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FIXED POINT THEOREMS OVER $\mathfrak{B}$-METRIC-LIKE SPACES AND APPLICATIONS IN ELECTRIC CIRCUIT EQUATIONS*

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Abstract In this article, we introduce the notion of α-admissible crooked mapping with respect to θ with its special cases, which are α-admissible crooked mapping with respect to θ^* and α^*-admissible crooked mapping with respect to θ. We present the notion of $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction and establish new fixed point results over $\mathfrak{b}$-metric-like space. The study includes illustrative examples to support our results. Furthermore, we apply our results to prove the existence and uniqueness solution of the electric circuit equation, which is in second-order differential equation form.

Keywords Fixed point, $\mathfrak{b}$-metric-like space, α-admissible crooked mapping with respect to θ, $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction, electric circuit equation.

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1. Introduction

Consider Ξ as a nonempty set, and let $\mathcal{H} : \Xi \to \Xi$ be a self-mapping of Ξ. A solution to the equation $\mathcal{H}(\omega) = \omega$ is referred to as a fixed point of H. Theorems addressing the existence and construction of solutions to operator equations, specifically $\mathcal{H}(\omega) = \omega$ is considered the most important part of fixed point theory. This theory stands as a prominent research domain within nonlinear analysis. Among its pivotal theorems, Banach's and Brouwer's fixed point theorems hold paramount importance. Particularly, Banach's fixed point theorem serves as a crucial tool in the metric theory of fixed points. Banach's fixed point theorem and its generalization

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play a pivotal role in applications of many diverse fields such as physics, particularly in resolving electric circuit equations (as documented in references [9, 13, 19]). Numerous publications have focused on investigating and solving practical and theoretical problems through the application of the Banach contraction principle and its generalization (refer to [2, 7, 12, 16–18]). In 1993, Czerwik [6] introduced a noteworthy extension of this fundamental principle by introducing the concept of $\mathfrak{b}$-metric spaces. Recently, Hussain et al. [1] delved into the topological structure of this space and provided various fixed point results in $\mathfrak{b}$-metric-like space. A substantial body of work has been dedicated to exploring fixed points of mappings utilizing specific contractive conditions in $\mathfrak{b}$-metric-like space (see [3–5, 9, 11, 24]).

On the other hand, the initial conceptualization of α-admissible mapping in metric spaces was attributed to Samet et al. [15]. Subsequently, in 2013, Salimi et al. [14] broadened this notion to α-admissible mapping with respect to θ.

The basic aim of this work is to establish novel fixed point results and apply them in demonstrating the existence and uniqueness solution of the electric circuit equation, which is in the second-order differential equation form. To accomplish this goal, we reformulate or modify the work of Salimi et al. [14] by presenting a fresh concept which is α-admissible crooked mapping with respect to θ. Utilizing this concept, we introduce the notion of $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction over a $\mathfrak{b}$-metric-like space. The paper is organized as follows. In Section 2, we present the basic concepts. In Section 3, we introduce the concept of α-admissible crooked mapping with respect to θ, along with illustrative examples to support this result. Additionally, we define the concept of $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction to establish new fixed point results over $\mathfrak{b}$-metric-like spaces, with non-trivial examples provided for illustration. In Section 4, we apply these results to demonstrate the existence and uniqueness of solutions to the electric circuit equation, which is formulated as a second-order differential equation. Our findings offer a novel perspective that complements or opposite with some concepts in the existing literature, providing an innovative approach to proving the existence and uniqueness of solutions for the electric circuit equation.

2. Preliminaries

Let $\Xi \neq \emptyset$ with a parameter $\mathfrak{z} \geq 1$ and let $D_{\mathfrak{b}} : \Xi \times \Xi \to [0, \infty)$ be a mapping satisfying the following conditions, $\forall\, \omega, \nu, \mu \in \Xi$.

($\mathfrak{b}_1$) $D_{\mathfrak{b}}(\omega, \nu) = 0 \quad \Leftrightarrow \quad \omega = \nu$.

($\mathfrak{b}_2$) $D_{\mathfrak{b}}(\omega, \nu) = 0 \quad \Rightarrow \quad \omega = \nu$.

($\mathfrak{b}_3$) $D_{\mathfrak{b}}(\omega, \nu) = D_{\mathfrak{b}}(\nu, \omega)$.

($\mathfrak{b}_4$) $D_{\mathfrak{b}}(\omega, \nu) \leq \mathfrak{z}\,[D_{\mathfrak{b}}(\omega, \mu) + D_{\mathfrak{b}}(\mu, \nu)]$.

Definition 2.1 ([6])**.** A pair $(\Xi, D_{\mathfrak{b}})$ satisfying ($\mathfrak{b}_1$), ($\mathfrak{b}_3$) and ($\mathfrak{b}_4$) is called $\mathfrak{b}$-metric space ($\mathfrak{b}$ − **ms** for short).

Definition 2.2 ([1])**.** A pair $(\Xi, D_{\mathfrak{b}})$ satisfying ($\mathfrak{b}_2$), ($\mathfrak{b}_3$) and ($\mathfrak{b}_4$) is called $\mathfrak{b}$-metric like space ($\mathfrak{b}$ − **mls** for short).

Definition 2.3 ([1])**.** Let $(\Xi, D_{\mathfrak{b}})$ be a $\mathfrak{b}$-mls with $\mathfrak{z} \geq 1$.

1. $\{\omega_{\mathfrak{n}}\}$ is convergent to $\omega \in \Xi$, if $\lim\limits_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}}, \omega) = \lim\limits_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega, \omega)$.

2. $\{\omega_{\mathfrak{n}}\}$ is Cauchy sequence, if $\lim_{\mathfrak{n},\mathfrak{m}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}})$ exists and finite.
3. $(\Xi, D_{\mathfrak{b}})$ is complete, if every Cauchy sequence in Ξ, there is $\omega \in \Xi$, $\lim_{\mathfrak{n},\mathfrak{m}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}}) = D_{\mathfrak{b}}(\omega,\omega) = \lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega)$.

Lemma 2.1 ([1]). *Let $(\Xi, D_{\mathfrak{b}})$ be a $\mathfrak{b}$-mls with $\mathfrak{z} \geq 1$. Assume that $\{\omega_{\mathfrak{n}}\}$ is convergent to $\omega \in \Xi$ and $D_{\mathfrak{b}}(\omega,\omega) = 0$. Then, for all $\nu, \in \Xi$, we have*

$$\mathfrak{z}^{-1} D_{\mathfrak{b}}(\omega,\nu) \leq \liminf_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\nu) \leq \limsup_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\nu) \leq \mathfrak{z} D_{\mathfrak{b}}(\omega,\nu).$$

Lemma 2.2 ([23]). *Let $(\Xi, D_{\mathfrak{b}})$ be a $\mathfrak{b}$-mls with $\mathfrak{z} \geq 1$. Then for all $\omega, \nu, \in \Xi$ and $\{\omega_{\mathfrak{n}}\} \subset \Xi$, we have*

a. $D_{\mathfrak{b}}(\omega,\nu) = 0 \quad \Rightarrow \quad D_{\mathfrak{b}}(\omega,\omega) = D_{\mathfrak{b}}(\nu,\nu) = 0.$

b. *If* $\lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1}) = 0 \quad \Rightarrow \quad \lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}+1},\omega_{\mathfrak{n}+1}) = 0.$

c. $\omega \neq \nu \quad \Rightarrow \quad D_{\mathfrak{b}}(\omega,\nu) > 0.$

Lemma 2.3 ([22]). *Let $\{\omega_{\mathfrak{n}}\}$ be a sequence in a complete $\mathfrak{b}$-mls $(\Xi, D_{\mathfrak{b}})$ with $\mathfrak{z} \geq 1$ such that*

$$\lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1}) = 0.$$

If $\lim_{\mathfrak{n},\mathfrak{m}\to+\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}}) \neq 0$, then there is $\varepsilon > 0$ and sequences of positive integers $\{\mathfrak{n}(\mathfrak{i})\}_{\mathfrak{i}=1}^{\infty}$ and $\{\mathfrak{m}(\mathfrak{i})\}_{\mathfrak{i}=1}^{\infty}$ with $\mathfrak{n}_{\mathfrak{i}} > \mathfrak{m}_{\mathfrak{i}} > \mathfrak{i}$ such that

$$D_{\mathfrak{b}}(\omega_{\mathfrak{m}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}}) \geq \varepsilon, \;\; D_{\mathfrak{b}}(\omega_{\mathfrak{m}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}-1}) < \varepsilon,$$
$$\varepsilon/\mathfrak{z}^2 \leq \limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{m}_{\mathfrak{i}}-1},\omega_{\mathfrak{n}_{\mathfrak{i}}-1}) \leq \varepsilon\mathfrak{z}, \;\; \varepsilon/\mathfrak{z} \leq \limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}-1},\omega_{\mathfrak{m}_{\mathfrak{i}}}) \leq \varepsilon$$

and

$$\varepsilon/\mathfrak{z} \leq \limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{m}_{\mathfrak{i}}-1},\omega_{\mathfrak{n}_{\mathfrak{i}}}) \leq \varepsilon\mathfrak{z}^2.$$

In this study, we will need some functions, which we know as follows:

(1) Φ be the set of all continuous and strictly increasing functions $F : (0,\infty) \to \mathbb{R}$.

(2) Ψ be the set of each function $\psi : (0,+\infty) \to (0,+\infty)$ satisfying the following condition: $\lim_{\mathfrak{s}\to\iota^+} \inf \psi(\mathfrak{s}) > 0, \; \forall \; \iota > 0$.
For more information on both Φ and Ψ (see [20, 21]).

(3) $\Im$ be the set of functions $\beta : [0,\infty) \to [0,1)$ such that

$$\lim_{\mathfrak{n}\to\infty} \beta(\iota_{\mathfrak{n}}) = 1 \quad \Rightarrow \quad \lim_{\mathfrak{n}\to\infty} \iota_{\mathfrak{n}} = 0.$$

(4) Γ be the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$.

Definition 2.4 ([8, 10]). The continuous function $\mathfrak{F} : [0,\infty) \times [0,\infty) \to \mathbb{R}$ that satisfies the below conditions:

($\mathfrak{c}_1$) $\mathfrak{F}(\iota,\mathfrak{s}) \leq \iota, \; \forall \; \iota,\mathfrak{s} \geq 0$,

($\mathfrak{c}_2$) $\mathfrak{F}(\iota,\mathfrak{s}) = \iota \Rightarrow \iota = 0$ or $\mathfrak{s} = 0, \; \forall \; \iota,\mathfrak{s} \geq 0$,

($\mathfrak{c}_3$) $\mathfrak{F}(0,0) = 0, \; \forall \; \iota,\mathfrak{s} \geq 0$, is called the C-class function and the set that contains all $\mathfrak{F}$ is denoted by C.

Example 2.1. The below functions $\mathfrak{F}$ belong to C, $\forall\ \iota, \mathfrak{s} \in [0, \infty)$

(1) $\mathfrak{F}(\iota, \mathfrak{s}) = \iota - \mathfrak{s}$.

(2) $\mathfrak{F}(\iota, \mathfrak{s}) = k\iota, \quad k \in (0, 1)$.

(3) $\mathfrak{F}(\iota, \mathfrak{s}) = \frac{\iota}{(1+\mathfrak{s})^{\mathfrak{r}}}, \quad \mathfrak{r} \in (0, \infty)$.

Definition 2.5 ([15]). Let $\alpha : \Xi \times \Xi \to (0, \infty)$ be a mapping. Then $\mathcal{H} : \Xi \to \Xi$ is called α-admissible mapping if:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1.$$

In 2013 [14] extended the above concept as follows

Definition 2.6 ([14]). Let $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$. Then $\mathcal{H} : \Xi \to \Xi$ is called α-admissible mapping with respect to θ if:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq \theta(\omega, \nu) \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\mathcal{H}\omega, \mathcal{H}\nu).$$

Example 2.2. Let $\Xi = \mathbb{R}$ and $\mathfrak{z} \geq 1$. Define $\mathcal{H} : \Xi \to \Xi$ by

$$\mathcal{H}(\omega) = \begin{cases} \mathfrak{z}\omega, & \text{if} \quad \omega \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

and $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$ by

$$\alpha(\omega, \nu) = \begin{cases} e^{\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

$$\theta(\omega, \nu) = \begin{cases} e^{-\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}. \end{cases}$$

Then $\mathcal{H}$ is an α-admissible mapping with respect to θ.

3. Main results

In this section, we present our main results, which include the introduction of novel fixed points. This is achieved by introducing two new concepts: the α-admissible crooked mapping with respect to θ and the $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction. Furthermore, we provide illustrative examples to aid in the clarification.

Definition 3.1. Let $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$. Then $\mathcal{H} : \Xi \to \Xi$ is called α-admissible crooked mapping with respect to θ if:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq \theta(\omega, \nu) \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu).$$

Example 3.1. Let $\Xi = \mathbb{R}$ and $\mathfrak{z} \geq 1$. Define $\mathcal{H} : \Xi \to \Xi$ by

$$\mathcal{H}(\omega) = \begin{cases} -\mathfrak{z}\omega, & \text{if} \quad \omega \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

and $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$ by

$$\alpha(\omega, \nu) = \begin{cases} e^{\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

$$\theta(\omega, \nu) = \begin{cases} e^{-\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly $\alpha(\omega, \nu) \geq \theta(\omega, \nu)$, $\omega, \nu \in [0, \infty)$. Since $\theta(\mathcal{H}\omega, \mathcal{H}\nu) = e^{-\mathcal{H}\omega} = e^{\mathfrak{z}\omega}$ and $\alpha(\mathcal{H}\omega, \mathcal{H}\nu) = e^{\mathcal{H}\omega} = e^{-\mathfrak{z}\omega}$. Then we get, $\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)$, $\omega, \nu \in [0, \infty)$. Hence, $\mathcal{H}$ is an α-admissible crooked mapping with respect to θ.

Remark 3.1. (1) If $\theta(\omega, \nu) = 1$ in Definition 3.1 we get the following condition:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \leq 1.$$

In this case $\mathcal{H}$ is called α-admissible crooked mapping with respect to θ^*.

(2) If $\alpha(\omega, \nu) = 1$ in Definition 3.1 we get the following condition:

$$\omega, \nu \in \Xi, \quad \theta(\omega, \nu) \leq 1 \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1.$$

In this case $\mathcal{H}$ is called α^*-admissible crooked mapping with respect to θ.

In Example 3.1 if we take $\theta(\omega, \nu) = 1$ then we get $\alpha(\omega, \nu) = e^{\omega} \geq 1$, $\omega, \nu \in [0, \infty)$. Since $\theta(\mathcal{H}\omega, \mathcal{H}\nu) = 1$, then $\alpha(\mathcal{H}\omega, \mathcal{H}\nu) = e^{-\mathfrak{z}\omega} \leq 1$, $\omega, \nu \in [0, \infty)$. And if we take $\alpha(\omega, \nu) = 1$ then we get $\theta(\omega, \nu) = e^{-\omega} \leq 1$, $\omega, \nu \in [0, \infty)$. Since $\alpha(\mathcal{H}\omega, \mathcal{H}\nu) = 1$, then $\theta(\mathcal{H}\omega, \mathcal{H}\nu) = e^{\mathfrak{z}\omega} \geq 1$, $\omega, \nu \in [0, \infty)$.

Remark 3.2. Let $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$ and $\mathcal{H} : \Xi \to \Xi$, then

(1) either $\mathcal{H}$ is an α-admissible mapping with respect to θ or α-admissible crooked mapping with respect to θ,

(2) $\mathcal{H}$ is both α-admissible mapping with respect to θ and α-admissible crooked mapping with respect to θ if and only if $\alpha(\omega, \nu) = \theta(\omega, \nu)$.

Definition 3.2. Let $(\Xi, D_{\mathfrak{b}})$ be a $\mathfrak{b}$-mls with parameter $\mathfrak{z} \geq 1$. A mapping $\mathcal{H} : \Xi \to \Xi$ is called an $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction if there is $F \in \Phi$, $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\mathfrak{F} \in C$, $\psi \in \Psi$, $\gamma \in \Gamma$ and $\forall\ \omega, \nu \in \Xi$, $D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ such that

$$\alpha(\omega, \nu)\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)\alpha(\mathcal{H}\omega, \mathcal{H}\nu),$$

and

$$\theta(\mathcal{H}\omega, \mathcal{H}\nu)\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)\theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$$

implies

$$\psi(D_{\mathfrak{b}}(\omega, \nu)) + F\left(\mathfrak{z} D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu), \gamma\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right)\right) \tag{3.1}$$

where

$$\Lambda(\omega, \nu) = \max\left\{ D_{\mathfrak{b}}(\omega, \nu), D_{\mathfrak{b}}(\omega, \mathcal{H}\omega), D_{\mathfrak{b}}(\nu, \mathcal{H}\nu), \frac{D_{\mathfrak{b}}(\omega, \mathcal{H}\nu) + D_{\mathfrak{b}}(\nu, \mathcal{H}\omega)}{2\mathfrak{z}}, \right.$$
$$\left. \min\left\{ \frac{D_{\mathfrak{b}}(\omega, \mathcal{H}\omega) D_{\mathfrak{b}}(\nu, \mathcal{H}\nu)}{1 + D_{\mathfrak{b}}(\omega, \nu)}, \frac{D_{\mathfrak{b}}(\nu, \mathcal{H}\nu)[1 + D_{\mathfrak{b}}(\omega, \mathcal{H}\omega)]}{1 + D_{\mathfrak{b}}(\omega, \nu)} \right\} \right\}.$$

Theorem 3.1. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H}: \Xi \to \Xi$ be an $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction. If the below conditions are satisfied:*

(h_1) *$\mathcal{H}$ is an α-admissible crooked mapping with respect to θ.*

(h_2) *There is $\omega_0 \in \Xi$ such that $\alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0)$.*

(h_3) *If $\{\mathfrak{n}_\mathfrak{i}\}$ and $\{\mathfrak{m}_\mathfrak{i}\}$ are sequences of positive integers with $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, then $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq \theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1})$.*

(h_4) *$\mathcal{H}$ is continuous,*

(h_4') *or if $\{\omega_\mathfrak{n}\}$ is a sequence in Ξ such that $\omega_\mathfrak{n} \to \omega \in \Xi$ and $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq \theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})$, then there is a subsequence $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ of $\{\omega_\mathfrak{n}\}$ with $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq \theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega)$.*

(h_5) *For all $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ fixed points of $\mathcal{H}$, we have $\alpha(\omega, \nu) \geq \theta(\omega, \nu)$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. Define a sequence $\{\omega_\mathfrak{n}\}$ by $\omega_{\mathfrak{n}+1} = \mathcal{H}^{\mathfrak{n}+1}\omega_0 = \mathcal{H}\omega_\mathfrak{n}$ $\forall$ $\mathfrak{n} \in \mathbb{N}$. For some $\mathfrak{n}$, if $\omega_{\mathfrak{n}+1} = \omega_\mathfrak{n}$, then $\mathcal{H}\omega_\mathfrak{n} = \omega_\mathfrak{n}$. Thus, $\omega_\mathfrak{n}$ is a fixed point of $\mathcal{H}$, proof is completed. So suppose that $D_\mathfrak{b}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) > 0$, $\forall$ $\mathfrak{n} \in \mathbb{N}$. By (h_2), $\exists$ $\omega_0 \in \Xi$ such that $\alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0)$, then we get

$$\alpha(\omega_0, \omega_1) = \alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0) = \theta(\omega_0, \omega_1)$$

then

$$\theta(\mathcal{H}\omega_0, \mathcal{H}\omega_1) \geq \alpha(\mathcal{H}\omega_0, \mathcal{H}\omega_1)$$

which implies

$$\alpha(\omega_0, \omega_1)\theta(\mathcal{H}\omega_0, \mathcal{H}\omega_1) \geq \theta(\omega_0, \omega_1)\alpha(\mathcal{H}\omega_0, \mathcal{H}\omega_1). \tag{3.2}$$

Since,

$$\theta(\omega_1, \omega_2) = \theta(\mathcal{H}\omega_0, \mathcal{H}\omega_1) \geq \alpha(\mathcal{H}\omega_0, \mathcal{H}\omega_1) = \alpha(\omega_1, \omega_2),$$

then

$$\alpha(\mathcal{H}\omega_1, \mathcal{H}\omega_2) \geq \theta(\mathcal{H}\omega_1, \mathcal{H}\omega_2)$$

which implies

$$\theta(\omega_1, \omega_2)\alpha(\mathcal{H}\omega_1, \mathcal{H}\omega_2) \geq \alpha(\omega_1, \omega_2)\theta(\mathcal{H}\omega_1, \mathcal{H}\omega_2),$$

or

$$\theta(\mathcal{H}\omega_0, \mathcal{H}\omega_1)\alpha(\mathcal{H}^2\omega_0, \mathcal{H}^2\omega_1) \geq \alpha(\mathcal{H}\omega_0, \mathcal{H}\omega_1)\theta(\mathcal{H}^2\omega_0, \mathcal{H}^2\omega_1). \tag{3.3}$$

From (3.2), (3.3) and continuing in this manner, we obtain

$$\begin{cases} \alpha(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\theta(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n}) \geq \theta(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\alpha(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n}), \\ \theta(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n})\alpha(\mathcal{H}^2\omega_{\mathfrak{n}-1}, \mathcal{H}^2\omega_\mathfrak{n}) \geq \alpha(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n})\theta(\mathcal{H}^2\omega_{\mathfrak{n}-1}, \mathcal{H}^2\omega_\mathfrak{n}). \end{cases} \tag{3.4}$$

Using (3.4), and applying (3.1), we get

$$\begin{aligned} &\psi(D_\mathfrak{b}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})) + F(\mathfrak{z}D_\mathfrak{b}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})) \\ =&\psi(D_\mathfrak{b}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})) + F(\mathfrak{z}D_\mathfrak{b}(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n})) \\ \leq&F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\right)\Lambda(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), \gamma\left(\Lambda(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\right)\Lambda(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\right)\right), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
&\Lambda(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})\\
&=\max\Bigg\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\mathcal{H}\omega_{\mathfrak{n}-\mathfrak{1}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\mathcal{H}\omega_{\mathfrak{n}}),\\
&\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\mathcal{H}\omega_{\mathfrak{n}})+D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\mathcal{H}\omega_{\mathfrak{n}-\mathfrak{1}})}{2\mathfrak{z}},\\
&\min\left\{\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\mathcal{H}\omega_{\mathfrak{n}-\mathfrak{1}})D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\mathcal{H}\omega_{\mathfrak{n}})}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})},\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\mathcal{H}\omega_{\mathfrak{n}})[1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\mathcal{H}\omega_{\mathfrak{n}-\mathfrak{1}})]}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})}\right\}\Bigg\}\\
&=\max\Bigg\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}}),\\
&\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}+\mathfrak{1}})+D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}})}{2\mathfrak{z}},\\
&\min\left\{\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})},\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})[1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})]}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})}\right\}\Bigg\}\\
&\le\max\Bigg\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}}),\frac{\mathfrak{z}[D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})+D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})]}{2\mathfrak{z}},\\
&\min\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})\}\Bigg\}\\
&\le\max\Bigg\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}),D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})\Bigg\}.
\end{aligned}$$

Now, if $D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})<D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})$, then from (3.5), $(\mathfrak{c}_{\mathfrak{1}})$ and $\beta\in\Im$ we get

$$\begin{aligned}
&\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}))+F(\mathfrak{z}D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}}))\\
&\le F\left(\mathfrak{F}\left(\beta\left(D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1}),\gamma\left(D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)\right)\\
&\le F\left(\beta\left(D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)\\
&\le F\left(D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right),
\end{aligned}\tag{3.6}$$

which is a contradiction. Therefore,

$$D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})<D_{\mathfrak{b}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}}).\tag{3.7}$$

Hence, $\{D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\}$ is a decreasing sequence of positive numbers. So, there is $\mathfrak{r}\ge 0$ such that $D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\to\mathfrak{r}$ as $\mathfrak{n}\to\infty$. Suppose that $\mathfrak{r}>0$. Using (3.5) and (3.7), we get

$$\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}))+F(\mathfrak{z}D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}}))\le F\left(D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}})\right).\tag{3.8}$$

Take the limit as $\mathfrak{n}\to\infty$, then

$$\liminf_{\mathfrak{n}\to\infty}\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}-\mathfrak{1}},\omega_{\mathfrak{n}}))+F(\mathfrak{z}\mathfrak{r})\le F\left(\mathfrak{r}\right),$$

which is a contradiction, then $\mathfrak{r}=0$. Hence,

$$\lim_{\mathfrak{n}\to\infty}D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+\mathfrak{1}})=0.\tag{3.9}$$

Now, we prove that $\lim\limits_{\mathfrak{n},\mathfrak{m}\to+\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}})=0$, using proof by contradiction, so we assume that $\lim\limits_{\mathfrak{n},\mathfrak{m}\to+\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}})>0$. By Lemma 2.3, there is $\varepsilon>0$ and sequences of positive integers $\{\mathfrak{n}_\mathfrak{i}\}$, $\{\mathfrak{m}_\mathfrak{i}\}$ with $\mathfrak{n}_\mathfrak{i}>\mathfrak{m}_\mathfrak{i}>\mathfrak{i}$ such that

$$\begin{aligned}
&D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}})\geq\varepsilon, D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}-1})<\varepsilon,\\
&\varepsilon/\mathfrak{z}^2\leq\limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\leq\varepsilon\mathfrak{z},\\
&\varepsilon/\mathfrak{z}\leq\limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}_\mathfrak{i}-1},\omega_{\mathfrak{m}_\mathfrak{i}})\leq\varepsilon,\\
&\varepsilon/\mathfrak{z}\leq\limsup_{\mathfrak{i}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}})\leq\varepsilon\mathfrak{z}^2.
\end{aligned}$$

Since $0<\varepsilon\leq D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}})=D_{\mathfrak{b}}(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})$. By (h_3), we have

$$\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\geq\theta(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1}),$$

and by (h_1) implies that

$$\theta(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})\geq\alpha(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1}). \tag{3.10}$$

Then

$$\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\theta(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})\geq\theta(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\alpha(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1}.\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1}). \tag{3.11}$$

Since

$$\theta(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}})=\theta(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})\geq\alpha(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})=\alpha(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}}),$$

then by (h_1), we get

$$\begin{aligned}
\alpha(\mathcal{H}^2\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}^2\omega_{\mathfrak{n}_\mathfrak{i}-1})&=\alpha(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}})\\
&\geq\theta(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}})\\
&=\theta(\mathcal{H}^2\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}^2\omega_{\mathfrak{n}_\mathfrak{i}-1}),
\end{aligned}$$

which implies

$$\begin{aligned}
&\theta(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})\alpha(\mathcal{H}^2\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}^2\omega_{\mathfrak{n}_\mathfrak{i}-1})\\
\geq&\alpha(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1})\theta(\mathcal{H}^2\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}^2\omega_{\mathfrak{n}_\mathfrak{i}-1}).
\end{aligned} \tag{3.12}$$

From (3.11), (3.12) and using (3.1), we get

$$\begin{aligned}
&\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1}))+F(\mathfrak{z}D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}},\omega_{\mathfrak{n}_\mathfrak{i}})) \qquad (3.13)\\
=&\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1}))+F(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega_{\mathfrak{m}_\mathfrak{i}-1},\mathcal{H}\omega_{\mathfrak{n}_\mathfrak{i}-1}))\\
\leq&F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\right)\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1}),\right.\right.\\
&\left.\left.\gamma\left(\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\right)\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\right)\right)\\
\leq&F\left(\beta\left(\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\right)\right)\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1}))\\
\leq&F\left(\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})\right)),
\end{aligned}$$

where

$$\Lambda(\omega_{\mathfrak{m}_\mathfrak{i}-1},\omega_{\mathfrak{n}_\mathfrak{i}-1})$$

$$= \max\Bigg\{ D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1}), D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \mathcal{H}\omega_{\mathfrak{m}_i-1}), D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \mathcal{H}\omega_{\mathfrak{n}_i-1}),$$
$$\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \mathcal{H}\omega_{\mathfrak{n}_i-1}) + D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \mathcal{H}\omega_{\mathfrak{m}_i-1})}{2\mathfrak{z}},$$
$$\min\Bigg\{ \frac{D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \mathcal{H}\omega_{\mathfrak{m}_i-1}) D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \mathcal{H}\omega_{\mathfrak{n}_i-1})}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})},$$
$$\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \mathcal{H}\omega_{\mathfrak{n}_i-1})[1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \mathcal{H}\omega_{\mathfrak{m}_i-1})]}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})} \Bigg\}\Bigg\}$$
$$= \max\Bigg\{ D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1}), D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{m}_i}), D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \omega_{\mathfrak{n}_i}),$$
$$\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i}) + D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \omega_{\mathfrak{m}_i})}{2\mathfrak{z}},$$
$$\min\Bigg\{ \frac{D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{m}_i}) D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \omega_{\mathfrak{n}_i})}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})},$$
$$\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_i-1}, \omega_{\mathfrak{n}_i})[1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{m}_i})]}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})} \Bigg\}\Bigg\}.$$

Using Lemma 2.3 and by (3.9), we get

$$\limsup_{i\to\infty} \Lambda(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1}) \le \max\left\{ \varepsilon\mathfrak{z}, 0, 0, \frac{\varepsilon\mathfrak{z}^2 + \varepsilon}{2\mathfrak{z}}, \min\left\{ \frac{(0)(0)}{1+\varepsilon\mathfrak{z}}, \frac{(0)[1+0]}{1+\varepsilon\mathfrak{z}} \right\}\right\} \tag{3.14}$$
$$\le \max\left\{ \varepsilon\mathfrak{z}, 0, 0, \frac{\varepsilon\mathfrak{z}^2 + \varepsilon\mathfrak{z}^2}{2\mathfrak{z}}, \min\left\{ \frac{(0)(0)}{1+\varepsilon\mathfrak{z}}, \frac{(0)[1+0]}{1+\varepsilon\mathfrak{z}} \right\}\right\}$$
$$\le \max\left(\varepsilon\mathfrak{z}, 0, 0, \varepsilon\mathfrak{z}, 0\right)$$
$$\le \varepsilon\mathfrak{z}.$$

Taking $\lim\limits_{i\to\infty}$ along (3.13) and using (3.14), we have

$$\liminf_{i\to\infty} \psi(D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})) + F(\varepsilon\mathfrak{z}) \le \liminf_{i\to\infty} \psi(D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})) \tag{3.15}$$
$$+ F\left(\mathfrak{z} \limsup_{i\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i}, \omega_{\mathfrak{n}_i})\right)$$
$$\le F\left(\limsup_{i\to\infty} \Lambda\left(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1}\right)\right)$$
$$\le F(\varepsilon\mathfrak{z}),$$

which is a contradiction since $\liminf\limits_{i\to\infty} \psi(D_{\mathfrak{b}}(\omega_{\mathfrak{m}_i-1}, \omega_{\mathfrak{n}_i-1})) > 0$ and $\varepsilon > 0$. Therefore, our assumption was wrong, and hence,

$$\lim_{\mathfrak{n},\mathfrak{m}\to+\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}}, \omega_{\mathfrak{m}}) = 0. \tag{3.16}$$

This means $\{\omega_{\mathfrak{n}}\}$ is a Cauchy sequence. Since Ξ is complete, $\exists\ \omega^* \in \Xi$ such that

$$\lim_{\mathfrak{n}\to\infty} D_{\mathfrak{b}}(\omega_{\mathfrak{n}}, \omega^*) = 0. \tag{3.17}$$

Next, we prove that $\mathcal{H}\omega^* = \omega^*$. By (h_4), gives $\mathcal{H}\omega_{\mathfrak{n}} \to \mathcal{H}\omega^*$ as $\mathfrak{n} \to \infty$.

Thus,

$$\mathcal{H}\omega^* = \lim_{\mathfrak{n}\to\infty} \mathcal{H}\omega_{\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty} \omega_{\mathfrak{n}+\mathfrak{1}} = \omega^*.$$

Now, we will use (h_4') to prove that $\mathcal{H}\omega^* = \omega^*$, so assume that $D_{\mathfrak{b}}(\mathcal{H}\omega^*, \omega^*) > 0$. Since there is $\omega^* \in \Xi$ such that $\omega_{\mathfrak{n}} \to \omega^*$ as $\mathfrak{n} \to \infty$, then by (h_4'), there is a subsequence $\{\omega_{\mathfrak{n}_{\mathfrak{i}}}\}$ of $\{\omega_{\mathfrak{n}}\}$ with

$$\alpha(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*) \geq \theta(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*),$$

and by (h_1), we have

$$\theta(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) \geq \alpha(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*). \tag{3.18}$$

Then, we obtain

$$\alpha(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\theta(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) \geq \theta(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\alpha(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*). \tag{3.19}$$

Since

$$\theta(\omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}, \omega^*) = \theta(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) \geq \alpha(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) = \alpha(\omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}, \omega^*)$$

and by (h_1), we get

$$\alpha(\mathcal{H}^2\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}^2\omega^*) = \alpha(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}, \mathcal{H}\omega^*) \geq \theta(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}, \mathcal{H}\omega^*) = \theta(\mathcal{H}^2\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}^2\omega^*)$$

which implies

$$\theta(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*)\alpha(\mathcal{H}^2\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}^2\omega^*) \geq \alpha(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*)\theta(\mathcal{H}^2\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}^2\omega^*). \tag{3.20}$$

From (3.19), (3.20) and using (3.1), we get

$$\begin{aligned} & \psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)) + F(\mathfrak{z} D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}, \mathcal{H}\omega^*)) \\ =& \psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)) + F(\mathfrak{z} D_{\mathfrak{b}}(\mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*)) \\ \leq& F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\right)\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*), \gamma\left(\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\right)\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\right)\right) \\ \leq& F\left(\beta\left(\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\right)\right)\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)) \\ \leq& F\left(\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)\right)), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} & \Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*) \\ =& \max\Bigg\{ D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*), D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}}), D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*), \\ & \frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) + D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}})}{2\mathfrak{z}}, \\ & \min\left\{ \frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}})D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)}, \frac{D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)[1 + D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega_{\mathfrak{n}_{\mathfrak{i}}})]}{1 + D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*)} \right\} \Bigg\} \\ =& \max\Bigg\{ D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega^*), D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}}), D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*), \\ & \frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}}, \mathcal{H}\omega^*) + D_{\mathfrak{b}}(\omega^*, \omega_{\mathfrak{n}_{\mathfrak{i}}+\mathfrak{1}})}{2\mathfrak{z}}, \end{aligned}$$

$$\min\left\{\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}+1})D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)},\frac{D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)[1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}+1})]}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)}\right\}\Bigg\}$$

$$\begin{aligned}\leq \max\Bigg\{&D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*),D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}+1}),D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*),\\ &\frac{\mathfrak{z}[D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)+D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)]+D_{\mathfrak{b}}(\omega^*,\omega_{\mathfrak{n}_{\mathfrak{i}}+1})}{2\mathfrak{z}},\\ &\min\left\{\frac{D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}+1})D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)},\frac{D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)[1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega_{\mathfrak{n}_{\mathfrak{i}}+1})]}{1+D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)}\right\}\Bigg\}.\end{aligned}$$

Taking $\limsup_{\mathfrak{i}\to\infty}\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)$, we get

$$\begin{aligned}\limsup_{\mathfrak{i}\to\infty}\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)\leq \max\Bigg\{&0,0,D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*),\frac{\mathfrak{z}[0+D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)]+0}{2\mathfrak{z}},\\ &\min\{0,D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)\}\Bigg\}\\ \leq& D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*).\end{aligned}\tag{3.22}$$

Also taking $\lim_{\mathfrak{i}\to\infty}$ along (3.21) and using (3.22), we get

$$\begin{aligned}&\liminf_{\mathfrak{i}\to\infty}\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*))+\limsup_{\mathfrak{i}\to\infty}F(\mathfrak{z}D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}+1},\mathcal{H}\omega^*))\\ \leq&\limsup_{\mathfrak{i}\to\infty}F\left(\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)\right)\\ \leq& F\left(\limsup_{\mathfrak{i}\to\infty}\Lambda(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*)\right)\\ \leq& F(D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)),\end{aligned}\tag{3.23}$$

from Lemma 2.1, we have

$$F\left(\mathfrak{z}^{-1}D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)\right)\leq\limsup_{\mathfrak{i}\to\infty}F(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}+1},\mathcal{H}\omega^*)),$$

then

$$\begin{aligned}\liminf_{\mathfrak{i}\to\infty}\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*))+F\left(D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)\right)\leq&\liminf_{\mathfrak{i}\to\infty}\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*))\\ &+\limsup_{\mathfrak{i}\to\infty}F(\mathfrak{z}D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}+1},\mathcal{H}\omega^*))\\ \leq& F\left(D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)\right),\end{aligned}\tag{3.24}$$

which is a contradiction since $\liminf_{\mathfrak{i}\to\infty}\psi(D_{\mathfrak{b}}(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega^*))>0$ and hence

$$D_{\mathfrak{b}}(\omega^*,\mathcal{H}\omega^*)=0\quad\Rightarrow\quad\mathcal{H}\omega^*=\omega^*.$$

Uniqueness

For uniqueness, suppose that there are two fixed points $\omega^*,\nu^*\in\Xi$ of $\mathcal{H}$ such that $\omega^*\neq\nu^*$, then by (h_5), we have

$$\alpha(\omega^*,\nu^*)\geq\theta(\omega^*,\nu^*),$$

and by (h_1), we get

$$\theta(\mathcal{H}\omega^*, \mathcal{H}\nu^*) \geq \alpha(\mathcal{H}\omega^*, \mathcal{H}\nu^*). \tag{3.25}$$

Then

$$\alpha(\omega^*, \nu^*)\theta(\mathcal{H}\omega^*, \mathcal{H}\nu^*) \geq \theta(\omega^*, \nu^*)\alpha(\mathcal{H}\omega^*, \mathcal{H}\nu^*). \tag{3.26}$$

Since,

$$\theta(\omega^*, \nu^*) = \theta(\mathcal{H}\omega^*, \mathcal{H}\nu^*) \geq \alpha(\mathcal{H}\omega^*, \mathcal{H}\nu^*) = \alpha(\omega^*, \nu^*),$$

by (h_1), we get

$$\alpha(\mathcal{H}^2\omega^*, \mathcal{H}^2\nu^*) = \alpha(\mathcal{H}\omega^*, \mathcal{H}\nu^*) \geq \theta(\mathcal{H}\omega^*, \mathcal{H}\nu^*) = \theta(\mathcal{H}^2\omega^*, \mathcal{H}^2\nu^*),$$

which implies

$$\theta(\mathcal{H}\omega^*, \mathcal{H}\nu^*)\alpha(\mathcal{H}^2\omega^*, \mathcal{H}^2\nu^*) \geq \alpha(\mathcal{H}\omega^*, \mathcal{H}\nu^*)\theta(\mathcal{H}^2\omega^*, \mathcal{H}^2\nu^*). \tag{3.27}$$

From (3.26),(3.27) and using (3.1), we obtain

$$\begin{aligned}
&\psi(D_{\mathfrak{b}}(\omega^*, \nu^*)) + F(\mathfrak{z}D_{\mathfrak{b}}(\omega^*, \nu^*)) \\
=&\psi(D_{\mathfrak{b}}(\omega^*, \nu^*)) + F(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega^*, \mathcal{H}\nu^*)) \\
\leq&F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega^*, \nu^*)\right)\Lambda(\omega^*, \nu^*), \gamma\left(\Lambda(\omega^*, \nu^*)\right)\Lambda(\omega^*, \nu^*)\right)\right) \\
\leq&F\left(\beta\left(\Lambda(\omega^*, \nu^*)\right)\Lambda(\omega^*, \nu^*)\right) \\
\leq&F\left(\Lambda(\omega^*, \nu^*)\right),
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
&\Lambda(\omega^*, \nu^*) \\
=&\max\Bigg\{D_{\mathfrak{b}}(\omega^*, \nu^*), D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*), D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*), \frac{D_{\mathfrak{b}}(\omega^*, \mathcal{H}\nu^*) + D_{\mathfrak{b}}(\nu^*, \mathcal{H}\omega^*)}{2\mathfrak{z}}, \\
&\min\left\{\frac{D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*)}{1 + D_{\mathfrak{b}}(\omega^*, \nu^*)}, \frac{D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*)[1 + D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)]}{1 + D_{\mathfrak{b}}(\omega^*, \nu^*)}\right\}\Bigg\} \\
\leq&\max\Bigg\{D_{\mathfrak{b}}(\omega^*, \nu^*), D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*), D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*), \\
&\frac{\mathfrak{z}[D_{\mathfrak{b}}(\omega^*, \nu^*) + D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*)] + \mathfrak{z}[D_{\mathfrak{b}}(\nu^*, \omega^*) + D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)]}{2\mathfrak{z}}, \\
&\min\left\{\frac{D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*)}{1 + D_{\mathfrak{b}}(\omega^*, \nu^*)}, \frac{D_{\mathfrak{b}}(\nu^*, \mathcal{H}\nu^*)[1 + D_{\mathfrak{b}}(\omega^*, \mathcal{H}\omega^*)]}{1 + D_{\mathfrak{b}}(\omega^*, \nu^*)}\right\}\Bigg\} \\
\leq&\max\left\{D_{\mathfrak{b}}(\omega^*, \nu^*), 0, 0, \frac{\mathfrak{z}[D_{\mathfrak{b}}(\omega^*, \nu^*)] + \mathfrak{z}[D_{\mathfrak{b}}(\nu^*, \omega^*)]}{2\mathfrak{z}}, 0\right\} \\
=&D_{\mathfrak{b}}(\nu^*, \omega^*).
\end{aligned}$$

Then, Equation (3.28) becomes

$$\psi(D_{\mathfrak{b}}(\omega^*, \nu^*)) + F(\mathfrak{z}D_{\mathfrak{b}}(\omega^*, \nu^*)) \leq F\left(D_{\mathfrak{b}}(\omega^*, \nu^*)\right). \tag{3.29}$$

This is a contradiction, and thus, $D_{\mathfrak{b}}(\omega^*, \nu^*) = 0 \quad \Rightarrow \quad \omega^* = \nu^*$. □

Example 3.2. Let $\Xi = \mathbb{R}$ and $D_{\mathfrak{b}} : \Xi \times \Xi \to [0, \infty)$ be defined by $D_{\mathfrak{b}}(\omega, \nu) = |\omega - \nu|^2$ for all $\omega, \nu \in \Xi$. Then $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls with parameter $\mathfrak{z} = 2$. Consider the maps $\mathcal{H} : \Xi \to \Xi$ given by

$$\mathcal{H}(\omega) = \begin{cases} -\dfrac{\omega}{4}, & \text{if} \quad \omega \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

and $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$ by

$$\alpha(\omega, \nu) = \begin{cases} e^{\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}, \end{cases}$$

$$\theta(\omega, \nu) = \begin{cases} e^{-\omega}, & \text{if} \quad \omega, \nu \in [0, \infty), \\ 0, & \text{otherwise}. \end{cases}$$

Clearly $\mathcal{H}$ is an α-admissible crooked mapping with respect to θ and we get

$$\alpha(\omega, \nu)\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)\alpha(\mathcal{H}\omega, \mathcal{H}\nu),$$

and

$$\theta(\mathcal{H}\omega, \mathcal{H}\nu)\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)\theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu).$$

Now, we prove that $\mathcal{H}$ is an $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction. Consider $F(\mathfrak{t}) = \mathfrak{t}$, $\psi(\mathfrak{t}) = \frac{\mathfrak{t}}{8}$, $\beta(\mathfrak{t}) = \frac{1}{2} \in [0, 1)$, $\mathfrak{F}(\mathfrak{t}, \mathfrak{s}) = \mathfrak{t}$ and $\gamma(\mathfrak{t}) = 0$. Then, we have

$$\begin{aligned}
\psi(D_{\mathfrak{b}}(\omega, \nu)) + F\left(\mathfrak{z} D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu)\right) &= \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) + F\left(2 D_{\mathfrak{b}}(-\frac{\omega}{4}, -\frac{\nu}{4})\right) \\
&= \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) + F\left(2\left|-\frac{\omega}{4} - (-\frac{\nu}{4})\right|^2\right) \\
&= \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) + F\left(2\left|\frac{\omega}{4} - \frac{\nu}{4}\right|^2\right) \\
&= \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) + \frac{1}{8}|\omega - \nu|^2 \\
&= \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) + \frac{1}{8} D_{\mathfrak{b}}(\omega, \nu) \\
&= \frac{1}{4} D_{\mathfrak{b}}(\omega, \nu) \\
&\leq \frac{1}{2} D_{\mathfrak{b}}(\omega, \nu) \\
&\leq F\left(\frac{1}{2} D_{\mathfrak{b}}(\omega, \nu)\right) \\
&\leq F\left(\frac{1}{2} \Lambda(\omega, \nu)\right) \\
&\leq F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu), \gamma\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right)\right).
\end{aligned}$$

Then $\mathcal{H}$ is an $(\beta\gamma, \alpha\theta, \psi F)$-rational contraction and the conditions of Theorem 3.1 are confirmed and $\mathcal{H}$ has $0 \in \Xi$ as a unique fixed point.

We assume that $\Lambda(\omega, \nu)$ is as stated in Definition 3.2. Now, we present some results:

Corollary 3.1. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H}:\Xi\to\Xi$. Assume that the below conditions hold:*

(h_1) *For all $\omega,\nu\in\Xi$ and $\alpha:\Xi\times\Xi\to[0,\infty)$,*

$$\alpha(\omega,\nu)\geq 1 \quad\Rightarrow\quad \alpha(\mathcal{H}\omega,\mathcal{H}\nu)\leq 1 \quad \text{and} \quad \alpha(\omega,\nu)\leq 1 \quad\Rightarrow\quad \alpha(\mathcal{H}\omega,\mathcal{H}\nu)\geq 1.$$

(h_2) *If there is $F\in\Phi$, $\alpha:\Xi\times\Xi\to[0,\infty)$, $\beta\in\Im$, $\mathfrak{F}\in C$, $\psi\in\Psi$, $\gamma\in\Gamma$ and $\forall\,\omega,\nu\in\Xi$, $D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$ such that $\alpha(\omega,\nu)\geq\alpha(\mathcal{H}\omega,\mathcal{H}\nu)$ and $\alpha(\mathcal{H}^2\omega,\mathcal{H}^2\nu)\geq\alpha(\mathcal{H}\omega,\mathcal{H}\nu)$ implies*

$$\begin{aligned}&\psi(D_{\mathfrak{b}}(\omega,\nu))+F\left(\mathfrak{z}\alpha(\omega,\nu)D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\\ \leq &F\Big(\mathfrak{F}\left(\beta\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu),\gamma\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu)\right)\Big).\end{aligned}$$

(h_3) *There is $\omega_0\in\Xi$ such that $\alpha(\omega_0,\mathcal{H}\omega_0)\geq 1$.*

(h_4) *If $\{\mathfrak{n}_{\mathfrak{i}}\}$ and $\{\mathfrak{m}_{\mathfrak{i}}\}$ are sequences of positive integers with $\mathfrak{n}_{\mathfrak{i}}>\mathfrak{m}_{\mathfrak{i}}>\mathfrak{i}$, then $\alpha(\omega_{\mathfrak{m}_{\mathfrak{i}}-1},\omega_{\mathfrak{n}_{\mathfrak{i}}-1})\geq 1$.*

(h_5) *$\mathcal{H}$ is continuous,*

(h_5') *or if $\{\omega_{\mathfrak{n}}\}$ is a sequence in Ξ such that $\omega_{\mathfrak{n}}\to\omega\in\Xi$ and $\alpha(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\geq 1$, then there is a subsequence $\{\omega_{\mathfrak{n}_{\mathfrak{i}}}\}$ of $\{\omega_{\mathfrak{n}}\}$ with $\alpha(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega)\geq 1$.*

(h_6) *For all $\omega,\nu\in\Xi$ $(\omega\neq\nu)$ fixed points of $\mathcal{H}$, we have $\alpha(\omega,\nu)\geq 1$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. Consider $\theta:\Xi\times\Xi\to[0,\infty)$ as $\theta(\omega,\nu)=1$, $\omega,\nu\in\Xi$ in Theorem 3.1. □

Corollary 3.2. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H}:\Xi\to\Xi$. Assume that the below conditions hold:*

(h_1) *For all $\omega,\nu\in\Xi$ and $\theta:\Xi\times\Xi\to[0,\infty)$,*

$$\theta(\omega,\nu)\leq 1 \quad\Rightarrow\quad \theta(\mathcal{H}\omega,\mathcal{H}\nu)\geq 1 \quad \text{and} \quad \theta(\omega,\nu)\geq 1 \quad\Rightarrow\quad \theta(\mathcal{H}\omega,\mathcal{H}\nu)\leq 1.$$

(h_2) *If there is $F\in\Phi$, $\theta:\Xi\times\Xi\to[0,\infty)$, $\beta\in\Im$, $\mathfrak{F}\in C$, $\psi\in\Psi$, $\gamma\in\Gamma$ and $\forall$ $\omega,\nu\in\Xi$, $D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$ such that $\theta(\mathcal{H}\omega,\mathcal{H}\nu)\geq\theta(\omega,\nu)$ and $\theta(\mathcal{H}\omega,\mathcal{H}\nu)\geq\theta(\mathcal{H}^2\omega,\mathcal{H}^2\nu)$ implies*

$$\begin{aligned}&\psi(D_{\mathfrak{b}}(\omega,\nu))+F\left(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\\ \leq &F\left(\theta(\omega,\nu)\mathfrak{F}\left(\beta\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu),\gamma\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu)\right)\right).\end{aligned}$$

(h_3) *There is $\omega_0\in\Xi$ such that $\theta(\omega_0,\mathcal{H}\omega_0)\leq 1$.*

(h_4) *If $\{\mathfrak{n}_{\mathfrak{i}}\}$ and $\{\mathfrak{m}_{\mathfrak{i}}\}$ are sequences of positive integers with $\mathfrak{n}_{\mathfrak{i}}>\mathfrak{m}_{\mathfrak{i}}>\mathfrak{i}$, then $\theta(\omega_{\mathfrak{m}_{\mathfrak{i}}-1},\omega_{\mathfrak{n}_{\mathfrak{i}}-1})\leq 1$.*

(h_5) *$\mathcal{H}$ is continuous,*

(h_5') *or if $\{\omega_{\mathfrak{n}}\}$ is a sequence in Ξ such that $\omega_{\mathfrak{n}}\to\omega\in\Xi$ and $\theta(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\leq 1$, then there is a subsequence $\{\omega_{\mathfrak{n}_{\mathfrak{i}}}\}$ of $\{\omega_{\mathfrak{n}}\}$ with $\theta(\omega_{\mathfrak{n}_{\mathfrak{i}}},\omega)\leq 1$.*

(h_6) *For all $\omega,\nu\in\Xi$ $(\omega\neq\nu)$ fixed points of $\mathcal{H}$, we have $\theta(\omega,\nu)\leq 1$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. Consider $\alpha : \Xi \times \Xi \to [0, \infty)$ as $\alpha(\omega, \nu) = 1$, $\omega, \nu \in \Xi$ in Theorem 3.1. □

Corollary 3.3. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H} : \Xi \to \Xi$. Assume that the below conditions hold:*

- (h_1) *$\mathcal{H}$ is an α-admissible crooked mapping with respect to θ.*
- (h_2) *If there is $F \in \Phi$, $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, $\gamma \in \Gamma$ and $\forall\ \omega, \nu \in \Xi$, $D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ such that*
$$\alpha(\omega, \nu)\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)\alpha(\mathcal{H}\omega, \mathcal{H}\nu)$$
and
$$\theta(\mathcal{H}\omega, \mathcal{H}\nu)\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)\theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$$
implies
$$\begin{aligned} &\psi(D_{\mathfrak{b}}(\omega, \nu)) + F\left(\mathfrak{z} D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \\ \leq &F\left(\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu) - \gamma\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right). \end{aligned}$$
- (h_3) *There is $\omega_0 \in \Xi$ such that $\alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0)$.*
- (h_4) *If $\{\mathfrak{n}_\mathfrak{i}\}$ and $\{\mathfrak{m}_\mathfrak{i}\}$ are sequences of positive integers with $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, then $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq \theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1})$.*
- (h_5) *$\mathcal{H}$ is continuous,*
- (h_5') *or if $\{\omega_\mathfrak{n}\}$ is a sequence in Ξ such that $\omega_\mathfrak{n} \to \omega \in \Xi$ and $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq \theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})$, then there is a subsequence $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ of $\{\omega_\mathfrak{n}\}$ with $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq \theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega)$.*
- (h_6) *For all $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ fixed points of $\mathcal{H}$, we have $\alpha(\omega, \nu) \geq \theta(\omega, \nu)$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. This result can be demonstrated by employing the Theorem 3.1 with $\mathfrak{F}(\iota, \mathfrak{s}) = \iota - \mathfrak{s}$. □

Corollary 3.4. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H} : \Xi \to \Xi$. Assume that the below conditions hold:*

- (h_1) *$\mathcal{H}$ is an α-admissible crooked mapping with respect to θ.*
- (h_2) *If there is $F \in \Phi$, $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, $\gamma \in \Gamma$ and $\forall\ \omega, \nu \in \Xi$, $D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ such that*
$$\alpha(\omega, \nu)\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)\alpha(\mathcal{H}\omega, \mathcal{H}\nu)$$
and
$$\theta(\mathcal{H}\omega, \mathcal{H}\nu)\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)\theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$$
implies
$$\psi(D_{\mathfrak{b}}(\omega, \nu)) + F\left(\mathfrak{z} D_{\mathfrak{b}}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(\frac{\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)}{\left(1 + \gamma\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right)^{\mathfrak{r}}}\right), \mathfrak{r} \in (0, \infty).$$
- (h_3) *There is $\omega_0 \in \Xi$ such that $\alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0)$.*
- (h_4) *If $\{\mathfrak{n}_\mathfrak{i}\}$ and $\{\mathfrak{m}_\mathfrak{i}\}$ are sequences of positive integers with $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, then $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq \theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1})$.*

(h_5) *$\mathcal{H}$ is continuous,*

(h_5') *or if $\{\omega_\mathfrak{n}\}$ is a sequence in Ξ such that $\omega_\mathfrak{n} \to \omega \in \Xi$ and $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq \theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})$, then there is a subsequence $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ of $\{\omega_\mathfrak{n}\}$ with $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq \theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega)$.*

(h_6) *For all $\omega, \nu \in \Xi$ ($\omega \neq \nu$) fixed points of $\mathcal{H}$, we have $\alpha(\omega, \nu) \geq \theta(\omega, \nu)$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. This result can be demonstrated by employing the Theorem 3.1 with $\mathfrak{F}(\iota, \mathfrak{s}) = \frac{\iota}{(1+\mathfrak{s})^\mathfrak{r}}$, $\quad \mathfrak{r} \in (0, \infty)$. □

Corollary 3.5. *Let $(\Xi, D_\mathfrak{b})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H} : \Xi \to \Xi$. Assume that the below conditions hold:*

(h_1) *$\mathcal{H}$ is an α-admissible crooked mapping with respect to θ.*

(h_2) *If there is $F \in \Phi$, $\alpha, \theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$ and $\forall\ \omega, \nu \in \Xi$, $D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ such that*

$$\alpha(\omega, \nu)\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)\alpha(\mathcal{H}\omega, \mathcal{H}\nu)$$

and

$$\theta(\mathcal{H}\omega, \mathcal{H}\nu)\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)\theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$$

implies

$$\psi(D_\mathfrak{b}(\omega, \nu)) + F\left(\mathfrak{z} D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right).$$

(h_3) *There is $\omega_0 \in \Xi$ such that $\alpha(\omega_0, \mathcal{H}\omega_0) \geq \theta(\omega_0, \mathcal{H}\omega_0)$.*

(h_4) *If $\{\mathfrak{n}_\mathfrak{i}\}$ and $\{\mathfrak{m}_\mathfrak{i}\}$ are sequences of positive integers with $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, then $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq \theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1})$.*

(h_5) *$\mathcal{H}$ is continuous,*

(h_5') *or if $\{\omega_\mathfrak{n}\}$ is a sequence in Ξ such that $\omega_\mathfrak{n} \to \omega \in \Xi$ and $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq \theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})$, then there is a subsequence $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ of $\{\omega_\mathfrak{n}\}$ with $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq \theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega)$.*

(h_6) *For all $\omega, \nu \in \Xi$ ($\omega \neq \nu$) fixed points of $\mathcal{H}$, we have $\alpha(\omega, \nu) \geq \theta(\omega, \nu)$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. This result can be demonstrated by employing the Theorem 3.1 with $\mathfrak{F}(\iota, \mathfrak{s}) = \iota$ and in this case, by ($\mathfrak{c}_2$) $\gamma(\iota) = 0$. □

Corollary 3.6. *Let $(\Xi, D_\mathfrak{b})$ be a complete $\mathfrak{b}$-mls and $\mathcal{H} : \Xi \to \Xi$. Assume that the below conditions hold:*

(h_1) *For all $\omega, \nu \in \Xi$ and $\alpha : \Xi \times \Xi \to [0, \infty)$,*

$$\alpha(\omega, \nu) \geq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \leq 1 \quad \text{and} \quad \alpha(\omega, \nu) \leq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1.$$

(h_2) *If there is $F \in \Phi$, $\alpha : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, $\gamma \in \Gamma$ and $\forall\ \omega, \nu \in \Xi$, $D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ such that $\alpha(\omega, \nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)$ and $\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)$ implies*

$$\begin{aligned}&\psi(D_\mathfrak{b}(\omega, \nu)) + F\left(\mathfrak{z}\alpha(\omega, \nu) D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu)\right)\\ \leq &F\left(\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu) - \gamma\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right).\end{aligned}$$

(h_3) *There is* $\omega_0 \in \Xi$ *such that* $\alpha(\omega_0, \mathcal{H}\omega_0) \geq 1$.

(h_4) *If* $\{\mathfrak{n}_\mathfrak{i}\}$ *and* $\{\mathfrak{m}_\mathfrak{i}\}$ *are sequences of positive integers with* $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, *then* $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq 1$.

(h_5) $\mathcal{H}$ *is continuous,*

(h_5') *or if* $\{\omega_\mathfrak{n}\}$ *is a sequence in* Ξ *such that* $\omega_\mathfrak{n} \to \omega \in \Xi$ *and* $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq 1$, *then there is a subsequence* $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ *of* $\{\omega_\mathfrak{n}\}$ *with* $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq 1$.

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$, *we have* $\alpha(\omega, \nu) \geq 1$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Taking $\mathfrak{F}(\iota, \mathfrak{s}) = \iota - \mathfrak{s}$ in Corollary 3.1. □

Corollary 3.7. *Let* $(\Xi, D_\mathfrak{b})$ *be a complete* $\mathfrak{b}$*-mls and* $\mathcal{H} : \Xi \to \Xi$. *Assume that the below conditions hold:*

(h_1) *For all* $\omega, \nu \in \Xi$ *and* $\alpha : \Xi \times \Xi \to [0, \infty)$,

$$\alpha(\omega,\nu) \geq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \leq 1 \quad \text{and} \quad \alpha(\omega,\nu) \leq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1.$$

(h_2) *If there is* $F \in \Phi$, $\alpha : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, *and* $\forall\ \omega, \nu \in \Xi$, $D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ *such that* $\alpha(\omega, \nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)$ *and* $\alpha(\mathcal{H}^2\omega, \mathcal{H}^2\nu) \geq \alpha(\mathcal{H}\omega, \mathcal{H}\nu)$ *implies*

$$\psi(D_\mathfrak{b}(\omega,\nu)) + F\left(\mathfrak{z}\alpha(\omega,\nu) D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(\beta\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu)\right).$$

(h_3) *There is* $\omega_0 \in \Xi$ *such that* $\alpha(\omega_0, \mathcal{H}\omega_0) \geq 1$.

(h_4) *If* $\{\mathfrak{n}_\mathfrak{i}\}$ *and* $\{\mathfrak{m}_\mathfrak{i}\}$ *are sequences of positive integers with* $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, *then* $\alpha(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \geq 1$.

(h_5) $\mathcal{H}$ *is continuous,*

(h_5') *or if* $\{\omega_\mathfrak{n}\}$ *is a sequence in* Ξ *such that* $\omega_\mathfrak{n} \to \omega \in \Xi$ *and* $\alpha(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \geq 1$, *then there is a subsequence* $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ *of* $\{\omega_\mathfrak{n}\}$ *with* $\alpha(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \geq 1$.

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$, *we have* $\alpha(\omega, \nu) \geq 1$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Taking $\mathfrak{F}(\iota, \mathfrak{s}) = \iota$ and $\gamma(\iota) = 0$ in Corollary 3.1. □

Corollary 3.8. *Let* $(\Xi, D_\mathfrak{b})$ *be a complete* $\mathfrak{b}$*-mls and* $\mathcal{H} : \Xi \to \Xi$. *Assume that the below conditions hold:*

(h_1) *For all* $\omega, \nu \in \Xi$ *and* $\theta : \Xi \times \Xi \to [0, \infty)$,

$$\theta(\omega,\nu) \leq 1 \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1 \quad \text{and} \quad \theta(\omega,\nu) \geq 1 \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \leq 1.$$

(h_2) *If there is* $F \in \Phi$, $\theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, $\gamma \in \Gamma$ *and* $\forall$ $\omega, \nu \in \Xi$, $D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ *such that* $\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)$ *and* $\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$ *implies*

$$\begin{aligned}&\psi(D_\mathfrak{b}(\omega,\nu)) + F\left(\mathfrak{z} D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu)\right)\\ \leq& F\left(\theta(\omega,\nu)\left(\beta\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu) - \gamma\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu)\right)\right).\end{aligned}$$

(h_3) *There is* $\omega_0 \in \Xi$ *such that* $\theta(\omega_0, \mathcal{H}\omega_0) \leq 1$.

(h_4) *If* $\{\mathfrak{n}_\mathfrak{i}\}$ *and* $\{\mathfrak{m}_\mathfrak{i}\}$ *are sequences of positive integers with* $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, *then* $\theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \leq 1$.

(h_5) $\mathcal{H}$ *is continuous,*

(h_5') *or if* $\{\omega_\mathfrak{n}\}$ *is a sequence in* Ξ *such that* $\omega_\mathfrak{n} \to \omega \in \Xi$ *and* $\theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \leq 1$, *then there is a subsequence* $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ *of* $\{\omega_\mathfrak{n}\}$ *with* $\theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \leq 1$.

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$, *we have* $\theta(\omega, \nu) \leq 1$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Taking $\mathfrak{F}(\iota, \mathfrak{s}) = \iota - \mathfrak{s}$ in Corollary 3.2. □

Corollary 3.9. *Let* $(\Xi, D_\mathfrak{b})$ *be a complete* $\mathfrak{b}$*-mls and* $\mathcal{H} : \Xi \to \Xi$. *Assume that the below conditions hold:*

(h_1) *For all* $\omega, \nu \in \Xi$ *and* $\theta : \Xi \times \Xi \to [0, \infty)$,

$$\theta(\omega, \nu) \leq 1 \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1 \quad \textit{and} \quad \theta(\omega, \nu) \geq 1 \quad \Rightarrow \quad \theta(\mathcal{H}\omega, \mathcal{H}\nu) \leq 1.$$

(h_2) *If there is* $F \in \Phi$, $\theta : \Xi \times \Xi \to [0, \infty)$, $\beta \in \Im$, $\psi \in \Psi$, *and* $\forall\ \omega, \nu \in \Xi$, $D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$ *such that* $\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\omega, \nu)$ *and* $\theta(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\mathcal{H}^2\omega, \mathcal{H}^2\nu)$ *implies*

$$\psi(D_\mathfrak{b}(\omega, \nu)) + F\left(\mathfrak{z} D_\mathfrak{b}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(\theta(\omega, \nu)\beta\left(\Lambda(\omega, \nu)\right)\Lambda(\omega, \nu)\right).$$

(h_3) *There is* $\omega_0 \in \Xi$ *such that* $\theta(\omega_0, \mathcal{H}\omega_0) \leq 1$.

(h_4) *If* $\{\mathfrak{n}_\mathfrak{i}\}$ *and* $\{\mathfrak{m}_\mathfrak{i}\}$ *are sequences of positive integers with* $\mathfrak{n}_\mathfrak{i} > \mathfrak{m}_\mathfrak{i} > \mathfrak{i}$, *then* $\theta(\omega_{\mathfrak{m}_\mathfrak{i}-1}, \omega_{\mathfrak{n}_\mathfrak{i}-1}) \leq 1$.

(h_5) $\mathcal{H}$ *is continuous,*

(h_5') *or if* $\{\omega_\mathfrak{n}\}$ *is a sequence in* Ξ *such that* $\omega_\mathfrak{n} \to \omega \in \Xi$ *and* $\theta(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \leq 1$, *then there is a subsequence* $\{\omega_{\mathfrak{n}_\mathfrak{i}}\}$ *of* $\{\omega_\mathfrak{n}\}$ *with* $\theta(\omega_{\mathfrak{n}_\mathfrak{i}}, \omega) \leq 1$.

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$, *we have* $\theta(\omega, \nu) \leq 1$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Taking $\mathfrak{F}(\iota, \mathfrak{s}) = \iota$ and $\gamma(\iota) = 0$ in Corollary 3.2. □

4. Application

In this section, we will apply our main results to find the solution of the second-order differential equation representing the electric circuit equation. Its components are a resistor (R), an electromotive force (E), a capacitor (C), an inductor (L), and a voltage (V) in series, as illustrated in Figure 1.

If the current I is the rate of change of charge q with respect to time $\mathfrak{t}$, $I = \frac{dq}{d\mathfrak{t}}$. We have the following relations:

(v_1) $V = IR$;

(v_2) $V = \frac{q}{C}$;

(v_3) $V = L\frac{dI}{d\mathfrak{t}}$.

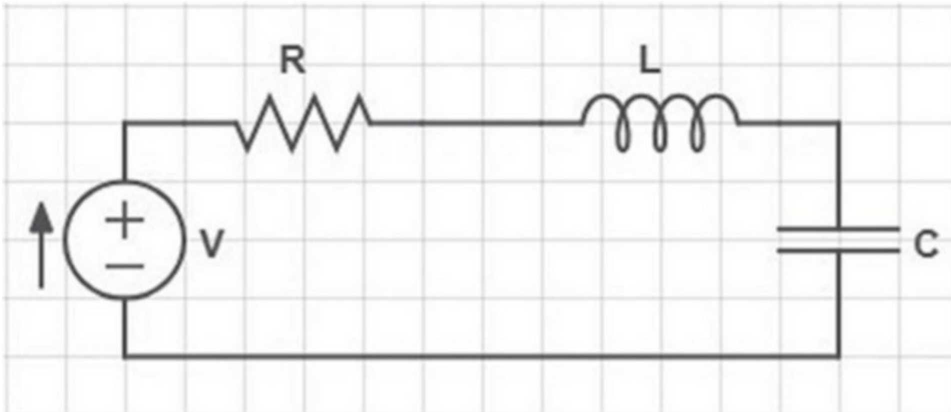

Figure 1. Electric circuit.

According to Kirchhoff's voltage law, the sum of these voltage drops equals the supplied voltage, i.e.,

$$IR + \frac{q}{C} + L\frac{dI}{d\mathfrak{t}} = V(\mathfrak{t}).$$

Or

$$\begin{cases} L\dfrac{d^2q}{d\mathfrak{t}^2} + R\dfrac{dq}{d\mathfrak{t}} + \dfrac{q}{C} = V(\mathfrak{t}), \\ q(0) = 0, \quad q'(0) = 0. \end{cases} \tag{4.1}$$

The Green function linked to equation (4.1) is expressed as:

$$\Upsilon(\mathfrak{t}, \mathfrak{s}) = \begin{cases} -\mathfrak{t}e^{\mu(\mathfrak{s}-\mathfrak{t})}, & 0 \leq \mathfrak{t} \leq \mathfrak{s} \leq 1, \\ -\mathfrak{s}e^{\mu(\mathfrak{s}-\mathfrak{t})}, & 0 \leq \mathfrak{s} \leq \mathfrak{t} \leq 1, \end{cases} \tag{4.2}$$

where a constant $\mu = \frac{R}{2L} > 0$. Problem (4.1) is equivalent to the following nonlinear integral equation

$$\omega(\mathfrak{t}) = \int_0^{\mathfrak{t}} \Upsilon(\mathfrak{t}, \mathfrak{s})\mathcal{K}(\mathfrak{s}, \omega(\mathfrak{s}))d\mathfrak{s}, \quad \forall\ \mathfrak{t} \in [0, 1]. \tag{4.3}$$

Let $\Xi = \mathcal{C}([0, 1])$ be the set of all continuous functions defined on $[0, 1]$, endowed with the metric $D_{\mathfrak{b}} : \Xi \times \Xi \to [0, \infty)$ defined as

$$D_{\mathfrak{b}} = (\|\omega\|_\infty + \|\nu\|_\infty)^m \quad \text{for all} \quad \omega, \nu \in \Xi, \tag{4.4}$$

where $\|\omega\|_\infty = \sup_{\mathfrak{t}\in[0,1]} \{\ |\omega(\mathfrak{t})|e^{-2\mathfrak{t}\mu m}\}$ and $m > 1$. It is clear that $(\Xi, D_{\mathfrak{b}})$ is a complete $\mathfrak{b}$-mls with parameter $\mathfrak{z} = 2^{m-1}$.

Theorem 4.1. *Let $(\Xi, D_{\mathfrak{b}})$ be a complete b-mls as described above and $\mathcal{H} : \Xi \to \Xi$ be a self map. Assume that the below condition holds:*

There exists increasing function $\mathcal{K} : [0, 1] \times \mathbb{R} \to \mathbb{R}$,

$$|\mathcal{K}(\mathfrak{s}, \omega(\mathfrak{s}))| + |\mathcal{K}(\mathfrak{s}, \nu(\mathfrak{s}))| \leq \mu^2 \left(\frac{1}{\mathfrak{z}^2} e^{-D_{\mathfrak{b}}(\omega,\nu)}\Lambda(\omega, \nu)\right)^{\frac{1}{m}}, \tag{4.5}$$

where

$$\Lambda(\omega,\nu)=\max\Bigg\{D_{\mathfrak{b}}(\omega,\nu),D_{\mathfrak{b}}(\omega,\mathcal{H}\omega),D_{\mathfrak{b}}(\nu,\mathcal{H}\nu),\frac{D_{\mathfrak{b}}(\omega,\mathcal{H}\nu)+D_{\mathfrak{b}}(\nu,\mathcal{H}\omega)}{2\mathfrak{z}},$$
$$\min\left\{\frac{D_{\mathfrak{b}}(\omega,\mathcal{H}\omega)D_{\mathfrak{b}}(\nu,\mathcal{H}\nu)}{1+D_{\mathfrak{b}}(\omega,\nu)},\frac{D_{\mathfrak{b}}(\nu,\mathcal{H}\nu)[1+D_{\mathfrak{b}}(\omega,\mathcal{H}\omega)]}{1+D_{\mathfrak{b}}(\omega,\nu)}\right\}\Bigg\},$$

for all $\mathfrak{s}\in[0,1]$ *and* $\omega,\nu,\mu\in\mathbb{R}^+$. *Then, equation* (4.1) *has a unique solution.*

Proof. Let $D_{\mathfrak{b}}$ be a function given by (4.4) and consider the self maps $\mathcal{H}:\Xi\to\Xi$ defined by

$$\mathcal{H}\omega(\mathfrak{t})=\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\mathcal{K}(\mathfrak{s},\omega(\mathfrak{s}))d\mathfrak{s},\quad \mathfrak{t}\in[0,1]. \tag{4.6}$$

If $\omega,\nu\in\Xi$ then, we get $D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$. Define $\alpha,\theta:\Xi\times\Xi\to[0,\infty)$ by

$$\alpha(\omega,\nu)=\theta(\omega,\nu)=1,\quad \text{for all}\quad \omega,\nu\in\Xi.$$

Therefore, $\mathcal{H}$ is an α-admissible crooked mapping with respect to θ, then we obtain

$$\alpha(\omega,\nu)\theta(\mathcal{H}\omega,\mathcal{H}\nu)\geq\theta(\omega,\nu)\alpha(\mathcal{H}\omega,\mathcal{H}\nu)$$

and

$$\theta(\mathcal{H}\omega,\mathcal{H}\nu)\alpha(\mathcal{H}^2\omega,\mathcal{H}^2\nu)\geq\alpha(\mathcal{H}\omega,\mathcal{H}\nu)\theta(\mathcal{H}^2\omega,\mathcal{H}^2\nu).$$

Now, we have

$$\begin{aligned}
&(|\mathcal{H}\omega(\mathfrak{t})|+|\mathcal{H}\nu(\mathfrak{t})|)^m\\
&=\left(\left|\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\mathcal{K}(\mathfrak{s},\omega(\mathfrak{s}))d\mathfrak{s}\right|+\left|\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\mathcal{K}(\mathfrak{s},\nu(\mathfrak{s}))d\mathfrak{s}\right|\right)^m\\
&\leq\left(\int_0^{\mathfrak{t}}|\Upsilon(\mathfrak{t},\mathfrak{s})\mathcal{K}(\mathfrak{s},\omega(\mathfrak{s}))|\,d\mathfrak{s}+\int_0^{\mathfrak{t}}|\Upsilon(\mathfrak{t},\mathfrak{s})\mathcal{K}(\mathfrak{s},\nu(\mathfrak{s}))|\,d\mathfrak{s}\right)^m\\
&\leq\left(\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\left(|\mathcal{K}(\mathfrak{s},\omega(\mathfrak{s}))|+|\mathcal{K}(\mathfrak{s},\nu(\mathfrak{s}))|\right)d\mathfrak{s}\right)^m\\
&\leq\left(\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\mu^2\left(\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\Lambda(\omega,\nu)\right)^{\frac{1}{m}}d\mathfrak{s}\right)^m\\
&\leq\left(\int_0^{\mathfrak{t}}\Upsilon(\mathfrak{t},\mathfrak{s})\mu^2e^{-2\mu\mathfrak{s}}e^{2\mu\mathfrak{s}}\left(\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\Lambda(\omega,\nu)\right)^{\frac{1}{m}}d\mathfrak{s}\right)^m\\
&\leq\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\Lambda(\omega,\nu)e^{-2\mu\mathfrak{s}m}\left(\mu^2\int_0^{\mathfrak{t}}e^{2\mu\mathfrak{s}}\Upsilon(\mathfrak{t},\mathfrak{s})d\mathfrak{s}\right)^m\\
&\leq\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty\left(\mu^2\int_0^{\mathfrak{t}}e^{2\mu\mathfrak{s}}\Upsilon(\mathfrak{t},\mathfrak{s})d\mathfrak{s}\right)^m\\
&\leq\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty\left(\mu^2e^{2\mu\mathfrak{t}}[-\frac{1}{\mu^2}(2\mu\mathfrak{t}-\mu\mathfrak{t}e^{-\mu\mathfrak{t}}+e^{-\mu\mathfrak{t}}-1)]\right)^m\\
&\leq\frac{1}{\mathfrak{z}^2}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty e^{2\mu\mathfrak{t}m}\left(1-2\mu\mathfrak{t}+\mu\mathfrak{t}e^{-\mu\mathfrak{t}}-e^{-\mu\mathfrak{t}}\right)^m,
\end{aligned}$$

then, we have

$$\begin{aligned}&\mathfrak{z}\left(|\mathcal{H}\omega(\mathfrak{t})|+|\mathcal{H}\nu(\mathfrak{t})|\right)^m e^{-2\mu\mathfrak{t}m}\\ \leq&\frac{1}{\mathfrak{z}}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty\left(1-2\mu\mathfrak{t}+\mu\mathfrak{t}e^{-\mu\mathfrak{t}}-e^{-\mu\mathfrak{t}}\right)^m,\end{aligned}$$

which leads to

$$\mathfrak{z}\left(\|\mathcal{H}\omega(\mathfrak{t})\|_\infty+\|\mathcal{H}\nu(\mathfrak{t})\|_\infty\right)^m \leq \frac{1}{\mathfrak{z}}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty\left(1-2\mu\mathfrak{t}+\mu\mathfrak{t}e^{-\mu\mathfrak{t}}-e^{-\mu\mathfrak{t}}\right)^m,$$

since $(1-2\mu\mathfrak{t}+\mu\mathfrak{t}e^{-\mu\mathfrak{t}}-e^{-\mu\mathfrak{t}})\leq 1$, then, we get

$$\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\leq\frac{1}{\mathfrak{z}}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty. \tag{4.7}$$

Taking ln along (4.7), we have

$$\begin{aligned}\ln\left(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)&\leq\ln\left(\frac{1}{\mathfrak{z}}e^{-D_{\mathfrak{b}}(\omega,\nu)}\|\Lambda(\omega,\nu)\|_\infty\right)\\&\leq -D_{\mathfrak{b}}(\omega,\nu)+\ln\left(\frac{1}{\mathfrak{z}}\|\Lambda(\omega,\nu)\|_\infty\right).\end{aligned}$$

Therefore,

$$D_{\mathfrak{b}}(\omega,\nu)+\ln\left(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\leq\ln\left(\frac{1}{\mathfrak{z}}\|\Lambda(\omega,\nu)\|_\infty\right). \tag{4.8}$$

If we take $F(\mathfrak{t})=\ln(\mathfrak{t})$, $\psi(\mathfrak{t})=\mathfrak{t}$, $\beta(\mathfrak{t})=\frac{1}{\mathfrak{z}}=\frac{1}{2^{m-1}}\in[0,1)$, $m>1$ and $\mathfrak{F}(\mathfrak{t},\mathfrak{s})=\mathfrak{t}$, $\gamma(\mathfrak{t})=0$, then Equation (4.8) equivalent

$$\psi(D_{\mathfrak{b}}(\omega,\nu))+F\left(\mathfrak{z}D_{\mathfrak{b}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\leq F\left(\mathfrak{F}\left(\beta\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu),\gamma\left(\Lambda(\omega,\nu)\right)\Lambda(\omega,\nu)\right)\right).$$

Then, $\mathcal{H}$ is an $(\beta\gamma,\alpha\theta,\psi F)$-rational contraction. Hence, by Theorem 3.1, $\mathcal{H}$ has a unique fixed point, which is a unique solution to the equation (4.1). This completes the proof. □

5. Conclusions

In this paper, we introduced the notion of an α-admissible crooked mapping with respect to θ and established novel fixed point theorems over a $\mathfrak{b}$-metric-like space. The proofs were based on the application of $(\beta\gamma,\alpha\theta,\psi F)$-rational contraction, a concept also introduced in this article. To reinforce our findings, illustrative examples were presented. Furthermore, we applied the derived results to validate the existence and uniqueness of solutions for the electric circuit equation. Our contributions extend and enrich the existing literature in this domain, presenting a novel perspective on confirming the existence and uniqueness of solutions for the electric circuit equation.

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