13) For $p=x=0$,

$$\varphi(0,y,0)=691200y-371200y^3=\varrho_7(y)\leq\varrho_7(0.7878)=363040.$$

(14) For $p=0, x=1$,

$$\varphi(0,y,1)=648000-604800y^2+320000y^3-43200y^4\leq 648000.$$

The equation (2.6) can be written as

$$\begin{aligned}\varphi(p,y,x)=&(675p^6-5400p^4+10800p^2)y^4+(-500p^6-46200p^4+285600p^2\\&-371200)y^3+(-27000p^4+108000p^2)y^2+(43200p^4-345600p^2\\&+691200)y-27000p^4+108000p^2+[(-2700p^5+21600p^3-43200p)y^4\\&+(4500p^5+72000p^3-36000p)y^3+(2700p^5-21600p^3+43200p)y^2\\&+(-4500p^5-72000p^3+360000p)y]x+[(-2700p^4+21600p^2\\&-43200)y^4+(70200p^4-453600p^2+691200)y^3+(-64800p^4\\&+410400p^2-604800)y^2+(-70200p^4+453600p^2-691200)y\\&+67500p^4-432000p^2+64800]x^2.\end{aligned}$$

Now considering the interior region of $(0,2)\times(0,1)\times(0,1)$. Note that all real solutions (p, y, x) of the system of equation

$$\begin{cases} \frac{\partial \varphi}{\partial p} = (4050p^5 - 21600p^3 + 21600p^2)y^4 + (-3000p^5 - 184800p^3 + 571200p)y^3 \\ \quad +(-108000p^3 + 216000p)y^2 + (172800p^3 - 691200p)y - 108000p^3 \\ \quad +216000p + [(-13500p^4 + 64800p^2 - 43200)y^4 + (22500p^4 + 216000p^2 \\ \quad -360000)y^3 + (13500p^4 - 64800p^2 + 43200)y^2 + (-22500p^4 - 216000p^2 \\ \quad +360000)y]x + [(-10800p^3 + 43200p)y^4 + (280800p^3 - 907200p)y^3 \\ \quad +(-259200p^3 + 820800p)y^2 + (-280800p^3 + 907200p)y \\ \quad +270000p^3 - 864000p]x^2 \\ = 0, \\ \frac{\partial \varphi}{\partial y} = (2700p^6 - 21600p^4 + 43200p^2)y^3 + (-1500p^6 - 138600p^4 + 856800p^2 \\ \quad -1113600)y^2 + (-54000p^4 + 216000p^2)y + 43200p^4 - 345600p^2 + 691200 \\ \quad +[(-10800p^5 + 86400p^3 - 172800p)y^3 + (13500p^5 + 216000p^3 \\ \quad -1080000p)y^2 + (5400p^5 - 43200p^3 + 86400p)y - 4500p^5 - 72000p^3 \\ \quad +360000p]x + [(-10800p^4 + 86400p^2 - 172800)y^3 + (210600p^4 \\ \quad -1360800p^2 + 2073600)y^2 + (-129600p^4 + 820800p^2 - 1209600)y \\ \quad -70200p^4 + 453600p^2 - 691200]x^2 \\ = 0, \\ \frac{\partial \varphi}{\partial x} = (-2700p^5 + 21600p^3 - 43200p)y^4 + (4500p^5 + 72000p^3 - 360000p)y^3 \\ \quad +(2700p^5 - 21600p^3 + 43200p)y^2 + (-4500p^5 - 72000p^3 + 360000p)y \\ \quad +\big[(-5400p^4 + 43200p^2 - 86400)y^4 + (140400p^4 - 907200p^2 + 1382400)y^3 \\ \quad +(-129600p^4 + 820800p^2 - 1209600)y^2 + (-140400p^4 + 907200p^2 \\ \quad -1382400)y + 135000p^4 - 864000p^2 + 1296000\big]x \\ = 0. \end{cases}$$

After a numerical calculation, there is no optimal point in $(0, 2) \times (0, 1) \times (0, 1)$.

The sharp bound for this Hankel determinant is determined by

$$|H_{3,1}(g)| = \frac{25}{576},$$

with an extremal function

$$g(\zeta) = \int_0^\zeta (1 + \frac{5}{6}t^3 + \frac{1}{6}t^{15})dt = \zeta + \frac{5}{24}\zeta^4 + \frac{1}{96}\zeta^{16}.$$

□

Theorem 2.2. *$g \in \mathcal{B}T_{4l}$, then*

$$|H_{2,3}(g)| = |b_3b_5 - b_4^2| \leq \frac{25}{576}.$$

The bound is sharp.

Proof. Let $g \in \mathcal{B}T_{4l}$. From (2.3), we get

$$b_3b_5 - b_4^2 = \frac{1}{552960}(-200p_1^4p_2 + 200p_1^3p_3 + 25p_1^6 + 400p_1^2p_2^2 + 6400p_2p_4 \\ +5600p_1p_2p_3 - 3200p_1^2p_4 - 3200p_2^3 - 6000p_3^2). \tag{2.7}$$

Using Lemma 1.1, we have

$$b_3b_5 - b_4^2 = \frac{1}{552960}\big(\lambda_1(p,y) + \lambda_2(p,y)\delta + \lambda_3(p,y)\delta^2 + \lambda_4(p,y,\delta)\sigma\big), \tag{2.8}$$

where

$$\begin{aligned}
\lambda_1(p,y) &= 25(4-p^2)^2p^2y^4,\\
\lambda_2(p,y) &= -100py^2\left(4-p^2\right)^2\left(1-|y|^2\right),\\
\lambda_3(p,y) &= 100\left(4-p^2\right)^2\left(1-|y|^2\right)(-15-|y|^2),\\
\lambda_4(p,y,\delta) &= 1600\left(4-p^2\right)^2 y\left(1-|y|^2\right)\left(1-|\delta|^2\right).
\end{aligned}$$

Now, using $y = |y|, x = |\delta|$ and taking $|\sigma| \leq 1$, we yield

$$\begin{aligned}
|b_3b_5 - b_4^2| &\leq \frac{1}{552960}\big(|\lambda_1(p,y)| + |\lambda_2(p,y)|x + |\lambda_3(p,y)|x^2 + |\lambda_4(p,y,\delta)|\big)\\
&\leq \frac{1}{552960}\psi(p,y,x),
\end{aligned} \tag{2.9}$$

where

$$\psi(p,y,x) = \rho_1(p,y) + \rho_2(p,y)x + \rho_3(p,y)x^2 + \rho_4(p,y)(1-x^2) \tag{2.10}$$

and

$$\begin{aligned}
\rho_1(p,y) &= 25(4-p^2)^2p^2y^4,\\
\rho_2(p,y) &= 100py^2\left(4-p^2\right)^2\left(1-y^2\right),\\
\rho_3(p,y) &= 100\left(4-p^2\right)^2\left(1-y^2\right)(15+y^2),\\
\rho_4(p,y) &= 1600\left(4-p^2\right)^2 y\left(1-y^2\right).
\end{aligned}$$

Let $\Pi : [0,2] \times [0,1]^2$ be the closed cuboid. The function ψ can be written as

$$\begin{aligned}
\psi(p,y,x) = {} & (25p^6 - 200p^4 + 400p^2)y^4 + (-1600p^4 + 12800p^2 - 25600)y^3\\
& +(1600p^4 - 12800p^2 + 25600)y + [(-100p^5 + 800p^3 - 1600p)y^4\\
& +(100p^5 - 800p^3 + 1600p)y^2]x + [(-100p^4 + 800p^2 - 1600)y^4\\
& +(1600p^4 - 12800p^2 + 25600)y^3 + (-1400p^4 + 11200p^2 - 22400)y^2
\end{aligned}$$

$$+(-1600p^4+12800p^2-25600)y+1500p^4-12000p^2+24000]x^2.$$

By taking the partial derivative, we achieve

$$\begin{aligned}\frac{\partial\psi}{\partial p}=&(150p^5-800p^3+800p)y^4+(-6400p^3+25600p)y^3+(6400p^3-25600p)y\\&+[(-500p^4+2400p^2-1600)y^4+(500p^4-2400p^2+1600)y^2]x+[(-400p^3\\&+1600p)y^4+(6400p^3-25600p)y^3+(-5600p^3+22400p)y^2+(-6400p^3\\&+25600p)y+6000p^3-24000p]x^2,\\\frac{\partial\psi}{\partial y}=&(100p^6-800p^4+1600p^2)y^3+(-4800p^4+38400p^2-76800)y^2+1600p^4\\&-12800p^2+25600+[(-400p^5+3200p^3-6400p)y^3+(200p^5-1600p^3\\&+3200p)y]x+[(-400p^4+3200p^2-6400)y^3+(4800p^4-38400p^2+76800)y^2\\&+(-2800p^4+22400p^2-44800)y-1600p^4+12800p^2-25600]x^2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\psi}{\partial x}=&(-100p^5+800p^3-1600p)y^4+(100p^5-800p^3+1600p)y^2+[(-200p^4\\&+1600p^2-3200)y^4+(3200p^4-25600p^2+51200)y^3+(-2800p^4+22400p^2\\&-44800)y^2+(-3200p^4+25600p^2-51200)y+3000p^4-24000p^2+48000]x.\end{aligned}$$

By setting $\frac{\partial\psi}{\partial p}=\frac{\partial\psi}{\partial y}=\frac{\partial\psi}{\partial x}=0$ and by utilizing numerical computations, we obtain

$$\begin{cases}p_1=2,\\y_1=y_1,\\x_1=x_1,\end{cases}\quad\begin{cases}p_2=-2,\\y_2=y_2,\\x_2=x_2,\end{cases}\quad\begin{cases}p_3=\frac{2\sqrt{3}}{3}\\y_3=1,\\x_3=-\frac{23\sqrt{3}}{3},\end{cases}\quad\begin{cases}p_4=-\frac{2\sqrt{3}}{3},\\y_4=1,\\x_4=\frac{23\sqrt{3}}{3},\end{cases}$$

$$\begin{cases}p_5=0,\\y_5=-1,\\x_5=\frac{\sqrt{2}}{2},\end{cases}\quad\begin{cases}p_6=0,\\y_6=-1,\\x_6=-\frac{\sqrt{2}}{2},\end{cases}\quad\begin{cases}p_7=\frac{2\sqrt{3}}{3},\\y_7=-1,\\x_7=\frac{-\sqrt{3}+\sqrt{4803}}{96},\end{cases}\quad\begin{cases}p_8=\frac{2\sqrt{3}}{3},\\y_8=-1,\\x_8=\frac{-\sqrt{3}-\sqrt{4803}}{96},\end{cases}$$

$$\begin{cases}p_9=-\frac{2\sqrt{3}}{3},\\y_9=-1,\\x_9=\frac{\sqrt{3}+\sqrt{4803}}{96},\end{cases}\quad\begin{cases}p_{10}=-\frac{2\sqrt{3}}{3},\\y_{10}=-1,\\x_{10}=\frac{\sqrt{3}-\sqrt{4803}}{96},\end{cases}\quad\begin{cases}p_{11}=0,\\y_{11}=\frac{\sqrt{3}}{3},\\x_{11}=0,\end{cases}\quad\begin{cases}p_{12}=0,\\y_{12}=-\frac{\sqrt{3}}{3},\\x_{12}=0,\end{cases}$$

$$\begin{cases}p_{13}=-1.3095,\\y_{13}=19.2008,\\x_{13}=3.1572,\end{cases}\quad\begin{cases}p_{14}=1.3095,\\y_{14}=19.2008,\\x_{14}=-3.1572.\end{cases}$$

Therefore, the function ψ has no optimal point in Π.
(1) For $p=2$,

$$\psi(2,y,x)=0.$$

(2) For $p=0$,

$$\begin{aligned}\psi(0,y,x)=&1600(1-y^2)(15+y^2)x^2+25600y(1-y^2)(1-x^2)\\ =&25600y-25600y^3+(-1600y^4+25600y^3-22400y^2\\ &-25600y+24000)x^2\\ =&\gamma_1(y,x).\end{aligned}$$

The critical points of γ_1 satisfy

$$\begin{cases}\dfrac{\partial\gamma_1}{\partial y}=25600-76800y^2+(-6400y^3+76800y^2-44800y-25600)x^2=0,\\ \dfrac{\partial\gamma_1}{\partial x}=(-3200y^4+51200y^3-44800y^2-51200y+48000)x=0.\end{cases}$$

Utilizing numerical computations, we yiled

$$\begin{cases}y_1=-0.5774,\\ x_1=0,\end{cases}\quad\begin{cases}y_2=0.5774,\\ x_2=0,\end{cases}\quad\begin{cases}y_3=-1,\\ x_3=0.7071,\end{cases}\quad\begin{cases}y_4=1,\\ x_4=-0.7071.\end{cases}$$

Therefore, the γ_1 has no optimal points in $(0,1)\times(0,1)$.
(3) For $y=0$,

$$\psi(p,0,x)=1500(4-p^2)x^2\le 6000.$$

(4) For $y=1$,

$$\psi(p,1,x)=25p^6-200p^4+400p^2=\gamma_2(p)\le\gamma_2(1.1547)=237.0370.$$

(5) For $x=0$,

$$\begin{aligned}\psi(p,y,0)&=(25p^6-200p^4+400p^2)y^4+(1600p^4-12800p^2+25600)(y-y^3)\\ &=\gamma_3(p,y).\end{aligned}$$

The optimal points of γ_3 satisfy

$$\begin{cases}\dfrac{\partial\gamma_3}{\partial p}=(150p^5-800p^3+800p)y^4+(6400p^3-25600p)(y-y^3)=0,\\ \dfrac{\partial\gamma_3}{\partial y}=(100p^6-800p^4+1600p^2)y^3+(-4800p^4+38400p^2-76800)y^2+1600p^4\\ \qquad -12800p^2+25600\\ \quad =0.\end{cases}$$

A computation reveals that there is no optimal point in $(0,2)\times(0,1)$.

(6) For $x = 1$,

$$\begin{aligned}\psi(p,y,1) &= (25p^6 - 100p^5 - 300p^4 + 800p^3 + 1200p^2 - 1600p - 1600)y^4 + (100p^5 \\ &\quad -1400p^4 - 800p^3 + 11200p^2 + 1600p - 22400)y^2 \\ &\quad +1500p^4 - 12000p^2 + 24000 \\ &= \gamma_4(p,y).\end{aligned}$$

Consider

$$\begin{cases} \dfrac{\partial \gamma_4}{\partial p} = (150p^5 - 500p^4 - 1200p^3 + 2400p^2 + 2400p - 1600)y^4 + (500p^4 - 5600p^3 \\ \qquad -2400p^2 + 22400p + 1600)y^2 + 6000p^4 - 24000p \\ \quad = 0, \\ \dfrac{\partial \gamma_4}{\partial y} = (100p^6 - 400p^5 - 1200p^4 + 3200p^3 + 4800p^2 - 6400p - 6400)y^3 + (200p^5 \\ \qquad -2800p^4 - 1600p^3 + 22400p^2 + 3200p - 44800)y \\ \quad = 0. \end{cases}$$

Thus, we have

$$\begin{cases} p_1 = 0, \\ y_1 = 0, \end{cases} \quad \begin{cases} p_2 = 2, \\ y_2 = 0, \end{cases} \quad \begin{cases} p_3 = -2, \\ y_3 = y_3, \end{cases} \quad \begin{cases} p_4 = -2.8439, \\ y_4 = 1.4760, \end{cases} \quad \begin{cases} p_5 = -2.8439, \\ y_5 = -1.4760. \end{cases}$$

Thus, the γ_4 has on critical point in $(0,2) \times (0,1)$.

(7) For $y = x = 0$, or $p = 2, y = 1$, or $p = 2, y = 0$, or $p = 2, x = 0$, or $p = 2, x = 1$, or $p = 0, y = 1$,

$$\psi(p,0,0) = \psi(2,1,x) = \psi(2,0,x) = \psi(2,y,0) = \psi(2,y,1) = \psi(0,1,x) = 0.$$

(8) For $y = 0, x = 1$,

$$\psi(p,0,1) = 1500(4 - p^2) \le 6000.$$

(9) For $y = x = 1$, or $y = 1, x = 0$,

$$\psi(p,1,1) = \psi(p,1,0) = 25p^6 - 200p^4 + 400p^2 = \gamma_2(p) \le \gamma_2(1.1547) = 237.0370.$$

(10) For $p = x = 0$,

$$\psi(0,y,0) = 25600y - 25600y^2 = \gamma_5(y) \le \gamma_5(\frac{1}{2}) = 6400.$$

(11) For $p = 0$ and $x = 1$,

$$\psi(0,y,1) = 24000 - 22400y^2 - 1600y^4 = \gamma_6(y) \le \gamma_6(0) = 24000.$$

(12) For $p = y = 0$,

$$\psi(0,0,x) = 24000x^2 \le 24000.$$

Then, the sharp bound for this Hankel determinant is determined by

$$|H_{2,3}(g)| = \frac{25}{576},$$

with an extremal function

$$g(\zeta) = \int_0^\zeta (1 + \frac{5}{6}t^3 + \frac{1}{6}t^{15})dt = \zeta + \frac{5}{24}\zeta^4 + \frac{1}{96}\zeta^{16}.$$

□

Acknowledgements

We thank the referees for their time and comments.

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