Journal of Applied Analysis and Computation
Volume 15, Number 2, April 2025, 973–992
Website:http://www.jaac-online.com
DOI:10.11948/20240241

EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR DISCONTINUOUS ELLIPTIC PROBLEMS WITH CRITICAL GROWTH*

Ziqing Yuan†

Abstract This paper concerns the following elliptical problem with discontinuous nonlinearity

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = f(u) + |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ u > 0, \end{cases}$$

where $N \geq 3$, $\epsilon > 0$ and $f(u)$ is a discontinuous function. We obtain the existence and concentration results of this problem. Our results generalize some recent results on this kind of problems. In order to obtain these results, a suitable truncation, concentration compactness principle, new analytic technique and variational method are used.

Keywords Elliptic problem, concentration, variational method, discontinuous nonlinearity.

MSC(2010) 35Q51, 34A36, 49J52.

1. Introduction

In this paper, we will concern the existence and concentration behavior of positive solutions for the following problem

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = f(u) + |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ u > 0, \end{cases} \tag{1.1}$$

where $\epsilon > 0$, $N \geq 3$ and $f(x)$ is defined by

$$f(u) = \begin{cases} g(u), & u \in [0, a], \\ (1+\delta)g(u), & u \in [a, +\infty), \end{cases}$$

$g(u) \in C(\mathbb{R}, \mathbb{R})$, which can be rewritten as

$$\begin{cases} -\Delta u + V(\epsilon x)u = f(u) + |u|^{2^*-2}u, & x \in \mathbb{R}^N, \\ u > 0. \end{cases} \tag{1.2}$$

†The corresponding author.
Department of Mathematics, Shaoyang University, Shaoyang, Hunan 422000, China
*The author was supported by the Natural Science Foundation of Hunan Provincial (Grant No. 2023JJ30559) and the National Natural Science Foundation of China (Grant No. 11901126).
Email: junjyuan@sina.com(Z. Yuan)

Assume that $V(x)$ and $g(u)$ satisfy the following basic assumptions:

(V1) there exists an open and bounded set Ω compactly contained in $\mathbb{R}^N$ such that $0 < \varrho = \inf_{x\in\mathbb{R}^N} V(x) \le V_0 = \inf_{x\in\Omega} V(x) < \min_{x\in\partial\Omega} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty$;

(g1) for all $t \in \mathbb{R}$, there exist $C > 0$ and $s \in (2, 2^*)$ such that $|g(t)| \le C(1+|t|^{s-1})$;

(g2) for all $t \in \mathbb{R}$, there is $\zeta \in (2, 2^*)$ such that

$$0 < \zeta G(t) = \zeta \int_0^t g(s)ds \le tg(t),$$

where $2^* = \begin{cases} \frac{2N}{N-2}, & \text{if } N > 2, \\ +\infty, & \text{if } N \le 2. \end{cases}$

(g3) there is $\varpi > 0$, which will be fixed later, such that $g(t) \ge \varpi$ for all $t \ge 2a$;

(g4) $\limsup_{t\to 0} \frac{g(t)}{t} = 0$;

(g5) $\frac{g(t)}{t}$ is increasing for $t > 0$.

It is easy to see that there exist lots of functions verifying (g1)-(g5), for example, set $g(t) = \sum_{i=1}^k \frac{\varpi|t|^{q_i-1}}{(2a)^{q_i-1}}$ if $t \ge 0$ and $g(t) = 0$ if $t \le 0$. Then $g(t)$ satisfies (g1)-(g5).

We say that u is a weak solution of problem (1.2), if $u \in H^1(\mathbb{R}^N)$ and

$$-\Delta u + V(\epsilon x)u \in [\underline{f}_\delta(u(x)) + |u|^{2^*-2}u, \overline{f}_\delta(u(x)) + |u|^{2^*-2}u] \ \text{ a.e. in } \ \mathbb{R}^N,$$

where $\underline{f}_\delta(t) = \lim_{\delta\to 0^+} f(t-\delta)$ and $\overline{f}_\delta(t) = \lim_{\delta\to 0^+} f(t+\delta)$.

Our main result is the following:

Theorem 1.1. *If $(V1)$ and $(g1)-(g6)$ hold, then there exist $\epsilon^*, \delta^*, a^* > 0$ such that problem (1.2) has a positive solution $u_{\epsilon,\delta,a}$ for $\epsilon \in (0,\epsilon^*), \delta \in (0,\delta^*)$, and $a \in (0, a^*)$. Furthermore, if $\theta_{\epsilon,\delta,a} \in \mathbb{R}^N$ denotes a maximum point of $u_{\epsilon,\delta,a}$, then $\lim_{(\epsilon,\delta,a)\to(0,0,0)} V(\epsilon\theta_{\epsilon,\delta,a}) = V_0$.*

We would like to point out that this kind of equation in (1.1) arises from the problem of deriving standing waves solutions of the nonlinear Schrödinger equation

$$i\epsilon\frac{\partial\Psi}{\partial t} = -\epsilon^2\Delta\Psi + (V(x)+E) - |\Psi|^{-1}h(|\Psi|)\Psi \ \text{ in } \mathbb{R}^N, \tag{1.3}$$

where $h(s) = f(s) + s^{2^*-1}$. A standing wave solution to problem (1.3) is one in the form where $\Psi(x,t) = exp(-i\epsilon^{-1}Et)u(x)$. In this case u is a solution of (1.2). For the case where $\delta = 0$, the right-hand side function is a continuous function with critical term. Then its energy functional is differentiable and there exist many results on discussing this type of problem (1.2), see [4, 10, 12, 19, 21, 22, 26, 27, 31] and references therein.

However, in this paper, the parameter δ is stipulated to be non-zero, thereby rendering the right-hand side nonlinearity of equation (1.2) discontinuous and its associated energy functional non-differentiable. This characteristic presents a significant challenge in the pursuit of understanding the solutions to equation (1.2). The academic interest in nonlinear partial differential equations featuring discontinuous nonlinearities has burgeoned, as numerous free boundary problems in mathematical physics can be articulated in this framework. Prominent examples include the seepage surface problem, the obstacle problem, and the Elenbass equation, as detailed in references [5–7]. A plethora of scholarly contributions have addressed problems of discontinuity. Notably, Corvellec et. al. [11], Alves et. al. [1–3], Grossi et. al. [17], Yuan and Yu [29], Yuan and Wang [28], Liu et. al. [24], Chang et. al. [8], Yue et. al. [30], Iannizzotto and Papageorgiou [18], and their respective bibliographies, have all made significant strides. These studies have harnessed a variety of methodologies, including variational methods for non-differentiable functionals, fixed point theory, global branching, lower and upper solution techniques, and the theory of multivalued mappings. These diverse approaches collectively enrich our analytical toolkit and deepen our comprehension of these complex mathematical phenomena.

To the best of our knowledge, there exist few papers involving the existence of solution for elliptic problems with discontinuous nonlinearity and critical growth by variational methods for non-differentiable energy functional. Noting that the classical critical points theory and variational methods for C^1 functional are not suitable for problem (1.2), inspired by the methods used in [5–7], we need to use variational methods and nonsmooth analysis for non-differentiable functionals. On the other hand, unlike [13], their solutions found are C^2, while in this paper, we don't have this regularity, since the nonlinearity is discontinuous. Then, some new arguments are needed to overcome the without of regularity of solutions. Furthermore, in order to obtain some estimates involving the mountain levels, the authors in [13] used a characterization of the Mountain Pass level involving infimum of energy functional on Nehari Manifolds. But the Nehari Manifolds is not well known yet for non-differentiable functionals, and so, we need to develop some some new arguments to derive good estimates involving the Mountain Pass levels. Also, in [13] del Pino and Felmer obtained the complete treatment (concentration and existence behavior of solutions) under condition (V1) with $\delta = 0$. They derived bound state solutions, but not ground state solutions. Of course, it is reasonable, since under condition (V1) some problems don't have any ground state solution. For this reason, we cannot find minimax critical points of the energy functional of problem (1.2). In order to solve this difficulty, we modify the nonlinearity to apply the Mountain Pass Lemma. Then we establish the existence of positive solutions. Finally, since the energy functional of (1.2) contains critical growth term, and the working space $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $p \in [2, 2^*]$, is not compact, we adapt a penalization method and the concentration compactness principle by Lions [20, Lemma 2.1] to overcome these difficulties.

This paper is organized as follows. In Section 2, some results involving locally Lipschitz continuous functionals are provided. In Section 3, the existence of solutions for an auxiliary problem are proved. In Section 4, based on Theorem 3.1, Theorem 1.1 is proved.

2. Preliminaries

We firstly give some notations. $(X, \|\cdot\|)$ denotes a (real) Banach space and $(X^*, \|\cdot\|_*)$ denotes its topological dual. C and $C_i (i = 1, 2, ...)$ denote estimated constant(the concrete values may be different from line to line). '$\to$' means the stronger convergence in X and '$\rightharpoonup$' stands for the weak convergence in X. $|u|_p$ denotes the norm of $L^p(\mathbb{R}^N)$.

Definition 2.1. ([15]) A function $J\colon X \to \mathbb{R}$ is locally Lipschitz if for every $v \in X$ there exist a neighborhood U of v and $L > 0$ such that for every ν, $\eta \in U$

$$|J(\nu) - J(\eta)| \le L\|\nu - \eta\|.$$

Definition 2.2. ([15]) Let $J : X \to \mathbb{R}$ be a locally Lipschitz function. The generalized derivative of J in v along the direction ν is defined by

$$J^0(v;\nu) = \limsup_{\eta\to u, \tau\to 0^+} \frac{J(\eta+\tau\nu) - J(\eta)}{\tau},$$

where v, $\nu \in X$.

It is easy to see that the function $\nu \mapsto J^0(v;\nu)$ is sublinear, continuous and so is the support function of a nonempty, convex and w^*-compact set $\partial J(v) \subset X^*$, defined by

$$\partial J(v) = \{v^* \in X^* : \langle v^*, \nu\rangle_X \le J^0(v;\nu) \text{ for all } \nu \in X\},$$

$m(v_n) = \inf_{v_n^* \in \partial J(v_n)} \|v_n^*\|_{X^*}$. If $J \in C^1(X)$, then

$$\partial J(v) = \{J'(v)\}.$$

Clearly, these definitions extend those of the Gâteaux directional derivative and gradient. A critical point of J is an element $v_0 \in E$ such that $0 \in \partial J(v_0)$ and a critical value of J is a real number c such that $J(v_0) = c$ for some critical point $v_0 \in E$.

Proposition 2.1. ([6,9]) *Let $\{v_n\} \subset X$ and $\{v_n^*\} \subset X^*$ with $v_n^* \in \partial J(v_n)$. If $v_n \to v$ in X and $v_n^* \rightharpoonup v^*$ in X^*, then $v^* \in \partial J(v)$.*

Proposition 2.2. ([6,9]) *Let $\Psi(v) = \int_{\mathbb{R}^N} G(v)dx$, where $G(t) = \int_0^t g(s)ds$. Then, $\Psi \in Lip_{loc}(L^{p+1}(\mathbb{R}^N), \mathbb{R})$, $\partial\Psi(v) \subset L^{\frac{p+1}{p}}(\mathbb{R}^N)$ and if $\rho \in \partial\Psi(v)$, it satisfies*

$$\rho(x) \in [\underline{g}(v(x)), \bar{g}(v(x))] \quad a.e. \quad in \quad \mathbb{R}^N.$$

3. An auxiliary problem

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space and

$$E = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx < +\infty\right\},$$

equipped with the inner product and norm

$$\langle u, v\rangle := \int_{R^N} (\nabla u \nabla v + V(x)uv)dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

It follows from (g4) and (g5) that

$$\lim_{t\to 0}\left[\frac{f(t)}{t} + t^{2^*-2}\right] = 0 \text{ and } \lim_{t\to +\infty}\left[\frac{f(t)}{t} + t^{2^*-2}\right] = +\infty,$$

which means that for a small enough, there exists $b > a > 0$ such that

$$\frac{(1+\delta)g(b)}{b} + b^{2^*-2} = \frac{V_0}{k} \quad (k > 1),$$

where V_0 is defined by (V1) and $k > 1 + \delta$. By (g3) we can choose $b > 0$ such that $a < b < 2a$. Based on the above facts, we define the function

$$\hat{f}(t) = \begin{cases} 0, & \text{if } \ t < 0, \\ f(t) + t^{2^*-1}, & \text{if } \ 0 < t < b, \\ \dfrac{V_0}{k} t, & \text{if } \ t \geq b. \end{cases}$$

Fixing $\Omega \subset \mathbb{R}^N$ be a bounded domain and using the function $\hat{f}$, we give the function

$$h(x,t) = \chi_\Omega(x)(f(t) + t^{2^*-1}) + (1 - \chi_\Omega(x))\hat{f}(t), \tag{3.1}$$

where χ_Ω is the characteristic function related to Ω, and consider the auxiliary problem

$$\begin{cases} -\Delta u + V(\epsilon x)u = h(\epsilon x, u), \\ u \in H^1(\mathbb{R}^N), u(x) > 0, \forall x \in \mathbb{R}^N. \end{cases} \tag{3.2}$$

From hypotheses $(g1) - (g4)$, h satisfies the following conditions for $x \in \mathbb{R}^N$.

(h1) $h(x,t) = 0$ for all $t < 0$ and $\limsup_{|t|\to 0} \frac{h(x,t)}{|t|} = 0$;

(h2) $h(x,t) = f(t) + t^{2^*-1}$ for all $x \in \Omega$, $t > 0$, or $x \in \Omega^c$ and $t \in [0,b]$;

(h3) $h(x,t) \leq (1+\delta)g(t) + t^{2^*-1}$ for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$;

(h4) $0 < \zeta H(x,t) = \zeta \int_0^t h(x,s)\mathrm{d}x \leq \underline{h}(x,t)t$ for all $x \in \Omega, t > 0$ and $0 < 2H(x,t) \leq \underline{h}(x,t)t \leq \overline{h}(x,t)t \leq \frac{1}{k}V_0 t^2$ for all $x \notin \Omega$, $t \geq 0$.

Set

$$I_\epsilon(u) := I_{\epsilon,a,\delta}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)u^2) - \int_{\mathbb{R}^N} H(\epsilon x, u),$$

$Q_\epsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)u^2)$, and $\Psi_\epsilon(u) := \int_{\mathbb{R}^N} H(\epsilon x, u)$.

Lemma 3.1. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ϵ. Then $\{u_n\}$ is bounded in H_ϵ.*

Proof. Set $\{u_n^*\} \subset H^{-1}(\mathbb{R}^N)$ be such that $I_\epsilon(u_n) \to c$, $m(u_n) = \|u_n^*\|_* = o_n(1)$. Due to $u_n^* \in \partial I_\epsilon(u_n)$, there exists $\xi_n^* \in \partial\Psi_\epsilon(u_n)$ satisfying

$$\langle u_n^*, \eta\rangle = \langle Q'_\epsilon(u_n), \eta\rangle - \langle \xi_n^*, \eta\rangle \tag{3.3}$$

for all $\eta \in H^1(\mathbb{R}^N)$. From $(h4)$ and $k > 1$, we have

$$\begin{aligned} c + o_n(1) &= I_\epsilon(u_n) - \frac{1}{\varsigma}\langle u_n^*, \eta\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\varsigma}\right)\|u_n\|^2 + \int_{\Omega_\epsilon^c} \left(\frac{1}{\varsigma}\xi_n u_n - H(\epsilon x, u_n)\right) \\ &\geq \left(\frac{1}{2} - \frac{1}{\varsigma}\right)\|u_n\|^2 + \frac{2-\varsigma}{\varsigma}\int_{\Omega_\epsilon^c} G(\epsilon x, u_n)|u_n|^2 \end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{\varsigma}\right)\|u_n\|^2+\frac{2-\varsigma}{2k\varsigma}\int_{\Omega_\epsilon^c}V(\epsilon x)|u_n|^2$$
$$=\frac{\varsigma-2}{2\varsigma}\int_{\mathbb{R}^N}\left[|\nabla u_n|^2+\left(1-\frac{1}{k}\right)V(\epsilon x)u_n^2\right]$$
$$\geq C\|u_n\|^2,$$

where $\xi_n \in [\underline{h}(\epsilon x, u_n), \overline{h}(\epsilon x, u_n)]$, which deduces that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Lemma 3.2. *Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ϵ. Then for each $\sigma > 0$, there is $\rho = \rho(\sigma) > 0$ such that*

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N\setminus B_\rho(0)}[|\nabla u_n|^2+V(\epsilon x)|u_n|^2]<\sigma.$$

Proof. Put u_n^*, ξ_n^* and ξ_n be the same as that used in the proof of Lemma 3.1, and $\varphi_R \in C^\infty(\mathbb{R}^N,[0,1])$ such that $\varphi_R(x) = 0$ in $B_R(0)$, $\varphi_R(x) = 1$ in $B_{2R}(0)^c$ and $|\nabla\varphi_R(x)| \leq \frac{C}{R}$ in $\mathbb{R}^N$, where C is a constant independent on R. Recalling that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\langle u_n^*, \varphi_R u_n\rangle = o_n(1)$, by (3.3), we have

$$\int_{\mathbb{R}^N}\varphi_R[|\nabla u_n|^2+V(\epsilon x)|u_n|^2]\leq\int_{\mathbb{R}^N}\xi_n\varphi_R u_n-\int_{\mathbb{R}^N}u_n\nabla\varphi_R\nabla u_n+o_n(1).$$

Fixed $\rho > 0$ such that $\Omega_\epsilon \subset B_{\frac{\rho}{2}}(0)$, by $\xi_n \in [\underline{h}(\epsilon x, u_n), \overline{h}(\epsilon x, u_n)]$, $(h4)$, and

$$\int_{\mathbb{R}^N}\xi_n\varphi_R u_n\leq\frac{1}{k}\int_{\mathbb{R}^N}\varphi_R V(\epsilon x)|u_n|^2,$$

we obtain

$$\left(1-\frac{1}{k}\right)\int_{\mathbb{R}^N}\varphi_R[|\nabla u_n|^2+V(\epsilon x)|u_n|^2]\leq\frac{C}{R}|u_n|_2|\nabla u_n|_2+o_n(1)<\sigma$$

for some R sufficiently large. □

Denote by $\mathscr{D}^{1,2}(\mathbb{R}^N)$ the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\|_2 = \left(\int_{\mathbb{R}^N}|\nabla u|^2\right)^{\frac{1}{2}}$ and set S be the best constant for Sobolev embedding $\mathscr{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Lemma 3.3. *If (V1) and (g1)-(g4) hold, then I_ϵ satisfies the $(PS)_c$ condition in $H^1(\mathbb{R}^N)$ for $c < \left(\frac{1}{2}-\frac{1}{\varsigma}\right)S^{\frac{N}{2}}$.*

Proof. According to Lemma 3.1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Choosing a subsequence, we may suppose that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$,

$$|\nabla u_n|^2\rightharpoonup|\nabla u|^2+\mu\ \ \textit{and}\ \ |u_n|^{2^*}\rightharpoonup|u|^{2^*}+\nu\ (\text{weak}^*-\text{sense of measure}).$$

Thanks to the concentration compactness principle by Lions [23, Lemma 2.1], we derive at most countable index set $\daleth$, and sequences $\{x_i\} \subset \mathbb{R}^N$, $\{\mu_i\}$, $\{\nu_i\} \subset [0,\infty)$, such that

$$\nu=\sum_{i\in\daleth}\nu_i\delta_{x_i},\quad\mu\geq\sum_{i\in\daleth}\mu_i\delta_{x_i}\quad\text{and}\quad S\nu_i^{\frac{2}{2^*}}\leq\mu_i \tag{3.4}$$

for all $i \in \daleth$, where δ_{x_i} is the Dirac mass at x_i.

We claim that $\daleth = \emptyset$. Proceeding by contradiction, suppose that $\daleth \neq \emptyset$ and fix $i \in \daleth$. Set u_n^*, ξ_n and ξ_n^* be the same as that used in the proof of Lemma 3.1. Consider $\eta \in C_0^\infty(\mathbb{R}^N, [0,1])$, such that $\eta \equiv 1$ in $B_1(0)$; $\eta \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla \eta|_{L^\infty} \leq 2$. Defining $\eta_R(x) = \eta(\frac{x-x_i}{R})$, where $R > 0$, we obtain that $\{\eta_R u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, which means that

$$\int_{\mathbb{R}^N} (\nabla u_n \nabla(\eta_R u_n) + V(\epsilon x)\eta_R u_n^2) = \int_{\Omega_\epsilon} \xi_n \eta_R u_n + \int_{\Omega_\epsilon^c} \xi_n \eta_R u_n + o_n(1).$$

From (h3) and (h4), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \eta_R |\nabla u_n|^2 \leq & C \int_{\mathbb{R}^N} |u_n|^q \eta_R + \int_{\mathbb{R}^N} |u_n|^{2^*} \eta_R + C \int_{\mathbb{R}^N} |u_n|^2 \eta_R \\ & - \int_{\mathbb{R}^N} (u_n \nabla u_n \nabla \eta_R + V(\epsilon x)\eta_R u_n^2). \end{aligned} \tag{3.5}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, the support of η_R is contained in $B_{2R}(x_i)$,

$$\begin{aligned} &\lim_{R\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} (u_n \nabla u_n \nabla \eta_R + V(\epsilon x)\eta_R u_n^2) = 0, \\ &\lim_{R\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 \eta_R = 0 \quad \textit{and} \quad \lim_{R\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^q \eta_R u_n = 0, \end{aligned}$$

which follows from (3.5) that

$$\int_{\mathbb{R}^N} \eta_R d\mu \leq \int_{\mathbb{R}^N} \eta_R d\nu + o_R(1).$$

Letting $R \to 0$ and by the standard theory of Radon measures, we infer that $\nu_i \geq \mu_i \geq S\nu_i^{\frac{2}{2^*}}$, i.e., $\nu_i > S^{\frac{N}{2}}$. Once that $\{u_n\}$ is a sequence of $(PS)_c$, Proceeding as in the proof of Lemma 3.1, we have that

$$\begin{aligned} c &= I_\epsilon(u_n) - \frac{1}{\varsigma}\langle u_n^*, u_n\rangle + o_n(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{\varsigma}\right) \int_{\mathbb{R}^N} |\nabla u_n|^2 \eta_R + C_0 \int_{\mathbb{R}^N} V(\epsilon x)|u_n|^2 \eta_R + o_n(1) \\ &\geq \left(\frac{1}{2} - \frac{1}{\varsigma}\right) S^{\frac{N}{2}}, \end{aligned}$$

which leads to a contradiction. Hence, it follows that $\daleth$ is empty and $u_n \to u$ in $L^{2^*}_{loc}(\mathbb{R}^N)$. By (h2), (g1) and (3.1), we derive a constant $C = C(a) > 0$ such that

$$|h(\epsilon x, t)| \leq C|t|^{2^*-1} \tag{3.6}$$

for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$. Since $\|u_n\|^2 = \int_{\mathbb{R}^N} \xi_n u_n + o_n(1)$, and $\xi_n \in [\underline{h}(\epsilon x, u_n), \bar{h}(\epsilon x, u_n)]$, the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^N)$, infers that $\{\xi_n\}$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$. Then, passing to a subsequence if necessary

$$\|u_n\| \to \lambda \text{ in } \mathbb{R},\ \xi_n \rightharpoonup \xi \text{ in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N) \text{ and } u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^N), \tag{3.7}$$

it follows from (3.3) that

$$\|u\|^2 = \int_{\mathbb{R}^N} \xi u. \tag{3.8}$$

Observing that

$$\left|\int_{B_R(0)}\xi_n u_n-\int_{B_R(0)}\xi u\right|\le |u_n-u|_{L^{2^*}(B_R(0))}|\xi_n|_{L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)}+\left|\int_{B_R(0)}(\xi_n-\xi)u\right|,$$

from (3.6), (3.7) and Riesz representation theorem, one has

$$\int_{B_R(0)}\xi_n u_n\to\int_{B_R(0)}\xi u. \tag{3.9}$$

Lemma 3.2 means that

$$\limsup_{n\to\infty}\left|\int_{B_R^c(0)}\xi_n u_n\right|=o_R(1). \tag{3.10}$$

Combining (3.9) and (3.10), we infer that

$$\int_{\mathbb{R}^N}\xi_n u_n\to\int_{\mathbb{R}^N}\xi u, \tag{3.11}$$

from which it follows that $\|u_n\|^2=\|u\|^2+o_n(1)$. Consequence, $u_n\to u$ in $H^1(\mathbb{R}^N)$. □

Lemma 3.4. *If hypotheses (V1) and (g1)-(g4) hold, fixed $\epsilon^*>0$, $a>0$ small, for each $\epsilon\in(0,\epsilon^*)$, then there exist $\hat{\gamma}_0>0$ and $v_0\in H^1(\mathbb{R}^N)$, which are independent of ϵ^* and a, such that*

(i) $\max_{t\in[0,\hat{\gamma}_0]}I_\epsilon(tv_0)<\left(\frac{1}{2}-\frac{1}{\varsigma}\right)S^{\frac{N}{2}}$;

(ii) *there are $r,\alpha>0$ such that $I_\epsilon(u)\ge\alpha$ for $u\in H^1(\mathbb{R}^N)$, $\|u\|=r$;*

(iii) $I_\epsilon(\hat{\gamma}v_0)<0$ *and* $\hat{\gamma}_0 v_0\in B_r(0)^c$.

Proof. Without loss of generality, we may assume that $0\in\Omega$ and $V_0=V(0)$. Fixed $\epsilon^*\in(0,1)$, set $v_0\in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N}(|\nabla v_0|^2+V_\infty|v_0|^2)=1$, $v_0\ge0$, supp$v_0\subset B_R(0)\subset\Omega$. Since $V(\epsilon x)\le V_\infty$, for all $x\in B_R(0)$ and $\epsilon\in(0,\epsilon^*)$, one has $\epsilon x\in\Omega$, then, from (h2), we derive

$$I_\epsilon(tv_0)\le\mathcal{L}(t)-\int_{B_R(0)}\int_0^{tv_0(x)}f(s)dsdx\le\mathcal{L}(t) \tag{3.12}$$

for all $t\ge0$, where $\mathcal{L}(t)=\frac{t^2}{2}-\frac{t^{2^*}}{2^*}\int_{B_R(0)}|v_0|^{2^*}$.

Since the function $\mathcal{L}(t)$ is increasing in $(0,t^*)$ for some $t^*>0$ and $\lim_{t\to0}\mathcal{L}(t)=0$, there is $\hat{\gamma}>0$, independent on ϵ^* and δ such that $\hat{\gamma}<t^*$ and $\max_{t\in[0,\hat{\gamma}]}I_\epsilon(tv_0)\le\mathcal{L}(\hat{\gamma}_0)<\left(\frac{1}{2}-\frac{1}{\varsigma}\right)S^{\frac{N}{2}}$, which proves (i).

Since $I_\epsilon(0)=0$, from $(g1),(h3)$ and choosing $\|u\|=r<\frac{\hat{\gamma}_0}{2}$, there is $\alpha>0$ such that $I_\epsilon(u)\ge\alpha$ for $u\in H^1(\mathbb{R}^N),\|u\|=r$. (ii) is proved.

We now prove (iii). It follows from (g3) that

$$\begin{aligned}\int_{B_R(0)}\int_0^{\hat{\gamma}_0v_0(x)}f(s)dsdx&\ge\int_{B_R(0)}\int_0^{\hat{\gamma}_0v_0(x)}g(s)dsdx\\&\ge\int_{B_R(0)}\int_{2a}^{\hat{\gamma}_0v_0(x)}ldsdx\\&=\int_{B_R(0)}(\hat{\gamma}_0v_0-2a)dx.\end{aligned}$$

From (3.12) we have

$$\begin{aligned} I_\epsilon(\hat{\gamma}_0 v_0) &\le \mathcal{L}(\hat{\gamma}_0) - \int_{B_R(0)} (\hat{\gamma}_0 v_0 - 2a)dx \\ &= \frac{1}{2}\hat{\gamma}_0^2 - \frac{\hat{\gamma}_0^{2^*}}{2^*}\int_{B_R(0)} |v_0|^{2^*} + 2a\omega_N R^N - \hat{\gamma}_0 \int_{B_R(0)} v_0 dx \\ &< 0 \end{aligned}$$

for both $\hat{\gamma}_0 > 0$ and $a > 0$ small enough. □

Remark 3.1. (i) The above lemma shows the restriction of the constant a given in (g3).

(ii) From the proof of this lemma, we know that the set $\{u(x) : u(x) > a\}$ has positive measure, otherwise, we cannot ensure $I_\epsilon(\hat{\gamma}_0 v_0) < 0$.

The following result establishes the existence of a ground solution to (3.2), which means that there exists a function u_ϵ such that $I_\epsilon(u_\epsilon) = c_\epsilon := c_{\epsilon,a,\delta}$, and $0 \in \partial I_\epsilon(u_\epsilon)$, where c_ϵ denotes the mountain pass level associated to I_ϵ.

Theorem 3.1. *If (g1)-(g4) and (V1) hold, then there exist $\epsilon^*, a > 0$ small such that for all $\epsilon \in (0, \epsilon^*)$, problem (3.2) has a positive solution u_ϵ satisfying*

(i) *u_ϵ is a weak solution of problem (3.2) for all $\epsilon \in (0, \epsilon^*)$;*

(ii) *The set $|\Lambda_{\epsilon,a}| = \{x \in \mathbb{R}^N : u_\epsilon(x) = a\}$ has null measure;*

(iii) *The set $\{x \in \mathbb{R}^N : u_\epsilon(x) > a\}$ has positive measure.*

Proof. (i) Set a, V_0 and $\hat{\gamma}_0$ be the same as that in Lemma 3.4. It follows from Lemma 3.4 that I_ϵ has the Mountain Pass geometry, and there exist sequences $\{u_n\} \subset H^1(\mathbb{R}^N)$, $\{u_n^*\} \subset \partial I_\epsilon(u_n)$ and $\{\xi_n^*\} \subset \partial \Psi_\epsilon(u_n)$ such that $u_n^* = Q'_\epsilon - \xi_n^*$ in $H^{-1}(\mathbb{R}^N)$,

$$\|u_n^*\|_* = o_n(1), \; I_\epsilon(u_n) = c_\epsilon + o_n(1),$$

where $c_\epsilon = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\epsilon(\gamma(t))$ and $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \textit{ and } \gamma(1) = \gamma_0 v_0\}$. Noting that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\xi_n \in [\underline{h}(\epsilon x, u_n), \bar{h}(\epsilon x, u_n)]$. From (3.6) we derive that $\{\xi_n\}$ is bounded in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$. According to Lemma 3.3, it follows that $u_n \to u_\epsilon$ in $H^1(\mathbb{R}^N)$ and $\xi_n \rightharpoonup \xi_\epsilon$ in $L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$. Therefore,

$$\int_{\mathbb{R}^N} (\nabla u_\epsilon \nabla \eta + V(\epsilon x)\eta) = \int_{\mathbb{R}^N} \xi_\epsilon \eta \tag{3.13}$$

for all $\eta \in H^1(\mathbb{R}^N)$, where $\xi_\epsilon \in [\underline{h}(\epsilon x, u_\epsilon), \bar{h}(\epsilon x, u_\epsilon)]$. Once that $\xi_\epsilon \in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$, by the elliptic regularity theory, $u_\epsilon \in W^{2,\frac{2^*}{2^*-1}}(\mathbb{R}^N)$ and

$$-\Delta u_\epsilon + V(\epsilon x)u_\epsilon \in [\underline{h}(\epsilon x, u_\epsilon), \bar{h}(\epsilon x, u_\epsilon)] \text{ a.e. in } \mathbb{R}^N. \tag{3.14}$$

Taking a test function u_ϵ^-, we derive $u_\epsilon = u_\epsilon^+ \ge 0$. By Harnack inequality [16, Theorem 8.20], one deduces that $u_\epsilon > 0$. Consequently, u_ϵ is a positive solution of (3.2) and (i) is proved.

We now assume that $|\Lambda_{\epsilon,a}| := \{x \in \mathbb{R}^N : u_\epsilon(x) = a\}$ has positive measure. It follows from Stampachia Theorem [25] that $-\Delta u_\epsilon(x) = 0$ a.e. in $\Lambda_{\epsilon,a}$. (3.14) infers that

$$V(\epsilon x)a \in [\underline{h}(\epsilon x, u_\epsilon(x)), \bar{h}(\epsilon x, u_\epsilon(x))] \text{ a.e. in } \Lambda_{\epsilon,a}. \tag{3.15}$$

Once that $a < b$ and

$$|h(x,t)| \leq \frac{V_0 t}{k} \tag{3.16}$$

for all $x \in \mathbb{R}^N$, $t \in [0,a]$, it follows from (3.15) that $1 \leq \frac{1}{k}$, which contradicts to $k > 1$. Hence $|\Lambda_{\epsilon,a}| = 0$, which shows (ii).

Next, we prove (iii). If the conclusion was false, i.e., $|\{x \in \mathbb{R}^N : u_\epsilon(x) > a\}| = 0$, then

$$u_\epsilon(x) \leq a \tag{3.17}$$

a.e. in $\mathbb{R}^N$. This, combining (3.13), (3.16), infers that $\left(1 - \frac{1}{k}\right)\|u_\epsilon\|^2 \leq 0$, which is a contradiction to $I_\epsilon(u_\epsilon) = c_\epsilon > 0$. Hence $u_\epsilon \geq 0$. □

4. Proof of Theorem 1.1

The following Lemma is very important to show that the solution proved in Theorem 3.1 is a solution of the original problem (1.2) for a small enough.

Lemma 4.1. *If u_ϵ is the solution found in Theorem 3.1 for $\epsilon \in (0, \epsilon^*)$, then* $\max_{t\geq 0} I_\epsilon(tu_\epsilon) = I_\epsilon(u_\epsilon)$.

Proof. We firstly give $A : [0, +\infty) \to \mathbb{R}$ the locally Lipschitz continuous function defined by

$$A(t) = I_\epsilon(tu_\epsilon), \quad \forall t \geq 0.$$

It is straightforward to prove that there exist $\sigma, t_0 > 0$ such that

$$A(t) > 0,\ \forall t \in (0,\sigma) \text{ and } A(t) < 0\ \ \forall t \geq t_0,$$

from which it follows that A has a maximum value. Hereafter, set $t_* > 0$ be a number where A attains its maximum, i.e.,

$$A(t_*) = \max_{t\geq 0} A(t).$$

We now claim that the number t_* is equal to 1. Indeed, noting that A is locally Lipschitz continuous function, we have that A is a.e. differentiable. Write $\tilde{\Omega}$ be the set of these points, where A' does't exist, then we derive $|\tilde{\Omega}| = 0$. Now it needs to prove that

(i) $A'(t) > 0$, $\forall t \in (0,1) \cap \tilde{\Omega}^c$;

(ii) $A'(t) < 0$, $\forall t \in (1, +\infty) \cap \tilde{\Omega}^c$.

From (i) and (ii) we know that A has a global maximum value at $t = 1$. Furthermore, $t = 1$ is the unique point where the global maximum is attained.

In the following, we firstly prove (i). Without loss of generality, we assume that $b < 1$. By the Chain Rule for locally Lipschitz continuous function, we obtain that there exists $u_t^* \in \partial I_\epsilon(tu)$ such that

$$A'(t) = \langle u_t^*, u\rangle,$$

or equivalently, there exists $\xi_t \in \partial\Psi_\epsilon(tu)$ verifying

$$A'(t) = t\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2) - \int_{\mathbb{R}^N} \xi_t u.$$

Thanks to $0 \in \partial I_\epsilon(u_\epsilon)$ and $|\Lambda_{\epsilon,a}| = 0$, it follows that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + V(\epsilon x)|u|^2) = \int_{\mathbb{R}^N} h(\epsilon x, u)u,$$

and so

$$A'(t) = t\int_{\mathbb{R}^N} h(\epsilon x, u)u - \int_{\mathbb{R}^N} \xi_t u.$$

According to Proposition 2.1 and Proposition 2.2, one has

$$\xi_t(x) \in [\underline{h}(\epsilon x, tu), \bar{h}(\epsilon x, tu)] \text{ a.e. in } \mathbb{R}^N,$$

thus, from $|\Lambda_{\epsilon,a}| = 0$, (g5) and the boundedness of $\bar{h}(\epsilon x, u)$ at $u = a$, we have

$$\begin{aligned} A'(t) &\geq t\left(\int_{\mathbb{R}^N} \underline{h}(\epsilon x, u)u - \int_{\mathbb{R}^N} \frac{\bar{h}(\epsilon x, tu)}{t} u\right) \\ &\geq t\Bigg(\int_{\{u\leq a\}} \left(\frac{g(u)}{u} - \frac{g(tu)}{tu}\right) u^2 + (1+\delta)\int_{\{a<u<b\}} \left(\frac{g(u)}{u} - \frac{g(tu)}{tu}\right) u^2 \\ &\quad + \int_{\{u\leq b\}} \left(u^{2^*-2} - t^{2^*-2}u^{2^*-2}\right) u^2 + \frac{V_0}{k}\int_{\{b<u<1\}} (u - tu)\Bigg) \\ &> 0, \quad \forall t \in (0,1), \end{aligned}$$

which deduces that

$$A'(t) > 0, \forall t \in (0,1) \cap \tilde{\Omega}^c.$$

(ii) can be proved by the same method as that employed in (i). It follows from (i) and (ii) that Lemma 4.1 is proved. □

As we know, it is crucial to discuss the problem

$$\begin{cases} -\Delta u + V_0 u = (1+\delta)g(u) + u^{2^*-1}, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ u(x) > 0, \ \forall x \in \mathbb{R}^N. \end{cases} \tag{4.1}$$

The energy functional associated to problem (4.1) is defined by

$$I_{V_0}(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0 u^2) - (1+\delta)\int_{\mathbb{R}^N} G(u_+) - \frac{1}{2^*}\int_{\mathbb{R}^N} u_+^{2^*},$$

where $u_+ = \max\{u, 0\}$. Then $I_0 \in C^1(H^1(\mathbb{R}^N, \mathbb{R}))$ and

$$\begin{aligned} I'_{V_0}(u)v = &\int_{\mathbb{R}^N} (\nabla u \nabla v + V_0 uv) - (1+\delta)\int_{\mathbb{R}^N} g(u_+)v \\ &- \int_{\mathbb{R}^N} u_+^{2^*-1} v, \quad \forall u, v \in H^1(\mathbb{R}^N). \end{aligned}$$

There exists a positive function $u_0 \in H^1(\mathbb{R}^N)$ such that $I'_{V_0}(u_0) = 0$ and $I_{V_0}(u_0) = c_{V_0}$, where c_{V_0} is the Mountain Pass level. Define the Nehari manifold corresponding to I_{V_0}

$$\mathscr{N}_{V_0} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I'_{V_0}(u)u = 0\},$$

then $c_{V_0} = \inf_{u\in\mathscr{N}_{V_0}} I_{V_0}(u)$.

Lemma 4.2. $\lim_{\epsilon_n, a_n, \delta_n \to 0} c_{\epsilon_n} = c_{V_0}$.

Proof. Firstly, for simplicity, we denote by $u_n = u_{\epsilon_n,a_n,\delta_n}$, $I_n = I_{\epsilon_n,a_n,\delta_n}$ and $c_n = c_{\epsilon_n,a_n,\delta_n}$ where c_n denotes the Mountain Pass level associated to I_n. Take $\epsilon_n, a_n, \delta_n \to 0$ as $n \to \infty$. For any $\rho > 0$, set $\eta \in C_0^\infty(\Omega)$ satisfying $0 \leq \eta(x) \leq 1$, $\eta(x) = 1$ for all $x \in B_1(0)$, and $\eta(x) = 0$ for all $x \in B_2^c(0) \subset\subset \Omega$. Moreover, for each $\rho > 1$. We denote by η_ρ and by u_ρ the functions, $\eta_\rho(x) = \eta(\frac{x}{\rho})$ and $u_\rho(x) = \eta_\rho(x)u_0(x)$. It is not difficult to show that $u_\rho \to u_0$ in $H^1(\mathbb{R}^N)$ as $\rho \to +\infty$. For each $R > 0$, there is $t_\rho > 0$ such that

$$I_{V_0}(t_\rho u_\rho) = \max_{t\geq 0} I_{V_0}(tu_\rho),$$

hence $I'_{V_0}(t_\rho u_\rho) = 0$ and

$$\frac{1}{t_\rho}\int_{\mathbb{R}^N}(|\nabla u_\rho|^2 + V_0 u_\rho^2) = (1+\delta)\int_{\mathbb{R}^N}\frac{g(t_\rho u_\rho)}{t_\rho}u_\rho + t_\rho^{2^*-3}\int_{\mathbb{R}^N} u_\rho^{2^*},$$

which means by $I'_{V_0}(u_0) = 0$ that $t_\rho \to 1$ as $\rho \to \infty$. Then, we can see that $t_\rho u_\rho \to u_0$ in $H^1(\mathbb{R}^N)$ and $I_{V_0}(t_\rho u_\rho) \to I_{V_0}(u_0)$ as $\rho \to \infty$. From a simple computation it follows that there exists $t_* > 0$ such that $I_\epsilon(t_* t_\rho u_\rho) < 0$ uniformly for $\epsilon, a > 0$ small enough. Taking $\gamma(t) = tt_* t_\rho u_\rho$ for $t \in [0,1]$, and from the definition of c_ϵ, we derive

$$c_\epsilon \leq \max_{t\in[0,1]} I_\epsilon(\hat{\gamma}(t)) \leq \max_{t\geq 0} I_\epsilon(\hat{\gamma}(t)) = I_\epsilon(\hat{t}t_\rho u_\rho)$$

for some $\hat{t} = \hat{t}(\epsilon, a, \rho) > 0$. A straightforward computation means that for each given $\rho > 0$, there exist positive constants C_1 and C_2 such that $C_1 < \hat{t} < C_2$ for $\epsilon, a > 0$ small enough. Once that $V(0) = V_0$, For any $\nu > 0$, there exists $\epsilon_0 > 0$ such that $0 < V(\epsilon_n x) - V_0 < \nu$ for $\epsilon_n \in (0, \epsilon_0)$ and $x \in B_{2R}(0)$. Consequently,

$$\int_{\mathbb{R}^N} V(\epsilon_n x)t_\rho^2 u_\rho^2 < \int_{\mathbb{R}^N}(V_0+\nu)t_\rho^2 u_\rho^2.$$

Then,

$$c_n \leq I_n(\hat{t}t_\rho u_\rho) \leq I_{V_0}(\hat{t}t_\rho u_\rho) + \frac{\hat{t}}{2}\nu\int_{B_{2R}(0)} t_\rho^2 u_\rho^2 + \delta\int_{\mathbb{R}^N} G(\hat{t}t_\rho u_\rho),$$

which leads to

$$\limsup_{n\to 0} c_n \leq c_{V_0}. \tag{4.2}$$

We now verify

$$\liminf_{n\to\infty} c_n \geq c_{V_0}. \tag{4.3}$$

Indeed, we proceed by contradiction, and assume that there is an integer N large and $\sigma > 0$ small such that

$$c_n \leq c_{V_0} - \sigma \text{ for all } n > N.$$

By Theorem 3.1 and the definition of c_{ϵ_n}, we derive

$$c_n = I_n(u_n) = \max_{t>0} I_n(tu_n) < c_{V_0} - \sigma$$

for any fixed $n > N$. From the definition of c_{V_0} one has that $c_{V_0} \leq \max_{t>0} I_{V_0}(tu_{\epsilon_n})$. Then, from the fact that, for all given $n > N, x \in \mathbb{R}^N$, $V_0 \leq V(\epsilon_n x)$, $(1 + \delta)\int_{\mathbb{R}^N} G(u) + \frac{1}{2^*}\int_{\mathbb{R}^N} u^{2^*} \geq \int_{\mathbb{R}^N} H(\epsilon x, u)$, it follows that

$$c_{V_0} - \sigma > \max_{t>0} I_n(tu_n) \geq \max_{t>0} I_{V_0}(tu_n) \geq c_{V_0},$$

which derives a contradiction. Then (4.3) is true. Finally, (4.2) and (4.3) deduce this lemma. □

Lemma 4.3. *Assume the same hypotheses of Theorem 3.1. Let $\{z_n\} \subset \mathscr{N}_{V_0}$ be such that $I_{V_0}(z_n) \to c_{V_0}$ and $z_n \to z$ in $H^1(\mathbb{R}^N)$. Then, there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $z_n(\cdot + \tilde{y}_n) \to z_0 \in \mathscr{N}_{V_0}$ with $I_{V_0}(z_0) = c_{V_0}$. Furthermore, if $z \neq 0$, then $\{\tilde{y}_n\}$ can be taken identically zero, thus , for this case, $z_n \to z$ in $H^1(\mathbb{R}^N)$.*

Proof. Similar as the method used in Lemma 3.1, we obtain that the sequence $\{z_n\}$ is bounded in $H^1(\mathbb{R}^N)$, thus passing to a subsequence if necessary, still denoted by $\{z_n\}$, we can suppose that there is $z \in H^1(\mathbb{R}^N)$ satisfying

$$z_n \rightharpoonup z \text{ in } H^1(\mathbb{R}^N).$$

From the Ekeland's Variational Principle [14], we can suppose that $\{z_n\}$ satisfying the following result

$$I_{V_0}(z_n) \to c_{V_0} \text{ and } I'_{V_0}(z_n) \to 0. \tag{4.4}$$

Based on (4.4), we divide our proof into two cases: $z \neq 0$ and $z = 0$.

Case 1. $z \neq 0$. Using the same argument as in the proof of Lemmas 3.3 and 3.4, we can also derive that

$$\begin{aligned} &\nabla z_n(x) \to \nabla z(x) \ \text{ a.e. in } \mathbb{R}^N, \\ &z_n^{2^*}(x) \to z^{2^*}(x) \ \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{4.5}$$

From the fact that $\langle I'_{V_0}(z_n), z_n\rangle = 0$, (4.4), it follows that $\langle I'_{V_0}(z), z\rangle = 0$, hence

$$c_{V_0} \leq I_{V_0}(z) = I_{V_0}(z) - \frac{1}{\varsigma}\langle I'_{V_0}(z), z\rangle,$$

which infers that

$$\begin{aligned} c_{V_0} \leq &\left(\frac{1}{2} - \frac{1}{\varsigma}\right)\int_{\mathbb{R}^N}(|\nabla z|^2 + z^2) + (1+\delta)\int_{\mathbb{R}^N}\left[\frac{1}{\varsigma}g(z)z - G(z)\right] \\ &+ \left(\frac{1}{\varsigma} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} z^{2^*} \\ :=& D. \end{aligned}$$

Then, by Fatou's Lemma

$$\begin{aligned} c_{V_0} &\leq D \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{2} - \frac{1}{\varsigma}\right)\int_{\mathbb{R}^N}(|\nabla z_n|^2 + z_n^2) + (1+\delta)\int_{\mathbb{R}^N}\left[\frac{1}{\varsigma}g(z_n)z_n - G(z_n)\right] \\ &\quad + \left(\frac{1}{\varsigma} - \frac{1}{2^*}\right)\int_{\mathbb{R}^N} z_n^{2^*} \\ &\leq c_{V_0}. \end{aligned}$$

Consequently,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla z_n|^2+z_n^2)\to\int_{\mathbb{R}^N}(|\nabla z|^2+z^2). \tag{4.6}$$

It follows from (4.5) and (4.6) that $z_n \to z$ in $H^1(\mathbb{R}^N)$.

Case 2. $z = 0$. For this case, there are $\rho, \beta > 0$ and $\{y_n\} \subset \mathbb{R}^N$ such that

$$\limsup_{n\to\infty}\int_{B_\rho(y_n)}|z_n|^2\geq\beta.$$

Indeed, if this case is not true, we have

$$\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_\rho(y)}|z_n|^2=0,$$

and by Lions result [23]

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|z_n|^q=0\quad\forall s\in(2,2^*).$$

By the above result and the fact that $\{z_n\} \subset \mathscr{N}_{V_0}$, we have

$$\int_{\mathbb{R}^N}(|\nabla z_n|^2+V_0z_n^2)=\int_{\mathbb{R}^N}z_n^{2^*}. \tag{4.7}$$

Suppose that $\int_{\mathbb{R}^N} z_n^{2^*} \to l$ as $n \to \infty$. Using the definition of the constant S, we obtain

$$S\left(\int_{\mathbb{R}^N}z_n^{2^*}\right)^{\frac{2}{2^*}}\leq\int_{\mathbb{R}^N}|\nabla z_n|^2\leq\int_{\mathbb{R}^N}(|\nabla z_n|^2+V_0z_n^2)=\int_{\mathbb{R}^N}z_n^{2^*},$$

which implies that $l \geq Sl^{\frac{2}{2^*}}$, i.e., $l > S^{\frac{N}{2}}$, which is a contradiction to $c_{V_0} < \left(\frac{1}{2}-\frac{1}{\varsigma}\right)S^{\frac{N}{2}}$. According to the Sobolev embedding, we have that $|y_n| \to \infty$. Defining $v_n = z_n(x + y_n)$, one derives

$$I_{V_0}(v_n)\to c_{V_0}\quad\text{and}\quad I'_{V_0}(v_n)\to 0.$$

It is obvious to see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, and there is a $v \in H^1(\mathbb{R}^N)$ with $v \neq 0$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Proceeding the same argument as in Case 1, we derive that $v_n \to v$ in $H^1(\mathbb{R}^N)$.

We now verify (4.4). From Ekeland's Variational Principle, it follows that there is a sequence $\{z_n\} \subset \mathscr{N}_{V_0}$ verifying

$$z_n=u_n+o_n(1),\quad I_{V_0}(z_n)\to c_{V_0}\quad\text{and}\quad I'_{V_0}(z_n)-\gamma_nJ'_{V_0}(z_n)=o_n(1),$$

where γ_n is a real number and $J_{V_0}(z) = I'_{V_0}(z)z,\ \forall z \in H^1(\mathbb{R}^N)$. Then there exists $\sigma > 0$ such that

$$|\langle J'_{V_0}(z_n),z_n\rangle|\geq\sigma,\ \forall n\in\mathbb{N}.$$

Indeed, by the definition of J_{V_0} and (g5), one has

$$\begin{aligned}-\langle J'_{V_0}(z_n),z_n\rangle&=(1+\delta)\int_{\mathbb{R}^N}(g'(z_n)z_n^2-g(z_n)z_n)+(2^*-2)\int_{\mathbb{R}^N}z_n^{2^*}\\&\geq(2^*-2)\int_{\mathbb{R}^N}z_n^{2^*}\\&>0.\end{aligned}$$

Since $I'_{V_0}(z_n)z_n = o_n(1)$, we have $J'_{V_0}(z_n)z_n = o_n(1)$, which derives $\gamma_n = o_n(1)$. Therefore,

$$I_{V_0}(z_n) \to c_{V_0} \quad \text{and} \quad I'_{V_0}(z_n) \to 0.$$

Consequently, we have

$$I_{V_0}(z_n) \to c_{V_0} \quad \text{and} \quad I'_{V_0}(z_n) \to 0.$$

Thus the proof is proved. □

Lemma 4.4. *Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence with $0 \in \partial J_{\epsilon_n}(u_n)$ and $I_{V_0}(u_n) \to c_{V_0}$, where $\epsilon_n, a_n, \delta_n \to 0^+$. Then, there is a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $w_n(x) := u_n(x + \tilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^N, \mathbb{R})$. Furthermore, up to a subsequence $\epsilon_n \tilde{y}_n \to y_0 \in \tilde{\Lambda}$, where $\tilde{\Lambda} = \{x \in \Omega : V(x) = V_0\}$.*

Proof. Lemma 3.1 means that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. We claim that there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and constants $\rho, \sigma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_\rho(\tilde{y}_n)} |u_n|^2 \ge \sigma > 0. \tag{4.8}$$

Indeed, suppose that (4.8) were false. Then, from Lion's result (see [23]), it follows that

$$\int_{\mathbb{R}^N} |u_n|^q = o_n(1)$$

as $n \to \infty$, for all $2 < q < 2^*$, which infers that

$$\int_{\mathbb{R}^N} G(u_n) = \int_{\mathbb{R}^N} u_n g(u_n) = o_n(1).$$

Thus

$$\int_{\mathbb{R}^N} H(\epsilon x, u_n) \le \frac{1}{2^*} \int_{\Omega \cup \{u_n \le a\}} (u_n^+)^{2^*} + \frac{V_0}{2k} \int_{\Omega^c \cap \{u_n \le a\}} u_n^2 + o_n(1) \tag{4.9}$$

and

$$\int_{\mathbb{R}^N} u_n h(\epsilon x, u_n) \le \int_{\Omega \cup \{u_n \le a\}} (u_n^+)^{2^*} + \frac{V_0}{k} \int_{\Omega^c \cap \{u_n \le a\}} u_n^2 + o_n(1). \tag{4.10}$$

This, combining $0 \in \partial I_{\epsilon_n}(u_n)$, we conclude that

$$\|u_n\|^2 - \frac{V_0}{k} \int_{\Omega^c \cap \{u_n \le a\}} u_n^2 + o_n(1) = \int_{\Omega \cup \{u_n \le a\}} (u_n^+)^{2^*}. \tag{4.11}$$

Set $m \ge 0$ be such that

$$\|u_n\|^2 - \frac{V_0}{k} \int_{\Omega^c \cap \{u_n \le a\}} u_n^2 \to m.$$

It is easily seen that $m > 0$. Otherwise, we can derive $u_n \to 0$, which is a contradiction to $c_{V_0} > 0$. By (4.11), it follows that

$$\int_{\Omega \cup \{u_n \le a\}} (u_n^+)^{2^*} \to m.$$

From the fact that $I_\epsilon(u_n) \to c_{V_0}$ and (4.9), one has

$$m \leq N c_{V_0}$$

and thus $m > 0$. Since

$$\|u_n\|^2 - \frac{V_0}{k} \int_{\Omega^c \cap \{u_n \leq a\}} u_n^2 \geq S \left(\int_{\Omega \cup \{u_n \leq a\}} (u_n^+)^{2^*} \right)^{\frac{2}{2^*}},$$

passing to the limit in the above inequality, it follows that

$$m \geq S m^{\frac{2}{2^*}},$$

which deduces that

$$c_{V_0} \geq \frac{1}{N} S^{\frac{N}{2}},$$

which is impossible. Hence (4.8) is true, and along to a subseuence

$$v_n := u_n(\cdot + \tilde{y}_n) \rightharpoonup v \neq 0 \text{ in } H^1(\mathbb{R}^N).$$

In the following, we take $t_n > 0$ such that $t_n v_n \in \mathscr{N}_{V_0}$. It follows from Lemma 4.1 that

$$c_{V_0} \leq I_{V_0}(t_n v_n) \leq \max_{t \geq 0} I_{\epsilon_n}(t u_n) = I_{\epsilon_n}(u_n) = c_{V_0} + o_n(1),$$

which means $I_{V_0}(t_n v_n) \to c_{V_0}$, thus $t_n v_n \not\to 0$ in $H^1(\mathbb{R}^N)$. Because $\{v_n\}$ and $\{t_n v_n\}$ are bounded in $H^1(\mathbb{R}^N)$ and $t_n v_n \not\to 0$ in $H^1(\mathbb{R}^N)$, the suquence $\{t_n\}$ is bounded. Passing to a subsequence if necessary, $t_n \to t_0 \geq 0$, for some t_0 independent of ϵ, a and δ. For ϵ, a and δ small enough, if $t_0 = 0$, we have $t_n v_n \to 0$ in $H^1(\mathbb{R}^N)$, which cannot occur. Therefore $t_0 > 0$, and $\{t_n v_n\}$ verifies

$$I_{V_0}(t_n v_n) \to c_{V_0}, \quad t_n v_n \rightharpoonup t_0 v \neq 0 \ \text{ in } \ H^1(\mathbb{R}^N).$$

It follows from Lemma 4.3 that $t_n v_n \to t_0 v$, or equivalently, $v_n \to v$ in $H^1(\mathbb{R}^N)$, with $v \neq 0$, which shows the first part results of this lemma. What is left is to show $\epsilon_n \tilde{y}_n \to y_0 \in \tilde{\Lambda}$. Set $y_n := \epsilon_n \tilde{y}_n$ and we assert that $\{y_n\}$ has a bounded subsequence. Indeed, if this were false, then $\{y_n\} \to \infty$. Take $R > 0$ such that $\Omega \subset B_R(0)$. We assume that $|y_n| > 2R$, for any $\hat{x} \in B_{R/\epsilon_n}(0)$, then $|\epsilon_n \hat{x} + y_n| \geq |y_n| - |\epsilon_n \hat{x}| > R$. Put

$$\eta_R(x) = \begin{cases} 0, & \text{if } |x| \leq R, \\ 1, & \text{if } |x| \leq 2R, \end{cases}$$

and $|\nabla \eta_\rho(x)| \leq C R^{-1}$ for all $x \in \mathbb{R}^N$. Applying $0 \in \partial I_{\epsilon_n}(u_n)$, we have

$$\begin{aligned} V_0 \left(1 - \frac{1}{k}\right) \int_{\mathbb{R}^N} v_n^2 \eta_R &\leq \int_{\mathbb{R}^N} \left[|\nabla v_n|^2 + \left(V(x) - \frac{V_0}{k} \right) v_n^2 \right] \eta_R \\ &= - \int_{\mathbb{R}^N} v_n \nabla \eta_R \nabla v_n + \int_{\mathbb{R}^N} h(\epsilon \hat{x} + y_n, v_n) \eta_R + o_n(1). \end{aligned}$$

It follows from (h4) that

$$V_0 \left(1 - \frac{1}{k}\right) \int_{\mathbb{R}^N} v_n^2 \eta_R \leq \frac{C}{R} \|v_n\|^2 + o_n(1),$$

which is impossible as R sufficiently large. This means that $\{y_n\}$ has a bounded subsequence. Hence, up to a subsequence we derive

$$y_n \to y_0 \in \mathbb{R}^N.$$

If $y_0 \notin \bar{\Omega}$, we can proceed as above and infer that $v_n \to 0$. Therefore, $y_0 \in \bar{\Omega}$.

In order to show that $V(y_0) = V_0$, we proceed by contradiction, and assume that $V(y_0) > V_0$. Once that $t_n v_n \to t_0 v$ in $H^1(\mathbb{R}^N, \mathbb{R})$, from Fatou's Lemma and the invariance of $\mathbb{R}^N$, we have

$$\begin{aligned} c_{V_0} &= I_{V_0}(t_0 v) \\ &< \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla(t_0 v)|^2 + V(y_0)|t_0 v|^2) - (1+\delta)\int_{\mathbb{R}^N} G(t_0 v) - \frac{1}{2^*}\int_{\mathbb{R}^N}(t_0 v)^{2^*} \\ &\le \liminf_{n\to\infty}\left[\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla(t_n v_n)|^2 + V(\epsilon_n\hat{x} + \tilde{y}_n)|t_n v_n|^2)\int_{\mathbb{R}^N} G(t_n v_n) \right. \\ &\quad \left. -\frac{1}{2^*}\int_{\mathbb{R}^N}(t_n v_n)^{2^*}\right] \\ &\le \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \\ &\le \liminf_{n\to\infty} I_{\epsilon_n}(u_n) \\ &= c_{V_0}, \end{aligned}$$

which is absurd. Hence $V(y_0) = V_0$ and $y_0 \in \bar{\Omega}$. The condition (V1) means that $y_0 \notin \partial\Omega$, which deduces that $y_0 \in \tilde{\Lambda}$. The proof is completed. □

The following Lemma comes from Lemma 3.7 in [1] and it can show that u_ϵ obtained in Theorem 3.1 is a solution of the original problem (1.2).

Lemma 4.5. *Let $\epsilon_n \to 0^+, \delta_n \to 0^+$, $u_n := u_{\epsilon_n, a_n, \delta_n}$ be a solution of (3.2) with $I_{\epsilon_n}(u_n) \to c_{V_0}$. Then $u_n \in L^\infty(\mathbb{R}^N)$ and given $\tau > 0$, there are $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|u_n|_{L^\infty(B_R(\tilde{y}_n)^c)} < \tau \quad \text{for all } n \ge n_0,$$

where $\{\tilde{y}_n\}$ is the sequence given in Lemma 4.4.

Proof of Theorem 1.1. Our first goal is to show that there exist $\hat{\epsilon} > 0$, $\hat{\delta} > 0$ and $\hat{a} > 0$ such that for all $\epsilon \in (0, \hat{\epsilon})$, $\delta \in (0, \hat{\delta})$, and $a \in (0, \hat{a})$, the solution $u_{\epsilon,\delta,a}$ of problem (3.2), given by Theorem 3.1, satisfies the inequality

$$|u_{\epsilon,\delta,a}|_{L^\infty(\mathbb{R}^N\setminus\Omega_\epsilon)} < b. \tag{4.12}$$

Assume that (4.12) were false, then for some sequence $\epsilon_n \to 0^+$, $\delta_n \to 0^+$ and $a_n \to 0^+$, the sequence $u_n = u_{\epsilon_n,\delta_n,a_n}$ satisfies

$$|u_n|_{L^\infty(\mathbb{R}^N\setminus\Omega_{\epsilon_n})} \ge b. \tag{4.13}$$

From Lemmas 4.4-4.5 and $I_{\epsilon_n}(u_n) \to c_{V_0}$, it follows that there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $\epsilon_n \tilde{y}_n \to y_0 \in \tilde{\Lambda}$. Putting $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Omega$, one has

$$B_{r/\epsilon_n}(y_0/\epsilon_n) = \frac{1}{\epsilon_n} B_r(y_0) \subset \Omega_{\epsilon_n}.$$

Furthermore, for any $\hat{x} \in B_{r/\epsilon_n}(\tilde{y}_n)$, there holds

$$|\hat{x} - \frac{y_0}{\epsilon_n}| \le |\hat{x} - \tilde{y}_n| + |\tilde{y}_n - \frac{y_0}{\epsilon_n}| < \frac{1}{\epsilon_n}(r + o_n(1)) < \frac{2r}{\epsilon_n}$$

for n large enough. For these values of n, $B_{r/\epsilon_n}(\tilde{y}_n) \subset \Omega_{\epsilon_n}$, from which it follows that $\mathbb{R}^N \setminus \Omega_{\epsilon_n} \subset \mathbb{R}^N \setminus B_{r/\epsilon_n}(\tilde{y}_n)$. On the other hand, applying Lemma 4.5 with $\tau = b$, there exists n_0 such that $r/\epsilon_n > R$ and

$$|u_n|_{L^\infty(\mathbb{R}^N \setminus \Omega_{\epsilon_n})} \le |u_n|_{L^\infty(\mathbb{R}^N \setminus B_{r/\epsilon_n}(\tilde{y}_n))} \le |u_n|_{L^\infty(\mathbb{R}^N \setminus B_R(\tilde{y}_n))} < b, \ \forall n \ge n_0,$$

which contradicts to (4.13) and this proves (4.12).

Then from the definition of h and (4.12), we derive that $h(\epsilon x, u(x)) \equiv f(u(x)) + u^{2^*-1}$, which proves that $u = u_{\epsilon,\delta,a}$ is a solution of problem (1.2). In order to study the behavior of the maximum points of $\{u_n\}$, it follows from (H1) that there is $\tau > 0$ such that

$$h(\epsilon x, t)t^2 \le \frac{V_0}{2}t^2 \tag{4.14}$$

for all $x \in \mathbb{R}^N$ and $t \le \tau$. According to Lemma 4.5, there exist $R > 0$ and $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))^c} < \tau. \tag{4.15}$$

Passing to a subsequence if necessary, we may suppose that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))^c} \ge \tau. \tag{4.16}$$

Otherwise, we can choose a subsequence such that $|u_n|_{L^\infty(\mathbb{R}^N)} < \tau$. Therefore, recalling the fact that u_n is a solution of (1.2) and (4.14), one derives

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_0|u_n|^2) \le \|u_n\|^2_{\epsilon_n} \le \int_{\mathbb{R}^N} h(\epsilon_n x, u_n)u_n^2 \le \frac{V_0}{2}\int_{\mathbb{R}^N} |u_n|^2,$$

which means that $\|u_n\| = 0$, and this makes no sense. Consequently, (4.16) holds.

From (4.15) and (4.16), it follows that the maximum point $\hat{y}_n \in \mathbb{R}^N$ of u_n belongs to $B_R(\tilde{y}_n)$, thus, $\hat{y}_n = \tilde{y}_n + \tilde{x}_n$ with $|\tilde{x}_n| \le R$. This means that

$$\epsilon_n \hat{y}_n = \epsilon_n \tilde{y}_n + \epsilon_n \tilde{x}_n \to y_0 \in \hat{\Lambda}.$$

Consequently,

$$\lim_{n\to\infty} V(\epsilon_n \hat{y}_n) = V(y_0) = V_0,$$

which completes our proof. □

References

[1] C. O. Alves, G. F. Figueiredo and R. G. Nascimento, *On existence and concentration of solutions for an elliptic problem with discontinuous nonlinearity via penalization method*, Z. Angew. Math. Phys., 2014, 65(1), 19–40.

[2] C. O. Alves, J. V. Gonvalves and J. A. Santos, *Strongly nonlinear multivalued elliptic equations on a bounded domain*, J. Global Optim., 2014, 58(3), 565–593.

[3] C. O. Alves, A. Santos and R. Nemer, *Multiple solutions for a problem with discontinuous nonlinearity,* Annali di Matematica Pura ed Applicata , 2018, 197(3), 883–903.

[4] X, Cabré and A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations*, J. Funct. Anal., 2006, 238(2), 709–733.

[5] K. Chang, *On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms*, Sci. Sinica, 1978, 21, 139–158.

[6] K. Chang, *Variational methods for nondifferentiabe functionals and their applications to partial differential inequalities*, J. Math. Anal. Appl., 1981, 80, 102–129.

[7] K. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities*, Comm. Pure Appl. Math., 1978, 33, 139–158.

[8] K. C. Chang, S. Shao, et al., *Nonsmooth critical point theory and applications to the spectral graph theory*, Science China Mathematics, 2021, 64(1), 1–32.

[9] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wley, New York, 1983.

[10] A. Clop, A. Gentile and A. Passarelli di Napoli, *Higher differentiability results for solutions to a class of non-homogeneous elliptic problems under subquadratic growth conditions*, Bulletin of Mathematical Sciences, 2023, 13(2), 2350008.

[11] J. N. Corvellec, V. V. Motreanu and C. Saccon, *Doubly resonant semilinear elliptic problems via nonsmooth critical point theory*, J. Differential Equations, 2010, 248(8), 2064–2091.

[12] C. Chu, X. Liu and Y. Xie, *Sign-changing solutions for semilinear elliptic equation with variable exponent*, J. Math. Anal. Appl., 2022, 507(1), 125748.

[13] M. del Pino and P. L. Felmer, *Local Mountain Pass for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differ. Equ., 1996, 4, 121–137.

[14] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., 1974, 47, 324–353.

[15] L. Gasiński and N. Papageorgiou, *Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems*, Chapman and Hall/CRC Press, Boca Raton, FL, 2005.

[16] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[17] M. Grossi, A. Pistoia and J. Wei, *Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory*, Calc. Var. Partial Differential Equations, 2000, 11(2), 143–175.

[18] A. Iannizzotto and N. S. Papageorgiou, *Existence of three nontrivial solutions for nonlinear Neumann hemivariational inequalities*, Nonlinear Anal., 2009, 70(9), 3285–3297.

[19] C. Ji and V. D. Radulescu, *Concentration phenomena for nonlinear magnetic Schrödinger equations with critical growth*, Israel J. Math., 2021, 241, 465–500.

[20] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.

[21] Y. Li, B. Zhang and X. Han, *Existence and concentration behavior of positive solutions to Schrödinger-Poisson-Slater equations*, Advances in Nonlinear Analysis, 2023, 12, 20220293.

[22] X. Lin and X. Tang, *On concave perturbations of a periodic elliptic problem in R2 involving critical exponential growth*, Advances in Nonlinear Analysis, 2023, 12, 169–181.

[23] P. L. Lions, *The concentration compactness principle in the calculus of variations. The locally compact case. Part II*, Ann. Inst. H. Poincaré, 1984, 1, 223–283.

[24] Z. Liu, V. D. Radulescu and Z. Q. Yuan, *Concentration of solutions for fractional Kirchhoffff equations with discontinuous reaction*, Z. Angew. Math. Phys., 2022, 73, 211.

[25] C. B. Morrey, *Multiple Integrals in Calculus of Variations*, Springer Verlag, Berlin, 1966.

[26] D. Naimen, *Concentration profile, energy, and weak limits of radial solutions to semilinear elliptic equations with Trudinger-Moser critical nonlinearities*, Calc. Var. Partial Differential Equations, 2021, 60(2), 1–54.

[27] C. Wang and J. Su, *The semilinear elliptic equations with double weighted critical exponents*, Journal of Mathematical Physics, 2022, 63(4), 041505.

[28] Z. Q. Yuan and Y. Wang, *Positive solutions to discontinuous quasilinear problems with critical exponents*, Monatshefte für Mathmatic, 2022, 199(3), 695–713.

[29] Z. Q. Yuan and J. S. Yu, *Existence of solutions for Dirichlet elliptic problems with discontinuous nonlinearity,* Nonlinear Anal., 2020, 197, 111848.

[30] Y. Yue, T. Yu and Z. Bai, *Infinitely many nonnegative solutions for a fractional differential inclusion with oscillatory potential*, Appl. Math. Lett., 2019, 88, 64–72.

[31] J. Zhang, W. Zhang and V. D. Radulescu, *Double phase problems with competing potentials: Concentration and multiplication of ground states*, Math. Z., 2022, 301, 4037–4078.