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SOLUTION OF CERTAIN PERIODIC BOUNDARY VALUE PROBLEM IN RELATIONAL METRIC SPACE VIA RELATIONAL ALMOST φ-CONTRACTIONS*

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Abstract This article is comprised of outcomes on fixed points for almost Matkowski contractions via locally $\mathcal{J}$-transitive relations. Our outcomes sharpen, unify, enrich and improve many fixed point theorems of the existing literatures. Several examples are furnished to demonstrate the credibility of our results. By implementing our outcomes, we ascertain a unique solution for certain boundary value problem.

Keywords Fixed points, binary relations, almost contractions, boundary value problems.

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1. Introduction

A significant and essential result of metric fixed point theory is classical Banach contraction principle (abbreviated as, BCP). In fact, BCP ensures the existence of a unique fixed point of a contraction in a complete metric space. Additionally, this result provides an iterative method for calculating the unique fixed point. Throughout the foregoing century, BCP has been expanded and generalized by numerous authors. A common generalisation of this finding is to expand the standard contraction to φ-contraction by means of a proper auxiliary function $\varphi : [0,\infty) \to [0,\infty)$. A variety of generalisations has been developed through effectively modifying φ,

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resulting in a huge number of articles on this topic. Matkowski [18] invented a new class of φ-contraction, that incorporated the concept of comparison functions which has been further studied in [1, 20] besides several others.

One of the noted generalizations is almost contraction, which is introduced by Berinde [13] and refined by Păcurar [21]. A map $\mathcal{J}$ on a metric space $(\mathbb{U}, \sigma)$ constitutes an almost contraction if $\exists\ c \in (0,1)$ and a nonnegative real number L allow for

$$\sigma(\mathcal{J}u, \mathcal{J}v) \leq c\sigma(u,v) + L\min\{\sigma(u, \mathcal{J}v), \sigma(v, \mathcal{J}u)\}, \quad \forall\ u, v \in \mathbb{U}.$$

Alam and Imdad [4] investigated an innovative variation of BCP in relational metric space. Nowadays, multiple findings regarding fixed points are proven in relational metric spaces through the use of various contractivity conditions, e.g., [2, 3, 5–7, 9–11, 14–17, 23]. In these outcomes, the contraction condition requires to be satisfied with regard to the elements that are linked through underlying relation. Due to this fact, relation-theoretic contractions continue to be marginally weaker than ordinary contractions.

The paper aims to establish the fixed point findings pursuant to almost φ-contractions employing a comparison function in the setup of metric space concerning locally $\mathcal{J}$-transitive. To exhibit utility of our outcomes, we provided two illustrative examples. Our findings are implemented to ascertain the unique solution of a BVP (i.e., boundary value problems).

2. Preliminaries

From now on, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ will symbolise, respectively, the set of: positive integers, nonnegative integers and real numbers. Also, $F(\mathcal{J})$ will denote the set of fixed points of mapping $\mathcal{J}$. Any subset of $\mathbb{U}^2$ is called a binary relation (or simply, a relation) on the set $\mathbb{U}$. In this instance, $\mathbb{U}$ represents a set, σ denotes a metric on $\mathbb{U}$, Λ continues to be a relation on $\mathbb{U}$, and $\mathcal{J} : \mathbb{U} \to \mathbb{U}$ represents a function. Λ induces a new relation $\Lambda^{-1} := \{(p,q) \in \mathbb{U}^2 : (q,p) \in \Lambda\}$, which is named as the transpose of Λ. The relation $\Lambda^s := \Lambda \cup \Lambda^{-1}$ constitutes symmetric closure of Λ. For a subset $\mathbb{E} \subseteq \mathbb{U}$, a relation $\Lambda|_{\mathbb{E}} := \Lambda \cap \mathbb{E}^2$ constitutes the restriction of Λ on $\mathbb{E}$.

Definition 2.1. [4] A pair $p, q \in \mathbb{U}$ constitutes Λ-comparative if $(p,q) \in \Lambda$ or $(q,p) \in \Lambda$. Usually, it is denoted by $[p,q] \in \Lambda$.

Remark 2.1. [4] $(p,q) \in \Lambda^s \iff [p,q] \in \Lambda$.

Definition 2.2. [4] A sequence $\{u_n\} \subset \mathbb{U}$ verifying $(u_n, u_{n+1}) \in \Lambda$, $\forall\ n \in \mathbb{N}$, is called Λ-preserving.

Definition 2.3. [4] We speak of Λ as $\mathcal{J}$-closed if

$$p, q \in \mathbb{U}; (p,q) \in \Lambda \implies (\mathcal{J}p, \mathcal{J}q) \in \Lambda.$$

Definition 2.4. [5] $(\mathbb{U}, \sigma)$ is known as Λ-complete if any Λ-preserving Cauchy sequence in $\mathbb{U}$ converges.

Definition 2.5. [5] $\mathcal{J}$ is referred as Λ-continuous at $w \in \mathbb{U}$ if for any Λ-preserving sequence $\{u_n\} \subset \mathbb{U}$ with $u_n \xrightarrow{\sigma} w$,

$$\mathcal{J}(u_n) \xrightarrow{\sigma} \mathcal{J}(w).$$

Further, any Λ-continuous function at every point of Λ constitutes Λ-continuous.

Definition 2.6. [4] Λ constitutes σ-self-closed if each Λ-preserving convergent sequence in $\mathbb{U}$ admits a subsequence, whose each term is Λ-comparative with the limit.

Definition 2.7. [22] We speak of a subset $\mathbb{V} \subseteq \mathbb{U}$ as Λ-directed if for any $p, q \in \mathbb{V}$, $\exists\, z \in \mathbb{U}$ with $(p, z) \in \Lambda$ and $(q, z) \in \Lambda$.

Definition 2.8. [6] Λ is referred as locally $\mathcal{J}$-transitive if for each Λ-preserving sequence $\{u_n\} \subset \mathcal{J}(\mathbb{U})$ (with range $\mathbb{E} = \{u_n : n \in \mathbb{N}\}$), $\Lambda|_{\mathbb{E}}$ is transitive.

Definition 2.9. [12] A function $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if

(i) φ is increasing and

(ii) $\lim\limits_{n\to\infty} \varphi^n(t) = 0,\ \forall\, t > 0.$

We'll symbolize the family of comparison functions by $\mathcal{F}_{\text{com}}$.

Proposition 2.1. [12] *Every $\varphi \in \mathcal{F}_{\text{com}}$ verifies that $\varphi(t) < t$, $\forall\, t > 0$ and $\varphi(0) = 0$.*

Proposition 2.2. [6] *$\mathcal{J}$-closedness implies $\mathcal{J}^n$-closedness, for each $n \in \mathbb{N}_0$.*

The following fact can be proposed taking advantage of symmetry of metric σ.

Proposition 2.3. *For any $\varphi \in \mathcal{F}_{\text{com}}$ and a nonnegative real number L, the following predictions are equivalent:*

(I) $\sigma(\mathcal{J}u, \mathcal{J}v) \leq \varphi(\sigma(u, v)) + L \min\{\sigma(u, \mathcal{J}v), \sigma(v, \mathcal{J}u)\},\ \forall\, u, v \in \mathbb{U}$ *with* $(u, v) \in \Lambda$,

(II) $\sigma(\mathcal{J}u, \mathcal{J}v) \leq \varphi(\sigma(u, v)) + L \min\{\sigma(u, \mathcal{J}v), \sigma(v, \mathcal{J}u)\},\ \forall\, u, v \in \mathbb{U}$ *with* $[u, v] \in \Lambda$.

3. Main results

The existence of fixed point for an almost Matkowski contraction map in relational metric space is assured by the following result.

Theorem 3.1. *Assuming $(\mathbb{U}, \sigma)$ serves as metric space, Λ as a relation on $\mathbb{U}$ while $\mathcal{J} : \mathbb{U} \to \mathbb{U}$ as a mapping. Also,*

(i) $(\mathbb{U}, \sigma)$ *is Λ-complete,*

(ii) $\exists\, u_0 \in \mathbb{U}$ *satisfying* $(u_0, \mathcal{J}u_0) \in \Lambda$,

(iii) *Λ remains locally $\mathcal{J}$-transitive and $\mathcal{J}$-closed,*

(iv) *$\mathbb{U}$ is Λ-continuous, or Λ continues σ-self-closed,*

(v) *$\exists\, \varphi \in \mathcal{F}_{\text{com}}$ and a nonnegative real number L verifying*

$$\sigma(\mathcal{J}u, \mathcal{J}v) \leq \varphi(\sigma(u, v)) + L \min\{\sigma(u, \mathcal{J}v), \sigma(v, \mathcal{J}u)\},\ \forall\, (u, v) \in \Lambda.$$

Then, $\mathcal{J}$ owns a fixed point.

Proof. Starting from $u_0 \in \mathbb{U}$, one can define a sequence $\{u_n\} \subset \mathbb{U}$ verifying

$$u_n := \mathcal{J}^n(u_0) = \mathcal{J}(u_{n-1}), \quad \forall\, n \in \mathbb{N}. \tag{3.1}$$

By $\mathcal{J}$-closedness of Λ and Proposition 2.2, we attain

$$(\mathcal{J}^n u_0, \mathcal{J}^{n+1} u_0) \in \Lambda,$$

so that

$$(u_n, u_{n+1}) \in \Lambda, \quad \forall\, n \in \mathbb{N}_0. \tag{3.2}$$

It follows that $\{u_n\}$ remains Λ-preserving sequence.

Define $\sigma_n := \sigma(u_n, u_{n+1})$. If $\sigma_{n_0} = \sigma(u_{n_0}, u_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by (3.1), one finds $\mathcal{J}(u_{n_0}) = u_{n_0}$. Thus, u_{n_0} serves as a fixed point of $\mathcal{J}$ and hence the task is accomplished.

When $\sigma_n > 0$, $\forall\, n \in \mathbb{N}_0$, one uses (v), (3.1) and (3.2) to get

$$\begin{aligned}\sigma_n &= \sigma(u_n, u_{n+1})\\ &= \sigma(\mathcal{J}u_{n-1}, \mathcal{J}u_n)\\ &\le \varphi(\sigma(u_{n-1}, u_n)) + L\min\{\sigma(u_n, \mathcal{J}u_{n-1}), \sigma(u_{n-1}, \mathcal{J}u_n)\}\end{aligned}$$

which again employing (3.1) becomes

$$\sigma_n \le \varphi(\sigma_{n-1}), \quad \forall\, n \in \mathbb{N}_0. \tag{3.3}$$

By monotonicity of φ in (3.3), we attain

$$\sigma_n \le \varphi(\sigma_{n-1}) \le \varphi^2(\sigma_{n-2}) \le \cdots \le \varphi^n(\sigma_0)$$

so that

$$\sigma_n \le \varphi^n(\sigma_0), \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

Letting $n \to \infty$ in (3.4) and employing property (ii) of φ, we attain

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.5}$$

Choose $\varepsilon > 0$. Then owing to (3.5), we can find $n \in \mathbb{N}_0$ allow for

$$\sigma_n < \varepsilon - \varphi(\varepsilon). \tag{3.6}$$

Now, we seek to verify that $\{u_n\}$ is Cauchy. Implementing monotonicity of φ, (3.3) and (3.6), we attain

$$\begin{aligned}\sigma(u_n, u_{n+2}) &\le \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2})\\ &= \sigma_n + \sigma_{n+1}\\ &\le \sigma_n + \varphi(\sigma_n)\\ &< \varepsilon - \varphi(\varepsilon) + \varphi[\varepsilon - \varphi(\varepsilon)]\\ &\le \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon)\\ &= \varepsilon.\end{aligned}$$

By monotonicity of φ, locally $\mathcal{J}$-transitivity of Λ, prediction (v), (3.1) and (3.6), we find

$$\sigma(u_n, u_{n+3}) \le \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+3})$$

$$
\begin{aligned}
&= \sigma_n + \sigma(\mathcal{J}u_n, \mathcal{J}u_{n+2}) \\
&< \varepsilon - \varphi(\varepsilon) + \varphi(\sigma(u_n, u_{n+2})) \\
&\leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) \\
&= \varepsilon.
\end{aligned}
$$

Using induction, one finds

$$\sigma(u_n, u_{n+p}) < \varepsilon, \quad \forall\, p \in \mathbb{N}.$$

It turns out that $\{u_n\}$ continues to be Cauchy and Λ-preserving. Accordingly by virtue of Λ-completeness of $\mathbb{U}$ $\exists$ $w \in \mathbb{U}$, confirming $u_n \xrightarrow{\sigma} w$.

Finally, we will use prediction (iv) to prove that $w \in F(\mathcal{J})$. Assuming that $\mathcal{J}$ is Λ-continuous. Because of Λ-preserving property of $\{u_n\}$, $u_n \xrightarrow{\sigma} w$ and Λ-continuity of $\mathcal{J}$, we conclude that $u_{n+1} = \mathcal{J}(u_n) \xrightarrow{\sigma} \mathcal{J}(w)$. Due to limit's uniqueness property, one obtains $\mathcal{J}(w) = w$. Considering that Λ is σ-self-closed. Then $\exists$ a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ satisfying $[u_{n_k}, w] \in \Lambda$, $\forall\, k \in \mathbb{N}$. By utilizing (v), Proposition 2.3 and $[u_{n_k}, w] \in \Lambda$, one obtains

$$
\begin{aligned}
\sigma(u_{n_k+1}, \mathcal{J}w) &= \sigma(\mathcal{J}u_{n_k}, \mathcal{J}w) \\
&\leq \varphi(\sigma(u_{n_k}, w)) + L \min\{\sigma(w, \mathcal{J}\vartheta_{n_k}), \sigma(\vartheta_{n_k}, \mathcal{J}w)\} \\
&= \varphi(\sigma(u_{n_k}, w)) + L \min\{\sigma(w, u_{n_k+1}), \sigma(\vartheta_{n_k}, \mathcal{J}w)\}.
\end{aligned}
$$

Using Proposition 2.1 (whether $\sigma(u_{n_k}, w)$ is zero or non-zero) and $u_{n_k} \xrightarrow{\sigma} w$, one gets

$$\sigma(u_{n_k+1}, \mathcal{J}w) \to 0 \text{ as } k \to \infty$$

yielding thereby $u_{n_k+1} \xrightarrow{\sigma} \mathcal{J}(w)$. Due to limit's uniqueness property, one gets $\mathcal{J}(w) = w$. Therefore, $w \in F(\mathcal{J})$ in both scenarios. □

The following findings on uniqueness of fixed point is obtained by modifying the almost contractivity condition together with two additional hypotheses.

Theorem 3.2. *As a supplement to the presumptions* (i)-(iv) *of Theorem 3.1, if the following circumstances are met:*

(vi) $\exists$ $\varphi_0 \in \mathcal{F}_{\text{com}}$ *and a nonnegative real number* L_0 *verifying*

$$\sigma(\mathcal{J}u, \mathcal{J}v) \leq \varphi_0(\sigma(u, v)) + L_0 \min\{\sigma(u, \mathcal{J}u), \sigma(v, \mathcal{J}v)\}\ \forall\ (u, v) \in \Lambda,$$

(vii) $\mathcal{J}(\mathbb{U})$ *is* Λ*-directed,*

then $\mathcal{J}$ *possesses a unique fixed point.*

Proof. Using the techniques similar to Theorem 3.1, one can conclude that $F(\mathcal{J}) \neq \emptyset$. Take $u, v \in F(\mathcal{J})$, *i.e.*,

$$\mathcal{J}(u) = u \text{ and } \mathcal{J}(v) = v. \tag{3.7}$$

As $u, v \in \mathcal{J}(\mathbb{U})$, by hypothesis (vii), $\exists$ $z \in \mathbb{U}$ with

$$(u, z) \in \Lambda \quad \text{and} \quad (v, z) \in \Lambda. \tag{3.8}$$

Denote $\mu_n := \sigma(u, \mathcal{J}^n z)$. Using (3.7), (3.8) and assumption (vi), one obtains

$$\begin{aligned}\mu_n &= \sigma(u, \mathcal{J}^n z)\\ &= \sigma(\mathcal{J}u, \mathcal{J}(\mathcal{J}^{n-1}z))\\ &\leq \varphi_0(\sigma(u, \mathcal{J}^{n-1}z)) + L_0 \min\{\sigma(u, \mathcal{J}u), \sigma(v, \mathcal{J}v)\}\\ &= \varphi_0(\mu_{n-1})\end{aligned}$$

so that

$$\mu_n \leq \varphi_0(\mu_{n-1}). \tag{3.9}$$

Using monotonicity of φ_0 in (3.9), we get

$$\mu_n \leq \varphi_0(\mu_{n-1}) \leq \varphi_0^2(\mu_{n-2}) \leq \cdots \leq \varphi_0^n(\mu_0)$$

so that

$$\mu_n \leq \varphi_0^n(\mu_0). \tag{3.10}$$

If $\mu_0 = 0$, then by Proposition 2.1, one concludes $\mu_n = 0$ yielding thereby $\lim_{n\to\infty} \mu_n = 0$. Otherwise, in case $\mu_0 > 0$, using the limit in (3.10) and the property of φ_0, we attain

$$\lim_{n\to\infty} \mu_n \leq \lim_{n\to\infty} \varphi_0^n(\mu_0) = 0.$$

Thus in each case, one has

$$\lim_{n\to\infty} \sigma(u, \mathcal{J}^n z) = 0. \tag{3.11}$$

Similarly, one can find

$$\lim_{n\to\infty} \sigma(v, \mathcal{J}^n z) = 0. \tag{3.12}$$

By using (3.11) and (3.12), one finds

$$\sigma(u, v) = \sigma(u, \mathcal{J}^n z) + \sigma(\mathcal{J}^n z, v) \to 0 \quad \text{as} \quad n \to \infty$$

that is, $u = v$. Therefore, $\mathcal{J}$ possesses a unique fixed point. □

4. Illustrative examples

With the objective of exhibiting Theorems 3.1 and 3.2, the next few examples are utilized.

Example 4.1. Consider $\mathbb{U} = [0, 1] \cup [4, 5]$ with Euclidean metric $\sigma(u, v) = |u - v|$. On $\mathbb{U}$, consider a relation $\Lambda := \{(u, v) \in \mathbb{U}^2 : u \geq v\}$. Then, $(\mathbb{U}, \sigma)$ being complete serves as Λ-complete. Let $\mathcal{J} : \mathbb{U} \to \mathbb{U}$ be a mapping defined by

$$\mathcal{J}(u) = \begin{cases} 0, & \text{if } u = 1,\\ 1/9, & \text{otherwise.}\end{cases}$$

Obviously, Λ is $\mathcal{J}$-closed and $\mathcal{J}$ is Λ-continuous. Let $\varphi(t) = t/9$. Then $\varphi \in \mathcal{F}_{\text{com}}$. Now, for any $L \geq 1/7$ and $\forall\ u, v \in \mathbb{U}$ verifying $(u, v) \in \Lambda$, one gets

$$\sigma(\mathcal{J}u, \mathcal{J}v) \leq \varphi(\sigma(u, v)) + L \min\{\sigma(u, \mathcal{J}v), \sigma(v, \mathcal{J}u)\}.$$

Therefore, the respective premises of Theorem 3.1. Moreover, all the premises of Theorem 3.2 hold. Consequently, $\mathcal{J}$ owns a unique fixed point: $w = 1/9$.

Example 4.2. Consider $\mathbb{U} = [0,4]$ with Euclidean metric $\sigma(u,v) = |u-v|$. On $\mathbb{U}$, define a relation $\Lambda = (0,4] \times (0,4]$. Clearly, the metric space $(\mathbb{U}, \sigma)$ being complete is Λ-complete. Let $\mathcal{J} : \mathbb{U} \to \mathbb{U}$ be a mapping defined by

$$\mathcal{J}(u) = \begin{cases} 1, & \text{if } 0 \le u < 3, \\ 4, & \text{if } 3 \le u \le 4. \end{cases}$$

Then, Λ is $\mathcal{J}$-closed. Also, $\mathcal{J}$ is Λ-continuous. Also, $\mathcal{J}$ satisfies the contractivity condition (v) for an arbitrary $\varphi \in \mathcal{F}_{\text{com}}$ and for an arbitrary $L \ge 0$. Hence, by Theorem 3.1, $F(\mathcal{J}) \neq \emptyset$. Indeed, here $\mathcal{J}$ owns two fixed points, namely: $w = 1$ and $w = 4$.

The set $\mathcal{J}(\mathbb{U}) = \{1, 4\}$ is not Λ-directed. Thus, this example does not allow for the practical use of Theorem 3.2, which guarantees the uniqueness of fixed point.

5. Applications

Over the course of this section, the existence and uniqueness of the solution of the following BVP will be addressed.

$$\begin{cases} \varsigma'(\delta) = \Upsilon(\delta, \varsigma(\delta)), & \delta \in [0,a], \\ \varsigma(0) = \varsigma(a) \end{cases} \tag{5.1}$$

where the function $\Upsilon : [0,a] \times \mathbb{R} \to \mathbb{R}$ remains continuous. In the sequel, $\mathcal{C}[0,a]$ and $\mathcal{C}'[0,a]$ respectively, refer to denote the class of: real valued continuous functions on the interval $[0,a]$ and real valued differentiable continuous functions on $[0,a]$. Recall that a function $\tilde{\varsigma} \in \mathcal{C}'[0,a]$ constitutes a lower solution of (5.1) (c.f. [19]) if

$$\begin{cases} \tilde{\varsigma}'(\delta) \le \Upsilon(\delta, \tilde{\varsigma}(\delta)), & \delta \in [0,a], \\ \tilde{\varsigma}(0) \le \tilde{\varsigma}(a). \end{cases}$$

Now, we present the existence cum uniqueness theorems in order to deliver a solution for Problem (5.1).

Theorem 5.1. *Along with the Problem* (5.1)*, if* $\exists\ \lambda > 0$ *and* $\exists\ \varphi \in \mathcal{F}_{\text{com}}$ *satisfying for* $p \le q$ $(p, q \in \mathbb{R})$ *that*

$$0 \le [\Upsilon(\delta, q) + \lambda q] - [\Upsilon(\delta, p) + \lambda p] \le \lambda \varphi(q - p), \tag{5.2}$$

then the existence of a lower solution of Problem (5.1) *ensures the existence of the unique solution of the problem.*

Proof. A version of Problem (5.1) is

$$\begin{cases} \varsigma'(\delta) + \lambda\varsigma(\delta) = \Upsilon(\delta, \varsigma(\delta)) + \lambda\varsigma(\delta), & \forall\ \delta \in [0,a], \\ \varsigma(0) = \varsigma(a). \end{cases} \tag{5.3}$$

Evidently (5.3) equates to the following integral equation:

$$\varsigma(\delta) = \int_0^a G(\delta, \xi)[\Upsilon(\xi, \varsigma(\xi)) + \lambda\varsigma(\xi)]d\xi \tag{5.4}$$

where $G(\delta,\xi)$ remains Green function given by

$$G(\delta,\xi)=\begin{cases}\dfrac{e^{\lambda(a+\xi-\delta)}}{e^{\lambda a}-1}, & 0\le\xi<\delta\le a,\\ \dfrac{e^{\lambda(\xi-\delta)}}{e^{\lambda a}-1}, & 0\le\delta<\xi\le a.\end{cases}$$

Set $\mathbb{U}:=\mathcal{C}[0,a]$. Introduce a function $\mathcal{J}:\mathbb{U}\to\mathbb{U}$ by

$$(\mathcal{J}\varsigma)(\delta)=\int_0^a G(\delta,\xi)[\Upsilon(\xi,\varsigma(\xi))+\lambda\varsigma(\xi)]d\xi,\quad \forall\ \delta\in[0,a]. \tag{5.5}$$

On $\mathbb{U}$, define a metric σ by

$$\sigma(\varsigma,\omega)=\sup_{\delta\in[0,a]}|\varsigma(\delta)-\omega(\delta)|,\quad \forall\ \varsigma,\omega\in\mathbb{U}. \tag{5.6}$$

On $\mathbb{U}$, define a relation Λ by

$$\Lambda=\{(\varsigma,\omega)\in\mathbb{U}\times\mathbb{U}:\varsigma(\delta)\le\omega(\delta),\ \forall\ \delta\in[0,a]\}. \tag{5.7}$$

It is now necessary to examine all the conditions of Theorem 3.2.
(i) Clearly, $(\mathbb{U},\sigma)$ is Λ-complete metric space.
(ii) If $\tilde\varsigma\in\mathcal{C}'[0,a]$ remains a lower solution of (5.1), then we have

$$\tilde\varsigma'(\delta)+\lambda\tilde\varsigma(\delta)\le\Upsilon(\delta,\tilde\varsigma(\delta))+\lambda\tilde\varsigma(\delta),\quad\forall\ \delta\in[0,a].$$

Taking the multiplication with e^{ks}, one gets

$$(\tilde\varsigma(\delta)e^{k\delta})'\le[\Upsilon(\delta,\tilde\varsigma(\delta))+\lambda\tilde\varsigma(\delta)]e^{ks},\quad\forall\ \delta\in[0,a]$$

implying thereby

$$\tilde\varsigma(\delta)e^{k\delta}\le\tilde\varsigma(0)+\int_0^\delta[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]e^{\lambda\xi}d\xi,\quad\forall\ \delta\in[0,a]. \tag{5.8}$$

Owing to $\tilde\varsigma(0)\le\tilde\varsigma(a)$, one gets

$$\tilde\varsigma(0)e^{\lambda a}\le\tilde\varsigma(a)e^{\lambda a}\le\tilde\varsigma(0)+\int_0^a[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]e^{\lambda\xi}d\xi$$

so that

$$\tilde\varsigma(0)\le\int_0^a\frac{e^{\lambda\xi}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi. \tag{5.9}$$

By (5.8) and (5.9), one finds

$$\begin{aligned}\tilde\varsigma(\delta)e^{k\delta}&\le\int_0^a\frac{e^{\lambda\xi}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi+\int_0^\delta e^{\lambda\xi}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi\\ &=\int_0^\delta\frac{e^{\lambda(a+\xi)}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi+\int_\delta^a\frac{e^{\lambda\xi}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi\end{aligned}$$

so that

$$\tilde\varsigma(\delta)\le\int_0^\delta\frac{e^{\lambda(a+\xi-\delta)}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi+\int_\delta^a\frac{e^{\lambda(\xi-\delta)}}{e^{\lambda a}-1}[\Upsilon(\xi,\tilde\varsigma(\xi))+\lambda\tilde\varsigma(\xi)]d\xi$$

$$
\begin{aligned}
&= \int_0^a G(\delta,\xi)[\Upsilon(\xi,\tilde{\varsigma}(\xi)) + \lambda\tilde{\varsigma}(\xi)]d\xi \\
&= (\mathcal{J}\tilde{\varsigma})(\delta), \quad \forall\ \delta \in [0,a]
\end{aligned}
$$

which implies that $(\tilde{\varsigma}, \mathcal{J}\tilde{\varsigma}) \in \Lambda$.

(iii) Take $\varsigma, \omega \in \mathbb{U}$ with $(\varsigma, \omega) \in \Lambda$. By (5.2), one gets

$$
\Upsilon(\delta,\varsigma(\delta)) + \lambda\varsigma(\delta) \le \Upsilon(\delta,\omega(\delta)) + \lambda\omega(\delta), \quad \forall\ \delta \in [0,a]. \tag{5.10}
$$

Using (5.5), (5.10) and the fact that $G(\delta,\xi) > 0$, $\forall\ \delta,\xi \in [0,a]$, we find

$$
\begin{aligned}
(\mathcal{J}\varsigma)(\delta) &= \int_0^a G(\delta,\xi)[\Upsilon(\xi,\varsigma(\xi)) + \lambda\varsigma(\xi)]d\xi \\
&\le \int_0^a G(\delta,\xi)[\Upsilon(\xi,\omega(\xi)) + \lambda\omega(\xi)]d\xi \\
&= (\mathcal{J}\omega)(\delta), \quad \forall\ \delta \in [0,a],
\end{aligned}
$$

which on using (5.7) implies that $(\mathcal{J}\varsigma, \mathcal{J}\omega) \in \Lambda$ and hence Λ is $\mathcal{J}$-closed.

(iv) If $\{\varsigma_n\} \subset \mathbb{U}$ remains Λ-preserving sequence converging to $\varsigma \in \mathbb{U}$, then convergence theory in $\mathbb{R}$ yields that $\varsigma_n(\delta) \le \varsigma(\delta)$, $\forall\ n \in \mathbb{N}$ and $\forall\ \delta \in [0,a]$. Making use of (5.7), one concludes that $(\varsigma_n, \varsigma) \in \Lambda$, $\forall\ n \in \mathbb{N}$. Therefore, Λ is σ-self-closed.

(v) and (vi) Take $\varsigma, \omega \in \mathbb{U}$ verifying $(\varsigma, \omega) \in \Lambda$. Then by (5.2), (5.5) and (5.6), one has

$$
\begin{aligned}
\sigma(\mathcal{J}\varsigma, \mathcal{J}\omega) &= \sup_{\delta\in[0,a]} |(\mathcal{J}\varsigma)(\delta) - (\mathcal{J}\omega)(\delta)| \\
&= \sup_{\delta\in[0,a]} \big((\mathcal{J}\omega)(\delta) - (\mathcal{J}\varsigma)(\delta)\big) \\
&\le \sup_{\delta\in[0,a]} \int_0^a G(\delta,\xi)[\Upsilon(\xi,\omega(\xi)) + \lambda\omega(\xi) - \Upsilon(\xi,\varsigma(\xi)) - \lambda\varsigma(\xi)]d\xi \\
&\le \sup_{\delta\in[0,a]} \int_0^a G(\delta,\xi)\lambda\varphi(\omega(\xi) - \varsigma(\xi))d\xi.
\end{aligned} \tag{5.11}
$$

Now, $0 \le \omega(\xi) - \varsigma(\xi) \le \sigma(\varsigma, \omega)$. Employing the monotonic property of φ, one gets $\varphi(\omega(\xi) - \varsigma(\xi)) \le \varphi(\sigma(\varsigma, \omega))$. It turns out, (5.11) converts to

$$
\begin{aligned}
\sigma(\mathcal{J}\varsigma, \mathcal{J}\omega) &\le \lambda\varphi(\sigma(\varsigma,\omega)) \sup_{\delta\in[0,a]} \int_0^a G(\delta,\xi)d\xi \\
&= \lambda\varphi(\sigma(\varsigma,\omega)) \sup_{\delta\in[0,a]} \frac{1}{e^{\lambda a} - 1}\left[\frac{1}{\lambda}e^{\lambda(a+\xi-\delta)}\Big|_0^\delta + \frac{1}{\lambda}e^{\lambda(\xi-\delta)}\Big|_\delta^a\right] \\
&= \lambda\varphi(\sigma(\varsigma,\omega)) \frac{1}{\lambda(e^{\lambda a} - 1)}(e^{\lambda a} - 1) \\
&= \varphi(\sigma(\varsigma,\omega))
\end{aligned}
$$

so that

$$
\begin{gathered}
\sigma(\mathcal{J}\varsigma, \mathcal{J}\omega) \le \varphi(\sigma(\varsigma,\omega)) + L\min\{\sigma(\varsigma, \mathcal{J}\omega), \sigma(\omega, \mathcal{J}\varsigma)\}, \\
\forall\ \varsigma, \omega \in \mathbb{U} \text{ satisfying } (\varsigma, \omega) \in \Lambda
\end{gathered}
$$

and

$$\sigma(\mathcal{J}\varsigma, \mathcal{J}\omega) \leq \varphi(\sigma(\varsigma, \omega)) + L_0 \min\{\sigma(\varsigma, \mathcal{J}\varsigma), \sigma(\omega, \mathcal{J}\omega)\},$$
$$\forall\ \varsigma, \omega \in \mathbb{U} \text{ satisfying } (\varsigma, \omega) \in \Lambda.$$

where $L \geq 0$ and $L_0 \geq 0$ are arbitrary.

(vii) Let $\mathcal{J}(u), \mathcal{J}(v) \in \mathcal{J}(\mathbb{U})$ (whereas $u, v \in \mathbb{U}$) be arbitrary. Write $\omega := \max\{\mathcal{J}u, \mathcal{J}v\}$. Then we have $(\mathcal{J}u, \omega) \in \Lambda$ and $(\mathcal{J}v, \omega) \in \Lambda$. Hence, $\mathcal{J}(\mathbb{U})$ is Λ-directed.

Therefore, by Theorem 3.2, $\mathcal{J}$ has a unique fixed point, which equivalently, forms the unique solution of Problem (5.1). □

6. Conclusions

The present article described the metrical fixed point results by means of a locally finitely $\mathcal{J}$-transitive relation over φ-contraction via a comparison function. It should be emphasized that Theorems 3.1 and 3.2 extended, enriched, improved, unified and modified a number of prior results in the following ways:

- By putting $L = 0$ in Theorem 3.2, one gets the findings of Arif et al. [10].
- For $\Lambda = \preceq$ and $L = 0$, our outcomes reduces to the outcomes of Agarwal et al. [1].
- By adopting $\varphi(t) = ct$ and $0 \leq c < 1$, the results of Khan [16] can be inferred. Take note that the locally $\mathcal{J}$-transitivity on the underlying relation is not required in this case.
- For $\varphi(t) = ct$, $0 \leq c < 1$ and $L = 0$, one deduces the outcomes of Alam and Imdad [4]. Note that, herein the requirement of locally $\mathcal{J}$-transitivity on underlying relation is relaxed.
- Under universal relation $\Lambda = \mathbb{U}^2$, we get corresponding metrical fixed point theorems under almost φ-contractions.
- Under universal relation $\Lambda = \mathbb{U}^2$ and for $L = 0$, Theorem 3.2 reduces to classical Matkowski fixed point Theorem [18].

For future works, some variants of Theorems 3.1 and 3.2 can be proved for almost Boyd-Wong contractions. Our findings can be extended to a couple of self-maps by investigating coincidence point results. An application of our results is given to certain BVP. Further applications of these findings include integral equations and matrix equations involving certain boundary conditions.

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Conflicts of interest

The authors declare no conflict of interest.

References

[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces,* Appl. Anal., 2008, 87(1), 109–116.

[2] A. Alam, M. Arif and M. Imdad, *Metrical fixed point theorems via locally finitely T-transitive binary relations under certain control functions,* Miskolc Math. Notes, 2019, 20(1), 59–73.

[3] A. Alam, R. George and M. Imdad, *Refinements to relation-theoretic contraction principle,* Axioms, 2022, 11, 316, 6 pp.

[4] A. Alam and M. Imdad, *Relation-theoretic contraction principle,* J. Fixed Point Theory Appl., 2015, 17(4), 693–702.

[5] A. Alam and M. Imdad, *Relation-theoretic metrical coincidence theorems,* Filomat, 2017, 31(14), 4421–4439.

[6] A. Alam and M. Imdad, *Nonlinear contractions in metric spaces under locally T-transitive binary relations,* Fixed Point Theory, 2018, 19(1), 13–24.

[7] S. F. Aldosary, I. Uddin and S. Mujahid, *Relational Geraghty contractions with an application to a singular fractional boundary value problem,* J. Appl. Anal. Comp., 2024, 14(6), 3480–3495.

[8] B. Almarri, S. Mujahid and I. Uddin, *New fixed point results for Geraghty contractions and their applications,* J. Appl. Anal. Comp., 2023, 13(5), 2788–2798.

[9] N. H. Altaweel and F. A. Khan, *Relation-theoretic fixed point theorems involving certain auxiliary functions with applications,* Symmetry, 2022, 14, 2614, 12 pp.

[10] M. Arif, M. Imdad and A. Alam, *Fixed point theorems under locally T-transitive binary relations employing Matkowski contractions,* Miskolc Math. Notes, 2022, 23(1), 71–83.

[11] M. Arif, I. A. Khan, M. Imdad and A. Alam, *Employing locally finitely T-transitive binary relations to prove coincidence theorems for nonlinear contractions,* J. Funct. Spaces, 2020, 2020, 6574695, 12 pp.

[12] V. Berinde, *Iterative Approximation of Fixed Points,* Springer-Verlag, Heidelberg, Germany, 2007.

[13] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration,* Nonlinear Anal. Forum., 2004, 9(1), 43–53.

[14] N. H. E. Eljaneid, F. A. Khan, H. I. A. Mohammed and A. Alam, *Relational quasi-contractions and related fixed point theorems,* J. Math., 2022, 2022, 4477660, 6 pp.

[15] A. Hossain, F. A. Khan and Q. H. Khan, *A relation-theoretic metrical fixed point theorem for rational type contraction mapping with an application,* Axioms, 2021, 10, 316, 11 pp.

[16] F. A. Khan, *Almost contractions under binary relations,* Axioms, 2022, 11, 441, 7 pp.

[17] F. A. Khan, F. Sk, M. G. Alshehri, Q. H. Khan and A. Alam, *Relational Meir-Keeler contractions and common fixed point theorems,* J. Funct. Spaces, 2022, 2022, 3550923, 9 pp.

[18] J. Matkowski, *Integrable solutions of functional equations,* Dissertationes Math., 1975, 127, 68 pp.

[19] J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order., 2005, 22(3), 223–239.

[20] D. O'Regan and A. Petruşel, *Fixed point theorems for generalized contractions in ordered metric spaces,* J. Math. Anal. Appl., 2008, 341(2), 1241–1252.

[21] M. Păcurar, *Remark regarding two classes of almost contractions with unique fixed point,* Creative Math. Inf., 2010, 19(2), 178–183.

[22] B. Samet and M. Turinici, *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications,* Commun. Math. Anal., 2012, 13(2), 82–97.

[23] F. Sk, F. A. Khan, Q. H. Khan and A. Alam, *Relation-preserving generalized nonlinear contractions and related fixed point theorems,* AIMS Math., 2022, 7(4), 6634–6649.