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THE COLLISION-AVOIDING FINITE-TIME FLOCKING OF A CUCKER-SMALE MODEL WITH PINNING CONTROL AND EXTERNAL PERTURBATION

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Abstract The Cucker-Smale model plays a vital role in analyzing flocking behavior. To investigate the impact of pinning control and external perturbation on finite-time flocking behavior, a modified Cucker-Smale model that incorporates these factors is proposed in this paper. Initially, by imposing appropriate restrictions on external perturbation, the system can achieve finite-time flocking, and the upper bound of settling time is derived explicitly. Subsequently, a new sufficient condition is given to ensure collision-avoiding during the flocking process. The results show that the convergence time depends on control parameters and the convergence speed of the perturbation. Lastly, numerical simulations are provided to illustrate the derived results.

Keywords Cucker-Smale model, pinning control, external perturbation, finite-time flocking, collision-avoiding.

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1. Introduction

Among many different types of dynamical behavior in the model, a noteworthy one is flocking, where the individual state eventually converges to a desired common state, such as bird flocks, locusts swarming, fish schools, and more. This phenomenon has captivated researchers for decades due to its inherent self-organizing and emergent properties.

Recently, some admirable results have been derived about flocking behavior. In 1995, Vicsek et al. proposed a novel type model in which each particle adjusts its velocity in response to its neighbors' states [27]. The results showed that motion becomes easily organized at larger densities and lower particle noise levels. Based on the Vicsek model, a second-order dynamic model was first proposed by Cucker and Smale in 2007 [14]. Different from [27], each agent modifies their velocity based on a weighted average of the discrepancies with others' velocities. The Cucker-Smale

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(in short C-S) model is taken as

$$\begin{cases} \dfrac{\mathrm{d}x_i}{\mathrm{d}t} = v_i, \quad i = 1, \cdots, N, \\ \dfrac{\mathrm{d}v_i}{\mathrm{d}t} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(||x_i - x_j||)(v_j - v_i), \end{cases}$$

where $x_i \in \mathbb{R}^d$ denote the position of ith particle and $v_i \in \mathbb{R}^d$ indicate velocity, and $||\cdot||$ denotes Euclidean norm. The communication rate ψ quantifies the strength of influence between two particles. For different parameters, conditional flocking and unconditional flocking are achieved in both continuous and discontinuous models, respectively. Very recently, the C-S model has attracted the interest of many scholars in various fields.

It is noted that some studies only focus on maintaining bounded position and consistent velocity, without considering collision-avoiding. However, it is crucial to ensure that there are no collisions between any two agents, which is often referred to as maintaining a safety distance. To the best of our knowledge, the collision-avoiding issue in the C-S model can be studied through two primary methods. The first is to introduce inter-particle bonding forces [6, 8, 9, 11–13]. The second is to apply the singular communication weight [2, 5, 31]. Currently, there is a lack of research on the modeling of collision-avoiding problems with global external perturbations.

In most studies, a control strategy is implemented for each agent. However, when dealing with significantly large group sizes, controlling each individual becomes impractical. To tackle this challenge, the pinning control scheme was proposed and relevant results were obtained [7, 20, 26]. The major advantage of pinning control lies in its capacity to drive the system towards the desired state by only controlling a small proportion of nodes, and it also effectively reduces a lot of unnecessary resource consumption. For instance, it is remarkable that synchronization can be achieved by controlling only a single controller in both continuous and discontinuous systems [29]. In [30], two different pinning strategies were compared: randomly pinning and selective pinning. The findings showed that the pinning strategy based on the highest connection is more efficient than randomly pinning.

In fact, external perturbation is also a non-negligible factor in the model, such as wind interference, electromagnetic interference, etc. These disruptions can potentially undermine flocking behavior partially or completely, affecting its rate of convergence or even rendering it nonexistent. For example, under the random effect of strong winds or water currents, birds or fishes will separate and fail to form a flock or school. Recently, some fruitful results about noise were received [3, 15, 17, 18]. However, some measurable disturbances are also worth studying, such as wind, water flow, and artificial thrust to avoid obstacles. Some researchers have obtained results in asymptotic flocking behavior with deterministic disturbances [21, 34]. Zhao et al. discussed collision avoidance and the effects of different types of disturbance on flocking behavior [34]. In [21], Lian et al. proved that the flocking of the model depends on the perturbed conditions and initial conditions.

For flocking behavior, the convergence time is a very significant index. Previous studies on the flocking of the C-S model mainly focus on asymptotic flocking [1, 4, 23], which means that the flocking can only occur as time approaches infinity. To conquer this drawback, the finite-time control technology was proposed [16].

Different from the asymptotic flocking, the finite-time flocking has an upper bound of the settling time. To date, lots of contributions have been devoted to the finite-time flocking issue of the C-S model [10, 22]. For example, Zhang et al. showed that the finite-time and fixed-time flocking of the C-S model can be reached under pinning control [33]. Furthermore, the finite-time flocking problem for a modified C-S model with unknown Hölder continuous intrinsic dynamics was investigated [24]. By utilizing the energy method, the occurrence of conditional finite-time flocking was demonstrated under specific conditions that depend on the initial data.

However, there are few studies addressing the collision-avoiding finite-time flocking issue of the leader-follower C-S model with pinning control and external perturbation. Inspired by the previously mentioned works, we propose a modified C-S model as follows.

The virtual leader is described by

$$\begin{cases} \dfrac{\mathrm{d}x_0(t)}{\mathrm{d}t} = v_0(t), \\ \dfrac{\mathrm{d}v_0(t)}{\mathrm{d}t} = g(t, v_0), \end{cases} \tag{1.1}$$

which are subject to initial condition $(x_0(0), v_0(0))$.

The ith follower can be described by

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = v_i(t),\ i = 1, \cdots, N, \\ \dfrac{\mathrm{d}v_i(t)}{\mathrm{d}t} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \psi(||x_i - x_j||)\mathrm{sig}(v_j - v_i)^\theta - \beta_i \mathrm{sig}(v_i - v_0)^\theta + g(t, v_i), \end{cases} \tag{1.2}$$

with some given initial conditions

$$(x_i(0), v_i(0)) := (x_{i0}, v_{i0}), \tag{1.3}$$

where $(x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d$, are position and velocity of the ith agent at time t, $0 < \theta < 1$, and $\mathrm{sig}(v_j - v_i)^\theta = (\mathrm{sign}(v_{j1} - v_{i1})\,|v_{j1} - v_{i1}|^\theta, \cdots, \mathrm{sign}(v_{jd} - v_{id})\,|v_{jd} - v_{id}|^\theta)^T$, K is the coupling strength. The connectivity function $\psi_{ij} = \psi(||x_i - x_j||)$ measures the interaction strength between agents depending on the distance between ith and jth agents. β_i is pinning control gain, given by,

$$\beta_i = \begin{cases} 0, & i = 1, 2, \cdots, l, \\ \beta > 0, & i = l+1, l+2, \cdots, N. \end{cases} \tag{1.4}$$

In addition, these functions are further assumed to satisfy the following conditions.

Assumption 1.1. [33] ψ_{ij} *is assumed to be non-increasing, satisfying*

$$\inf_{r \geq 0} \psi(r) \geq \psi^* > 0. \tag{1.5}$$

Assumption 1.2. *The external perturbation satisfies the following condition*

$$||g(t, v_i(t)) - g(t, v_0(t))|| \leq \alpha_i ||v_i - v_0||, \quad i = 1, 2, \cdots, N, \tag{1.6}$$

where α_i is a positive constant.

For Assumption 1.1, the existence of a lower bound means that communication among individuals always exists. For Assumption 1.2, the external force difference is related to its velocity difference.

Definition 1.1. System reaches a finite-time flocking if the solutions $\{x_i, v_i\}(i = 1, \cdots, N)$ satisfy

$$||v_i - v_j|| = 0, \forall t \geq T, \text{ and } \sup_{0 \leq t \leq +\infty} ||x_i - x_j|| < \infty, \tag{1.7}$$

for $x_i(0), v_i(0)$ and $1 \leq i, j \leq N$, where T is called the settling time. Moreover, if the minimum distance between particles meets

$$\inf_{t \geq 0} ||x_i(t) - x_j(t)|| > 0, i \neq j, \tag{1.8}$$

then we say that the system reaches the collision-avoiding finite-time flocking.

The remainder of this work is structured as follows. In Section 2, as proof of our main conclusion, we provide some key lemmas and definitions. In Section 2.1, we prove the finite-time flocking result and establish the upper bound for settling time by imposing appropriate restrictions on the external perturbation, and sufficient conditions are given to ensure that there is no collision during the flocking process. In Section 3, we demonstrate our main results by numerical simulations. Finally, the conclusions are presented in Section 4.

2. Finite-time flocking with collision-avoidance

In this part, we will provide some important definitions and lemma for our subsequent research.

2.1. Lemmas

We give the following important lemmas to better prove the main results in this Section.

Lemma 2.1. [19] *Let $a_1, a_2, \ldots, a_n > 0$, and $0 < p < r$. Then the following norm equivalence property holds*

$$\left(\sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}}, \tag{2.1}$$

and

$$\left(\frac{1}{n} \sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \geq \left(\frac{1}{n} \sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}}. \tag{2.2}$$

Lemma 2.2. [28] *If the graph $\mathcal{G}(A)$ is strongly connected, then the eigenvalue 0 of the graph Laplacian L_A is algebraically simple and all other eigenvalues are with positive real parts. If $\mathcal{G}(A)$ is also undirected then $\omega^T L_A \omega = \frac{1}{2} \sum\limits_{i,j=1}^{N} a_{ij}(\omega_j - \omega_i)^2$, where $\omega = (\omega_1, \ldots, \omega_N)^T \in \mathbb{R}^n$.*

Lemma 2.3. [32] *Let B be a $m\times m(m\geq 2)$ diagonal matrix with diagonal elements $b_i\geq 0$. Set $A=(a_{ij})_{m\times m}(a_{ij}\geq 0)$ to be a weighted adjacency matrix of a connected symmetric graph. Let $L=D-A$ the Laplace matrix of A, where $D=diag(d_1,d_2,\cdots,d_m)$ and $d_i=\sum\limits_{j=1}^{m}a_{ij}$. If there exists some $b_{i_0}>0$, then the minimal eigenvalue of $B+L$ has the following estimate:*

$$\lambda_{\min}(B+L)\geq\min\left\{\frac{\lambda_2}{4m},\frac{b_{i_0}}{2m}\right\}, \tag{2.3}$$

where λ_2 is the second eigenvalue of L, called the Fiedler number of A.

Lemma 2.4. [25] *Suppose there is a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U}\subset\mathcal{R}^n$ of the origin, and*

$$\dot{V}(x)\leq -lV^{\alpha}(x)+kV(x),\forall x\in\mathcal{U}\backslash\{0\}. \tag{2.4}$$

Then, the origin is finite-time stable. The set

$$\Omega=\left\{x|V^{1-\alpha}(x)<\frac{l}{k}\right\}\cap\mathcal{U}, \tag{2.5}$$

is contained in the domain of attraction of the origin. The settling time satisfies $T(x)\leq\frac{\mathrm{In}(1-\frac{k}{l}V^{1-\alpha}(x))}{k(\alpha-1)}$, $x\in\Omega$.

2.2. Finite-time flocking

In this section, by imposing some specific restrictions on the external perturbation, we prove that the system can achieve finite-time flocking.

Theorem 2.1. *For system (1.2), if $\inf\limits_{r\geq 0}\psi(r)\geq\psi^*>0$, $g_i(t)$ satisfies Assumption 1.1, and the initial state satisfies*

$$V^{\frac{1-\theta}{2}}(0)<\frac{1}{2\alpha}\lambda_1^{\frac{\theta+1}{2}}, \tag{2.6}$$

where λ_1 is the smallest eigenvalue of $(2L_A+D_\beta)$, L_A is the Laplacian matrix of $A=(a_{ij})\left(a_{ij}=\left(\frac{K\psi^*}{N}\right)^{\frac{2}{\theta+1}}\right)$, and $D_\beta=diag(\underbrace{0,\cdots,0}_{l},(2\beta)^{\frac{2}{\theta+1}},\cdots,(2\beta)^{\frac{2}{\theta+1}})$. Then the system can achieve finite-time flocking and an upper bound of the settling time is given by

$$T_1=\frac{\ln\left(1-2\alpha\lambda_1{}^{-\frac{\theta+1}{2}}V^{\frac{1-\theta}{2}}(0)\right)}{\alpha(\theta-1)}, \tag{2.7}$$

where $V(0)=\sum\limits_{i=1}^{N}\|\hat{v}_i(0)\|^2$ and $\hat{v}_i(0)=v_i(0)-v_0(0)$ is the initial value of $\hat{v}_i(t)$.

Proof. Let

$$\begin{aligned}\hat{x}_i&=x_i-x_0,\\ \hat{v}_i&=v_i-v_0,\\ \hat{g}_i&=g_i-g_0,\end{aligned} \tag{2.8}$$

the error system equation is written as

$$\begin{cases} \dfrac{\mathrm{d}\hat{x}_i(t)}{\mathrm{d}t} = \hat{v}_i(t),\ i = 1, \cdots, N, \\ \dfrac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t} = \dfrac{K}{N} \displaystyle\sum_{j=1}^{N} \psi_{ij} \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} - \beta_i \mathrm{sig} \hat{v}_i^{\theta} + \hat{g}(t, v_i). \end{cases} \tag{2.9}$$

We define the Lyapunov function

$$V(t) = \sum_{i=1}^{N} \|\hat{v}_i(t)\|^2, \tag{2.10}$$

the derivative of $V(t)$ along the trajectories of (2.9) gives

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N} \left\langle \hat{v}_i(t), \dot{\hat{v}}_i(t) \right\rangle \\ &= 2\sum_{i=1}^{N} \left\langle \hat{v}_i(t), \frac{K}{N} \sum_{j=1}^{N} \psi_{ij} \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} - \beta_i \mathrm{sig} \hat{v}_i^{\theta} + \hat{g}(t, v_i) \right\rangle \\ &= 2\frac{K}{N} \sum_{i=1}^{N} \left\langle \hat{v}_i(t), \sum_{j=1}^{N} \psi_{ij} \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle - 2\sum_{i=1}^{N} \left\langle \hat{v}_i(t), \beta_i \mathrm{sig} \hat{v}_i^{\theta} \right\rangle \\ &\quad + 2\sum_{i=1}^{N} \langle \hat{v}_i(t), \hat{g}(t, v_i) \rangle \\ &\leq 2\frac{K}{N} \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_i(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle - 2\sum_{i=1}^{N} \beta_i ||\hat{v}_i(t)||^{\theta+1} + 2\sum_{i=1}^{N} \alpha_i ||\hat{v}_i(t)||^2, \end{aligned} \tag{2.11}$$

using the symmetry indicates that

$$\begin{aligned} \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_i(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle &= \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_i(t) - \hat{v}_j(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle \\ &\quad + \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_j(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle \\ &= -\sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_j(t) - \hat{v}_i(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle \\ &\quad - \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_j(t), \mathrm{sig}(\hat{v}_i - \hat{v}_j)^{\theta} \right\rangle, \end{aligned} \tag{2.12}$$

which implies that

$$\begin{aligned} \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_i(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle &= -\frac{1}{2} \sum_{i,j=1}^{N} \psi_{ij} \left\langle \hat{v}_j(t) - \hat{v}_i(t), \mathrm{sig}(\hat{v}_j - \hat{v}_i)^{\theta} \right\rangle \\ &= -\frac{1}{2} \sum_{i,j=1}^{N} \psi_{ij} \sum_{k=1}^{d} |\hat{v}_{jk}(t) - \hat{v}_{ik}(t)|^{\theta+1}. \end{aligned} \tag{2.13}$$

By Lemma 2.1, we obtain

$$\left(\sum_{k=1}^{d} |\hat{v}_{jk}(t) - \hat{v}_{ik}(t)|^{\theta+1}\right)^{\frac{1}{\theta+1}} \geq \left(\sum_{k=1}^{d} |\hat{v}_{jk}(t) - \hat{v}_{ik}(t)|^{2}\right)^{\frac{1}{2}} = ||\hat{v}_j - \hat{v}_i||, \tag{2.14}$$

that is

$$\sum_{k=1}^{d} |\hat{v}_{jk}(t) - \hat{v}_{ik}(t)|^{\theta+1} \geq ||\hat{v}_j - \hat{v}_i||^{\theta+1}. \tag{2.15}$$

From (2.13) and (2.15), we can deduce that

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq -\frac{K}{N}\sum_{i,j=1}^{N} \psi_{ij}||\hat{v}_j - \hat{v}_i||^{\theta+1} - 2\sum_{i=1}^{N} \beta_i||\hat{v}_i(t)||^{\theta+1} + 2\sum_{i=1}^{N} \alpha_i||\hat{v}_i(t)||^2 \\
&\leq -\sum_{i=1}^{N}\left(\frac{K}{N}\sum_{j=1}^{N} \psi_{ij}||\hat{v}_j - \hat{v}_i||^{\theta+1} + 2\beta_i||\hat{v}_i(t)||^{\theta+1}\right) + 2\sum_{i=1}^{N} \alpha_i||\hat{v}_i(t)||^2 \\
&\leq -\sum_{i=1}^{N}\left[\sum_{j=1}^{N}\left(\left(\frac{K\psi_{ij}}{N}\right)^{\frac{1}{\theta+1}} ||\hat{v}_j - \hat{v}_i||\right)^{\theta+1} + \left((2\beta_i)^{\frac{1}{\theta+1}}||\hat{v}_i(t)||\right)^{\theta+1}\right] \\
&\quad + 2\sum_{i=1}^{N} \alpha_i||\hat{v}_i(t)||^2 \\
&\leq -\sum_{i=1}^{N}\left[\sum_{j=1}^{N}\left(\frac{K\psi_{ij}}{N}\right)^{\frac{2}{\theta+1}} ||\hat{v}_j - \hat{v}_i||^2 + (2\beta_i)^{\frac{2}{\theta+1}}||\hat{v}_i(t)||^2\right]^{\frac{\theta+1}{2}} \\
&\quad + 2\sum_{i=1}^{N} \alpha_i||\hat{v}_i(t)||^2.
\end{aligned} \tag{2.16}$$

Consequently, from $\inf_{r\geq 0} \psi_{ij}(r) \geq \psi^*$ it follows that

$$\begin{aligned}
\frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq -\sum_{i=1}^{N}\left[\sum_{j=1}^{N}\left(\frac{K\psi^*}{N}\right)^{\frac{2}{\theta+1}} ||\hat{v}_j - \hat{v}_i||^2 + (2\beta_i)^{\frac{2}{\theta+1}}||\hat{v}_i(t)||^2\right]^{\frac{\theta+1}{2}} \\
&\quad + 2\sum_{i=1}^{N} \alpha_i||\hat{v}_i(t)||^2.
\end{aligned} \tag{2.17}$$

Define a new matrix $A = (a_{ij})$ whose elements $a_{ij} = \left(\frac{K\psi^*}{N}\right)^{\frac{2}{\theta+1}}$, and the matrix A is the adjacency matrix of the graph $\mathcal{G}(A)$. Then L_A is regarded as the Laplacian matrix of $\mathcal{G}(A)$. Applying Lemma 2.2, we have

$$\sum_{i,j=1}^{N}\left(\frac{K\psi^*}{N}\right)^{\frac{2}{\theta+1}} ||\hat{v}_j - \hat{v}_i||^2 = 2\hat{v}^T L_A \hat{v}. \tag{2.18}$$

Let $D_\beta = diag(\underbrace{0, \cdots, 0}_{l}, (2\beta)^{\frac{2}{\theta+1}}, \cdots, (2\beta)^{\frac{2}{\theta+1}})$, we get

$$\sum_{i=1}^{N} (2\beta_i)^{\frac{2}{\theta+1}} ||\hat{v}_i(t)||^2 = \hat{v}^T D_\beta \hat{v}. \tag{2.19}$$

Then, the inequality (2.17) now reads,

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq -(2\hat{v}^T L_A \hat{v} + \hat{v}^T D_\beta \hat{v})^{\frac{\theta+1}{2}} + 2\sum_{i=1}^{N} \alpha_i ||\hat{v}_i(t)||^2 \\ &\leq -(\hat{v}^T (2L_A + D_\beta)\hat{v})^{\frac{\theta+1}{2}} + 2\sum_{i=1}^{N} \alpha_i ||\hat{v}_i(t)||^2 \\ &\leq -(\lambda_1 \hat{v}^T \hat{v})^{\frac{\theta+1}{2}} + 2\sum_{i=1}^{N} \alpha_i ||\hat{v}_i(t)||^2 \\ &\leq -\lambda_1^{\frac{\theta+1}{2}} V^{\frac{\theta+1}{2}}(t) + 2\alpha V(t), \end{aligned} \tag{2.20}$$

where $\alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ and λ_1 is the smallest eigenvalue of $(2L_A + D_\beta)$. According to Lemma 2.3, we know that $\lambda_1 > 0$ when $0 \leq l < N$. From Lemma 2.4 and (2.6), this implies that $V(t) < V(0)$. It immediately follows that

$$V(t) \equiv 0, \quad t \geq T_1, \tag{2.21}$$

where

$$T_1 = \frac{\ln\left(1 - 2\alpha\lambda_1^{-\frac{\theta+1}{2}} V^{\frac{1-\theta}{2}}(0)\right)}{\alpha(\theta - 1)}, \tag{2.22}$$

thus

$$v_i \equiv v_0, \quad \forall t \geq T_1. \tag{2.23}$$

We know that in finite time the speed of the follower can converge to the speed of the leader.

Then, we will prove that the distance is bounded. Let $X(t) = \sum_{i=1}^{N} \|\hat{x}_i(t)\|^2$,

$$\begin{aligned} \frac{\mathrm{d}X(t)}{\mathrm{d}t} &= 2\sum_{i=1}^{N} \langle \hat{x}_i, \hat{v}_i \rangle \\ &\leq 2X^{\frac{1}{2}}(t) V^{\frac{1}{2}}(t). \end{aligned} \tag{2.24}$$

It leads to

$$\left| \frac{\mathrm{d}X^{\frac{1}{2}}(t)}{\mathrm{d}t} \right| \leq V^{\frac{1}{2}}(t), \tag{2.25}$$

then

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + \int_0^t V^{\frac{1}{2}}(\tau) \mathrm{d}\tau. \tag{2.26}$$

From (2.23), we know that the velocity of all agents remains consistent in finite time and T_1 is the settling time. By Theorem 2.1, we obtain

$$\begin{aligned} X^{\frac{1}{2}}(t) &\leq X^{\frac{1}{2}}(0) + \int_0^{T_1} V^{\frac{1}{2}}(\tau)\mathrm{d}\tau \\ &\leq X^{\frac{1}{2}}(0) + T_1 V^{\frac{1}{2}}(0), \end{aligned} \tag{2.27}$$

that is

$$X^{\frac{1}{2}}(t) \leq X^{\frac{1}{2}}(0) + T_1 V^{\frac{1}{2}}(0), \tag{2.28}$$

i.e.,

$$\sup_{0\leq t\leq+\infty} \|x_i - x_j\| \leq \sup_{0\leq t\leq+\infty} \|x_i - x_0\| + \sup_{0\leq t\leq+\infty} \|x_j - x_0\| \leq 2X^{\frac{1}{2}}(t) < +\infty. \tag{2.29}$$

There is an upper bound on the distance between any two agents. Combining (2.23) and (2.29), the finite-time flocking behavior can be achieved. This concludes the proof of Theorem 2.1. □

Remark 2.1. Obviously, the eigenvalue λ_1 depends on both β and l. Additionally, from (2.20) and Assumption 1.1 it follows that the convergence time is related to λ_1, as will be shown in Section 3 .

Remark 2.2. Note that the external perturbations are introduced in the model, and in the absence of external perturbation, our conclusions degenerate to the results in [33].

Remark 2.3. We note that when $\psi^* = 0$, the matrix A becomes a zero matrix. In this case, when $l = 0$, $\lambda_1 = (2\beta)^{\frac{2}{\theta+1}}$, and if the system satisfies condition (2.6), equations (2.23) and (2.29) hold, leading to the system achieving flocking. However, this approach requires controlling all nodes, which is impractical and leads to unnecessary resource consumption when there are many agents in the group. When $0 < l < N$, $\lambda_1 = 0$, and conditions (2.6) and (2.23) do not hold, meaning that the followers in the system do not converge to the leader's velocity, and flocking cannot be achieved. Therefore, based on the above discussion and Theorem 2.1, the condition $\inf_{r\geq 0} \psi(r) \geq \psi^* > 0$ in Assumption 1.1 is more suitable for the model proposed in this paper.

2.3. Collision-avoidance

Aside from bounded position and consistent velocity, collision avoidance is an essential component of safe and effective operations in real-world applications like UAV cooperative operations and formation flying. We will provide sufficient conditions for avoiding collisions during flocking in this section.

Theorem 2.2. *Assuming the initial state of system* (1.2) *is non-collisional, and the initial state satisfies*

$$\min_{i\neq j} ||x_i(0) - x_j(0)|| > \sqrt{2}V^{\frac{1}{2}}(0)T_1, \tag{2.30}$$

then, the solution of system (1.2) *meets*

$$\inf_{t\geq 0} ||x_i(t) - x_j(t)|| > 0, i \neq j, \tag{2.31}$$

system (1.2) *can achieve a finite-time flocking with non-collision, where* $V^{\frac{1}{2}}(0) = \sqrt{\sum\limits_{i=1}^{N} \|\hat{v}_i(0)\|^2}$ *and* T_1 *is defined in Theorem 2.1.*

Proof. Let

$$X_{ij}(t) = ||x_i(t) - x_j(t)||, \tag{2.32}$$

and

$$V_{ij}(t) = ||v_i(t) - v_j(t)|| = ||\hat{v}_i(t) - \hat{v}_j(t)||, \tag{2.33}$$

where $i \neq j$ and $i, j \in 0, 1, \cdots, N$. By using Cauchy Schwarz's inequality, we have

$$\begin{aligned} V_{ij} &\leq ||\hat{v}_i(t)|| + ||\hat{v}_j(t)|| \\ &\leq \sqrt{2}\left(||\hat{v}_i(t)||^2 + ||\hat{v}_j(t)||^2\right)^{\frac{1}{2}} \\ &\leq \sqrt{2}\left(\sum_{i=1}^{N} ||\hat{v}_i(t)||^2\right)^{\frac{1}{2}} \\ &= \sqrt{2}V^{\frac{1}{2}}(t). \end{aligned} \tag{2.34}$$

For $X_{ij}(t)$, it is straightforward to get

$$\frac{\mathrm{d}}{\mathrm{d}t}X_{ij}^2(t) \leq 2||x_i(t) - x_j(t)||||v_i(t) - v_j(t)|| = 2X_{ij}(t)V_{ij}(t), \tag{2.35}$$

that is,

$$\left|\frac{\mathrm{d}X_{ij}(t)}{\mathrm{d}t}\right| \leq V_{ij}(t). \tag{2.36}$$

Integrating both sides of the inequality above leads to

$$\begin{aligned} |X_{ij}(t) - X_{ij}(0)| &= \left|\int_0^t \frac{\mathrm{d}X_{ij}(s)}{\mathrm{d}s}\mathrm{d}s\right| \\ &\leq \int_0^t \left|\frac{\mathrm{d}X_{ij}(s)}{\mathrm{d}s}\right| \mathrm{d}s \\ &\leq \int_0^t V_{ij}(s)\mathrm{d}s \\ &\leq \sqrt{2}\int_0^t V^{\frac{1}{2}}(s)\mathrm{d}s. \end{aligned} \tag{2.37}$$

Note that $V(t) = 0 \Rightarrow V_{ij} = 0$, for $t \geq T_1$. Since $V(t) < V(0)$ we have, for $t \geq 0$

$$\begin{aligned} |X_{ij}(t) - X_{ij}(0)| &\leq \sqrt{2}\int_0^{T_1} V^{\frac{1}{2}}(s)\mathrm{d}s \\ &\leq \sqrt{2}\int_0^{T_1} V^{\frac{1}{2}}(0)\mathrm{d}s \\ &\leq \sqrt{2}V^{\frac{1}{2}}(0)T_1, \end{aligned} \tag{2.38}$$

by triangle inequality, it immediately follows that

$$\begin{aligned} |X_{ij}(t)| &\geq |X_{ij}(0)| - |X_{ij}(t) - X_{ij}(0)| \\ &\geq \min_{i\neq j}|X_{ij}(0)| - \sqrt{2}V^{\frac{1}{2}}(0)T_1 \\ &> 0. \end{aligned} \tag{2.39}$$

Similarly, it is easy to know that

$$\|\hat{x}_i(t) - \hat{x}_i(0)\| \leq \sqrt{2}V^{\frac{1}{2}}(0)T_1.$$

Thus, we have

$$\|\hat{x}_i(t)\| \geq \min_{i\neq j} \|\hat{x}_i(0)\| - \sqrt{2}V^{\frac{1}{2}}(0)T_1 > 0,$$

i.e.,

$$\|x_i(t) - x_i(0)\| \geq \min_{i\neq j} \|x_i(0) - x_0(0)\| - \sqrt{2}V^{\frac{1}{2}}(0)T_1 > 0.$$

This completes the proof of Theorem 2.2. There is a lower bound for the distance between each agent, also known as the minimum safe distance in practical application, so that collision avoidance can be achieved. By maintaining a minimum safe distance, the follower can progressively adjust its speed to match that of the leader while avoiding any potential collisions. This further enhances the feasibility and practicality of achieving flocking in a controlled and safe manner. $\square$

3. Numerical simulations

The numerical simulations presented in this section lead to the verification of the theoretical results. From a graph theory perspective, it is worth noting that the graph topology used in all simulations is fully connected, which is different from the Vicsek model [27]. The initial positions are generated by random real numbers between 0 and 100, and the initial velocities are generated by random real numbers between 0 and 1, as shown in Table 1. By calculation, the following examples all satisfy the collision-avoiding condition (2.30). To more accurately describe the flocking behavior, we define the following two indicators: $\delta_v(t) = (1/N)\sum_{i=1}^{N}[v_i(t) - v_0(t)]^2$ and $\delta_x(t) = (1/N)\sum_{i=1}^{N}[x_i(t) - x_0(t)]^2$.

First, we set the simulation parameters as $N = 9, K = 0.5, \psi_{ij} = 1/(1 + ||x_i - x_j||^2)^\eta + \varepsilon, \eta = 0.3, \varepsilon = 0.5, \alpha = 0.2 \times 10^{-3}, \theta = 0.1, \beta = 0.02$ and l_0 is the pinning node number. In order to more accurately compare and illustrate the impact of external perturbation, we simulate the system (1.2) in the absence of perturbations. The position and velocity trajectories of the leader and follower are displayed in Figure 1, and it is evident that the system is capable of achieving flocking without collision, which is consistent with findings reported in [33]. Now, we give two types of external perturbation functions.

Example 3.1. We give the first type of perturbation functions

$$\begin{aligned} g(t, v_i) =&(g^1(t, v_i), g^2(t, v_i), g^3(t, v_i)) \\ =&(0.01v_i(t), 0.01v_i(t), 0.01v_i(t)),\ k = 1, 2, \cdots, N, \\ g(t, v_0) =&0. \end{aligned}$$

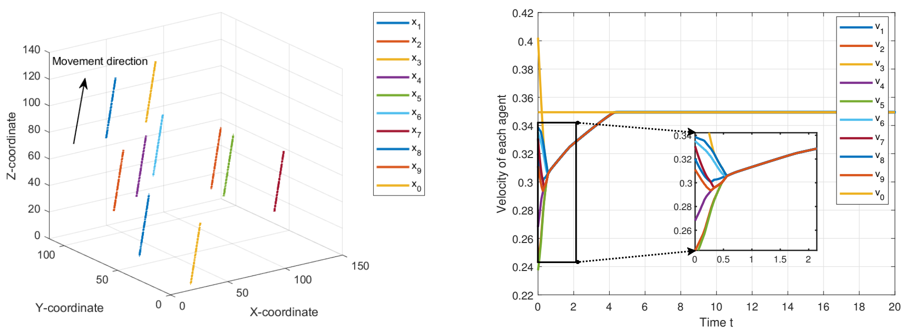
Figure 2 shows the position and velocity trajectory of agents with the first type of perturbation.

Example 3.2. We give the second type of perturbation functions

$$\begin{aligned} g(t, v_i) &= (g^1(t, v_i), g^2(t, v_i), g^3(t, v_i)) \\ &= (0.01e^{-t}v_i(t), 0.01e^{-t}v_i(t), 0.01e^{-t}v_i(t)), \quad i = 0, 1, \cdots, N. \end{aligned}$$

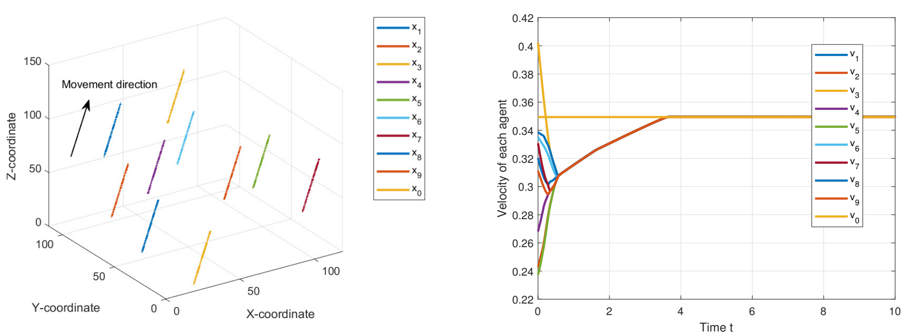
Table 1. The initial data.

Agents	Initial position	Initial velocity
0	(23.35098822, 6.35806232, 1.35717253)	(0.17892249, 0.12294130, 0.27377104)
1	(2.83716271, 63.82368204, 93.57764104)	(0.21779096, 0.00195207, 0.23472881)
2	(51.17941080, 17.05384731, 63.46187937)	(0.12425189, 0.05693580, 0.20066483)
3	(69.20627928, 97.70330225, 79.68958925)	(0.28116679, 0.08984982, 0.27332209)
4	(41.33477110, 76.65635198, 37.02955415)	(0.11090864, 0.15978806, 0.18450114)
5	(96.30840608, 53.83110701, 35.27054549)	(0.12827346, 0.03878263, 0.19590807)
6	(76.05092786, 98.22827001, 38.13956569)	(0.23094290, 0.15725701, 0.18537001)
7	(97.88899083, 8.12068720, 40.04691407)	(0.09561159, 0.17805203, 0.26184265)
8	(17.48891001, 48.29038502, 8.05104128)	(0.27520961, 0.08823129, 0.17659469)
9	(2.62736353, 56.22173639, 41.83644922)	(0.12418765, 0.17052620, 0.22872895)



(a) (b)

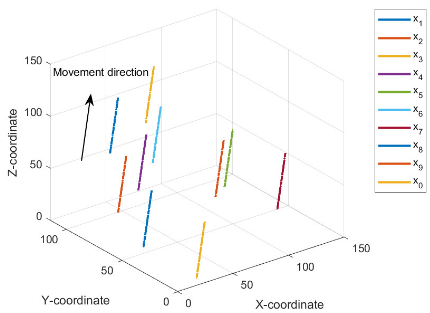
Figure 1. The flocking without external perturbation.



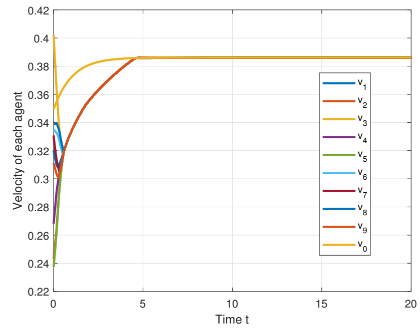
(a) (b)

Figure 2. The flocking with the first type of external perturbation.

Figure 3 shows the position and velocity trajectory of agents with the second type of perturbation.



(a)



(b)

Figure 3. The flocking with the second type of external perturbation.

From the two examples above, we can obtain that the two forms of external perturbations have distinct impacts on the position and velocity, but the velocity of different agents eventually tends to converge, and the system achieves the collision avoiding flocking.

Next, the influence of parameter β and l_0 on the eigenvalue $\lambda_1 = \lambda_{\min}(2L_A + D_\beta)$ is analyzed. Based on the definition of L_A, the connectivity function is $\psi(r) = 1/(1+r^2)^\eta + \varepsilon \geq 1/(1+d_{\max}^2)^\eta + \varepsilon = \psi^*$; here, $d_{\max}$ denotes the maximum distance between any two followers. Through simple calculations, we find that $d_{\max} = ||x_1 - x_7|| = 122.4876$ when $\beta = 0.02$. Similar to the conclusion of Lemma 2.3, Figure 4 shows that λ_1 is not only positive but also minimum when only one node is controlled. The result indicates that λ_1 is an increasing function of l_0 for the same control gain. Similarly, for a given pinning node number, a larger β yields a greater eigenvalue λ_1.

Then, in order to verify the influence of parameter β and l_0 on convergence speed, we conduct a simulation of the system (1.2) with the first type of perturbation for different values of β and l_0. For convenience, we present the computational results only for $\beta = 0.02$, but the following condition holds for other cases as well. Through a straightforward calculation, we deduce that $V^{\frac{1-\theta}{2}}(0) = 0.3860$ and $\frac{1}{2\alpha}\lambda_1^{\frac{\theta+1}{2}} = 29.6542$ satisfy (2.6). From Figure 5, we observe that approximately after $t \approx 4.3s$, the velocities of all the followers converge to that of the leader, and the settling time $T_1 \approx 72.7937s$ is calculated by equation (2.7). This implies that the system can achieve flocking without collisions in a finite time, and this result aligns with the assertion presented in Theorem 2.1. Figure 5 suggests that the larger the parameter β, the faster the flocking occurs. From Figure 6, it is evident that when the number of pinning nodes increases, the flocking converges more quickly. Consequently, it would be advised to increase β and l_0 if collision-avoiding flocking in real applications needs to be achieved in a shorter time. Combining Figure 4, Figure 5, and Figure 6, it is shown that the convergence speed of flocking increases as the eigenvalue λ_1 increases, which explains Remark 2.1.

Finally, we use the same parameters, and choose appropriate initial data and $N = 10, 30, 50$, respectively in Figure 7. We get the convergence time becomes shorter as N increases. This aligns with the findings presented in [24].

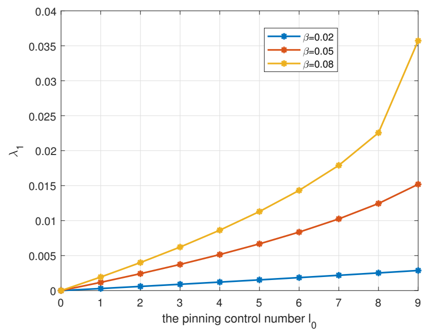
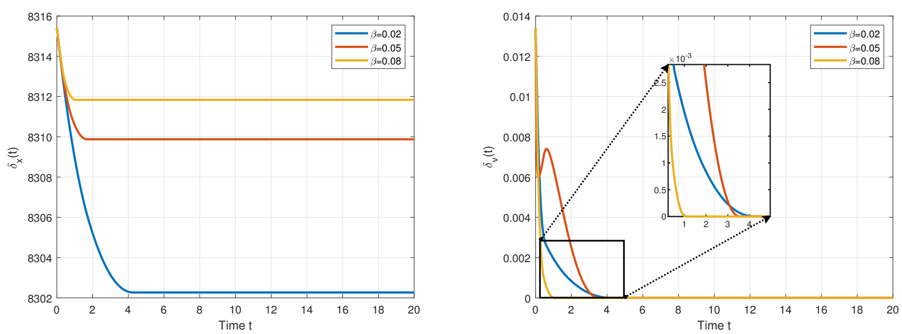
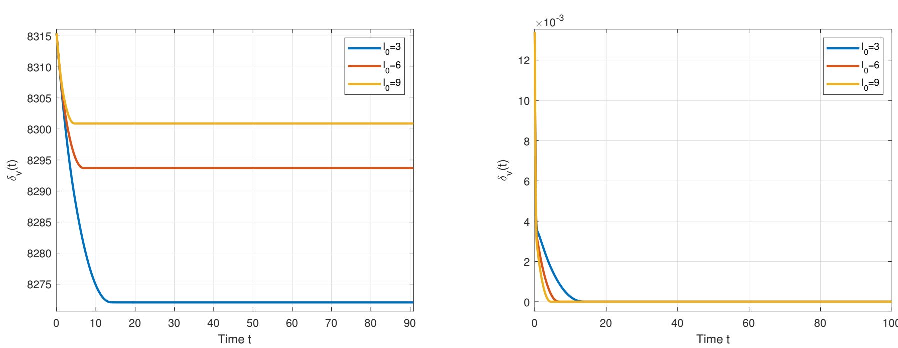


Figure 4. The eigenvalue λ_1 with different l_0 and β.



(a) (b)

Figure 5. The flocking for different control gain β.

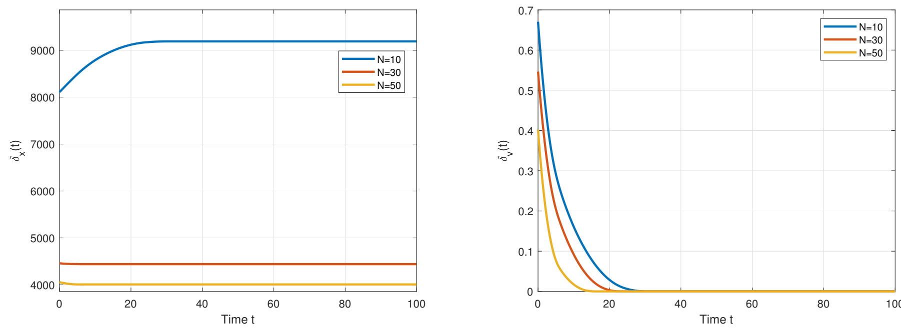


(a) (b)

Figure 6. The flocking with different l_0.

4. Conclusion

The collision-avoiding finite-time flocking of a leader-follower C-S model with pinning control and external perturbation is investigated in this paper. When the external perturbation function satisfies the specific conditions, the system can suc-



(a) (b)

Figure 7. The flocking with different N.

cessfully achieve a finite-time flocking; simultaneously, upper bounds on the relative position and settling time are derived by means of the appropriate Lyapunov function. Then, a sufficient condition is provided by applying inequality techniques to ensure that there are no collisions during the flocking process. To further validate the derived results, some numerical simulations are conducted. These simulations not only confirm the correctness of the theoretical results but also enable us to analyze the influence of parameters such as β and l_0, as well as different external perturbations, on the speed of convergence. When subjected to smaller external perturbations, we also observe that individuals quickly adjust their states to form flocks.

In the analysis of flocking with collision avoidance, it is necessary for the system to achieve finite-time flocking that $g(t)$ must satisfy Assumption 1.1; by introducing a pinning control strategy, a small percentage of nodes can be controlled to guide the system towards the required target state, resulting in reduced energy consumption and improved efficiency. If the model needs to quickly form a collision-avoiding flocking, such as in UAV formation flight or target tracking scenarios, the relevant parameters and perturbations can be adjusted. These findings are supported by Remark 2.1 and numerical simulations. This highlights the practical engineering significance of the main results presented in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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