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THE DISSIPATIVE CONDITION OF THE FIRST ORDER 3 × 3 HYPERBOLIC SYSTEM WITH CONSTANT COEFFICIENTS*

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Abstract In this paper, we study the dissipative property of the first order 3×3 hyperbolic system with constant coefficients. For the corresponding $n\times n$ system, when the coefficients matrices are symmetric, it has been studied in [16] and the well-know Kawashima-Shizuta condition is obtained. When $n=3$ and for asymmetric system, we give a sufficient condition for the system to be strictly dissipative.

Keywords 3×3 hyperbolic system, dissipative property, Kawashima-Shizuta condition.

MSC(2010) 35L40, 35L45, 35L65.

1. Introduction

In this paper, we study the dissipative property of the following first order 3×3 strictly hyperbolic system with constant coefficients

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + FU = 0, \tag{1.1}$$

where $x \in \mathbb{R}$, $U(x,t) = (u_1, u_2, u_3)^\top$, $A = (a_{ij})_{3\times 3}$ and $F = (f_{ij})_{3\times 3}$ are two constant matrices. Since (1.1) is strictly hyperbolic, A has three distinct real eigenvalues: $\mu_1 < \mu_2 < \mu_3$.

Consider the Cauchy problem of system (1.1) with the initial data

$$U_0(x) = (u_1(x,0),\ u_2(x,0),\ u_3(x,0))^\top \in L^1(\mathbb{R}) \cap C^\alpha(\mathbb{R}) \tag{1.2}$$

for some $\alpha \in \mathbb{R}^+$.

For $n \in \mathbb{R}_+$, $U(x,t) = (u_1, \cdots, u_n)^\top \in \mathbb{R}^n$, $A = (a_{ij})_{n\times n}$ and $F = (f_{ij})_{n\times n}$, (1.1) represents an $n \times n$ hyperbolic system. A more general case is the following quasilinear hyperbolic system

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + F(U) = 0, \tag{1.3}$$

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where $x \in \mathbb{R}$, $U \in \mathbb{R}^n$, $A(U)$ and $F(U)$ are two smooth function matrices vanishing at the origin. Generally speaking, for n dimension(n-d), the dissipative conditions of (1.1) can be directly generalized to system (1.3) when the initial data are small. An important dissipative condition of (1.1) is the strongly dissipative condition, namely, the matrix $P^{-1}\nabla FP$ is strictly row or column-diagonally dominant, where $P = (R_1, \cdots, R_n)$ is the $n \times n$ matrix composed of the right eigenvectors R_i $(i = 1, \cdots, n)$ of matrix A, and P^{-1} is the inverse matrix of P. Similarly, if the matrix $P^{-1}(0)\nabla F(0)P(0)$ is strictly row or column-diagonally dominant, then (1.3) is strongly dissipative, where $P(U) = (R_1(U), \cdots, R_n(U))$ is the $n \times n$ matrix composed of the right eigenvectors $R_i(U)$ $(i = 1, \cdots, n)$ of matrix $A(U)$, and $P^{-1}(U)$ is the inverse matrix of $P(U)$. Strongly dissipative condition can be further generalized to matrices positively diagonally similar to a strictly diagonally dominant matrix, see Section 2 below or [9, 14] for more details. In [2–4], strongly dissipative condition was used to study the global existence of weak solutions to systems of conservation laws.

Another important dissipative condition is the well-known Kawashima-Shizuta algebraic condition (see [11]), which can be used to study the decay properties of solutions to hyperbolic-parabolic coupled systems. Kawashima-Shizuta condition has several equivalent formulations (see [17]). For some applications of Kawashima-Shizuta condition, see [3, 6, 8, 10, 12, 13, 19]. Recently, systems with much weaker dissipations which violate Kawashima-Shizuta condition have attracted a lot of attentions, see [5, 7, 19, 22, 24].

In [23], for (1.1) in 2-d, we proposed a dissipative condition which can be regarded as a generalization of Kawashima-Shizuta condition to asymmetric system.

In [18], the authors obtained the pointwise estimates of the one-dimensional thermoelastic system with second sound, which is hyperbolic with a damping term. The higher dimensional systems were also studied by many authors (see [15, 19–21]).

However, strongly dissipative condition is somehow too strong as a dissipative condition (see [9]). Kawashima-Shizuta condition is weaker, but it is applicable only for system which is symmetric or symmetrizable. The main purpose of the present paper is to find sufficient conditions for system (1.1) to be strictly dissipative. In fact, for system (1.1), we will propose a new dissipative condition (see (2.9), (2.10)) which can be used for asymmetric system.

The rest of the paper is organized as follows. In Section 2, we review the concepts of strongly dissipative condition and Kawashima-Shizuta condition and give a new dissipative condition for the first order 3×3 hyperbolic system. We also explain the relations among these conditions. In Section 3, we verify that the new dissipative condition implies the strictly dissipative property of (1.1), even if it is not symmetric or symmetrizable. In Section 4, we give the pointwise estimates to the solution of Cauchy problem (1.1), (1.2). Finally, in Section 5, we discuss some critical cases.

2. Dissipative conditions

We first review the concept of strictly diagonally dominant of a matrix. A matrix $B = (b_{ij})_{n\times n}$ is called strictly row-diagonally dominant if

$$b_{ii} > \sum_{j\neq i} |b_{ij}|, \ i = 1, \cdots, n \tag{2.1}$$

or strictly column-diagonally dominant if

$$b_{ii} > \sum_{j \neq i} |b_{ji}|, \ \ i = 1, \cdots, n. \tag{2.2}$$

Denote μ_i $(i = 1, \cdots, n)$ as the n distinct eigenvalues of A, and R_i $(i = 1, \cdots, n)$ are the corresponding eigenvectors, $P = (R_1, \cdots, R_n)$, and P^{-1} is the inverse matrix of P. For (1.1) in n-d, let $R_i = (R_{i1}, \cdots, R_{in})^\top$ $(i = 1, \cdots, n)$ be right eigenvectors corresponding to the eigenvalues μ_i $(i = 1, \cdots, n)$ of matrix A respectively. Denote

$$P = (R_1, \cdots, R_n), \ \Lambda = P^{-1}AP = \mathrm{diag}\{\mu_1, \cdots, \mu_n\}, \ B = P^{-1}FP \triangleq (b_{ij})_{n \times n}. \tag{2.3}$$

Definition 2.1. The $n \times n$ hyperbolic system (1.1) is called strongly dissipative if $B = P^{-1}FP$ is strictly diagonally dominant.

If (1.1) is strongly dissipative, then (1.1) with initial data $U(x, 0) = \varphi(x)$ admits a unique global C^1 solution $U = U(x, t)$ for $t \geq 0$, which decays exponentially in time, provided that the C^1 norm of $\varphi(x)$ is suitably small (see [14]).

Definition 2.2. Two matrices B and $\tilde{B}$ are called to be positively diagonally similar if there exists a diagonal matrix $\gamma > 0$ such that $B = \gamma \tilde{B} \gamma^{-1}$.

Denote

$$\tilde{\tilde{B}} = (\tilde{\tilde{b}}_{ij})_{n \times n}, \quad \tilde{\tilde{b}}_{ij} = \begin{cases} b_{ii} & \text{for } \ i = j; \\ -|b_{ij}| & \text{for } \ i \neq j. \end{cases}$$

As stated in Theorem 2.1 of Chapter 4.2 in [14], $B = (b_{ij})_{n \times n}$ is positively diagonally similar to a strictly row (or column)-diagonally dominant matrix if and only if the real parts of all the eigenvalues of $\tilde{\tilde{B}}$ are positive. Thus strongly dissipative condition (in Definition 2.1) can be generalized to the matrices which are positively diagonally similar to a strictly diagonally dominant matrix.

Using the same notation in (2.3) for $n = 3$, by using of the transformation $U = PV$, $V = (v_1, v_2, v_3)^\top$, (1.1) can be rewritten as

$$\frac{\partial V}{\partial t} + \Lambda \frac{\partial V}{\partial x} + BV = 0. \tag{2.4}$$

Meanwhile, the initial data (1.2) are transformed into

$$V(x, 0) = P^{-1} U_0(x) \triangleq V_0(x). \tag{2.5}$$

Definition 2.3. ([16]). System (1.1) is called strictly dissipative if the real parts of all eigenvalues of matrix $F + \mathrm{i}\xi A$ (or equivalently $B + \mathrm{i}\xi\Lambda$, see Remark 2.1 below) are positive for any $\xi \in \mathbb{R} \setminus \{0\}$.

In [16], the authors proved that, when A and F are both real symmetric matrices and F is nonnegative definite, system (1.1) is strictly dissipative if and only if

$$FR_i \neq 0, \quad i = 1, 2, 3. \tag{2.6}$$

In fact, (2.6) is an equivalent form of Kawashima-Shizuta condition. The corresponding right eigenvectors to the eigenvalues μ_1, μ_2 and μ_3 of matrix Λ are obviously $\tilde{R}_1 = (0, 0, 1)^\top$, $\tilde{R}_2 = (0, 1, 0)^\top$ and $\tilde{R}_3 = (1, 0, 0)^\top$ respectively.

Remark 2.1. System (1.1) is strictly dissipative if and only if system (2.4) is strictly dissipative. In fact, with the transformation $U = PV$, we have $\Lambda\tilde{R}_i = P^{-1}AP\tilde{R}_i = \mu_i\tilde{R}_i$ $(i = 1,2,3)$, then $AP\tilde{R}_i = \mu_i P\tilde{R}_i$, i.e., $P\tilde{R}_i = R_i$. We also obtain $B\tilde{R}_i = P^{-1}FP\tilde{R}_i = P^{-1}FR_i$. Hence $FR_i \neq 0$ $(i = 1,2,3)$ if and only if $B\tilde{R}_i \neq 0$ $(i = 1,2,3)$.

For system (2.4), condition (2.6) shows when B is real symmetric and nonnegative definite, (2.4) is strictly dissipative if and only if all the right eigenvectors $\tilde{R}_i$ $(i = 1,2,3)$ of Λ are not in the kernel of B, namely,

$$B\tilde{R}_{4-i} = (b_{1i}, b_{2i}, b_{3i})^\top \neq 0, \ \ i = 1,2,3.$$

Hence for system (2.4), Kawashima-Shizuta condition takes the form

$$b_{1i}^2 + b_{2i}^2 + b_{3i}^2 \neq 0, \ \ i = 1,2,3. \tag{2.7}$$

Recall the strictly dissipative condition of (1.1) for $n = 2$. Denote $B = P^{-1}FP$. In [23], we have shown that the strictly hyperbolic system (1.1) for $n = 2$ is strictly dissipative if and only if B satisfies

$$a > 0, \quad d > 0; \quad ad \geq bc. \tag{2.8}$$

According to (2.8), a quite natural dissipative condition for system (2.4) can be given as follows.

Condition 1.

$$\begin{cases} b_{ii} > 0, \quad i = 1,2,3, \\ B_{ii} \triangleq b_{jj}b_{kk} - b_{jk}b_{kj} > 0, \quad i,j,k = 1,2,3 \ \text{ and } \ i \neq j \neq k \neq i, \\ |B| > 0. \end{cases} \tag{2.9}$$

Formally, (2.9) is stronger than (2.8) since there is no equality included in (2.9). The characteristic polynomial of B can be written as

$$\hat{\lambda}^3 - (b_{11} + b_{22} + b_{33})\hat{\lambda}^2 + (B_{11} + B_{22} + B_{33})\hat{\lambda} - |B| = 0.$$

Under condition (2.9), it can be easily verify if the eigenvalues of B are all real, then they must be nonnegative. Even so, however, (2.9) is far from sufficient to assure that the real parts of all the eigenvalues of $B + \mathrm{i}\xi\Lambda$ are positive for any $\xi \in \mathbb{R} \setminus \{0\}$.

Example 2.1. ([23]) For system (2.4) with

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 50 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix},$$

it can be easily verified that B satisfies (2.9). In [23], we have showed that there exists some $\xi_0 \in \mathbb{R} \setminus \{0\}$ such that $B + \mathrm{i}\xi_0\Lambda$ has a pure imaginary root $\lambda(\xi_0) = \frac{9\xi_0^3 - 34\xi_0}{5(\xi_0^2+3)}\mathrm{i}$.

In this paper, besides condition (2.9), we propose the following additional condition.

Condition 2.

$$\sqrt{|B|} < \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}. \tag{2.10}$$

In Section 3, we will prove that (2.9) and (2.10) are sufficient to assure that (1.1) is strictly dissipative.

The following lemma shows that conditions (2.9) and (2.10) are weaker than the strongly dissipation condition.

Lemma 2.1. *If* (1.1) *is strongly dissipative, then* (2.9) *and* (2.10) *hold true.*

Proof. Since (1.1) is strongly dissipative, assume that B is strictly row-diagonally dominant, there hold

$$b_{11} > |b_{12}| + |b_{13}|,\ b_{22} > |b_{21}| + |b_{23}|,\ b_{33} > |b_{31}| + |b_{32}|, \tag{2.11}$$

which imply that $b_{ii} > 0$ $(i = 1, 2, 3)$ and

$$b_{11}b_{22} > (|b_{12}| + |b_{13}|)(|b_{21}| + |b_{23}|) \geq |b_{12}||b_{21}| \geq b_{12}b_{21}.$$

Similarly, we have $b_{22}b_{33} > b_{23}b_{32}$ and $b_{11}b_{33} > b_{13}b_{31}$, thus both $(2.9)_1$ and $(2.9)_2$ hold. By Gerschgorin's disk theorem, the three eigenvalues of B lie in the union of the disks

$$|z - b_{ii}| \leq \sum_{j\neq i} |b_{ij}|, \quad i = 1, 2, 3.$$

Combining with (2.1), we obtain that the real parts of three eigenvalues $\hat{\lambda}_i$ $(i = 1, 2, 3)$ of B are all positive. Thus if $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ are all real numbers, then we have $|B| = \hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3 > 0$. If $\hat{\lambda}_1 > 0$ and $\hat{\lambda}_2 = a + b\mathrm{i}$, $\hat{\lambda}_3 = a - b\mathrm{i}$, for some $a > 0$, $b \in \mathbb{R}$, we have $|B| = \hat{\lambda}_1(a^2 + b^2) > 0$. Thus (2.9) holds true.

By using of (2.11), we have

$$\begin{aligned} b_{11}b_{22}b_{33} &> (|b_{12}| + |b_{13}|)(|b_{21}| + |b_{23}|)(|b_{31}| + |b_{32}|) \geq b_{12}b_{23}b_{31}, \\ b_{11}b_{22}b_{33} &> (|b_{12}| + |b_{13}|)(|b_{21}| + |b_{23}|)(|b_{31}| + |b_{32}|) \geq b_{21}b_{13}b_{32}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |B| &= b_{11}(b_{22}b_{33} - b_{23}b_{32}) + b_{12}b_{23}b_{31} - b_{12}b_{21}b_{33} + b_{21}b_{13}b_{32} - b_{31}b_{13}b_{22} \\ &< b_{11}B_{11} + b_{11}b_{22}b_{33} - b_{12}b_{21}b_{33} + b_{11}b_{22}b_{33} - b_{31}b_{13}b_{22} \\ &= b_{11}B_{11} + b_{22}B_{22} + b_{33}B_{33}, \end{aligned} \tag{2.12}$$

i.e., $\sqrt{|B|} < \sqrt{b_{11}B_{11} + b_{22}B_{22} + b_{33}B_{33}}$. By using of inequality $\sqrt{a+b+c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$ for $a,\ b,\ c \geq 0$, we have

$$\sqrt{|B|} < \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}.$$

Thus (2.10) holds true. The case of strictly column-diagonally dominant can be similarly discussed. This completes the proof of Lemma 2.1. □

Remark 2.2. (2.9) and (2.10) do not indicate strongly dissipative condition.

For example, for system (2.4) with $B = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$, we have $b_{ii} > 0$ $(i = 1, 2, 3)$, $B_{11} = 5$, $B_{22} = 2$, $B_{33} = 2 - \varepsilon$, and $|B| = 4 - 2\varepsilon$. It is easy to verify that both (2.9) and (2.10) hold when $\varepsilon \in (-\infty, 2)$. However, when $\varepsilon \in (-\infty, 0) \cup (0, 2)$, we have $b_{11} = 1 < |b_{21}| + |b_{31}| = 1 + |\varepsilon|$, and (2.1) does not hold.

Lemma 2.2. *If B is nonnegative definite and symmetric, then there holds*

$$\sqrt{|B|} \leq \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}. \tag{2.13}$$

Proof. Since B is nonpositive definite and symmetric, there hold $b_{ii} \geq 0$, $B_{ii} \geq 0$, $b_{ij} = b_{ji}$ $(i, j = 1, 2, 3)$ and $|B| \geq 0$. Then we have $b_{11}^2 b_{22}^2 b_{33}^2 \geq b_{12}^2 b_{23}^2 b_{31}^2 = b_{21}^2 b_{13}^2 b_{32}^2$ and

$$b_{11}b_{22}b_{33} \geq b_{12}b_{23}b_{31} = b_{21}b_{32}b_{13}. \tag{2.14}$$

Direct calculation gives

$$\begin{aligned}
\left(\sum_{i=1}^{3} \sqrt{b_{ii}B_{ii}}\right)^2 &= \sum_{i=1}^{3} b_{ii}B_{ii} + 2\sum_{\substack{j,k=1 \\ j\neq k}}^{3} \sqrt{b_{jj}B_{jj}b_{kk}B_{kk}} \\
&= 2b_{11}b_{22}b_{33} - 2b_{12}b_{23}b_{31} + |B| + 2\sum_{\substack{j,k=1 \\ j\neq k}}^{3} \sqrt{b_{jj}B_{jj}b_{kk}B_{kk}} \\
&\geq |B| + 2\sum_{\substack{j,k=1 \\ j\neq k}}^{3} \sqrt{b_{jj}B_{jj}b_{kk}B_{kk}} \\
&\geq |B|,
\end{aligned} \tag{2.15}$$

where in the second-to-last inequality, we have used the inequality (2.14). Thus (2.13) holds. □

To analyze the relation between conditions (2.9), (2.10) with Kawashima-Shizuta condition; We take some critical cases into account.

Lemma 2.3. *Suppose that B is nonnegative definite and symmetric, if (2.4) is strictly dissipative, then either*

$$b_{ii} > 0,\ B_{ii} = 0,\ i = 1, 2, 3,\ |B| = 0 \tag{2.16}$$

or

$$\begin{cases} b_{ii} > 0, \quad i = 1, 2, 3, \\ \textit{for some fixed } i, B_{ii} \geq 0 \textit{ and } B_{jj} > 0 \textit{ for } j \neq i, \\ |B| \geq 0 \end{cases} \tag{2.17}$$

holds.

Proof. Since B is nonpositive definite and symmetric, there hold $b_{ii} \geq 0$, $B_{ii} \geq 0$, $b_{ij} = b_{ji}$ $(i, j = 1, 2, 3)$ and $|B| \geq 0$. If (2.4) is strictly dissipative, to prove (2.16) or (2.17), we need to show that $b_{ii} \neq 0$ $(i = 1, 2, 3)$ and it is not true that $B_{11} = B_{22} = 0,\ B_{33} > 0$. Similarly, $B_{11} > 0,\ B_{22} = B_{33} = 0$ and $B_{22} > 0,\ B_{11} = B_{33} = 0$ are also impossible. In fact, if $B_{11} = B_{22} = 0,\ B_{33} > 0$, and B is symmetric, we get $b_{22}b_{33} = b_{23}^2$, $b_{11}b_{33} = b_{13}^2$ and $b_{11}b_{22} > b_{12}^2$, thus we have

$$\begin{aligned}0 &\leq |B| \\ &= b_{11}b_{22}b_{33} + 2b_{12}b_{13}b_{23} - b_{11}b_{23}^2 - b_{22}b_{13}^2 - b_{33}b_{12}^2 \\ &= 2b_{12}b_{13}b_{23} - b_{11}b_{22}b_{33} - b_{33}b_{12}^2 \\ &= \pm 2b_{12}b_{33}\sqrt{b_{11}b_{22}} - b_{11}b_{22}b_{33} - b_{33}b_{12}^2 \\ &= -b_{33}(b_{12} \pm \sqrt{b_{11}b_{22}})^2 \\ &< 0,\end{aligned}$$

which is obviously a contradiction.

If $b_{11} = 0$, we obtain $B_{22} = b_{11}b_{33} - b_{31}^2 \geq 0$, $B_{33} = b_{11}b_{22} - b_{21}^2 \geq 0$, then $b_{21} = b_{31} = 0$, which contradicts with (2.7) for $i = 1$. If the case of $b_{22} = 0$ or $b_{33} = 0$ can be similarly discussed.

When $b_{ii} > 0,\ B_{ii} = 0,\ i = 1, 2, 3$, we have $b_{11}b_{22} = b_{12}^2$, $b_{22}b_{33} = b_{23}^2$ and $b_{11}b_{33} = b_{13}^2$. By direct calculation, we get

$$0 \leq |B| = 2b_{12}b_{13}b_{23} - 2b_{11}b_{22}b_{33} = 2(\pm b_{11}b_{22}b_{33} - b_{11}b_{22}b_{33}) \leq 0.$$

Obviously, $|B| = 0$.

Thus if (2.4) is strictly dissipative, either (2.16) or (2.17) holds. □

Remark 2.3. By multiplying both sides of (2.4) from the left by $\mathrm{diag}\{m, n, p\}$ (m, $n, p \in \mathbb{R} \setminus \{0\}$), when $b_{ij}b_{ji} > 0$ $(i, j = 1, 2, 3,\ i \neq j)$, it is easy to verify that (2.4) is symmetrizable. Thus Kawashima-Shizuta condition is still applicable in this case. However, if one inequality of $b_{ij}b_{ji} < 0$ $(i, j = 1, 2, 3,\ i \neq j)$ holds, (2.4) is nonsymmetrizable and Kawashima-Shizuta condition fails. In this paper, we will show that (2.9) and (2.10) can still assure the dissipative property of (2.4) (or (1.1)) even if it is not symmetrizable.

3. Main result

Denote $\lambda_1(\xi)$, $\lambda_2(\xi)$ and $\lambda_3(\xi)$ as the three eigenvalues of matrix $B + \mathrm{i}\xi\Lambda$, where B and Λ are given in (2.4). The following theorem shows that conditions (2.9) and (2.10) are sufficient to assure the dissipative property of system (1.1). Here we need not require that B is symmetric or symmetrizable.

Theorem 3.1. *If $B = P^{-1}FP$ satisfies* (2.9) *and* (2.10)*, then system* (1.1) *is strictly dissipative in the sense of Definition 2.3.*

Simple calculation shows that $\lambda_1(\xi)$, $\lambda_2(\xi)$ and $\lambda_3(\xi)$ satisfy

$$\lambda^3 - \sum_{i=1}^{3}(b_{ii} + \mu_i\xi\mathrm{i})\lambda^2 + \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3}(B_{ii} + b_{ii}(\mu_j + \mu_k)\xi\mathrm{i} - \mu_j\mu_k\xi^2)\lambda$$

$$-\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3}(B_{ii}\mu_i\xi\mathrm{i}-b_{ii}\mu_j\mu_k\xi^2)-|B|+\mu_1\mu_2\mu_3\xi^3\mathrm{i}=0. \tag{3.1}$$

We divide the proof of Theorem 3.1 into several lemmas.

First we review Argument principle and a generalization of Argument principle in the complex analysis.

Theorem 3.2. (Argument Principle, [1]) *If $f(z)$ is a meromorphic in Ω with the zeros a_j and the poles b_k, then*

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{f'(z)}{f(z)}d &= \sum_j n(\gamma,a_j)-\sum_k n(\gamma,b_k)z\\ &= \frac{\triangle_\gamma \arg f(z)}{2\pi}\end{aligned}$$

for every cycle γ which is homologous to zeros in Ω and does not pass through any of the zeros or poles, where $n(\gamma,a_j)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{dz}{z-a_j}$ and $n(\gamma,b_k)=\frac{1}{2\pi\mathrm{i}}\int_\gamma\frac{dz}{z-b_k}$. $\triangle_\gamma \arg f(z)$ represents the change of $\arg f(z)$ after z travels around the positive direction of γ, which must be an integral multiple of 2π.

As a corollary of Theorem 3.2, we have

Lemma 3.1. *Assume that*

$$P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$$

is an $n-th$ order polynomial, and $P(z)$ has no zero on the imaginary axis, then its zeros are all in the right half plane $\mathrm{Re}z>0$ if and only if

$$\underset{y(-\infty\nearrow+\infty)}{\Delta\arg} P(\mathrm{i}y)=-n\pi.$$

Namely, as the point z goes from $-\infty$ to ∞ along the imaginary axis from top to bottom, $P(z)$ goes around the origin $\frac{n}{2}$ times.

Lemma 3.2. *If both (2.9) and (2.10) hold, then the real parts of three eigenvalues $\hat\lambda_i$ $(i=1,2,3)$ of B are all positive.*

Proof. Simple calculation shows that $\hat\lambda_1$, $\hat\lambda_2$ and $\hat\lambda_3$ satisfy

$$\hat\lambda^3-(\hat\lambda_1+\hat\lambda_2+\hat\lambda_3)\hat\lambda^2+(\hat\lambda_1\hat\lambda_2+\hat\lambda_2\hat\lambda_3+\hat\lambda_1\hat\lambda_3)\hat\lambda-\hat\lambda_1\hat\lambda_2\hat\lambda_3=0. \tag{3.2}$$

Since both (2.9) and (2.10) hold, we have

$$\begin{cases} a\triangleq\hat\lambda_1+\hat\lambda_2+\hat\lambda_3=\sum\limits_{i=1}^3 b_{ii}>0,\\ b\triangleq\hat\lambda_1\hat\lambda_2+\hat\lambda_2\hat\lambda_3+\hat\lambda_1\hat\lambda_3=\sum\limits_{i=1}^3 B_{ii}>0,\\ c\triangleq\hat\lambda_1\hat\lambda_2\hat\lambda_3=|B|>0\end{cases} \tag{3.3}$$

and

$$c=|B|$$

$$
\begin{aligned}
&< (\sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}})^2 \\
&\leq (b_{11} + b_{22} + b_{33})(B_{11} + B_{22} + B_{33}) \\
&= ab. \qquad (3.4)
\end{aligned}
$$

The second inequality in (3.4) is due to Cauchy's inequality.

If matrix B has a pure imaginary eigenvalue $\hat{\lambda} = m\mathrm{i}$ ($m \neq 0$), plug it into (3.2), then we have

$$m(m^2 - b) = 0,\ am^2 = c, \qquad (3.5)$$

which implies $c = ab$ since $m \neq 0$ and this contradicts with (3.4). Hence B has no pure imaginary root. Define

$$f(\hat{\lambda}) = \hat{\lambda}^3 - a\hat{\lambda}^2 + b\hat{\lambda} - c.$$

By Argument Principle, we have $\mathrm{Re}\hat{\lambda}_i > 0$ ($i = 1, 2, 3$). In fact, we obtain that the slope $\tan\theta = \frac{-(y^3 - by)}{ay^2 - c}$ of $f(\mathrm{i}y) = -(y^3 - by)\mathrm{i} + ay^2 - c$ has two asymptotes and three zeros. Then if $\hat{\lambda}$ goes from $-\infty$ to ∞ along the imaginary axis, $f(\hat{\lambda})$ goes around the origin $\frac{3}{2}$ times. Hence $\underset{y(-\infty\nearrow+\infty)}{\Delta\arg} f(\mathrm{i}y) = -3\pi$. By Lemma 3.1, we have $\mathrm{Re}\hat{\lambda}_i > 0$ ($i = 1, 2, 3$). The proof of Lemma 3.2 is complete. □

The following property is a corollary of Lemma 3.2:

Lemma 3.3. *If* (2.9) *holds, then the real parts of* $\lambda_1(\xi)$, $\lambda_2(\xi)$ *and* $\lambda_3(\xi)$ *are all positive near* $\xi = 0$.

Proof. Set $\lambda = a_0 + a_1\xi + a_2\xi^2 + O(\xi^3)$ near $\xi = 0$ and substitute it into (3.1), then we can get

$$a_0^3 - \sum_{i=1}^{3} b_{ii}a_0^2 + \sum_{i=1}^{3} B_{ii}a_0 - |B| = 0. \qquad (3.6)$$

Since (2.9) holds, we have $a_0 \neq 0$. Obviously, (3.6) has the same form as the characteristic polynomial of B in (3.2). Hence, by Lemma 3.2, we obtain $\mathrm{Re}a_0 > 0$, then $\mathrm{Re}\lambda_i(\xi) > 0$ ($i = 1, 2, 3$) near $\xi = 0$. □

Lemma 3.4. *The real parts of* $\lambda_1(\xi)$, $\lambda_2(\xi)$ *and* $\lambda_3(\xi)$ *are all positive near* $\xi = \infty$ *if and only if* $b_{ii} > 0,\ i = 1, 2, 3$.

Proof. Let us first consider the approximate expressions of $\lambda_i(\xi)$ ($i = 1, 2, 3$) near $\xi = \infty$. Set $\xi = \frac{1}{\eta}$, then (3.1) becomes

$$
\begin{aligned}
&\eta^3\lambda^3 - \sum_{i=1}^{3}(b_{ii}\eta^3 + \mu_i\eta^2\mathrm{i})\lambda^2 + \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3}(B_{ii}\eta^3 + b_{ii}(\mu_j + \mu_k)\eta^2\mathrm{i} - \mu_j\mu_k\eta)\lambda \\
&- \sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3}(B_{ii}\mu_i\eta^2\mathrm{i} - b_{ii}\mu_j\mu_k\eta) - |B|\eta^3 + \mu_1\mu_2\mu_3\mathrm{i} = 0. \qquad (3.7)
\end{aligned}
$$

Set $\lambda = a_0 \frac{\mathrm{i}}{\eta} + a_1 + a_2 \eta \mathrm{i} + O(\eta^2)$ and substitute it into (3.7), then one obtains

$$\begin{cases} 3a_0 a_1^2 - 3a_0^2 a_2 - \sum\limits_{i=1}^{3} (2a_0 a_1 b_{ii} + a_1^2 \mu_i - 2a_0 a_2 \mu_i + B_{ii}(\mu_i - a_0)) \\ + \sum\limits_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} (a_1 b_{ii}(\mu_j + \mu_k) - a_2 \mu_j \mu_k) = 0, \\ 3a_0^2 a_1 - \sum\limits_{i=1}^{3} (a_0^2 b_{ii} + 2a_0 a_1 \mu_i) + \sum\limits_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} (a_0 b_{ii}(\mu_j + \mu_k) + a_1 \mu_j \mu_k - b_{ii} \mu_j \mu_k) = 0, \\ (a_0 - \mu_1)(a_0 - \mu_2)(a_0 - \mu_3) = 0. \end{cases} \tag{3.8}$$

Choose $\lambda = \lambda_i(\xi)$ $(i = 1, 2, 3)$ in turn, we get $a_0 = \mu_i$ and $a_1 = b_{ii},\ i = 1, 2, 3$, respectively. Thus we conclude that $\mathrm{Re}\lambda_i(\xi) = b_{ii} + O(\xi^{-1}) > 0$ near $\xi = \infty$ if and only if $b_{ii} > 0,\ i = 1, 2, 3$. □

Lemma 3.5. *Under conditions* (2.9) *and* (2.10)*, one of the following three inequalities must hold:*

$$\sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} > |\sqrt{b_{33}B_{33}} - \sqrt{|B|}|; \tag{3.9}$$
$$\sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}} > |\sqrt{b_{11}B_{11}} - \sqrt{|B|}|; \tag{3.10}$$
$$\sqrt{b_{11}B_{11}} + \sqrt{b_{33}B_{33}} > |\sqrt{b_{22}B_{22}} - \sqrt{|B|}|. \tag{3.11}$$

Proof. The inequalities (3.9)-(3.11) are equivalent to

$$\begin{aligned} \sqrt{b_{33}B_{33}} - \sqrt{b_{11}B_{11}} - \sqrt{b_{22}B_{22}} < \sqrt{|B|} < \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}; \\ \sqrt{b_{11}B_{11}} - \sqrt{b_{22}B_{22}} - \sqrt{b_{33}B_{33}} < \sqrt{|B|} < \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}; \\ \sqrt{b_{22}B_{22}} - \sqrt{b_{11}B_{11}} - \sqrt{b_{33}B_{33}} < \sqrt{|B|} < \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}. \end{aligned}$$

Due to (2.10), the right haves of the three inequalities obviously hold. Thus if neither of the inequalities (3.9)-(3.11) holds, we have

$$\begin{aligned} \sqrt{b_{33}B_{33}} \geq \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{|B|}, \\ \sqrt{b_{11}B_{11}} \geq \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}} + \sqrt{|B|}, \\ \sqrt{b_{22}B_{22}} \geq \sqrt{b_{11}B_{11}} + \sqrt{b_{33}B_{33}} + \sqrt{|B|}. \end{aligned}$$

By summing up the three inequalities, we have $0 < 3\sqrt{|B|} + \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}} \leq 0$, which is clearly a contradiction. □

Lemma 3.6. *Under conditions* (2.9) *and* (2.10)*, one of the following three inequalities must hold:*

$$\sqrt{b_{33}B_{33}} + \sqrt{|B|} > |\sqrt{b_{11}B_{11}} - \sqrt{b_{22}B_{22}}|; \tag{3.12}$$
$$\sqrt{b_{22}B_{22}} + \sqrt{|B|} > |\sqrt{b_{33}B_{33}} - \sqrt{b_{11}B_{11}}|; \tag{3.13}$$
$$\sqrt{b_{11}B_{11}} + \sqrt{|B|} > |\sqrt{b_{22}B_{22}} - \sqrt{b_{33}B_{33}}|. \tag{3.14}$$

Proof. The inequalities (3.12)-(3.14) are equivalent to

$$\begin{aligned}
&\sqrt{b_{22}B_{22}}-\sqrt{b_{33}B_{33}}-\sqrt{|B|}<\sqrt{b_{11}B_{11}}<\sqrt{b_{22}B_{22}}+\sqrt{b_{33}B_{33}}+\sqrt{|B|};\\
&\sqrt{b_{11}B_{11}}-\sqrt{b_{22}B_{22}}-\sqrt{|B|}<\sqrt{b_{33}B_{33}}<\sqrt{b_{11}B_{11}}+\sqrt{b_{22}B_{22}}+\sqrt{|B|};\\
&\sqrt{b_{33}B_{33}}-\sqrt{b_{11}B_{11}}-\sqrt{|B|}<\sqrt{b_{22}B_{22}}<\sqrt{b_{11}B_{11}}+\sqrt{b_{33}B_{33}}+\sqrt{|B|}.
\end{aligned}$$

Obviously the right haves of the three inequalities are equivalent to the left haves in the three inequalities. Thus if neither of the inequalities (3.12)-(3.14) holds, we get the conclusion as in the proof of Lemma 3.5. $\square$

Lemma 3.7. *If $B=P^{-1}FP$ satisfies* (2.9) *and* (2.10) *for $\xi\in\mathbb{R}\backslash\{0\}$, then $B+\mathrm{i}\xi\Lambda$ has no pure imaginary eigenvalue.*

It suffices to prove that (3.1) has no pure imaginary solution. In fact, if there exists some fixed ξ such that $\lambda(\xi)=a(\xi)\mathrm{i}$ and $a(\xi)$ is a real number, by substituting it to (3.1), we obtain

$$\sum_{i=1}^{3}(a(\xi)-\mu_i\xi)B_{ii}=\prod_{i=1}^{3}(a(\xi)-\mu_i\xi), \tag{3.15}$$

$$\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{3} b_{ii}(a(\xi)-\mu_j\xi)(a(\xi)-\mu_k\xi)=|B|. \tag{3.16}$$

Denote $b_i=a(\xi)-\mu_i\xi$, $i=1,2,3$. (3.15) and (3.16) can be respectively written as

$$b_1B_{11}+b_2B_{22}+b_3B_{33}=b_1b_2b_3, \tag{3.17}$$

$$b_2b_3b_{11}+b_1b_3b_{22}+b_1b_2b_{33}=|B|. \tag{3.18}$$

We assume that (3.9) and (3.12) in Lemmas 3.5 and 3.6 hold. Taking (3.17) $\times$ $(b_1b_{22}+b_2b_{11})+$ (3.18) $\times$ $(b_1b_2-B_{33})$ to eliminate b_3, we get

$$\begin{aligned}
&b_{33}b_1^2b_2^2+b_{22}B_{11}b_1^2+b_{11}B_{22}b_2^2\\
&+(b_{11}B_{11}+b_{22}B_{22}-b_{33}B_{33}-|B|)b_1b_2+|B|B_{33}=0.
\end{aligned} \tag{3.19}$$

Define

$$\begin{aligned}
F(x,y)=\;&b_{33}x^2y^2+b_{22}B_{11}x^2+b_{11}B_{22}y^2\\
&+(b_{11}B_{11}+b_{22}B_{22}-b_{33}B_{33}-|B|)xy+|B|B_{33}.
\end{aligned} \tag{3.20}$$

We will verify that $F(x,y)>0$ for any $x,\ y\in\mathbb{R}$, which implies that (3.19) does not hold.

By direct calculation, we have

$$\begin{aligned}
&F_{xx}(x,y)=2(b_{22}B_{11}+b_{33}y^2)>0,\\
&F_{yy}(x,y)=2(b_{11}B_{22}+b_{33}x^2)>0,\\
&F_{xy}(x,y)=F_{yx}(x,y)=b_{11}B_{11}+b_{22}B_{22}-b_{33}B_{33}-|B|+4b_{33}xy.
\end{aligned}$$

We can verify that the sign of determinant of the Hessian $H_F\triangleq\begin{pmatrix}F_{xx}&F_{xy}\\ F_{yx}&F_{yy}\end{pmatrix}$ is undetermined, so we are not able to determine its extreme values directly. Thus we take a different approach to study the minimum value of $F(x,y)$.

Obviously, $F(0,y) = b_{11}B_{22}y^2 + |B|B_{33} = |B|B_{33} > 0$. Set $y = kx$ and denote $t = x^2$, then (3.20) becomes

$$f_k(t) \triangleq b_{33}k^2t^2 + g(k)t + |B|B_{33}, \tag{3.21}$$

where

$$g(k) = b_{11}B_{22}k^2 + (b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B|)k + b_{22}B_{11}.$$

To finish the proof, we need only to prove $f_k(t) > 0$ for any $t \geq 0$ and $k \in \mathbb{R}$ under conditions (2.9) and (2.10).

Obviously we have $f_k(0) = F(0,y) > 0$. When $t > 0$, let us first see two simple cases.

Case 1. When $k = 0$, by using of (2.9), we have $f_0(t) = b_{22}B_{11}t + |B|B_{33} > 0$ for any $t > 0$.

Case 2. When $k \neq 0$, $g(k)$ is a quadratic function. If

$$\Delta_g \triangleq (b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B|)^2 - 4b_{11}b_{22}B_{11}B_{22} \leq 0, \tag{3.22}$$

we have $g(k) \geq 0$, which implies $f_k(t) > 0$ for $t > 0$.

For the case of $\Delta_g > 0$, we have the following

Lemma 3.8. *Under conditions* (2.9), (2.10) *and* $\Delta_g > 0$, *there holds* $f_k(t) > 0$ *for* $t > 0$.

Proof. We divided the proof into two cases.

Case 1. $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| < 0$. In this case, $g(k)$ has two real roots k_1 and k_2 satisfying $0 < k_1 < k_2$. Thus for any $k \in (-\infty, k_1] \bigcup [k_2, +\infty)$, we have $g(k) \geq 0$, thus $f_k(t) > 0$ for $t > 0$.

For $k \in (k_1, k_2)$, we have $g(k) < 0$. Since we have assumed that (3.9) holds, by squaring it we get

$$b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|} + 2\sqrt{b_{11}b_{22}B_{11}B_{22}} > 0$$

or

$$-\left[b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|}\right] < 2\sqrt{b_{11}b_{22}B_{11}B_{22}}. \tag{3.23}$$

Denote

$$\begin{aligned} h_1(k) &= g(k) + 2k\sqrt{b_{33}B_{33}|B|} \\ &= b_{11}B_{22}k^2 + (b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|})k + b_{22}B_{11}. \end{aligned} \tag{3.24}$$

If $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|} \geq 0$, we have $h_1(k) > 0$ for $k > 0$.

Conversely, if $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|} < 0$, by squaring the two sides of (3.23), we get

$$\Delta_{h_1} \triangleq (b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|})^2 - 4b_{11}b_{22}B_{11}B_{22} < 0, \tag{3.25}$$

thus we get $h_1(k) > 0$, or $-g(k) < 2k\sqrt{b_{33}B_{33}|B|}$. Since $g(k) < 0$ for $k \in (k_1, k_2)$, we can further get $\Delta_{f_k} = g^2(k) - 4k^2 b_{33}B_{33}|B| < 0$, thus $f_k(t) > 0$ for $t > 0$.

Case 2. $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| \geq 0$. Similarly, in this case, $g(k)$ has two roots k_1 and k_2 satisfying $k_1 < k_2 < 0$. Thus for $k \in (-\infty, k_1] \bigcup [k_2, +\infty)$, we have $g(k) \geq 0$, thus $f_k(t) > 0$ for $t > 0$.

For $k \in (k_1, k_2)$, we have $g(k) < 0$. Since we have assumed that (3.12) holds, by squaring it we get

$$-b_{11}B_{11} - b_{22}B_{22} + b_{33}B_{33} + |B| + 2\sqrt{b_{33}B_{33}|B|} + 2\sqrt{b_{11}b_{22}B_{11}B_{22}} > 0$$

or

$$-\left[b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| - 2\sqrt{b_{33}B_{33}|B|}\right] > -2\sqrt{b_{11}b_{22}B_{11}B_{22}}. \quad (3.26)$$

Denote

$$\begin{aligned} h_2(k) &= g(k) - 2k\sqrt{b_{33}B_{33}|B|} \\ &= b_{11}B_{22}k^2 + (b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| - 2\sqrt{b_{33}B_{33}|B|})k + b_{22}B_{11}. \end{aligned} \quad (3.27)$$

If $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| - 2\sqrt{b_{33}B_{33}|B|} \leq 0$, we have $h_2(k) > 0$ for $k < 0$.

Conversely, if $b_{11}B_{11} + b_{22}B_{22} - b_{33}B_{33} - |B| - 2\sqrt{b_{33}B_{33}|B|} > 0$, by squaring both sides of (3.26), we get

$$\Delta_{h_2} \triangleq (b_{11}B_{11}+b_{22}B_{22}-b_{33}B_{33}-|B|-2\sqrt{b_{33}B_{33}|B|})^2-4b_{11}b_{22}B_{11}B_{22} < 0, \quad (3.28)$$

thus we have $h_2(k) > 0$, or $0 > g(k) > 2k\sqrt{b_{33}B_{33}|B|}$. Since $g(k) < 0$ for $k \in (k_1, k_2)$, we can further obtain $\Delta_{f_k} = g^2(k) - 4k^2 b_{33}B_{33}|B| < 0$, thus $f_k(t) > 0$ for $t > 0$. □

The conclusion of Theorem 3.1 follows from Lemma 3.3, Lemma 3.4 and Lemma 3.7.

4. Pointwise estimates on Green function

In this section, we establish the pointwise estimates of problem (1.1)-(1.2) under conditions (2.9) and (2.10).

Recall that $\lambda = a_0 + a_1\xi + a_2\xi^2 + O(\xi^3)$ near $\xi = 0$ as denoted in Lemma 3.3. Denote a_m^i $(m = 0, 1)$ as the coefficient of the m-th term in the approximate expressions of $\lambda_i(\xi)$ $(i = 1, 2, 3)$ near $\xi = 0$.

Theorem 4.1. *For any given nonnegative α, assume that $U_0 \in L^1(\mathbb{R}) \cap C^\alpha(\mathbb{R})$ with compact support and $B = P^{-1}FP$ satisfies (2.9) and (2.10). Then for any positive integer N, the solution $U(x,t)$ to (1.1)−(1.2) satisfies the following estimate for any $(x,t) \in \mathbb{R} \times \mathbb{R}^+$*

$$|D^\alpha U(x,t)| \leq Mt^{-\frac{\alpha+1}{2}} \left[B_N(x,t) + e^{-\epsilon t}\sum_{i=1}^{3} B_N(x + t\mathrm{Im}a_1^i, t)\right] \| U_0(x)\|_{L^1(\mathbb{R})}, \quad (4.1)$$

where $B_N(x,t) = (1 + \frac{x^2}{1+t})^{-N}$, $\epsilon = \frac{1}{2}\min\{a_0^1,\ a_0^2,\ a_0^3\}$, M depends only on α, N and the support of $U_0(x)$.

Theorem 4.1 describes the decay rates as well as the directions of decay of the solution. (4.1) show that the decay rates of the solution is $t^{-\frac{1}{2}}$. Moreover, along any direction except $x \neq 0$, $t\mathrm{Im}a_1^i$ $(i = 1, 2, 3)$, the solution decays very fast.

The method of proving Theorem 4.1 is based on a delicate analysis for the Fourier transform of the Green function of (1.1). In the sequel, we use $\hat{f}(\xi)$ to denote the Fourier transform of $f(x)$ and $\check{f}(\xi)$ to denote the inverse Fourier transform of $f(x)$, that is,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\mathrm{i}x\xi} f(x)dx, \quad \check{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mathrm{i}x\xi} f(\xi)d\xi.$$

Here the notation "i" denotes the imaginary unit satisfying $\mathrm{i}^2 = -1$.

Consider system (1.1) with initial data

$$U_0(x) = (u_1(x,0), u_2(x,0), u_3(x,0))^\top = \delta(x)I, \tag{4.2}$$

where I is the identity matrix and $\delta(x)$ is the Dirac function. The solution to (1.1) and (4.2), denoted as $G(x,t)$, is called the Green function of Cauchy problem (1.1) and (1.2). Taking Fourier transform with respect to x to (1.1) and (4.2), we get

$$\frac{\partial \hat{G}}{\partial t}(\xi, t) = -(F + \mathrm{i}\xi A)\hat{G}(\xi, t), \quad \hat{G}(\xi,\ 0) = I. \tag{4.3}$$

Since $P^{-1}(F + \mathrm{i}\xi A)P = B + \mathrm{i}\xi\Lambda$, $\lambda_i(\xi)$ $(i = 1, 2, 3)$ are also the eigenvalues of $F + \mathrm{i}\xi A$.

As is well known, the decay of the solution is mainly related to the properties of $\hat{G}(\xi, t)$ near $\xi = 0$ in the frequency space. By studying the decay property for the Fourier transform of the Green function, we can obtain the pointwise estimates (4.1). Exactly, we find three directions out of which the solution decays of any polynomial order, which shows the hyperbolic property of the problem. The proof of Theorem 4.1 is similar to Theorem 3.1 of Section 3 in [23], so we will not state it here for brevity.

5. Critical cases

Critical case implies that at least one "<" in (2.9) and (2.10) is replaced by "=". According to Lemma 3.4, $b_{ii} > 0$ $(i = 1, 2, 3)$ if and only if $\mathrm{Re}\lambda_i(\xi) > 0$ $(i = 1, 2, 3)$ near $\xi = \infty$, hence $b_{ii} = 0$ $(i = 1, 2, 3)$ are not critical cases. First, let us give some examples to explain the complexity of the critical situation of conditions (2.9) and (2.10).

Example 5.1. Consider system (2.4) with

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & m & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}.$$

If $m = 1$, we have $B_{11} = B_{22} = 0$. When $\xi = \frac{1}{3}$, the characteristic polynomial of $B + \mathrm{i}\xi\Lambda$ is

$$(\lambda - \mathrm{i})\left(\lambda^2 - \left(\frac{\mathrm{i}}{3} + 6\right)\lambda - \frac{5\mathrm{i}}{3} + 1\right) = 0,$$

which has a pure imaginary roots $\lambda = \mathrm{i}$.

If $m = \frac{3}{2}$, we have $B_{11} = B_{22} = 0$ and $\sqrt{|B|} = \sum_{i=1}^{3} \sqrt{b_{ii}B_{ii}} = \sqrt{\frac{3}{2}}$. When $\xi = \frac{\sqrt{3}}{6}$, we can similarly verify that $\lambda = \frac{\sqrt{3}}{2}\mathrm{i}$ is an eigenvalue of $B + \mathrm{i}\xi\Lambda$.

Remark 5.1. Example 5.1 shows that when B is asymmetric, conditions

$$b_{ii} > 0,\ B_{ii} > 0,\ B_{jj} = B_{kk} = 0, \quad i,j,k = 1,2,3,\ i \neq j \neq k \neq i, |B| \geq 0 \qquad (5.1)$$

and (2.13) do not imply that the characteristic polynomial of $B + \mathrm{i}\xi\Lambda$ has no pure imaginary solution. Hence (2.13) and (5.1) do not imply that the real parts of $\lambda_1(\xi)$, $\lambda_2(\xi)$ and $\lambda_3(\xi)$ are all positive for any $\xi \in \mathbb{R} \setminus \{0\}$.

Example 5.2. Consider system (2.4) with

$$\Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 48 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

By direct calculation, we have $\sqrt{|B|} = \sum_{i=1}^{3} \sqrt{b_{ii}B_{ii}} = 3\sqrt{6}$. The characteristic polynomial of $B + \mathrm{i}\xi\Lambda$ is

$$\lambda^3 - 6\lambda^2 + (11 + 2\xi\mathrm{i} + \xi^2)\lambda - (4\xi\mathrm{i} + 2\xi^2) - 54 = 0. \qquad (5.2)$$

We can verify that when $\xi = \sqrt{3}$ (or $-\sqrt{3}$), $\lambda = 2\sqrt{3}\mathrm{i}$ (or $-2\sqrt{3}\mathrm{i}$) is one of its roots.

Set $\lambda = a_0 + a_1\xi + a_2\xi^2 + O(\xi^3)$ near $\xi = 0$ and substitute it into (3.1), then we have

$$\begin{cases} a_0^3 - \sum\limits_{i=1}^{3} b_{ii}a_0^2 + \sum\limits_{i=1}^{3} B_{ii}a_0 - |B| = 0, \\ 3a_0^2a_1 - \sum\limits_{i=1}^{3}(a_0^2\mu_i\mathrm{i} + 2a_0a_1b_{ii} - a_1B_{ii} + B_{ii}\mu_i\mathrm{i}) + a_0 \sum\limits_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} b_{ii}(\mu_j + \mu_k)\mathrm{i} = 0, \\ 3(a_0a_1^2 + a_0^2a_2) - \sum\limits_{i=1}^{3}((2a_0a_2 + a_1^2)b_{ii} + 2a_0a_1\mu_i\mathrm{i} - a_2B_{ii}) \\ + \sum\limits_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} (a_1b_{ii}(\mu_j + \mu_k)\mathrm{i} + (b_{ii} - a_0)\mu_j\mu_k) = 0, \\ a_1^3 + 6a_0a_1a_2 - \sum\limits_{i=1}^{3}(2a_1a_2b_{ii} + (a_1^2 + 2a_0a_2)\mu_i\mathrm{i}) \\ + \sum\limits_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} (a_2b_{ii}(\mu_j + \mu_k)\mathrm{i} - a_1\mu_j\mu_k) + \mu_1\mu_2\mu_3\mathrm{i} = 0. \end{cases} \qquad (5.3)$$

Example 5.3. Consider system (2.4) with

$$\Lambda = \begin{pmatrix} -22 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 5 & 4 & 1 \end{pmatrix}.$$

By direct calculation, we have $|B| = 0$. Plugging it into $(5.3)_1$ yields $a_0 = 0$ or $a_0 = \frac{3\pm\sqrt{19}\mathrm{i}}{2}$. When $a_0 \neq 0$, we have $\mathrm{Re}a_0 = \frac{3}{2} > 0$. When $a_0 = 0$, by using of $(5.3)_{2,3}$, we have $a_1 = -15\mathrm{i}$, $a_2 = -\frac{55}{7} < 0$. Thus $\lambda = -\frac{55}{7}\xi^2 - 15\xi\mathrm{i} + O(\xi^3)$ near $\xi = 0$, the principal part is $-\frac{55}{7}\xi^2$, which is not strictly dissipative.

Nevertheless, we have

Proposition 5.1. (i) *If $B = P^{-1}FP$ satisfies*

$$b_{ii} > 0, \ i = 1, 2, 3, \tag{5.4}$$

$$\textit{for some fixed } i, \ B_{ii} = 0 \textit{ and } B_{jj} > 0 \textit{ for } j \neq i, \tag{5.5}$$

$$0 < \sqrt{|B|} \leq \sqrt{b_{11}B_{11}} + \sqrt{b_{22}B_{22}} + \sqrt{b_{33}B_{33}}, \tag{5.6}$$

then (1.1) *is strictly dissipative.*

(ii) *If $B = P^{-1}FP$ satisfies* (5.4) *and*

$$B_{ii} = 0, \ i = 1, 2, 3, \ |B| = 0, \tag{5.7}$$

then (1.1) *is strictly dissipative.*

Proof. Since $b_{ii} > 0$ $(i = 1, 2, 3)$, Lemma 3.4 implies that $\mathrm{Re}\lambda_i(\xi) > 0$ $(i = 1, 2, 3)$ near $\xi = \infty$.

(i) If B satisfies (5.5) and (5.6), we assume $B_{11} = 0$, $B_{22} > 0$ and $B_{33} > 0$ since other cases can be similarly discussed. Then we have $\sum_{i=1}^{3} b_{ii} > 0$, $\sum_{i=1}^{3} B_{ii} > 0$ and $|B| > 0$. By Lemma 3.3, (5.5) indicates that the real parts of $\lambda_1(\xi)$, $\lambda_2(\xi)$ and $\lambda_3(\xi)$ are all positive near $\xi = 0$.

In addition, we can show that $B + \mathrm{i}\xi\Lambda$ has no pure imaginary eigenvalue for $\xi \in \mathbb{R} \setminus \{0\}$ by similar discussion as in the proof of Lemma 3.7. Denote $f_k(t)$, $g(k)$, h_1 and h_2 as in Lemma 3.7. Since $B_{11} = 0$, we have

$$\begin{aligned} f_k(t) &= b_{33}k^2t^2 + g(k)t + |B|B_{33}, \\ h_1 &= g(k) + 2k\sqrt{b_{33}B_{33}|B|}, \\ h_2 &= g(k) - 2k\sqrt{b_{33}B_{33}|B|}, \end{aligned}$$

where $g(k) = b_{11}B_{22}k^2 + (b_{22}B_{22} - b_{33}B_{33} - |B|)k$. Since B satisfies (5.5) and (5.6), we have the same conclusions as Lemma 3.5 and Lemma 3.6. We still assume that (3.9) and (3.12) hold, which can be written as

$$\sqrt{b_{22}B_{22}} \geq |\sqrt{b_{33}B_{33}} - \sqrt{|B|}|, \tag{5.8}$$

$$\sqrt{b_{33}B_{33}} + \sqrt{|B|} \geq \sqrt{b_{22}B_{22}}. \tag{5.9}$$

Case 1. When $k = 0$, we have $f_0(t) = |B|B_{33} > 0$ for any $t > 0$.

Case 2. When $k \neq 0$, we have $\Delta_g \triangleq (b_{22}B_{22} - b_{33}B_{33} - |B|)^2 \geq 0$.

(1) If $\Delta_g = 0$, we have $f_k(t) > 0$ for any $t > 0$.

(2) If $\Delta_g > 0$ and $b_{22}B_{22} - b_{33}B_{33} - |B| < 0$, $g(k)$ has two real roots $0 = k_1 < k_2$. Thus for any $k \in (-\infty, 0] \bigcup [k_2, +\infty)$, we have $g(k) \geq 0$, and then $f_k(t) > 0$ for $t > 0$. For $k \in (0, k_2)$, we have $g(k) < 0$. By using of (5.8), we can get

$$b_{22}B_{22} - b_{33}B_{33} - |B| + 2\sqrt{b_{33}B_{33}|B|} \geq 0,$$

then we have $h_1(k) > 0$ for $k > 0$. We can further get $\triangle_{f_k} = g^2(k) - 4k^2 b_{33}B_{33}|B| < 0$, thus $f_k(t) > 0$ for $t > 0$.

If $\Delta_g > 0$ and $b_{22}B_{22} - b_{33}B_{33} - |B| > 0$, $g(k)$ has two roots $k_1 < k_2 = 0$. Thus for $k \in (-\infty, k_1] \bigcup [0, +\infty)$, we have $g(k) \geq 0$, and then $f_k(t) > 0$ for $t > 0$. For $k \in (k_1, 0)$, we have $g(k) < 0$. (5.9) can be written as

$$-b_{22}B_{22} + b_{33}B_{33} + |B| + 2\sqrt{b_{33}B_{33}|B|} \geq 0,$$

then we have $h_2(k) > 0$ for $k < 0$. We can further get $\triangle_{f_k} = g^2(k) - 4k^2 b_{33}B_{33}|B| < 0$, thus $f_k(t) > 0$ for $t > 0$.

To sum up, we obtain that the real parts of $\lambda_1(\xi)$, $\lambda_2(\xi)$ and $\lambda_3(\xi)$ are all positive for any $\xi \in \mathbb{R} \setminus \{0\}$.

(ii) If B satisfies (5.4) and (5.7), we show that $B + \mathrm{i}\xi\Lambda$ has no pure imaginary solution for any $\xi \in \mathbb{R} \setminus \{0\}$. In fact, plugging (5.7) into (3.15) and (3.16), we obtain

$$\begin{cases} \prod_{i=1}^{3} (a(\xi) - \mu_i \xi) = 0, \\ \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} b_{ii}(a(\xi) - \mu_j \xi)(a(\xi) - \mu_k \xi) = 0. \end{cases} \tag{5.10}$$

By using of $(5.10)_1$ and $\mu_i \neq \mu_j$ $(i \neq j)$, we obtain that for any fixed ξ, only one of the three equalities $a(\xi) - \mu_i \xi = 0$ $(i = 1, 2, 3)$ holds. Suppose that $a(\xi) - \mu_1 \xi = 0$ and $a(\xi) - \mu_2 \xi \neq 0$, $a(\xi) - \mu_3 \xi \neq 0$, by using of $(5.10)_2$, we have $(a(\xi) - \mu_2 \xi)(a(\xi) - \mu_3 \xi) b_{11} = 0$, which is impossible since $b_{11} > 0$.

Next we claim that $\mathrm{Re}\lambda_i(\xi) > 0$ $(i = 1, 2, 3)$ near $\xi = 0$. Denote $\alpha = \mu_3 - \mu_2 \neq 0$, $\beta = \mu_2 - \mu_1 \neq 0$. Note that $\alpha \neq -\beta$ since $\mu_1 \neq \mu_3$. Plugging (5.7) into $(5.3)_1$ yields $a_0^2(a_0 - \sum_{i=1}^{3} b_{ii}) = 0$. When $a_0 \neq 0$, we have $a_0 = \sum_{i=1}^{3} b_{ii} > 0$. When $a_0 = 0$, plug it and (5.7) into $(5.3)_{2,3,4}$, then we have

$$\begin{cases} \sum_{i=1}^{3} b_{ii} a_1^2 - \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} b_{ii}(\mu_j + \mu_k) a_1 \mathrm{i} - \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} b_{ii} \mu_j \mu_k = 0, \\ \sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} b_{ii}(2a_1 \mathrm{i} + (\mu_j + \mu_k)) a_2 + (a_1 \mathrm{i} + \mu_1)(a_1 \mathrm{i} + \mu_2)(a_1 \mathrm{i} + \mu_3) = 0. \end{cases} \tag{5.11}$$

It is easy to check that $(5.11)_1$ has two roots

$$a_1 = \frac{\sum_{\substack{i,j,k=1 \\ i \neq j \neq k \neq i}}^{3} b_{ii}(\mu_j + \mu_k) \pm \sqrt{\Delta(\alpha, \beta)}}{2 \sum_{i=1}^{3} b_{ii}} \mathrm{i}, \tag{5.12}$$

where $\Delta(\alpha,\beta) = (b_{11}+b_{22})^2\alpha^2 + 2(b_{22}^2+b_{22}b_{33}+b_{11}b_{22}-b_{11}b_{33})\alpha\beta + (b_{22}+b_{33})^2\beta^2$ is a quadratic form with respect to α and β. Denote $r = b_{11}+b_{22}+b_{33}$, the matrix of $\Delta(\alpha,\beta)$ can be written as

$$D = \begin{pmatrix} (b_{11}+b_{22})^2 & b_{22}r - b_{11}b_{33} \\ b_{22}r - b_{11}b_{33} & (b_{22}+b_{33})^2 \end{pmatrix},$$

which is obviously positive definite. Hence, (5.12) implies that a_1 is a pure imaginary number. Plugging (5.12) into $(5.11)_2$ yields

$$\begin{aligned} \pm\sqrt{\Delta(\alpha,\beta)}a_2 = \frac{1}{8r^3}&\Big((b_{11}+b_{22})\alpha - (b_{22}+b_{33})\beta \pm \sqrt{\Delta(\alpha,\beta)}\Big) \\ &\times\Big((b_{11}+b_{22})\alpha + (b_{22}+b_{33})\beta + 2b_{11}\beta \pm \sqrt{\Delta(\alpha,\beta)}\Big) \\ &\times\Big((b_{11}+b_{22})\alpha + (b_{22}+b_{33})\beta + 2b_{33}\alpha \mp \sqrt{\Delta(\alpha,\beta)}\Big). \end{aligned} \tag{5.13}$$

Denote $x = \frac{\alpha}{\beta}$ $(x \neq 0, -1)$, then $\Delta(\alpha,\beta)$ becomes

$$\Delta(x) = \beta^2\Big((b_{11}+b_{22})^2x^2 + 2(b_{22}r - b_{11}b_{33})x + (b_{22}+b_{33})^2\Big). \tag{5.14}$$

We consider the case of $\beta > 0$ since $\beta < 0$ can be similarly discussed. By using of (5.14), (5.13) becomes

$$Q_\pm(x) \triangleq \pm 8r^3\sqrt{\Delta(x)}a_2 = 2\beta^2[q_1(x) \pm q_2(x)], \tag{5.15}$$

where

$$\begin{aligned} q_1(x) = {}& b_{33}(b_{11}+b_{22})^2x^3 + b_{33}((b_{11}+2b_{22})r - 3b_{11}b_{33})x^2 \\ & -b_{11}((b_{22}+2b_{33})r - 3b_{11}b_{33})x - b_{11}(b_{22}+b_{33})^2, \end{aligned} \tag{5.16}$$

$$q_2(x) = \Big(b_{33}(b_{11}+b_{22})x^2 + 2b_{11}b_{33}x + b_{11}(b_{22}+b_{33})\Big)\sqrt{\Delta(x)}. \tag{5.17}$$

By direct calculations, we have $b_{11}^2b_{33}^2 - b_{11}b_{33}(b_{11}+b_{22})(b_{22}+b_{33}) = -b_{11}b_{22}b_{33}r < 0$. It is easy to verify that $q_2(x) > 0$ for any $x \in \mathbb{R}$.

By some tedious but direct calculations, we have

$$q_1^2(x) - q_2^2(x) = -4b_{11}b_{22}b_{33}rx^2(x^2 - x + 1) < 0. \tag{5.18}$$

Then we have $-q_2(x) < q_1(x) < q_2(x)$. By observing of (5.15), we have $Q_+(x) > 0$ and $Q_-(x) < 0$.

To sum up, we have $a_2 > 0$. Thus we have $\mathrm{Re}\lambda_i(\xi) > 0$ $(i = 1,2,3)$ near $\xi = 0$. This completes the proof of Proposition 5.1. □

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