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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF COEXISTENCE STATES TO A DIFFUSIVE HOLLING TYPE II PREDATOR-PREY MODEL WITH HUNTING COOPERATION*

Lina Zhang[1,†] and Xue Xiao[1]

Abstract In this paper, we delve into a diffusive Holling type II predator-prey model, incorporating the element of hunting cooperation, and examine it under Dirichlet boundary conditions. Our primary focus is on addressing two pivotal questions: Firstly, we endeavor to establish the existence of coexistence states across a range of hunting cooperation effects. This exploration aims to reveal how the predator and prey species can maintain their coexistence within the ecological system, regardless of the magnitude of cooperation among predators during hunting. Secondly, we are interested in elucidating the asymptotic behavior of these coexistence states as the cooperation parameter approaches infinity. This analysis will provide insights into how the ecological balance shifts as the predators' cooperation increases indefinitely, offering a deeper understanding of the long-term ecological implications of such cooperation.

Keywords Predator-prey, hunting cooperation, coexistence state, asymptotic behavior.

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1. Introduction

In predator-prey interactions, predators frequently collaborate and cooperate while hunting their prey. For example, lions [15, 20], wolves [21], chimpanzees [3] and African wild dogs [7] unite in their efforts to capture and dispatch their prey.

An earlier mathematical model addressing the phenomenon of cooperative hunting was put forth by Berec [2]. This model utilized ordinary differential equations to simulate the intricate predator-prey interactions. Berec's analysis revealed that cooperation among hunters introduces instability in the predator-prey system, broadening the range of parameters that permit limit cycle oscillations. Later on, Alves and Hilker [1] expressed the belief that hunting cooperation enhances the predator's attack rate. Therefore, they deemed it essential to incorporate a cooperation term in the formulation of the predator population's attack rate. As a result, they established the hunting cooperation models with Holling types I, II, III and IV func-

†The corresponding author.
1 College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
*Supported by the National Natural Science Foundation of China (12161080).
Email: linazhang@nwnu.edu.cn(L. Zhang), 13893263533@163.com(X. Xiao)

tional response, respectively, where the predator-prey model with type II functional response and hunting cooperation can be written in the form

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) - \dfrac{(\lambda + aP)NP}{1 + H(\lambda + aP)N}, \\ \dfrac{dP}{dt} = \dfrac{e(\lambda + aP)NP}{1 + H(\lambda + aP)N} - mP, \end{cases} \tag{1.1}$$

where N and P are prey and predator densities respectively. The parameter r is the per capita intrinsic growth rate of prey, K is the carrying capacity of prey, e is the conversion efficiency, m is the per capita mortality rate of predators, H is the handling time of predator population and is a dimensionless parameter, λ is the attack rate of the per predator on the prey and a describes the predator cooperation in hunting. The parameters r, K, e, m, H, λ are positive constants and a is a non-negative constant.

Alves and Hilker numerically investigated the existence and stability of the positive equilibria and have shown that the hunting cooperation can be beneficial to the predator population by increasing the attack rate. In the simulations of [1], by comparing the bifurcation diagrams of models (1.1) and (1.1) with $H = 0$ (a hunting cooperation model with type I functional response) at the same parameter values, the authors concluded that these two models have quite similar bifurcation behavior, and type II function response can promote the possibility of sustained oscillation. Afterwards, rigorous mathematical analysis about the stability of equilibria and the detailed behaviour of bifurcations for (1.1) was done in [9]. The authors not only discussed how hunting cooperation among predators affects the population dynamics of (1.1), but also investigated the corresponding reaction-diffusion model

$$\begin{cases} \dfrac{\partial N}{\partial t} - D_1\Delta N = rN\left(1 - \dfrac{N}{K}\right) - \dfrac{(\lambda + aP)NP}{1 + H(\lambda + aP)N}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial P}{\partial t} - D_2\Delta P = \dfrac{e(\lambda + aP)NP}{1 + H(\lambda + aP)N} - mP, & x \in \Omega,\ t > 0, \\ \dfrac{\partial N}{\partial \nu} = \dfrac{\partial P}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ N(x, 0) \geq 0,\ P(x, 0) \geq 0, & x \in \Omega, \end{cases} \tag{1.2}$$

where the positive constants D_1 and D_2 are diffusion rates, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, ν is the outward normal vector on $\partial\Omega$. Turing unstable region in model (1.2) is given precisely in [9]. In addition, Singh et al. in [22] also analyzed Turing instability by the linearization method and obtained complex pattern formations by using extensive numerical simulations for (1.2). The dynamics of model (1.2) with $H = 0$ have been studied in [6, 23, 24].

It is widely recognized that boundary conditions significantly influence the dynamics of reaction-diffusion systems. In the referenced literature [6, 9, 22–24], the authors have extensively discussed the system (1.2) under the homogeneous Neumann boundary conditions. This particular boundary condition suggests that the system is self-contained with zero population flux across the boundary.

However, another crucial boundary condition deserves equal attention - the Dirichlet boundary condition, characterized by $N = P = 0$ on $\partial\Omega$. This condition implies that the boundary is hostile to the survival of species, rendering it unsuitable for habitation. This concept is not merely theoretical; there is empirical evidence [5] to support its relevance. For instance, studies have shown that certain neurohemicals in the human brain, such as acetylcholine and triethylcholine, are capable of freely traversing boundaries within specific regions of the brain but are unable to persist on these boundaries due to the presence of a third chemical. From a mathematical perspective, there exist significant differences in the research methods for reaction-diffusion equations with different boundary conditions.

The significance of studying Dirichlet boundary conditions lies in their ability to provide deeper insights into the behavior of reaction-diffusion systems in hostile environments. Understanding how species interact and adapt under such conditions can have profound implications in various fields, including ecology, biology, and even neuroscience. Therefore, the exploration of Dirichlet boundary conditions represents a crucial area of research that holds significant potential for advancing our knowledge in these disciplines.

Motivated by the previous researches, this paper focuses on system (1.2) under the homogeneous Dirichlet boundary conditions. For the sake of brevity and clarity, we adopt the following rescaling techniques

$$\begin{aligned}
&u=\frac{r}{KD_1}N,\quad v=\frac{\lambda}{D_1}P,\quad \bar{t}=D_1t,\quad \mu=\frac{r}{D_1},\quad \alpha=\frac{aD_1}{\lambda^2},\\
&m=\frac{H\lambda KD_1}{r},\quad \beta=\frac{e\lambda KD_1}{rD_2},\quad \gamma=\frac{m}{D_2},\quad \tau=\frac{D_1}{D_2}
\end{aligned}$$

and then drop the upper bar of $\bar{t}$, we obtain the following simplified reaction-diffusion system

$$\begin{cases}
\dfrac{\partial u}{\partial t}-\Delta u=u(\mu-u)-\dfrac{(1+\alpha v)uv}{1+m(1+\alpha v)u}, & x\in\Omega,\ t>0,\\
\tau\dfrac{\partial v}{\partial t}-\Delta v=\dfrac{\beta(1+\alpha v)uv}{1+m(1+\alpha v)u}-\gamma v, & x\in\Omega,\ t>0,\\
u=v=0, & x\in\partial\Omega,\ t>0,\\
u(x,0)\geq 0,\ v(x,0)\geq 0.
\end{cases}\tag{1.3}$$

Notably, α exhibits a direct proportionality to a, highlighting its crucial role in determining the efficiency of hunting cooperation among predators.

In ecosystems, the coexistence of diverse species is of utmost significance. Hence, we pay more attention to whether the two species in the system (1.3) can coexist. From an ecological perspective, a positive steady-state solution of (1.3) means a coexistence state of prey and predator [4,8]. Therefore, the research on the steady-state problems is very important. In this paper, our primary focus lies on exploring two fundamental questions: (1) The existence of positive steady-state solutions of (1.3); (2) The asymptotic behavior of these positive steady-state solutions as the cooperation parameter α goes to infinity. To this end, we consider the steady-state system

$$\begin{cases}
-\Delta u=u(\mu-u)-\dfrac{(1+\alpha v)uv}{1+m(1+\alpha v)u}, & x\in\Omega,\\
-\Delta v=\dfrac{\beta(1+\alpha v)uv}{1+m(1+\alpha v)u}-\gamma v, & x\in\Omega,\\
u=v=0, & x\in\partial\Omega.
\end{cases}\tag{1.4}$$

Before presenting our main results, it is imperative to establish certain notations and fundamental concepts that will be frequently referenced throughout this paper. Let $\lambda_1(q) < \lambda_2(q) \le \lambda_3(q) \le \cdots$ be all eigenvalues of the following problem

$$-\triangle\phi + q(x)\phi = \lambda\phi, \quad x \in \Omega, \quad \phi = 0, \; x \in \; \partial\Omega,$$

where $q(x) \in C(\bar{\Omega})$. We know that $\lambda_1(q)$ is simple and $\lambda_1(q)$ is strictly increasing in the sense that $q_1 \le q_2$ and $q_1 \not\equiv q_2$ implies $\lambda_1(q_1) < \lambda_1(q_2)$. When $q(x) \equiv 0$, we denote $\lambda_i(0)$ by λ_i, and denote by φ_1 the eigenfunction corresponding to λ_1 with normalization $\|\varphi_1\|_\infty$=1 and positive in Ω.

We define $C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) |\ u = 0 \ \text{ on } \ \partial\Omega\}$. It is well-known that for any $\mu > \lambda_1$, the problem

$$-\triangle u = u(\mu - u), \quad x \in \Omega, \quad u = 0, \; x \in \; \partial\Omega \tag{1.5}$$

has a unique positive solution, say θ_μ. In addition, the mapping $\mu \to \theta_\mu$ is strictly increasing, continuously differentiable from (λ_1, ∞) to $C^2(\Omega) \cap C_0(\bar{\Omega})$ and that $\theta_\mu \to 0$ uniformly on $\bar{\Omega}$ as $\mu \to \lambda_1^+$. Moreover, we have $0 < \theta_\mu < \mu$ in Ω. Therefore, (1.4) has a semi-trivial solution $(\theta_\mu, 0)$ if $\mu > \lambda_1$.

To the first question, we establish the conditions for existence of coexistence states by using the fixed point index [12]. Our analysis reveals that the existence of coexistence states is determined by the principal eigenvalue associated with a operator (as stated in Theorem 3.1), which is independent of the hunting rate α.

Regarding the second question, we proceed with an asymptotic analysis of the positive solutions of (1.4) as $\alpha \to \infty$. Our findings reveal that excessively strong cooperation can lead solely to the extinction of predators, and the prey density $u(x)$ converges to θ_μ in $C(\bar{\Omega})$ (see Theorem 4.1). This paradoxical occurrence has also been observed in prior studies [1, 18, 19] involving Holling type I functional responses. As pointed out in [1], this phenomenon arises due to the following reasons: the beneficial effect of cooperation on predators was overcompensated by the decrease in prey density since a scarcer prey implies a smaller predator density in turn.

The highlights and advantages of this paper are presented as follows:

(1) In previous studies, the hunting cooperation models with diffusion have mostly focused on Neumann boundary conditions, yet research on Dirichlet boundary conditions is rarely involved. This paper not only fills the gap in this field but also presents significant differences in research methods and conclusions compared to studies under Neumann boundary conditions. This breakthrough research provides valuable insights and references for people to deeply understand the influence of boundary conditions on the dynamic behavior of species, and it will greatly enrich the research achievements in this field.

(2) This paper aims to establish a comprehensive and systematic research framework and method to deeply explore the existence and asymptotic behavior of coexistence solutions in the predator-prey model with hunting cooperation under Dirichlet boundary conditions. It is worth mentioning that the research method used in this article is not only applicable to the current model but can also be extended to other types of functional response functions, such as the classic Holling III and Ivlev types, demonstrating its strong universality and innovation.

(3) In this paper, we specifically selected the Holling II type functional response function as the research object because it has a wide application basis and representativeness in predator-prey relationships. Through in-depth research and analysis,

the conclusions drawn in this paper not only have high theoretical value but also demonstrate widespread applicability in practical applications, playing an important role in promoting the research development in this field.

The contents of this paper are structured as follows. Firstly, in Section 2, we present some preliminary results, including some estimates for positive solutions of (1.4), a sufficient condition for the non-existence of positive solutions and a necessary condition for the existence of positive solutions of (1.4). These results will play an important role in the subsequent sections. In Section 3, we discuss the existence of positive solutions of (1.4). Finally, in Section 4, we establish the limiting behavior of any positive solution of (1.4) as $\alpha \to \infty$, providing a comprehensive understanding of the solution's asymptotic characteristics.

2. Preliminaries and a priori estimates

This section aims to present some preliminary results, specifically focusing on establishing estimates for positive solutions of (1.4). These estimates will serve as the foundation for subsequent sections. Throughout this section, the parameters μ, m, β and γ are fixed, while the constant M will vary based on these parameters but not on α. Since these parameters are fixed, this dependence will not be explicitly stated.

Lemma 2.1. *Let* (u, v) *be a positive solution of* (1.4). *Then* (u, v) *satisfies*

$$u \le \mu, \qquad v \le \frac{\beta\mu(\mu+\gamma)}{\gamma} := M_0.$$

Proof. According to the maximum principle, it can be derived that $u(x) \le \mu$ on $\overline{\Omega}$. Set $w = \beta u + v$. Then

$$\begin{aligned} -\Delta w &= \beta u(\mu - u) - \gamma v \\ &= \beta u(\mu + \gamma - u) - \gamma(\beta u + v) \\ &\le \beta\mu(\mu+\gamma) - \gamma w. \end{aligned} \tag{2.1}$$

If we once again apply the maximum principle, it follows that

$$w = \beta u + v \le \frac{\beta\mu(\mu+\gamma)}{\gamma} \quad \text{in } \overline{\Omega},$$

which implies the desired result. □

Lemma 2.2. *Let* (u, v) *be a positive solution of* (1.4). *Then there exists a positive constant* M *such that*

(i) $\displaystyle\int_\Omega |\nabla u|^2 dx \le M, \int_\Omega |\nabla v|^2 dx \le M;$

(ii) $\displaystyle\alpha \int_\Omega \frac{uv^2}{1+m(1+\alpha v)u} dx \le M.$

Proof. (i) By multiplying the first equation of (1.4) with u and integrating over the domain Ω, we obtain

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega \left[u^2(\mu - u) - \frac{(1+\alpha v)u^2 v}{1+m(1+\alpha v)u} \right] dx \le \mu \int_\Omega u^2 \, dx.$$

According to Lemma 2.1, it follows that $\int_\Omega |\nabla u|^2 \, dx \le M$.

To demonstrate that $\int_\Omega |\nabla v|^2 dx \le M$, we proceed by multiplying the equation in (2.1) by w and integrating the result over Ω. This operation yields that

$$\int_\Omega |\nabla w|^2 dx = \int_\Omega (\beta u + v)[\beta u(\mu - u) - \gamma v] dx \le \beta\mu \int_\Omega u(\beta u + v) dx.$$

As a direct consequence of Lemma 2.1, we have the inequality $\int_\Omega |\nabla w|^2 dx \le M$. Consequently, it follows that

$$\int_\Omega |\nabla v|^2 \, dx = \int_\Omega |\nabla(\beta u + v) - \nabla(\beta u)|^2 \, dx \le 2\int_\Omega \left(|\nabla w|^2 + \beta^2 |\nabla u|^2\right) dx \le M.$$

Hence, u and v are uniformly bounded in $W_0^{1,2}(\Omega)$.

(ii) We know from [17] that the elliptic problem

$$-\Delta\phi = \beta\mu(\mu + \gamma) - \gamma\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega \tag{2.2}$$

has a maximal solution, say ϕ_0, such that $0 \le \phi \le \phi_0 \in C^{2,\tau}(\overline{\Omega})$ for all nonnegative solutions ϕ and a $\tau \in (0, 1)$. By (2.1), $0 \le \beta u(x) + v(x) \le \phi_0(x)$ in $\overline{\Omega}$. Noticing that $u(x) = \phi_0(x) = 0$ on $\partial\Omega$, it follows that

$$\left|\frac{\partial u}{\partial \nu}\right| \le \frac{1}{\beta}\left|\frac{\partial \phi_0}{\partial \nu}\right| \le M \text{ on } \partial\Omega, \tag{2.3}$$

where ν is the outward normal vector on $\partial\Omega$. Integrating the first equation in (1.4), we have

$$\begin{aligned}\alpha\int_\Omega \frac{uv^2}{1 + m(1+\alpha v)u} dx &= \int_\Omega [\Delta u + u(\mu - u)] dx - \int_\Omega \frac{1}{1 + m(1+\alpha v)u} dx \\ &\le \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dx + \int_\Omega u(\mu - u) dx.\end{aligned}$$

It follows that $\alpha \int_\Omega \frac{uv^2}{1 + m(1+\alpha v)u} dx \le M$ from Lemma 2.1 and (2.3). This completes the proof. □

We now present a sufficient condition that guarantees the non-existence of positive solutions, along with some crucial necessary conditions that underlie the existence of positive solutions to (1.4).

Lemma 2.3. *(i) If $\mu \le \lambda_1$, then* (1.4) *has no positive solution.*

(ii) Conditions $\mu > \lambda_1$ and

$$\lambda_1\left(\gamma - \frac{\beta\theta_\mu}{\frac{1}{1+\alpha M_0} + m\theta_\mu}\right) < 0$$

are necessary conditions for the existence of positive solutions of (1.4)*, where $M_0 = \frac{\beta\mu(\mu+\gamma)}{\gamma}$.*

Proof. (i) Suppose that (u, v) is a positive solution of (1.4) for $\mu \le \lambda_1$. Then (u, v) satisfies

$$-\Delta u + u\left[u + \frac{(1+\alpha v)v}{1 + m(1+\alpha v)u}\right] = \mu u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Hence,

$$\lambda_1\left(u+\frac{(1+\alpha v)v}{1+m(1+\alpha v)u}\right)=\mu.$$

By the comparison principle of eigenvalues, we have $\mu>\lambda_1$, a contradiction is obtained.

(ii) Let (u,v) be a positive solutions of (1.4), then $\mu>\lambda_1$ by (i). Since

$$-\Delta u=u(\mu-u)-\frac{(1+\alpha v)uv}{1+m(1+\alpha v)u}\le u(\mu-u)\ \text{ in }\ \Omega,\quad u=0\ \text{ on }\ \partial\Omega,$$

we obtain that u is a lower solution of

$$-\Delta u=u(\mu-u)\ \text{ in }\ \Omega,\quad u=0\ \text{ on }\ \partial\Omega.$$

Hence $u\le\theta_\mu$. Since v satisfies

$$-\Delta v=\frac{\beta(1+\alpha v)uv}{1+m(1+\alpha v)u}-\gamma v\ \text{ in }\ \Omega,\quad u=0\ \text{ on }\ \partial\Omega,$$

we have

$$0=\lambda_1\left(\gamma-\frac{\beta(1+\alpha v)u}{1+m(1+\alpha v)u}\right)>\lambda_1\left(\gamma-\frac{\beta\theta_\mu}{\frac{1}{1+\alpha M_0}+m\theta_\mu}\right),$$

where M_0 is given in Lemma 2.1. □

3. Existence of coexistence states

In this section, we employ the theory of fixed point index as a pivotal tool to establish conditions for the existence of coexistence states to (1.4). This theory serves as a fundamental framework for our proofs, enabling us to delve deeper into the intricacies of the coexistence states.

Let E be a real Banach space and W is the natural positive cone of E. For $y\in W$, define

$$W_y=\{x\in E: y+\xi x\in W \text{ for some } \xi>0\},$$
$$S_y=\{x\in\overline{W}_y: -x\in\overline{W}_y\}.$$

Let y_* be a fixed point of a compact operator $\mathcal{A}:W\to W$ and $\mathcal{L}=\mathcal{A}'(y_*)$ be the Fréchet derivative of $\mathcal{A}$ at y_*. We say that $\mathcal{L}$ has property α on $\overline{W}_{y_*}$, if there exists a $\eta\in(0,1)$ and a $y\in\overline{W}_{y_*}\setminus S_{y_*}$ such that $y-\eta\mathcal{L}y\in S_{y_*}$. For an open subset $U\subset W$, let $\text{index}_W(\mathcal{A},U)$ be the Leray-Schauder degree $\deg_W(I-\mathcal{A},U,0)$, where I is the identity map, the fixed of $\mathcal{A}$ at y_* in W is defined by $\text{index}_W(\mathcal{A},y_*):=\text{index}(\mathcal{A},U(y_*),W)$, where $U(y_*)$ is a small open neighborhood of y_* in W.

The following Lemma is from Lemma 4.1 of [13].

Lemma 3.1. *Assume that $I-\mathcal{L}$ is invertible on $\overline{W}_{y_*}$.*

(i) If $\mathcal{L}$ has property α on $\overline{W}_{y_*}$, then $\text{index}_W(\mathcal{A},y_*)=0$.

(ii) If $\mathcal{L}$ does not have property α on $\overline{W}_{y_*}$, then $\text{index}_W(\mathcal{A}, y_*) = (-1)^\sigma$*, where σ is the sum of algebraic multiplicities sum of the eigenvalues of $\mathcal{L}$ which are greater than 1.*

We also introduce the notations mentioned below.

(i) $X := C_0(\bar{\Omega}) \oplus C_0(\bar{\Omega})$.

(ii) $W := P_0 \oplus P_0$, where $P_0 := \{\phi \in C_0(\bar{\Omega}) : \phi \geq 0 \text{ in } \bar{\Omega}\}$.

(iii) $D := \{(u, v) \in X : u < \mu + 1,\ v < M_0 + 1\}$.

(iv) $D' := D \cap W$.

Take P as a sufficiently large positive constant such that

$$P > \max\{\mu + M_0(1 + \alpha M_0) + 1, \gamma + 1\}.$$

Define a positive compact operator $\mathcal{A}_\theta$: $X \to X$ by

$$\mathcal{A}_\theta(u, v) = (-\Delta + P)^{-1} \begin{pmatrix} \theta u \left(\mu - u - \dfrac{(1 + \alpha v)v}{1 + m(1 + \alpha v)u}\right) + Pu \\ \theta v \left(\dfrac{\beta(1 + \alpha v)u}{1 + m(1 + \alpha v)u} - \gamma\right) + Pv \end{pmatrix}$$

for $\theta \in [0, 1]$.

Observe that the coexistence state in W for (1.4) is equivalent to the existence of a positive fixed point in D' for $\mathcal{A} := \mathcal{A}_1$. Recall that (1.4) admits a trivial solution $(0, 0)$ and a semi-trivial solution $(\theta_\mu, 0)$ when $\mu > \lambda_1$. By using the properties of fixed point index, we will demonstrate the presence of a positive solution (u, v) in D'.

Lemma 3.2. *Assume that $\mu > \lambda_1$ holds, then* $\text{index}_{\text{W}}(\mathcal{A}, D') = 1$.

Proof. We can regard θ as a parameter. Assuming that (u_θ, v_θ) is a positive fixed point of $\mathcal{A}_\theta$ in D', it becomes evident that $u_\theta < \mu, v_\theta < M_0$ for every $\theta \in [0, 1]$. Consequently, $\mathcal{A}_\theta$ possesses no fixed point on the boundary $\partial D'$, ensuring that $\deg(\mathcal{A}_\theta, D', 0)$ is well defined and remains constant regardless of θ. Applying the homotopy invariance and normalization properties of the degree, we can deduce that

$$\text{index}_{\text{W}}(\mathcal{A}_0, D') = \text{index}_{\text{W}}(\mathcal{A}_1, D') = 1.$$

This leads to the desired result. □

Lemma 3.3. *Assume that $\mu > \lambda_1$ holds, then* $\text{index}_W(\mathcal{A}, (0, 0)) = 0$.

Proof. To demonstrate that the index is zero, lemma 3.1 shall be employed. In this case we have $y_* = (0, 0)$, $\overline{W}_{y_*} = W$, $S_{y_*} = \{(0, 0)\}$, and

$$\mathcal{L} := \mathcal{A}'(0, 0) = (-\Delta + P)^{-1} \begin{pmatrix} \mu + P & 0 \\ 0 & -\gamma + P \end{pmatrix}.$$

Firstly, we show that $I-\mathcal{L}$ is invertible on W. Suppose that there exists some function $(\phi,\psi)^\top \in W$ such that $\mathcal{L}(\phi,\psi)^\top=(\phi,\psi)^\top$, then we obtain that

$$\begin{cases} -\Delta\phi=\mu\phi, & x\in\Omega,\\ -\Delta\psi=-\gamma\psi, & x\in\Omega,\\ (\phi,\psi)=(0,0), & x\in\partial\Omega. \end{cases}$$

So μ and $-\gamma$ are eigenvalues of $-\Delta$ with corresponding eigenfunctions ϕ and ψ. Since $\mu>\lambda_1$ and $-\gamma<\lambda_1$, $\phi=0$ and $\psi=0$. This implies that $I-\mathcal{L}$ is invertible on W.

Furthermore, we show that $\mathcal{L}$ has property α on W. In fact, choosing $y=(\varphi_1,0)^\top$ and $\eta_1=\frac{\lambda_1+P}{\mu+P}$, where φ_1 is the eigenfunction corresponding to λ_1. It is easy to check that $\eta_1\in(0,1)$, $(\varphi_1,0)^\top\in\overline{W}_{(0,0)}\setminus S_{(0,0)}$ and $(\varphi_1,0)^\top-\eta_1\mathcal{L}(\varphi_1,0)^\top\in S_{(0,0)}$. Hence, $\mathcal{L}$ has property α and we can use lemma 3.1 to conclude $\text{index}_W(\mathcal{A},(0,0))=0$. □

Next, we calculate the fixed point for the point $(\theta_\mu,0)$, which will depend on the sign of $\lambda_1\left(-\frac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)$.

Lemma 3.4. *(i) If* $\lambda_1\left(-\dfrac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)<0$, *then* $\text{index}_{\mathrm{W}}(\mathcal{A},(\theta_\mu,0))=0$.

(ii) If $\lambda_1\left(-\dfrac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)>0$, *then* $\text{index}_{\mathrm{W}}(\mathcal{A},(\theta_\mu,0))=1$.

Proof. For $y_*=(\theta_\mu,0)$, we have $\overline{W}_{y_*}=C_0(\Omega)\oplus P_0$, and

$$\mathcal{L}:=\mathcal{A}'(\theta_\mu,0)=(-\Delta+P)^{-1}\begin{pmatrix} \mu-2\theta_\mu+P & -\dfrac{\theta_\mu}{1+m\theta_\mu}\\ 0 & \dfrac{\beta\theta_\mu}{1+m\theta_\mu}-\gamma+P \end{pmatrix}.$$

Firstly, we claim that $I-\mathcal{L}$ is invertible on $C_0(\Omega)\oplus C_0(\Omega)$ if $\lambda_1\left(-\frac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)\neq 0$. Suppose that there exists some function (ϕ,ψ) such that $\mathcal{L}(\phi,\psi)^\top=(\phi,\psi)^\top$, that is

$$\begin{cases} -\Delta\phi=(\mu-2\theta_\mu)\phi-\dfrac{\theta_\mu}{1+m\theta_\mu}\psi, & \text{in } \Omega,\\ -\Delta\psi=\left(\dfrac{\beta\theta_\mu}{1+m\theta_\mu}-\gamma\right)\psi, & \text{in } \Omega,\\ (\phi,\psi)=(0,0), & \text{on } \partial\Omega. \end{cases}$$

If $\psi\not\equiv 0$, then the second equation above implies that $0=\lambda_1\left(-\frac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)$, which contrary to $\lambda_1\left(-\frac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right)\neq 0$. Hence $\psi\equiv 0$. The first equation becomes

$$-\Delta\phi+(-\mu+2\theta_\mu)\phi=0 \ \text{ in } \ \Omega, \quad \phi=0 \ \text{ on } \ \partial\Omega.$$

If $\phi\not\equiv 0$, then $\lambda_1(-\mu+2\theta_\mu)=0$. Recall that θ_μ is a solution of (1.5), thus $\lambda_1(-\mu+\theta_\mu)=0$. Therefore,

$$0=\lambda_1(-\mu+2\theta_\mu)>\lambda_1(-\mu+\theta_\mu)=0.$$

This contradiction implies that $\phi \equiv 0$. Hence $(\phi, \psi) \equiv (0, 0)$ and $I - \mathcal{L}$ is invertible on $C_0(\Omega) \oplus C_0(\Omega)$.

Next we consider whether $\mathcal{L}$ has property α in $\overline{W}_{(\theta_\mu,0)}$. Here we have $S_{(\theta_\mu,0)} = C_0(\Omega) \oplus \{0\}$ and $\overline{W}_{(\theta_\mu,0)} \backslash S_{(\theta_\mu,0)} = C_0(\Omega) \oplus \{P_0 \setminus \{0\}\}$. Suppose that $\mathcal{L}$ has property α in $\overline{W}_{(\theta_\mu,0)}$. Then there is a $\eta_1 \in (0, 1)$ and $(\phi, \psi) \in \overline{W}_{(\theta_\mu,0)} \backslash S_{(\theta_\mu,0)}$ such that $(I - \eta_1 \mathcal{L})(\phi, \varphi)^\top \in S_{(\theta_\mu,0)}$. That is

$$
\begin{aligned}
&- \Delta\phi + P\phi - \eta_1 \left[(\mu - 2\theta_\mu + P)\phi - \frac{\theta_\mu}{1 + m\theta_\mu}\psi\right] = 0,\\
&- \Delta\psi + P\psi - \eta_1 \left(\frac{\beta\theta_\mu}{1 + m\theta_\mu} - \gamma + P\right)\psi = 0.
\end{aligned}
$$

From the second equation we have

$$
\lambda_1 \left(P - \eta_1 \left(\frac{\beta\theta_\mu}{1 + m\theta_\mu} - \gamma + P\right)\right) = 0. \tag{3.1}
$$

Consider η as a parameter and denote

$$
f(\eta) := \lambda_1 \left(P - \eta \left(\frac{\beta\theta_\mu}{1 + m\theta_\mu} - \gamma + P\right)\right).
$$

Obviously, $f(\eta)$ is decreasing with $\eta \in [0, \infty)$ and $f(0) = \lambda_1(P) > 0$.

(i) If $\lambda_1 \left(-\frac{\beta\theta_\mu}{1+m\theta_\mu} + \gamma\right) < 0$, then $f(1) < 0$. Due to the continuity and monotonicity of $f(\eta)$, there exists a unique $\eta_1 \in (0, 1)$ such that $f(\eta_1) = 0$. Hence (3.1) holds and $\mathcal{L}$ dose have property α in $\overline{W}_{(\theta_\mu,0)}$. From Lemma 3.1, it follows that $\text{index}_W(\mathcal{A}, (\theta_\mu, 0)) = 0$.

(ii) If $\lambda_1 \left(-\frac{\beta\theta_\mu}{1+m\theta_\mu} + \gamma\right) > 0$, then $f(1) > 0$. It follows from the continuity and monotonicity that $f(\eta) > 0$ for all $\eta \in (0, 1)$. Therefore, (3.1) dose not hold and $\mathcal{L}$ dose not have property α in $\overline{W}_{(\theta_\mu,0)}$.

To calculate $\text{index}_W(\mathcal{A}, (\theta_\mu, 0))$, suppose that $1/\eta$ is an eigenvalue of $\mathcal{L}$ with corresponding eigenvector $(\phi, \psi)^\top$. Then $\mathcal{L}(\phi, \psi)^\top = \frac{1}{\eta}(\phi, \psi)^\top$. That is

$$
\begin{aligned}
&(-\Delta + P)^{-1} \left[(\mu - 2\theta_\mu + P)\phi - \frac{\theta_\mu}{1 + m\theta_\mu}\psi\right] = \frac{1}{\eta}\phi,\\
&(-\Delta + P)^{-1} \left(\frac{\beta\theta_\mu}{1 + m\theta_\mu} - \gamma + P\right)\psi = \frac{1}{\eta}\psi.
\end{aligned}
$$

From the second equation we have

$$
-\Delta\psi + \left[P - \eta \left(\frac{\beta\theta_\mu}{1 + m\theta_\mu} - \gamma + P\right)\right]\psi = 0. \tag{3.2}
$$

Since $f(0) > 0$, $f(1) > 0$, $f(+\infty) < 0$ and $f(\eta)$ is decreasing with $\eta \in (0, +\infty)$, one can conclude that $\eta > 1$ in (3.2). Thus $\mathcal{L}$ has no eigenvalues greater than 1. Hence $\sigma = 0$ and we have $\text{index}_W(\mathcal{A}, (\theta_\mu, 0)) = (-1)^\sigma = 1$. □

Using Lemmas 3.2-3.4, we have the following theorem which gives the existence of coexistence states to system (1.4).

Theorem 3.1. *If* $\mu > \lambda_1$ *and* $\lambda_1\left(-\dfrac{\beta\theta_\mu}{1+m\theta_\mu}+\gamma\right) < 0$*, then* (1.4) *has at least one positive solution for any* $\alpha > 0$.

Proof. By using Lemmas 3.2-3.4 and the additivity of the index, we have

$$\text{index}_W(\mathcal{A}, D') - \text{index}_W(\mathcal{A}, (0,0)) - \text{index}_W(\mathcal{A}, (\theta_\mu, 0)) = 1 \neq 0.$$

Hence, there must be a coexistence state of (1.4) in D'. □

4. The asymptotic behavior of coexistence states

In this section, our focus is on exploring the asymptotic behavior of positive solutions of (1.4) as the hunting cooperation parameter α goes to infinity. Unless explicitly stated otherwise, we shall maintain the assumption that $\mu > \lambda_1$. For each $\alpha > 0$, let (u_α, v_α) represent any positive solution to (1.4). We present the following lemmas to aid in our analysis.

Lemma 4.1. *If* $\lim_{\alpha\to\infty} u_\alpha = 0$ *uniformly on* $\overline{\Omega}$*, then* $\lim_{\alpha\to\infty} v_\alpha = 0$ *uniformly on* $\overline{\Omega}$.

Proof. Assume that $\lim_{\alpha\to\infty} u_\alpha = 0$ uniformly on $\overline{\Omega}$. For any given constant $\epsilon > 0$, there exists a large $\overline{\alpha}$ such that $u_\alpha \leq \epsilon$ for all $\alpha > \overline{\alpha}$. Hence, it follows from (2.1) that

$$-\Delta(\beta u_\alpha + v_\alpha) \leq \beta\epsilon(\mu+\gamma) - \gamma(\beta u_\alpha + v_\alpha) \text{ for } \alpha > \overline{\alpha}.$$

Therefore, we have

$$v_\alpha(x) \leq \beta u_\alpha(x) + v_\alpha(x) \leq \frac{\beta\epsilon(\mu+\gamma)}{\gamma}$$

on $\overline{\Omega}$ for all $\alpha > \overline{\alpha}$. The arbitrariness of ϵ implies that $\lim_{\alpha\to\infty} v_\alpha = 0$ uniformly on $\overline{\Omega}$. □

Lemma 4.2. *Either* $\lim_{\alpha\to\infty} u_\alpha = 0$ *or* $\lim_{\alpha\to\infty} v_\alpha = 0$ *uniformly on* $\overline{\Omega}$.

Proof. To establish this assertion, we argue by contradiction. Suppose that there exists a certain $\epsilon_0 > 0$ and a positive sequence $\{\alpha_i\}_{i=1}^\infty$ satisfying $\alpha_i \to \infty$ such that

$$u_{\alpha_i}(x_i) \geq \epsilon_0 \text{ and } v_{\alpha_i}(x_i) \geq \epsilon_0 \text{ for some } x_i \in \overline{\Omega}.$$

It is clear that $x_i \notin \partial\Omega$ for all i. Without loss of generality, we may assume that $x_i \to x_0 \in \overline{\Omega}$ as $i \to \infty$. Since $\phi_0(x)$ is a maximal solution to (2.2) and $\beta u_{\alpha_i} + v_{\alpha_i}$ satisfies (2.1) with $\alpha = \alpha_i$, which implies that $\beta u_{\alpha_i}(x) + v_{\alpha_i}(x) \leq \phi_0(x)$ on $\overline{\Omega}$. If $x_0 \in \partial\Omega$, then

$$0 < (\beta+1)\epsilon_0 \leq \beta u_{\alpha_i}(x_i) + v_{\alpha_i}(x_i) \leq \phi_0(x_i).$$

However, this contradicts $\phi_0(x_i) \to 0$ as $i \to \infty$. Hence, $x_0 \notin \partial\Omega$.

We define

$$x = x_i + \frac{y}{\sqrt{\alpha_i}},\ U_i(y) = u_{\alpha_i}(x),\ V_i(y) = v_{\alpha_i}(x),\ \Omega_i = \left\{y : x_i + \frac{y}{\sqrt{\alpha_i}} \in \Omega\right\}.$$

Then (U_i, V_i) satisfies

$$\begin{cases} -\Delta U_i = \dfrac{1}{\alpha_i} U_i(\mu - U_i) - \dfrac{U_i V_i}{\alpha_i[1 + m(1 + \alpha_i V_i)U_i]} - \dfrac{U_i V_i^2}{1 + m(1 + \alpha_i V_i)U_i}, & y \in \Omega_i, \\ -\Delta V_i = \dfrac{\beta U_i V_i}{\alpha_i[1 + m(1 + \alpha_i V_i)U_i]} + \dfrac{\beta U_i V_i^2}{1 + m(1 + \alpha_i V_i)U_i} - \dfrac{\gamma}{\alpha_i} V_i, & y \in \Omega_i, \\ U_i = V_i = 0, & y \in \partial\Omega_i. \end{cases} \tag{4.1}$$

Obviously, $0 \in \Omega_i$ due to $x_i \in \Omega$ for all i. We can combine $x_0 \notin \partial\Omega$ and the definition of Ω_i to discover that $\text{dist}(0, \partial\Omega_i) \to \infty$ as $i \to \infty$,

$$U_i(0) = u_{\alpha_i}(x_i) \geq \epsilon_0 > 0 \text{ and } V_i(0) = v_{\alpha_i}(x_i) \geq \epsilon_0 > 0. \tag{4.2}$$

As stated in Lemma 2.1, it can be deduced from (4.1) that there exists a $M > 0$ (independent of i) such that $U_i, V_i \leq M$ on $\overline{\Omega}_i$ for all i. Additionally, utilizing (4.1), we find that $-\triangle U_i$ and $-\triangle V_i$ have a bound (independent of i) in $L^\infty(B_{2R}(0))$, where $B_{2R}(0)$ represents a ball centered at the origin with a radius of $2R$. By invoking the standard elliptic regularity arguments from [10] along with the Sobolev embedding theorems, we can conclude that, upon selecting a subsequences, $U_i \to U$ and $V_i \to V$ in $C^1(\overline{B}_R(0))$. Furthermore, $0 \leq U(y), V(y) \leq M$ in $B_R(0)$. Lastly, the pair (U, V) satisfies

$$-\Delta U = -\Delta V = 0, \quad \text{in} \quad B_R(0). \tag{4.3}$$

The arbitrariness of R in (4.3) implies that (U, V) solves

$$-\Delta U = -\Delta V = 0, \quad \text{in} \quad \mathbb{R}^N. \tag{4.4}$$

Since U, V are bounded and harmonic in $\mathbb{R}^N$, U and V are constants. The boundary condition in (4.1) implies that $U \equiv V \equiv 0$, which is in contradiction with (4.2). □

As a consequence of Lemma 4.1 and Lemma 4.2, we can see the asymptotic behavior of v_α as α tends to infinity.

Lemma 4.3. $\lim_{\alpha\to\infty} v_\alpha = 0$ *uniformly on* $\overline{\Omega}$.

Next, we consider the asymptotic behavior of u_α as α tends to infinity.

Lemma 4.4. *There exists a subsequence* $\{(u_{\alpha_i}, v_{\alpha_i})\}$ *of* $\{(u_\alpha, v_\alpha)\}$ *and nonnegative functions* $u \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$ *and* $w_0 \in C^1(\overline{\Omega})$ *such that* $u_{\alpha_i} \to u, v_{\alpha_i} \to 0$ *in* $W_0^{1,2}(\Omega)$ *and* $\beta u_{\alpha_i} + v_{\alpha_i} \to w_0$ *in* $C^1(\overline{\Omega})$ *as* $\alpha_i \to \infty$. *Moreover,* $w_0 = \beta u$ *a.e. in* Ω.

Proof. Based on Lemma 2.1 and Lemma 2.2 (i), subject to a subsequence, we may assume that

$$u_{\alpha_i} \to u, v_{\alpha_i} \to v \text{ weakly in } W_0^{1,2}(\Omega) \text{ and strongly in } L^2(\Omega) \text{ as } \alpha_i \to \infty \tag{4.5}$$

for some nonnegative functions $u, v \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$. The remaining proof is divided into four steps.

Step 1. we claim that $\int_\Omega |\nabla u_{\alpha_i}|^2 dx \to \int_\Omega |\nabla u|^2 dx$ as $i \to \infty$. In view of Lemma 2.2 (ii), we can conclude $\int_\Omega uv^2\, dx = 0$ as $\alpha \to \infty$. Hence, $uv^2 = 0$ a.e. in Ω. Thus,

$$uv = 0 \text{ and } \nabla u \cdot \nabla v = 0 \text{ a.e. in } \Omega. \tag{4.6}$$

By multiplying the second equation of (1.4) with $(u,v,\alpha)=(u_{\alpha_i},v_{\alpha_i},\alpha_i)$ by the limit u, and integrating over the domain Ω, we arrive at a new expression

$$\int_\Omega \nabla u\cdot\nabla v_{\alpha_i}\ dx=\int_\Omega \frac{\beta u(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx-\gamma\int_\Omega uv_{\alpha_i}\ dx. \tag{4.7}$$

It follows form (4.5) that

$$\int_\Omega \nabla u\cdot\nabla v\ dx=\beta\lim_{i\to\infty}\int_\Omega \frac{u(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx-\gamma\int_\Omega uv\ dx.$$

This equation, coupled with (4.6), leads to

$$\lim_{i\to\infty}\int_\Omega \frac{u(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx=0. \tag{4.8}$$

A similar calculation gives

$$\int_\Omega \nabla u\cdot\nabla u_{\alpha_i}\ dx=\int_\Omega uu_{\alpha_i}(\mu-u_{\alpha_i})\ dx-\int_\Omega \frac{u(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx.$$

From (4.5) and (4.8), we can see

$$\int_\Omega |\nabla u|^2\ dx=\int_\Omega u^2(\mu-u)\ dx,\quad i\to\infty. \tag{4.9}$$

Multiplying the first equation of (1.4) with $(u,v,\alpha)=(u_{\alpha_i},v_{\alpha_i},\alpha_i)$ by u_{α_i}, integrating the obtained equation in Ω, and then using (4.5) and (4.9), we get

$$\begin{aligned}\int_\Omega |\nabla u_{\alpha_i}|^2\ dx&=\int_\Omega u_{\alpha_i}^2(\mu-u_{\alpha_i})\ dx-\int_\Omega \frac{(1+\alpha_i v_{\alpha_i})u_{\alpha_i}^2v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx\\ &\le\int_\Omega u_{\alpha_i}^2(\mu-u_{\alpha_i})\ dx\\ &\to\int_\Omega u^2(\mu-u)\ dx\\ &=\int_\Omega |\nabla u|^2\ dx.\end{aligned}$$

Hence,

$$\limsup_{i\to\infty}\int_\Omega |\nabla u_{\alpha_i}|^2\ dx\le\int_\Omega |\nabla u|^2\ dx.$$

By the weak lower semi-continuity of the norm, we can gain

$$\int_\Omega |\nabla u_{\alpha_i}|^2\ dx\to\int_\Omega |\nabla u|^2\ dx\ \text{ as }\ i\to\infty.$$

Step 2. We further show that $v=0$ a.e. in Ω. Similar to (4.7), we obtain that

$$\int_\Omega \nabla v\cdot\nabla u_{\alpha_i}\ dx=\int_\Omega vu_{\alpha_i}(\mu-u_{\alpha_i})\ dx-\int_\Omega \frac{v(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx,$$

and so it follows from (4.6) that

$$\lim_{i\to\infty}\int_\Omega \frac{v(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx=0. \tag{4.10}$$

Moreover, we can also gain

$$\int_\Omega \nabla v\cdot\nabla v_{\alpha_i}\ dx=\int_\Omega \frac{\beta v(1+\alpha_i v_{\alpha_i})u_{\alpha_i}v_{\alpha_i}}{1+m(1+\alpha_i v_{\alpha_i})u_{\alpha_i}}\ dx-\gamma\int_\Omega v v_{\alpha_i}\ dx,$$

which combined with (4.5), (4.6) and (4.10), gives

$$\int_\Omega |\nabla v|^2\ dx+\gamma\int_\Omega v^2\ dx=0.$$

So $v=0$ a.e. in Ω.

Step 3. We prove that there exists a nonnegative function $w_0\in C^1(\overline{\Omega})$ such that $\beta u_{\alpha_i}+v_{\alpha_i}\to w_0$ in $C^1(\overline{\Omega})$. Moreover, $w_0=\beta u$ a.e. in Ω. In view of (2.1), Lemma 2.1 establishes that $\{-\Delta(\beta u_{\alpha_i}+v_{\alpha_i})\}$ is bounded in L^∞. Leveraging the regularity properties of elliptic equations, we deduce that $\{\beta u_{\alpha_i}+v_{\alpha_i}\}$ is bounded in $W^{2,p}(\Omega)$ for all $p\geq 1$, and even further, it is bounded in $C^{1,\tau}(\overline{\Omega})$ for all $\tau\in(0,1)$. Consequently, there exists a subsequence of $\{\beta u_{\alpha_i}+v_{\alpha_i}\}$ that converges in $C^1(\overline{\Omega})$. For simplicity, we assume that $\beta u_{\alpha_i}+v_{\alpha_i}\to w_0$ in $C^1(\overline{\Omega})$, where w_0 is a nonnegative function belonging to $C^1(\overline{\Omega})$. Due to (4.5) and Step 1, we can deduce that $v_{\alpha_i}=(\beta u_{\alpha_i}+v_{\alpha_i})-\beta u_{\alpha_i}\to w_0-\beta u$ in $W_0^{1,2}(\Omega)$. Therefore, as per Step 2, we conclude that $w_0-\beta u=v=0$ a.e. in Ω, and so $w_0=\beta u$ a.e. in Ω.

Step 4. It remains to demonstrate that $u_{\alpha_i}\to u, v_{\alpha_i}\to 0$ in $W_0^{1,2}(\Omega)$. Obviously, $u_{\alpha_i}\to u$ in $W_0^{1,2}(\Omega)$ follows directly from (4.5) and Step 1. Furthermore, Step 3's argument establishes that $v_{\alpha_i}\to w_0-\beta u=v=0$ in $W_0^{1,2}(\Omega)$. This conclusively verifies the proof of Lemma 4.4. □

Now, let's state the key theorem regarding the asymptotic behavior of coexistence states as the cooperation effect parameters, denoted as α, tends to infinity.

Theorem 4.1. *Assume that $\mu>\lambda_1$ and $\{(u_\alpha,v_\alpha)\}$ is a sequence of positive solutions to system* (1.4). *Then as $\alpha\to\infty$, $v_\alpha\to 0$ in $C(\overline{\Omega})$ and $u_\alpha\to\theta_\mu$ in $C(\overline{\Omega})$, where θ_μ is the unique positive solution of* (1.5).

Proof. From Lemma 4.3, it is evident that $v_\alpha\to 0$ in $C(\overline{\Omega})$. Furthermore, our objective is to demonstrate that, $u_n\to u_\infty$ $\beta u_\alpha+v_\alpha\to\beta\theta_\mu$ in $C^1(\overline{\Omega})$, and $u_\alpha\to\theta_\mu$ in $C(\overline{\Omega})$ as $\alpha\to\infty$.

In view of the proof outlined in Step 3 of Lemma 4.4, there exists a subsequence $\{(u_{\alpha_i},v_{\alpha_i})\}$ of $\{(u_\alpha,v_\alpha)\}$ such that $\beta u_{\alpha_i}+v_{\alpha_i}$ is bounded in $C^{1,\tau}(\overline{\Omega})$ for all $\tau\in(0,1)$. On the other hand, Lemma 4.4 asserts that $u_{\alpha_i}\to u$ in $W_0^{1,2}(\Omega)$, $\beta u_{\alpha_i}+v_{\alpha_i}\to w_0$ in $C^1(\overline{\Omega})$, and $w_0=\beta u$ a.e. in Ω, where $u\in L^\infty(\Omega)\cap W_0^{1,2}(\Omega)$ and $w_0\in C^1(\overline{\Omega})$ are nonnegative functions. Substituting $(u,v)=(u_{\alpha_i},v_{\alpha_i})$ into the first equation of (2.1), we obtain

$$\begin{aligned}&\int_\Omega \nabla(\beta u_{\alpha_i}+v_{\alpha_i})\cdot\nabla\phi\ dx\\ =&\beta\int_\Omega u_{\alpha_i}(\mu-u_{\alpha_i})\phi\ dx-\gamma\int_\Omega v_{\alpha_i}\phi\ dx,\quad \forall\phi\in W_0^{1,2}(\Omega).\end{aligned}$$

Letting $\alpha_i \to \infty$ and using Lemma 4.4, we conclude

$$\int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega u(\mu - u)\phi \, dx,$$

which indicates that u is a weak solution for (1.5). As a result of the elliptic regularity theory, u is a classical solution. Consequently, we arrive at the conclusion that $u = \theta_\mu$ in Ω, thus finalizing the proof. □

5. Conclusions

This paper has made significant contributions to the understanding of diffusive Holling type II predator-prey models under Dirichlet boundary conditions, particularly in incorporating the element of hunting cooperation. By addressing the existence of coexistence states across various levels of hunting cooperation and elucidating their asymptotic behavior as cooperation increases indefinitely, this study provides valuable insights into the dynamic behavior of predator-prey species within ecological systems. The unique focus on Dirichlet boundary conditions fills a gap in the existing research and reveals distinct differences compared to studies under Neumann boundary conditions.

Furthermore, the comprehensive and systematic research framework and method developed in this paper not only demonstrate strong applicability to the current model but also hold potential for extension to other functional response functions, showcasing its innovation and universality. The specific choice of the Holling II type functional response function ensures that the conclusions drawn have both theoretical significance and practical applicability, thereby advancing research in this crucial field of ecological modeling.

Recently, the predator-prey models of hunting cooperation combined with other factors have also been widely studied. R. Han et al. [11] studied the temporal as well as spatio-temporal dynamics of a prey-predator model with additive Allee effect in prey growth and hunting cooperation among the specialist predators. D. Pal et al. [16] considered a modified Leslie-Gower predator-prey model incorporating fear, intra-specific competition among predators, hunting cooperation of predators and cross-diffusion. B. Mondal et al. [14] investigated the predator-prey model with Crowley-Martin response function, considering the effect of hunting cooperation. These findings highlight the complex interplay between social interactions, cooperation, and system dynamics in predator-prey systems. These excellent research achievements are mainly based on the deep study of ODE model and realized by meticulous numerical simulation technology.

In the future work, we will innovatively introduce diffusion, cross-diffusion and Dirichlet boundary conditions to deeply explore the intricate and combined effects of hunting cooperation and other various factors.

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