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DISCRETE MULTI-GALERKIN METHODS FOR DERIVATIVE-DEPENDENT HAMMERSTEIN TYPE WEAKLY SINGULAR NONLINEAR FREDHOLM INTEGRAL EQUATIONS WITH GREEN'S KERNEL

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Abstract In this article, we study discrete multi-Galerkin and iterated discrete multi-Galerkin methods for solving the derivative-dependent nonlinear Hammerstein type Fredholm integral equations, where the nonlinear function within the integration is dependent on the derivative, and the kernel function is of Green's type. We achieve the error bounds by substituting all integrals in the multi-Galerkin method with numerical quadrature and obtain the superconvergence results for derivative-dependent Fredholm-Hammerstein integral equations using piecewise polynomials as basis functions. By applying the numerical quadrature rule, we prove that the iterated discrete multi-Galerkin method provides superior convergence rates over the discrete multi-Galerkin method with $\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)})$, where h represents the norm of the partitions. Numerical results are presented to validate the theoretical findings, with figures illustrating a comparison of the error analysis between the proposed methods and those discussed in [22].

Keywords Hammerstein type integral equation, piecewise polynomials, two-point boundary value problem, discrete multi-Galerkin methods, Green's kernel, derivative-dependent, superconvergence results.

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1. Introduction

Nonlinear BVPs arising in ordinary differential equations are prevalent across various fields such as mathematical models [1], diffusion problems [5], chemical reactions, Stellar Structure [11], engineering [13], nuclear physics [15], heat conduction [23], physiology [24], etc.

This paper deals with the following two-point boundary value problems (BVPs):

$$(z'(t))' = \psi(t, z(t), z'(t)), \tag{1.1}$$

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with the boundary conditions

$$z(0) = \delta_1, \quad \mu_1 z(1) + \lambda_1 z'(1) = \xi_1. \tag{1.2}$$

The reformulated integral equation is expressed as follows:

$$z(t) = \delta_1 + \frac{(\xi_1 - \delta_1\lambda_1)t}{\mu_1 + \lambda_1} + \int_0^1 \mathcal{K}(t,s)\psi(s, z(s), z'(s))ds, \ \forall\ t \in [0,1], \tag{1.3}$$

with kernel

$$\mathcal{K}(t,s) = \begin{cases} t(1 - \dfrac{\mu_1 s}{\mu_1 + \lambda_1}),\ 0 \le t \le s, \\ s(1 - \dfrac{\mu_1 t}{\mu_1 + \lambda_1}),\ s \le t \le 1, \end{cases}$$

where $\delta_1, \mu_1 > 0$, λ_1 and ξ_1 are constants.

In most cases, BVPs are often difficult to solve analytically. Therefore, we need to use some numerical approximate methods for their solution. The numerical methods to solve the BVPs, such as the decomposition, Adomian decomposition, and modified decomposition methods are well documented in the literature (see [2, 9, 17, 19, 39]), in which the researchers considered the BVP (1.1) with nonlinear function ψ independent of derivatives. Although these numerical methods provide several advantages, they require substantial computational effort mainly due to the calculation of indeterminate coefficients in more complex transcendental or nonlinear algebraic equations, which adds to the overall computational work (see [15, 24, 25]). Moreover, in some cases, the undetermined coefficients may not have a unique solution. This is a significant drawback of using these methods for addressing nonlinear BVPs. Hence instead of directly solving BVPs, it is feasible to solve an equivalent integral equation that leads to a derivative-dependent nonlinear Hammerstein type Fredholm integral equation:

$$z(t) = g(t) + \int_0^1 \mathcal{K}(t,s)\psi(s, z(s), z'(s))\, ds, \tag{1.4}$$

where the function $g(.)$, Green's kernel $\mathcal{K}(.,.)$ and $\psi(., z, z')$ are known, while $z(\cdot)$ is the unknown function to be determined in the Banach space $\mathbb{X}$.

In general, integral equation (1.4) represents the regenerate form of nonlinear BVPs. Several authors have discussed Hammerstein type second kind Fredholm integral equations, where the nonlinear function ψ is independent of derivative (see [20, 26, 27, 29, 32]). Atkinson et al. [6–8] explored the spectral and iterated spectral methods for solving Fredholm-Hammerstein integral equations with certain classes of kernels. Recently researchers studied the Volterra-Fredholm and systems of integral partial differential equations using hybrid functions [37, 38]. They also investigated the solutions of various types of partial integro-differential equations [33–36]. Chakraborty et al. [10] introduced the spectral methods for solving Volterra-Hammerstein integral equations of the second kind. In [31], Nigam et al. developed discrete spectral methods for Fredholm-Hammerstein integral equations. Meanwhile, Allouch et al. [4] proposed spectral methods to address the derivative-dependent nonlinear Hammerstein type integral equations with Green's kernels.

In [28], Mandal et al. studied the Galerkin method and its iterated version to solve derivative-dependent Fredholm-Hammerstein integral equations, utilizing piecewise polynomials as basis functions, and they achieved convergence results in infinity norm. However, the authors excluded considerations of errors arising from the inner product and integral operators. To address these types of errors, Kant et al. [22] presented the discrete Galerkin method and its iterated version for solving derivative-dependent Hammerstein type Fredholm integral equation with Green's kernel and achieved better convergence. Hence the primary objective of choosing these methods in this article is to obtain superconvergence results, including better convergence rates and improved accuracy by accounting for errors arising from integrals and inner products. These improvements are made in comparison to the discrete Galerkin and iterated discrete Galerkin methods presented in [22] and studied with collocation and multi-collocation methods [30].

The main motivation of this article is to achieve superconvergence results by addressing errors arising from integrals and inner products in derivative-dependent nonlinear Hammerstein type integral equations. Recently in [21], the authors obtained superconvergence results for derivative-dependent Fredholm-Hammerstein integral equations of the second kind using modified Galerkin and iterated modified Galerkin methods, which demonstrated enhanced performance compared to the Galerkin and iterated Galerkin methods. Motivated by this, our aim is to achieve a higher rate of convergence with reduced computational complexity for the same, thereby improving upon the discrete Galerkin and iterated discrete Galerkin methods discussed in [22].

In this article, we study the discrete multi-Galerkin method and its iterated version using piecewise polynomials to solve derivative-dependent nonlinear Hammerstein type Fredholm integral equations with Green's kernel defined by (1.4). We achieved order of convergence $\mathbb{O}(h^{\min(d+1,\ m+m_1, m+m_2)})$ and $\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)})$, respectively, where h represents the norm of the partitions and d is the degree of precision of the quadrature rule. Further, $m = \min\{r+1, k_1\}$, $m_1 = \min\{r+1, k_1, \mu+2\}$, $m_2 = \min\{r+1, k_1-1, \mu+1\}$, where r represents the degree of the piecewise polynomials, k_1 is the smoothness of the solution, and $k_1 \geq \mu \geq -1$. Hence, it shows that the iterated discrete multi-Galerkin method exhibits superior convergence rates compared to the discrete multi-Galerkin method.

This paper is structured in the following manner: In Sec 2, we develop the mathematical formulation of the discrete multi-Galerkin method and its iterated version for solving derivative-dependent nonlinear Hammerstein type Fredholm integral equations defined by (1.4). In Sec 3, we analyze the superconvergence results obtained by discrete multi-Galerkin and iterated discrete multi-Galerkin methods. In Sec 4, we provide numerical examples to confirm the theoretical findings. For the rest of the paper, we will use C as a generic constant.

2. Discrete Multi-Galerkin methods: Derivative-dependent nonlinear Hammerstein type Fredholm integral equations

Consider the derivative-dependent nonlinear Hammerstein type Fredholm integral equations

$$z(t) = g(t) + \int_0^1 \mathcal{K}(t,s)\psi(s, z(s), z'(s))\, ds, \tag{2.1}$$

where $g(.)$, $\mathcal{K}(.,.)$ and $\psi(.,z,z')$ are known sufficiently smooth functions, whereas $z(\cdot)$ is the unknown function to be found in the Banach space $\mathbb{X} = \mathbb{L}^\infty[0,1]$.

Take a certain $t \in [0,1]$, consider

$$\mathcal{K}_{1t}(s) = \mathcal{K}_t(s), \quad \forall\, t \in [0,s], \tag{2.2}$$
$$\mathcal{K}_{2t}(s) = \mathcal{K}_t(s), \quad \forall\, t \in [s,1]. \tag{2.3}$$

For each $t \in [0,1]$, we assume $\mathcal{K}_{1t} \in \mathbb{C}^{k_1}[0,t]$, $\mathcal{K}_{2t} \in \mathbb{C}^{k_1}[t,1]$ and $\mathcal{K}_t(s) = \mathcal{K}(t,s) \in \mathbb{C}^{k_1}(0,t) \cap \mathbb{C}^{k_1}(t,1) \cap \mathbb{C}^{\mu}(0,1)$, where $k_1 \geq 1$ and $k_1 \geq \mu \geq -1$. In general, consider $g \in \mathbb{C}^{k_1}[0,1]$. From Atkinson et al. ([7], Theorem 4.1 and Corollary 4.2), implies that $z \in \mathbb{C}^{k_1}([0,1])$. The Green's kernel, in the context of two-point BVP of (1.1) defined by

$$\mathcal{K}(t,s) = \begin{cases} t(1 - \dfrac{\mu_1 s}{\mu_1 + \lambda_1}),\ 0 \leq t \leq s, \\ s(1 - \dfrac{\mu_1 t}{\mu_1 + \lambda_1}),\ s \leq t \leq 1, \end{cases}$$

with $\mu = 0$ and $k_1 \geq 1$.

Denote

$$\|z\|_{k_1,\infty} = \max\{\|z^{(m)}\|_\infty : 0 \leq m \leq k_1\},$$

where $z^{(m)}$ represents the m^{th} - order derivative of z.

In this paper, we assume the following conditions for $g(t)$, $\mathcal{K}(t,s)$, and $\psi(s,z(s),z'(s))$:

(i) $g \in \mathbb{C}^{k_1}[0,1]$.

(ii) $\mathcal{K}(t,s) \in \mathbb{C}^{k_1}([0,1]\times[0,1])$.

(iii) $A_1 = \sup\limits_{t,s\in[0,1]} |\mathcal{K}(t,s)| < \infty$, $A_2 = \sup\limits_{t,s\in[0,1]} |l(t,s)| < \infty$.

(iv) The nonlinear function $\psi(s,z,z')$ is Lipschitz continuous in z and z', i.e., for each $z_1, z_2, z_1', z_2' \in \mathbb{X}$, $\exists$ a constant $C_1 > 0$ such that

$$|\psi(s,z_1,z_1') - \psi(s,z_2,z_2')| \leq C_1\{|z_1(s) - z_2(s)| + |z_1'(s) - z_2'(s)|\}, \forall\, s \in [0,1].$$

(v) The partial derivatives $\psi^{(0,1,0)}(s,z,z') \in \mathbb{C}([0,1]\times\mathbb{X}\times\mathbb{X})$, $\psi^{(0,0,1)}(s,z,z') \in \mathbb{C}([0,1]\times\mathbb{X}\times\mathbb{X})$ of ψ exists and Lipschitz continuous w.r.t the second and third variables, i.e., for each $z_1, z_2, z_1', z_2' \in \mathbb{X}$, $\exists$ constants $C_2, C_3 > 0$ such that

$$|\psi^{(0,1,0)}(s,z_1,z_1') - \psi^{(0,1,0)}(s,z_2,z_2')| \leq C_2\{|z_1(s) - z_2(s)| + |z_1'(s) - z_2'(s)|\},$$
$$|\psi^{(0,0,1)}(s,z_1,z_1') - \psi^{(0,0,1)}(s,z_2,z_2')| \leq C_3\{|z_1(s) - z_2(s)| + |z_1'(s) - z_2'(s)|\},$$

where $\forall\, s \in [0,1]$.

(vi) Let $A = A_1 + A_2$ and C_1 hold the properties such that $AC_1 < 1$.

For each $z \in \mathbb{X}$ and $t \in [0,1]$, let

$$\mathbb{K}z(t) = \int_0^1 \mathcal{K}(t,s) z(s)\, ds,$$

and

$$\mathbb{L}z(t) = \frac{d}{dt}(\mathbb{K}z)(t) = \int_0^1 l(t,s) z(s)\, ds,$$

where $l_t(s) = l(t,s) = \frac{\partial}{\partial t}\mathcal{K}(t,s)$ satisfies $l(t,s) \in \mathbb{C}^{k_1-1}(0,t) \cap \mathbb{C}^{k_1-1}(t,1) \cap \mathbb{C}^{\mu-1}$ $(0,1)$.

We consider

$$\|\mathbb{K}\|_\infty \leq A_1 \text{ and } \|\mathbb{L}\|_\infty \leq A_2, \tag{2.4}$$

where $\mathbb{K}$ and $\mathbb{L}$ are compact operators.

To approximate Eq. (2.1), we use the approximation approach proposed by Kumar and Sloan [27]. To begin, we let

$$\eta(s) = \psi(s, z(s), z'(s)),\ s \in [0,1]. \tag{2.5}$$

Let if $\psi(.,.,.) \in \mathcal{C}^{k_1}([0,1] \times [0,1] \times [0,1])$, then applying the product rule of differentiation implies that $\eta \in \mathcal{C}^{k_1}[0,1]$.

Then the Eq. (2.1) can be expressed in the following as

$$z(t) = g(t) + \int_0^1 \mathcal{K}(t,s)\eta(s)\, ds, \quad t \in [0,1]. \tag{2.6}$$

Consider the operator $\mathbb{K} : \mathbb{X} \to \mathbb{X}$ defined by

$$\mathbb{K}\eta(s) = \int_0^1 \mathcal{K}(t,s)\eta(s) ds. \tag{2.7}$$

The Eq. (2.6) is reduced in operator form as

$$z = g + \mathbb{K}\eta. \tag{2.8}$$

Further, introduce the nonlinear operator $\Omega : \mathbb{X} \to \mathbb{X}$ defined by

$$\Omega(z)(s) = \psi(s, z(s), z'(s)). \tag{2.9}$$

Hence the Eq. (2.5) becomes

$$\eta = \Omega(g + \mathbb{K}\eta). \tag{2.10}$$

Now assuming that $\mathbb{T}(z) = \Omega(g + \mathbb{K}z),\ z \in \mathbb{X}$, from Eq. (2.10), we have

$$\eta = \mathbb{T}\eta. \tag{2.11}$$

Using the assumption, we have $AC_1 < 1$ such that the operator $\mathbb{T}$ is a contraction mapping on the Banach space $\mathbb{X}$, then using the principle of the Banach contraction theorem, it proves that the Eq. (2.11) has an isolated solution η_0 in $\mathbb{X}$.

Let $\mathbb{X}_h$ be the approximation subspaces defined by

$$\mathbb{X}_h = \mathbb{P}_{m,\triangle} = \{z : z|_{[t_{j-1},\ t_j]} \in \mathbb{P}_m,\ j = 1, 2, ..., n\}, \tag{2.12}$$

where the space $\mathbb{P}_m$ includes all polynomials of degree not exceed m $(m \geq 1)$. Consider the partition of $[0,1]$ as $\Pi_n : 0 = t_0 < t_1 < \cdots < t_n = 1$, and $h = \max h_j$, represents the norm of partition defined as $h_j = \{t_j - t_{j-1},\ j = 1, 2, .., n\}$. Let $\triangle_j = \{[t_{j-1}, t_j] :\ 1 \leq j \leq n\}$, and define $\mathcal{C}_\triangle = \prod_{j=1}^{n} \mathcal{C}(\triangle_j)$, where $\mathcal{C}_\triangle$ constitutes a Banach space in the uniform norm, represented as $\|z\|_{\triangle,\infty} = \max_j \|z_j\|_\infty$.

Further, we define useful notations as

$$\mathcal{L} = \{1, 2, \ldots, M\}, \quad \Gamma = \{t_1, t_2, \ldots, t_M\}. \tag{2.13}$$

Then there exists integer $p > 0$ such that

$$\tau \leq p \leq 2\tau, \tag{2.14}$$

where τ represents total number of quadrature points in $[0, 1]$.

The introduction of index sets can be expressed as the form:

$$H_j = j + (i-1)\tau, \text{ where } 1 \leq i \leq m, \ 1 \leq j \leq \tau. \tag{2.15}$$

Consider the collection $\{\Gamma_i\}_{i=1}^m \subseteq \Gamma$ satisfying the following condition

$$\begin{aligned} &t_j \in \Gamma_i = \{t_i^1, t_i^2, \ldots, t_i^p\}, \quad \forall\, j \in H_i, \\ &\max\{|t - t_j|;\ t \in \Gamma_i,\ j \in H_j\} \leq s^h, \end{aligned}$$

here s is a provided constant (typically $s \leq 2$).

Now introduce the notation l_i^q for $q = 1, 2, \ldots, p$, denotes the elementary Lagrange-polynomials corresponding to the nodes in Γ_i.

Hence we have

$$(L_i y)(t) = \sum_{q=1}^{p} y(t_i^q) l_i^q(t), \tag{2.16}$$

denotes as Lagrange-interpolation polynomial for the function y at these nodes.

Define the collection of indices $\{\mathcal{B}_i\}_{i=1}^m$ such that

$$\begin{aligned} &H_i \subset \mathcal{B}_i = \{i_1, i_2, \ldots, i_p\} \subset H,\ H_i = \{t_q : q \in \mathcal{B}_i\}, \\ &t_i^q = t_{i_q},\ i = 1, 2, \ldots, m, \text{ and } q = 1, \ldots, p. \end{aligned}$$

Then Eq. (2.16) can be expressed in the form of Newton interpolation polynomial:

$$[L_i(y)](t) = \sum_{q=1}^{p-1} y[t_i^1, \ldots, t_i^{q+1}](t - t_i^1) \ldots (t - t_i^q). \tag{2.17}$$

Now use nested multiplication for the evaluation process. To do this, we use Eq. (2.17) to create $\mathbb{K}_h$ and $\mathbb{L}_h$ as discrete analogues of $\mathbb{K}$ and $\mathbb{L}$, respectively.

Defining some functions in the following manner:

$$\pi_i^{(1)}(t) = \begin{cases} 0, & t \leq \Gamma_{i-1}, \\ t - \Gamma_{i-1}, & \Gamma_{i-1} \leq t \leq \Gamma_i, \\ f_i, & \Gamma_i \leq t, \end{cases} \tag{2.18}$$

$$\pi_i^{(2)}(t) = \begin{cases} f_i, & t \leq \Gamma_{i-1}, \\ \Gamma_i - t, & \Gamma_{i-1} \leq t \leq \Gamma_i, \\ 0, & \Gamma_i \leq t, \end{cases} \tag{2.19}$$

$$s_{ij}^1(t) = \Gamma_{i-1} + \pi_i^{(1)}(t)\hat{t_j},$$

$$s_{ij}^2(t) = \Gamma_i + \pi_i^{(2)}(t)\hat{t_j} - 1,$$

where $1 \leq i \leq m$ and $1 \leq j \leq \tau$.

As introduced earlier notations, the discrete operators $\mathbb{K}_h : C[0,1] \to C[0,1]$ and $\mathbb{L}_h : C[0,1] \to C[0,1]$ defined as

$$(\mathbb{K}_h\eta)(t) = \sum_{\nu=1}^{2}\sum_{i=1}^{m}\pi_i^{(p)}(t)\sum_{j=1}^{\tau}\hat{w}_j\mathcal{K}(t, s_{ij}^{\nu}(t))(L_i\eta)(s_{ij}^{\nu}(t)), \tag{2.20}$$

and

$$(\mathbb{L}_h\eta)(t) = \sum_{\nu=1}^{2}\sum_{i=1}^{m}\pi_i^{(p)}(t)\sum_{j=1}^{\tau}\hat{w}_j l(t, s_{ij}^{\nu}(t))(L_i\eta)(s_{ij}^{\nu}(t)). \tag{2.21}$$

For ease of computation, simplify the definition for $t \in \Lambda_i$ as follows:

$$(\mathbb{K}_h\eta)(t) = \sum_{\nu=1}^{2}(L_i\eta)A_i^{\nu}(t) + \sum_{\eta=1,\eta\neq i}^{m}\sum_{j\in H_\eta}\mathcal{K}(t, t_j)\eta(t_j), \tag{2.22}$$

where

$$A_i^{\nu}(t) = \pi_i^{(\nu)}(t)\sum_{j=1}^{\tau}\hat{w}_j\mathcal{K}(t, s_{ij}^{\nu}(t))(L_i\eta)(s_{ij}^{\nu}(t)). \tag{2.23}$$

Proposition 2.1. *Consider $\eta_0 \in \mathcal{C}^{d+1}[0,1]$ as the unique solution of nonlinear Fredholm integral equations defined in* (2.8)*, then there hold:*

$$\|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty = \mathcal{O}(h^{d+1}),$$
$$\|(\mathbb{L}_h - \mathbb{L})\eta_0\|_\infty = \mathcal{O}(h^{d+1}),$$

here h represents the norm of partitions and d is the degree of precision.

Now define the inner product in discrete form as:

$$(\alpha, \beta)_h = \sum_{i=1}^{m}\sum_{j=1}^{\tau} w_i^j\alpha(y_i^j)\beta(y_i^j). \tag{2.24}$$

Consider $r = j + (i-1)\tau$, for each $1 \leq i \leq m$, $1 \leq j \leq \tau$, also $t_{r=j+(i-1)\tau} = y_i^j$ and $w_{r=j+(i-1)\tau} = w_i^j$, and set $M = m\tau$.

We define

$$(\alpha, \beta)_h = \sum_{r=1}^{M} w_r\alpha(y_r)\beta(y_r). \tag{2.25}$$

Discrete orthogonal projection operator: Consider $\mathbb{Q} : \mathbb{X} \to \mathbb{C}_{\Lambda_i}$ as the discrete orthogonal projection operator such that

$$\mathbb{Q}x = \sum_{j=1}^{\tau}\langle x, e_j\rangle_{\Lambda_{i,h}} e_j, \tag{2.26}$$

where

$$\langle \alpha, \beta \rangle_{\Lambda_{i,h}} = \sum_{j=1}^{\tau} w_j^i \alpha(x_j^i)\beta(x_j^i). \tag{2.27}$$

For convenient, define discrete projection operator $\mathbb{Q} : \mathbb{X} \to \mathbb{C}_\Lambda$ by

$$\mathbb{Q}_h z = (\mathbb{Q}z_1, \mathbb{Q}z_2, \ldots, \mathbb{Q}z_n), \tag{2.28}$$

where $\mathbb{Q}z_i$ denotes the discrete orthogonal projection operator that projects $z_i \in \mathbb{C}(\Lambda_i)$ onto the subspace of polynomials with degree at most r defined on Λ_i.

We can also define the above operator $\mathbb{Q}_h : \mathbb{X} \to \mathbb{X}_h$ as

$$\mathbb{Q}_h z = \sum_{j=1}^{M} \langle z, e_j \rangle_M e_j, \quad z \in X, \tag{2.29}$$

where

$$\mathbb{Q}_h z = (\mathbb{Q}z_1, \mathbb{Q}z_2, \ldots, \mathbb{Q}z_n), \tag{2.30}$$

and $\mathbb{Q}_h$ satisfies such that

$$\langle \mathbb{Q}_h z, z_h \rangle_M = \langle z, z_h \rangle_M, \; z \in X, z_h \in X_h. \tag{2.31}$$

Lemma 2.1. *Let $\eta_0 \in \mathcal{C}^{d+1}[0,1]$ be a unique solution of nonlinear Fredholm integral equations defined in* (2.8)*, the following result hold*

$$\|\mathbb{K}_h(\mathbb{I} - \mathbb{Q}_h)\|_\infty = \mathcal{O}(h^{m_1}) \quad \text{and} \quad \|\mathbb{L}_h(\mathbb{I} - \mathbb{Q}_h)\|_\infty = \mathcal{O}(h^{m_2}),$$

where h represents the norm of partitions, $m_1 = \min\{r+1, k_1, \mu+2\}$ and $m_2 = \min\{r+1, k_1 - 1, \mu+1\}$.

From Sloan [41] and Chatelin [12], we present some essential properties of $\mathbb{Q}_h$, which are indispensable for our convergence analysis.
(i.) $\forall\ z \in \mathbb{X}$,

$$\langle z - \mathbb{Q}_h z, z - \mathbb{Q}_h z \rangle_M = \min_{s \in \mathbb{X}_h} \langle z - s, z - s \rangle_M. \tag{2.32}$$

(ii.) $\forall\ z \in \mathbb{X}$, $\exists$ a constant $\rho > 0$ such that

$$\|\mathbb{Q}_h\|_\infty \le \rho < \infty, \tag{2.33}$$

and

$$\|\mathbb{Q}_h z - z\|_\infty \le \inf_{\psi \in \mathbb{X}_h} \|z - \psi\|_\infty \to 0 \text{ as } h \to 0. \tag{2.34}$$

(iii.) In general, $z \in \mathbb{C}^r[0,1]$,

$$\|\mathbb{Q}_h z - z\|_\infty \le C h^r \|z^r\|_\infty, \tag{2.35}$$

here C represents a generic constant independent of h.

To solve Eq. (2.10), using the discrete multi-Galerkin approximation, we aim to achieve the approximate solution $\eta_h^M \in \mathbb{X}$ such that

$$\eta_h^M = \mathbb{T}_h^M(\eta_h^M), \tag{2.36}$$

where the operator $\mathbb{T}_h^M : \mathbb{X} \to \mathbb{X}$ defined by (see [14, 16])

$$\mathbb{T}_h^M(u) = \mathbb{Q}_h\Omega(g + \mathbb{K}_h u) + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h u). \tag{2.37}$$

The iterated discrete multi-Galerkin method approximates as follows:

$$\tilde{\eta}_h^M = \Omega(g + \mathbb{K}_h\eta_h^M). \tag{2.38}$$

From Eq. (2.8), corresponding approximate solutions z_h^M and $\tilde{z}_h^M$ of z defined as:

$$z_h^M = g + \mathbb{K}_h(\eta_h^M), \tag{2.39}$$

$$\tilde{z}_h^M = g + \mathbb{K}_h(\tilde{\eta}_h^M). \tag{2.40}$$

Now applying $\mathbb{Q}_h$ and $(\mathbb{I} - \mathbb{Q}_h)$ to the Eq. (2.36), we have

$$\mathbb{Q}_h\eta_h^M = \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_h^M), \tag{2.41}$$

$$(\mathbb{I} - \mathbb{Q}_h)\eta_h^M = (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_h^M). \tag{2.42}$$

Eq. (2.42) can be expressed as

$$\eta_h^M = \mathbb{Q}_h\eta_h^M + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_h^M). \tag{2.43}$$

From Eqs. (2.41) and (2.43), we obtain

$$\mathbb{Q}_h\eta_h^M = \mathbb{Q}_h\Omega(g + \mathbb{K}_h(\mathbb{Q}_h\eta_h^M + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_h^M))). \tag{2.44}$$

Let $\mathcal{M}_h^M = \mathbb{Q}_h\eta_h^M$, we find $\mathcal{M}_h^M \in \mathbb{X}_h$, then above equation becomes

$$\mathcal{M}_h^M = \mathbb{Q}_h\Omega(g + \mathbb{K}_h(\mathcal{M}_h^M + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathcal{M}_h^M))). \tag{2.45}$$

Now find η_h^M from the Eq. (2.43) as

$$\eta_h^M = \mathcal{M}_h^M + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathcal{M}_h^M). \tag{2.46}$$

Note that Fréchet derivative of $\mathbb{T}_h^M$ at η_0 is defined as

$$\begin{aligned}\mathbb{T}_h^{M\prime}(\eta_0) = {} & \mathbb{Q}_h\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\ & + \mathbb{Q}_h\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h.\end{aligned} \tag{2.47}$$

3. Superconvergence results

This section focuses on the study of the existence of the discrete multi-Galerkin technique and its iterated version for addressing derivative-dependent nonlinear Hammerstein type Fredholm integral equations given by (2.1) and achieves their superconvergence results. For this, we will commence with the theorem from [42], which provides conditions that if one equation is solvable, then another related equation must also be solvable.

Theorem 3.1. [42] *Let $\widehat{\chi}$ and $\widetilde{\chi}$ be the continuous operators and Φ be the open set in the Banach space $\mathbb{X}$. Let $\tilde{\eta}_0 \in \Phi$ be the isolated solution of $\eta = \widetilde{\chi}\eta$. Then the given condition holds*
(a) The operator $\widehat{\chi}$ is Fréchet differentiable in a neighborhood of the point $\tilde{\eta}_0$, and the linear operator $(\mathbb{I} - \widehat{\chi}'(\tilde{\eta}_o))$ is continuously differentiable.
(b) Consider some $\delta > 0$ and constant $0 < q < 1$, the given conditions hold (assuming δ is sufficiently small such that $\|\eta - \tilde{\eta}_0\| \leq \delta$ contained in Φ)

$$\sup_{\|\eta-\tilde{\eta}_0\|\leq\delta} \|(\mathbb{I} - \widehat{\chi}'(\tilde{\eta}_o))^{-1}(\widehat{\chi}'(\eta) - \widehat{\chi}'(\tilde{\eta}_o))\| \leq q, \tag{3.1}$$

$$\beta = \|(\mathbb{I} - \widehat{\chi}'(\tilde{\eta}_o))^{-1}(\widehat{\chi}(\tilde{\eta}_o) - \widetilde{\chi}(\tilde{\eta}_o))\| \leq \delta(1-q). \tag{3.2}$$

Then the equation $\eta = \widehat{\chi}\eta$ possesses an exact solution $\hat{\eta}_0$ within $\|\eta - \tilde{\eta}_0\| \leq \delta$. Furthermore, the inequality

$$\frac{\beta}{1+q} \leq \|\hat{\eta}_0 - \tilde{\eta}_0\| \leq \frac{\beta}{1-q}, \tag{3.3}$$

holds.

Next, we will investigate the existence of discrete multi-Galerkin solutions and their convergence analysis.

Lemma 3.2. *Let $\eta_0 \in \mathbb{C}^{k_1}[0,1]$ be a unique solution of nonlinear Fredholm integral equations defined in* (2.8)*, then there hold:*

$$\|(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_1)}),$$

and

$$\|(\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_2)}),$$

where d denotes the degree of precision and $m = \min\{r+1, k_1\}$, $m_1 = \min\{r+1, k_1, \mu+2\}$ and $m_2 = \min\{r+1, k_1 - 1, \mu+1\}$.

Proof. (p.9, [22]), follows the proof of this Lemma. □

Theorem 3.3. *Consider $\eta_0 \in \mathbb{C}^{k_1}[0,1]$ as the exact solution of* (2.10) *and $\mathbb{T}_h^{M'}(\eta_0)$ be the fréchet derivative of $\mathbb{T}_h^M$ at η_0 given by* (2.47)*. Assume that the operator $\mathbb{T}_h^{M'}(\eta_0)$ do not have an eigenvalue 1. Then for small h, the operator $(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))$ exists and uniformly bounded on $\mathbb{X}$. Then there exists a constant $\mathcal{L} > 0$ such that*

$$\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty \leq \mathcal{L} < \infty.$$

Proof. From the Eqs. (2.11) and (2.47), we obtain

$$\begin{aligned}
&\|\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}'(\eta_0)\|_\infty \\
=&\|\mathbb{Q}_h\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \mathbb{Q}_h\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h \\
&- \Omega^{(0,1,0)}(g + \mathbb{K}\eta_0)\mathbb{K} - \Omega^{(0,0,1)}(g + \mathbb{K}\eta_0)\mathbb{L}\|_\infty.
\end{aligned} \tag{3.4}$$

From Eq. (3.4), we obtain

$$\{\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h\}$$

$$
\begin{aligned}
&= \{\Omega^{(0,1,0)}(g+\mathbb{K}_h\eta_0)\mathbb{K}_h - \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h + \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h\} \\
&\leq C_2[\{(\mathbb{K}_h-\mathbb{K})\eta_0\}\mathbb{K}_h + \{(\mathbb{L}_h-\mathbb{L})\eta_0\}\mathbb{K}_h] + \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h, \qquad (3.5)
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\{\Omega^{(0,0,1)}(g+\mathbb{K}_h\eta_0)\mathbb{L}_h\} \\
\leq & C_3[\{(\mathbb{K}_h-\mathbb{K})\eta_0\}\mathbb{L}_h + \{(\mathbb{L}_h-\mathbb{L})\eta_0\}\mathbb{L}_h] + \Omega^{(0,0,1)}(g+\mathbb{K}\eta_0)\mathbb{L}_h. \qquad (3.6)
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\{\Omega^{(0,1,0)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h\} \\
=&\{\Omega^{(0,1,0)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h - \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h\mathbb{Q}_h\} \\
\leq & C_2[\{(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\}\mathbb{K}_h\mathbb{Q}_h + \{(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0\}\mathbb{K}_h\mathbb{Q}_h] \\
&+ \Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h\mathbb{Q}_h, \qquad (3.7)
\end{aligned}
$$

similarly,

$$
\begin{aligned}
&\{\Omega^{(0,0,1)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h\} \\
\leq & C_3[\{(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\}\mathbb{L}_h\mathbb{Q}_h + \{(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0\}\mathbb{L}_h\mathbb{Q}_h] \\
&+ \Omega^{(0,0,1)}(g+\mathbb{K}\eta_0)\mathbb{L}_h\mathbb{Q}_h. \qquad (3.8)
\end{aligned}
$$

Substituting estimates (3.5)-(3.8) in Eq. (3.4), we obtain

$$
\begin{aligned}
&\|\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}'(\eta_0)\|_\infty \\
\leq & \rho(C_2A_1 + C_3A_2)[\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h-\mathbb{L})\eta_0\|_\infty] \\
&+ \rho(1+\rho)(C_2A_1 + C_3A_2)[\|(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0\|_\infty] \\
&+ \|\Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)(\mathbb{K}_h-\mathbb{K})\|_\infty + \|\Omega^{(0,0,1)}(g+\mathbb{K}\eta_0)(\mathbb{L}_h-\mathbb{L})\|_\infty \\
&+ \|\Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\mathbb{K}_h\mathbb{Q}_h\|_\infty + \|\Omega^{(0,0,1)}(g+\mathbb{K}\eta_0)\mathbb{L}_h\mathbb{Q}_h\|_\infty.
\end{aligned}
$$

Next, using boundedness of $\|\Omega^{(0,1,0)}(g+\mathbb{K}\eta_0)\|_\infty$, $\|\Omega^{(0,0,1)}(g+\mathbb{K}\eta_0)\|_\infty$, Proposition 2.1 and Lemma 3.2, we obtain

$$
\|\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}'(\eta_0)\|_\infty \to 0 \quad \text{as} \quad h \to 0. \qquad (3.9)
$$

This implies $\mathbb{T}_h^{M'}(\eta_0)$ is norm convergent to $\mathbb{T}'(\eta_0)$. According to Ahues et al. [3], $(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}$ is invertible and uniformly bounded for small h, i.e., $\exists$ a positive constant $\mathcal{L}$ independent of h such that $\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty \leq \mathcal{L} < \infty$.

Hence Proved the Theorem. □

Theorem 3.4. *Let $\eta_0 \in \mathbb{C}^{k_1}[0,1]$ be the unique solution of derivative-dependent Hammerstein Fredholm integral equations defined by Eq. (2.10). Assume that the linear operator $\mathbb{T}_h^{M'}$ do not have an eigenvalue 1, where $\mathbb{T}_h^{M'}$ represents the Fréchet derivative of $\mathbb{T}_h^M$, then for sufficiently small h and parameter $\delta > 0$, the Eq. (2.36) possesses isolated discrete multi-Galerkin approximation $\eta_h^M \in \mathbb{B}(\eta_0, \delta) = \{\eta : \|\eta - \eta_0\|_\infty < \delta\}$. Additionally, there exists a constant $0 < q < 1$, independent of h, such that*

$$
\frac{\beta_h}{1+q} \leq \|\eta_h^M - \eta_0\|_\infty \leq \frac{\beta_h}{1-q},
$$

with $\beta_h = \|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty$.

Proof. For every η, $\eta_0, u \in \mathbb{X}$, we consider

$$\begin{aligned}
&\|[\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}_h^{M'}(\eta)]u\|_\infty \\
=&\|[\{\mathbb{Q}_h\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \mathbb{Q}_h\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h\} \\
&- \{\mathbb{Q}_h\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta)\mathbb{K}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta)\mathbb{K}_h\mathbb{Q}_h \\
&+ \mathbb{Q}_h\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta)\mathbb{L}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta)\mathbb{L}_h\mathbb{Q}_h\}u]\|_\infty \\
\le&\|\mathbb{Q}_h[\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0) - \Omega^{(0,1,0)}(g + \mathbb{K}_h\eta)]\mathbb{K}_h u\|_\infty \\
&+ \|(\mathbb{I} - \mathbb{Q}_h)[\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta)]\mathbb{K}_h\mathbb{Q}_h u\|_\infty \\
&+ \|\mathbb{Q}_h[\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0) - \Omega^{(0,0,1)}(g + \mathbb{K}_h\eta)]\mathbb{L}_h u\|_\infty \\
&+ \|(\mathbb{I} - \mathbb{Q}_h)[\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta)]\mathbb{L}_h\mathbb{Q}_h u\|_\infty.
\end{aligned} \tag{3.10}$$

From Eqs. (2.4) and (2.33), we get

$$\|\mathbb{K}_h\eta\|_\infty \le A_1\|\eta\|_\infty, \tag{3.11}$$
$$\|\mathbb{L}_h\eta\|_\infty \le A_2\|\eta\|_\infty. \tag{3.12}$$

Also

$$\|\mathbb{K}_h\mathbb{Q}_h\eta\|_\infty \le A_1\rho\|\eta\|_\infty, \tag{3.13}$$
$$\|\mathbb{L}_h\mathbb{Q}_h\eta\|_\infty \le A_2\rho\|\eta\|_\infty. \tag{3.14}$$

Then using the estimates (3.11)-(3.14) in the Eq. (3.10), we get

$$\begin{aligned}
&\|[\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}_h^{M'}(\eta)]u\|_\infty \\
\le&\rho C_2[\{\|\mathbb{K}_h(\eta_0 - \eta)\|_\infty + \|\mathbb{L}_h(\eta_0 - \eta)\|_\infty\}\|\mathbb{K}_h u\|_\infty] \\
&+ (1+\rho)C_2[\{\|\mathbb{K}_h\mathbb{Q}_h(\eta_0 - \eta)\|_\infty + \|\mathbb{L}_h\mathbb{Q}_h(\eta_0 - \eta)\|_\infty\}\|\mathbb{K}_h\mathbb{Q}_h u\|_\infty] \\
&+ \rho C_3[\{\|\mathbb{K}_h(\eta_0 - \eta)\|_\infty + \|\mathbb{L}_h(\eta_0 - \eta)\|_\infty\}\|\mathbb{L}_h u\|_\infty] \\
&+ (1+\rho)C_3[\{\|\mathbb{K}_h\mathbb{Q}_h(\eta_0 - \eta)\|_\infty + \|\mathbb{L}_h\mathbb{Q}_h(\eta_0 - \eta)\|_\infty\}\|\mathbb{L}_h\mathbb{Q}_h u\|_\infty] \\
\le&\rho(C_2 + C_3)(A_1 + A_2)^2\|\eta_0 - \eta\|_\infty\|u\|_\infty \\
&+ (1+\rho)\rho^2(C_2 + C_3)(A_1 + A_2)^2\|\eta_0 - \eta\|_\infty\|u\|_\infty \\
=&[\rho(C_2 + C_3)(A_1 + A_2)^2 + (1+\rho)\rho^2(C_2 + C_3)(A_1 + A_2)^2]\|\eta_0 - \eta\|_\infty\|u\|_\infty \\
=&[\rho(C_2 + C_3)A^2 + (1+\rho)\rho^2(C_2 + C_3)A^2]\|\eta_0 - \eta\|_\infty\|u\|_\infty \\
=&[\rho(C_2 + C_3)A^2(1 + \rho + \rho^2)]\|\eta_0 - \eta\|_\infty\|u\|_\infty.
\end{aligned} \tag{3.15}$$

Hence

$$\|\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}_h^{M'}(\eta)\|_\infty \le [\rho(C_2 + C_3)A^2(1 + \rho + \rho^2)]\|\eta_0 - \eta\|_\infty. \tag{3.16}$$

From the Theorem 3.3, we obtain

$$\begin{aligned}
&\sup_{\|\eta-\eta_o\|\le\delta}\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^{M'}(\eta_0) - \mathbb{T}_h^{M'}(\eta)\|_\infty \\
\le&\mathcal{L}[\rho(C_2 + C_3)A^2(1 + \rho + \rho^2)]\delta \le q,
\end{aligned}$$

now we take δ such that $0 < q < 1$, ensures the validation of (3.1) according to Vainikko's Theorem 3.1 of [42].

Consider

$$\begin{aligned}\beta_h &= \|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty \\ &\leq \mathcal{L}\|\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)\|_\infty \\ &= \mathcal{L}\|\mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\|_\infty \\ &= \mathcal{L}\|\mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + \Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\|_\infty \\ &= \mathcal{L}\|\{\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\} - \{\mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) \\ &\quad - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0)\}\|_\infty \\ &\leq \mathcal{L}C_1\|\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\} \\ &\quad - \mathbb{Q}_h\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K}_h)\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L}_h)\eta_0\}\|_\infty \\ &\leq \mathcal{L}C_1\|(\mathbb{I} - \mathbb{Q}_h)\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\} + \mathbb{Q}_h\{(\mathbb{K}_h - \mathbb{K})\eta_0 \\ &\quad + (\mathbb{L}_h - \mathbb{L})\eta_0\}\|_\infty. \end{aligned} \tag{3.17}$$

Using Proposition 2.1 and Lemma 3.2 in above equation, we obtain

$$\begin{aligned}\beta_h \leq& \mathcal{L}C_1(1+\rho)\{\|(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\|_\infty\} \\ &+ \mathcal{L}C_1\rho\{\|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h - \mathbb{L})\eta_0\|_\infty\} \\ \to& 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{3.18}$$

Taking sufficiently small h such that $\beta_h \leq \delta(1-q)$, ensures the validation of (3.2) according to Vainikko's Theorem 3.1 of [42] hold.

Hence from Theorem 3.1, we have

$$\frac{\beta_h}{1+q} \leq \|\eta_h^M - \eta_0\|_\infty \leq \frac{\beta_h}{1-q},$$

with $\beta_h = \|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty$.

Hence proved the Theorem. □

Theorem 3.5. *Let $\eta_0 \in \mathbb{C}^{k_1}[0,1]$ be the unique solution of equation defined by (2.10) and assume that the linear operator $\mathbb{T}_h^{M'}$ do not have an eigenvalue 1, where $\mathbb{T}_h^{M'}$ represents the Fréchet derivative of $\mathbb{T}_h^M$. Let $\mathbb{Q}_h : \mathbb{X} \to \mathbb{X}_h$ be the discrete orthogonal projection operator defined by (2.29) and η_h^M be the discrete multi-Galerkin approximation of η_0. Then there holds*

$$\|\eta_h^M - \eta_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}),$$

and if z_h^M is the corresponding approximation of z_0, Then the following holds

$$\|z_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}),$$

where d denotes the degree of precision and $m = \min\{r+1, k_1\}$, $m_1 = \min\{r+1, k_1, \mu+2\}$ and $m_2 = \min\{r+1, k_1-1, \mu+1\}$.

Proof. By using the Theorem 3.4, we have

$$\frac{\beta_h}{1+q} \leq \|\eta_h^M - \eta_0\|_\infty \leq \frac{\beta_h}{1-q},$$

with $\beta_h = \|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty$.

Consider

$$\begin{aligned}
&\|\eta_h^M - \eta_0\|_\infty \\
&\le \frac{\beta_h}{1-q} \\
&\le \frac{1}{1-q}\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty \\
&\le C\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty\|(\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0))\|_\infty \\
&= C\mathcal{L}\|\mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\|_\infty \\
&= C\mathcal{L}\|\mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + \Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\|_\infty \\
&= C\mathcal{L}\|\{\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\} \\
&\quad - \{\mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0)\}\|_\infty \\
&\le C\mathcal{L}C_1\|\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\} \\
&\quad - \mathbb{Q}_h\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K}_h)\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L}_h)\eta_0\}\|_\infty \\
&\le C\mathcal{L}C_1\|(\mathbb{I} - \mathbb{Q}_h)\{(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\} \\
&\quad + \mathbb{Q}_h\{(\mathbb{K}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h - \mathbb{L})\eta_0\}\|_\infty \\
&\le C\mathcal{L}C_1(1+\rho)\{\|(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\|_\infty\} \\
&\quad + C\mathcal{L}C_1\rho\{\|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h - \mathbb{L})\eta_0\|_\infty\}. \qquad (3.19)
\end{aligned}$$

Now using the result of Proposition 2.1 and Lemma 3.2, we obtain

$$\begin{aligned}
&\|\eta_h^M - \eta_0\|_\infty \\
&= \{\mathbb{O}(h^{\min(d+1,\ m+m_1)}) + \mathbb{O}(h^{\min(d+1,\ m+m_2)})\} + \{\mathbb{O}(h^{d+1}) + \mathbb{O}(h^{d+1})\} \\
&= \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}). \qquad (3.20)
\end{aligned}$$

From Eqs. (2.8) and (2.40), we have

$$\begin{aligned}
\|z_h^M - z_0\|_\infty &= \|\mathbb{K}_h\eta_h^M - \mathbb{K}\eta_0\|_\infty \\
&\le \|\mathbb{K}_h\eta_h^M - \mathbb{K}_h\eta_0 + \mathbb{K}_h\eta_0 - \mathbb{K}\eta_0\|_\infty \\
&\le \|\mathbb{K}_h(\eta_h^M - \eta_0)\|_\infty + \|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty \\
&\le A_1\|\eta_h^M - \eta_0\|_\infty + \|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty \\
&= \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}) + \mathbb{O}(h^{d+1}) \\
&= \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}). \qquad (3.21)
\end{aligned}$$

Hence proved the Theorem. □

Next, we will investigate the superconvergence analysis for iterated discrete multi-Galerkin approximation.

Theorem 3.6. *Let $\eta_0 \in \mathbb{C}^{k_1}[0,1]$ be the exact solution of equation (2.10) and assume that the linear operator $\mathbb{T}_h^{M'}$ do not have an eigenvalue 1, where $\mathbb{T}_h^{M'}$ represents the Fréchet derivative of $\mathbb{T}_h^M$. Let $\mathbb{Q}_h : \mathbb{X} \to \mathbb{X}_h$ be the discrete orthogonal projection operator given by (2.29) and $\tilde{\eta}_h^M$ be the iterated discrete multi-Galerkin approximation of η_0. Then the following holds*

$$\|\tilde{\eta}_h^M - \eta_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}),$$

and if $\tilde{z}_h^M$ *is the corresponding approximation of* z_0*, Then the following holds*

$$\|\tilde{z}_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}),$$

where d *denotes the degree of precision and* $m = \min\{r+1, k_1\}$*,* $m_1 = \min\{r+1, k_1, \mu+2\}$ *and* $m_2 = \min\{r+1, k_1-1, \mu+1\}$*.*

Proof. We consider

$$\begin{aligned}\eta_h^M - \eta_0 &= \mathbb{T}_h^M(\eta_h^M) - \mathbb{T}(\eta_0)\\ &= \mathbb{T}_h^M(\eta_h^M) - \mathbb{T}_h^M(\eta_0) - \mathbb{T}_h^{M'}(\eta_0)(\eta_h^M - \eta_0) + \mathbb{T}_h^{M'}(\eta_0)(\eta_h^M - \eta_0)\\ &\quad + \mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0).\end{aligned} \tag{3.22}$$

This implies that

$$\begin{aligned}&(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))(\eta_h^M - \eta_0)\\ =&\mathbb{T}_h^M(\eta_h^M) - \mathbb{T}_h^M(\eta_0) - \mathbb{T}_h^{M'}(\eta_0)(\eta_h^M - \eta_0) + \mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0).\end{aligned} \tag{3.23}$$

Now applying the Mean Value Theorem, we obtain

$$\begin{aligned}&\eta_h^M - \eta_0\\ =&(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_h^M) - \mathbb{T}_h^M(\eta_0) - \mathbb{T}_h^{M'}(\eta_0)(\eta_h^M - \eta_0) + \mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\\ =&(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_h^M) - \mathbb{T}_h^M(\eta_0) - \mathbb{T}_h^{M'}(\eta_0)(\eta_h^M - \eta_0)]\\ &+ (\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\\ =&(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^{M'}(\eta_0 + \theta_1(\eta_h^M - \eta_0)) - \mathbb{T}_h^{M'}(\eta_0)](\eta_h^M - \eta_0)]\\ &+ (\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)],\end{aligned} \tag{3.24}$$

where $0 < \theta_1 < 1$.

Operating $\mathbb{K}_h$ on both side of Eq. (3.24), we have

$$\begin{aligned}&\|\mathbb{K}_h(\eta_h^M - \eta_0)\|_\infty\\ =&\|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty\|[\mathbb{T}_h^{M'}(\eta_0 + \theta_1(\eta_h^M - \eta_0)) - \mathbb{T}_h^{M'}(\eta_0)](\eta_h^M - \eta_0)\|_\infty\\ &+ \|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty.\end{aligned} \tag{3.25}$$

From Theorem 3.3, we have

$$\begin{aligned}\|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}u\|_\infty &\le A_1\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}u\|_\infty\\ &\le A_1\|(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty\|u\|_\infty\\ &\le A_1\mathcal{L}\|u\|_\infty.\end{aligned} \tag{3.26}$$

Now using the above estimate in Eq. (3.25), we obtain

$$\begin{aligned}\|\mathbb{K}_h(\eta_h^M - \eta_0)\|_\infty =&A_1\mathcal{L}\|[\mathbb{T}_h^{M'}(\eta_0 + \theta_1(\eta_h^M - \eta_0)) - \mathbb{T}_h^{M'}(\eta_0)](\eta_h^M - \eta_0)\|_\infty\\ &+ \|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty.\end{aligned} \tag{3.27}$$

Using the identity $(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1} = \mathbb{I} + (\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)$, in the second term of Eq. (3.27), we get

$$\|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty$$

$$
\begin{aligned}
&= \|\mathbb{K}_h\{\mathbb{I} + (\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)\}[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty \\
&= \|\mathbb{K}_h[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty + \|\mathbb{K}_h(\mathbb{I} - \mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty \\
&\leq \|\mathbb{K}_h[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty + A_1\mathcal{L}\|\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty. \qquad (3.28)
\end{aligned}
$$

Since

$$
\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0) = \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0),
$$

and

$$
\begin{aligned}
&\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)] \\
=&[\mathbb{Q}_h\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \mathbb{Q}_h\Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h + (\mathbb{I} - \mathbb{Q}_h)\Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h] \\
&\times \{\mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0) + (\mathbb{I} - \mathbb{Q}_h)\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)\} \\
=&\{\mathbb{Q}_h[\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + \Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h] \\
&+ (\mathbb{I} - \mathbb{Q}_h)[\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h + \Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h] \\
&\times \{[\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)] - [\mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0)]\} \\
\leq&[\rho\{\|\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h + \Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h\|_\infty\} \\
&+ (1 + \rho)\{\|\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h + \Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h\|_\infty\}] \\
&\times [C_1(1 + \rho)\{\|(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty\} \\
&+ C_1\rho\{\|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty + \|(\mathbb{L}_h - \mathbb{L})\eta_0\|_\infty\}]. \qquad (3.29)
\end{aligned}
$$

Hence using the estimates (3.28)-(3.29) in Eq. (3.27), we obtain

$$
\begin{aligned}
&\|\mathbb{K}_h(\eta_h^M - \eta_0)\|_\infty \\
=& A_1\mathcal{L}\|[\mathbb{T}_h^{M'}(\eta_0 + \theta_1(\eta_h^M - \eta_0)) - \mathbb{T}_h^{M'}(\eta_0)](\eta_h^M - \eta_0)\|_\infty \\
&+ \|\mathbb{K}_h[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty + A_1\mathcal{L}\|\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^M(\eta_0) - \mathbb{T}(\eta_0)]\|_\infty \\
\leq& CA_1\mathcal{L}\|\eta_h^M - \eta_0\|_\infty^2 + \|\mathbb{K}_h(\mathbb{I} - \mathbb{Q}_h)[(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0]\|_\infty \\
&+ \|\mathbb{K}_h\mathbb{Q}_h[(\mathbb{K}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h - \mathbb{L})\eta_0]\|_\infty + A_1\mathcal{L}\|\{\mathbb{Q}_h[\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h \\
&+ \Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h] + (\mathbb{I} - \mathbb{Q}_h)[\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h]\} \times \{[\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \Omega(g + \mathbb{K}\eta_0)] \\
&- [\mathbb{Q}_h\Omega(g + \mathbb{K}_h\mathbb{Q}_h\eta_0) - \mathbb{Q}_h\Omega(g + \mathbb{K}_h\eta_0)]\}\|_\infty \\
\leq& CA_1\mathcal{L}\|\eta_h^M - \eta_0\|_\infty^2 + \|\mathbb{K}_h(\mathbb{I} - \mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0]\|_\infty \\
&+ \|\mathbb{K}_h\mathbb{Q}_h[(\mathbb{K}_h - \mathbb{K})\eta_0 + (\mathbb{L}_h - \mathbb{L})\eta_0]\|_\infty + A_1\mathcal{L}\rho\|[\Omega^{(0,1,0)}(g + \mathbb{K}_h\eta_0)\mathbb{K}_h \\
&+ \Omega^{(0,0,1)}(g + \mathbb{K}_h\eta_0)\mathbb{L}_h]\|_\infty + A_1\mathcal{L}(1 + \rho)\|[\Omega^{(0,1,0)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h \\
&+ \Omega^{(0,0,1)}(g + \mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h]\|_\infty \times [C_1(1 + \rho)\{\|(\mathbb{K}_h\mathbb{Q}_h - \mathbb{K})\eta_0\|_\infty \\
&+ \|(\mathbb{L}_h\mathbb{Q}_h - \mathbb{L})\eta_0\|_\infty\} + C_1\rho\{\|(\mathbb{K}_h - \mathbb{K})\eta_0\|_\infty \\
&+ \|(\mathbb{L}_h - \mathbb{L})\eta_0\|_\infty\}]. \qquad (3.30)
\end{aligned}
$$

Operating $\mathbb{L}_h$ on both side of Eq. (3.24), we have

$$
\|\mathbb{L}_h(\eta_h^M - \eta_0)\|_\infty
$$

$$=\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty\|[\mathbb{T}_h^{M'}(\eta_0+\theta_1(\eta_h^M-\eta_0))-\mathbb{T}_h^{M'}(\eta_0)](\eta_h^M-\eta_0)\|_\infty$$
$$+\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty. \tag{3.31}$$

From Theorem 3.3, we have

$$\begin{aligned}\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}u\|_\infty &\le A_2\|(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}u\|_\infty\\ &\le A_2\|(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}\|_\infty\|u\|_\infty\\ &\le A_2\mathcal{L}\|u\|_\infty. \end{aligned}\tag{3.32}$$

Now using the above estimate in Eq. (3.31), we obtain

$$\begin{aligned}\|\mathbb{L}_h(\eta_h^M-\eta_0)\|_\infty =&A_2\mathcal{L}\|[\mathbb{T}_h^{M'}(\eta_0+\theta_1(\eta_h^M-\eta_0))-\mathbb{T}_h^{M'}(\eta_0)](\eta_h^M-\eta_0)\|_\infty\\ &+\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty.\end{aligned}\tag{3.33}$$

Now using the identity $(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}=\mathbb{I}+(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)$, in the second part of Eq. (3.33), we get

$$\begin{aligned}&\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty\\ &=\|\mathbb{L}_h\{\mathbb{I}+(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)\}[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty\\ &=\|\mathbb{L}_h[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty+\|\mathbb{L}_h(\mathbb{I}-\mathbb{T}_h^{M'}(\eta_0))^{-1}\mathbb{T}_h^{M'}(\eta_0)\}[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty\\ &\le\|\mathbb{L}_h[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty+A_2\mathcal{L}\|\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty.\end{aligned}\tag{3.34}$$

Since we have

$$\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)=\mathbb{Q}_h\Omega(g+\mathbb{K}_h\eta_0)+(\mathbb{I}-\mathbb{Q}_h)\Omega(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)-\Omega(g+\mathbb{K}\eta_0),$$

and

$$\begin{aligned}&\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\\ &\le[\rho\{\|\Omega^{(0,1,0)}(g+\mathbb{K}_h\eta_0)\mathbb{K}_h+\Omega^{(0,0,1)}(g+\mathbb{K}_h\eta_0)\mathbb{L}_h\|_\infty\}\\ &\quad+(1+\rho)\{\|\Omega^{(0,1,0)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h+\Omega^{(0,0,1)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h\|_\infty\}]\\ &\quad\times[C_1(1+\rho)\{\|(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h\mathbb{Q}_h-\mathbb{K})\eta_0\|_\infty\}\\ &\quad+C_1\rho\{\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h-\mathbb{L})\eta_0\|_\infty\}].\end{aligned}\tag{3.35}$$

Hence using the estimates (3.34) and (3.35) in Eq.(3.33), we obtain

$$\begin{aligned}&\|\mathbb{L}_h(\eta_h^M-\eta_0)\|_\infty\\ &=A_2\mathcal{L}\|[\mathbb{T}_h^{M'}(\eta_0+\theta_1(\eta_h^M-\eta_0))-\mathbb{T}_h^{M'}(\eta_0)](\eta_h^M-\eta_0)\|_\infty\\ &\quad+\|\mathbb{L}_h[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty+A_2\mathcal{L}\|\mathbb{T}_h^{M'}(\eta_0)[\mathbb{T}_h^{M}(\eta_0)-\mathbb{T}(\eta_0)]\|_\infty\\ &\le CA_2\mathcal{L}\|\eta_h^M-\eta_0\|_\infty^2+\|\mathbb{L}_h(\mathbb{I}-\mathbb{Q}_h)[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty\\ &\quad+\|\mathbb{L}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty+A_2\mathcal{L}\|\{\mathbb{Q}_h[\Omega^{(0,1,0)}(g+\mathbb{K}_h\eta_0)\mathbb{K}_h\\ &\quad+\Omega^{(0,0,1)}(g+\mathbb{K}_h\eta_0)\mathbb{L}_h]+(\mathbb{I}-\mathbb{Q}_h)[\Omega^{(0,1,0)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h\\ &\quad+\Omega^{(0,0,1)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h]\}\times\{[\Omega(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)-\Omega(g+\mathbb{K}\eta_0)]\\ &\quad-[\mathbb{Q}_h\Omega(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)-\mathbb{Q}_h\Omega(g+\mathbb{K}_h\eta_0)]\}\|_\infty\end{aligned}$$

$$\begin{aligned}
\leq & CA_2\mathcal{L}\|\eta_h^M-\eta_0\|_\infty^2+\|\mathbb{L}_h(\mathbb{I}-\mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty\\
&+\|\mathbb{L}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty+A_2\mathcal{L}\rho\|[\Omega^{(0,1,0)}(g+\mathbb{K}_h\eta_0)\mathbb{K}_h\\
&+\Omega^{(0,0,1)}(g+\mathbb{K}_h\eta_0)\mathbb{L}_h]\|_\infty+A_2\mathcal{L}(1+\rho)\|[\Omega^{(0,1,0)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{K}_h\mathbb{Q}_h\\
&+\Omega^{(0,0,1)}(g+\mathbb{K}_h\mathbb{Q}_h\eta_0)\mathbb{L}_h\mathbb{Q}_h]\|_\infty\\
&\times[C_1(1+\rho)\{\|(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0\|_\infty\}\\
&+C_1\rho\{\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h-\mathbb{L})\eta_0\|_\infty\}]. \qquad (3.36)
\end{aligned}$$

From the Eqs. (3.30) and (3.36), we get

$$\begin{aligned}
&\|\tilde{\eta}_h^M-\eta_0\|_\infty\\
=&\|\Omega(g+\mathbb{K}_h\eta_h^M)-\Omega(g+\mathbb{K}\eta_0)\|_\infty\\
=&\|\Omega(g+\mathbb{K}_h\eta_h^M)-\Omega(g+\mathbb{K}_h\eta_0)+\Omega(g+\mathbb{K}_h\eta_0)-\Omega(g+\mathbb{K}\eta_0)\|_\infty\\
\leq& C_1[\{\|\mathbb{K}_h(\eta_h^M-\eta_0)\|_\infty+\|\mathbb{L}_h(\eta_h^M-\eta_0)\|_\infty\}+\{\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h-\mathbb{L})\eta_0\|_\infty\}]\\
\leq& C(A_1+A_2)\mathcal{L}\|\eta_h^M-\eta_0\|_\infty^2+\|\mathbb{K}_h(\mathbb{I}-\mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty\\
&+\|\mathbb{K}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty+\|\mathbb{L}_h(\mathbb{I}-\mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\\
&+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty+\|\mathbb{L}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty\\
\leq& CA\mathcal{L}\|\eta_h^M-\eta_0\|_\infty^2+\|\mathbb{K}_h(\mathbb{I}-\mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty\\
&+\|\mathbb{K}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty+\|\mathbb{L}_h(\mathbb{I}-\mathbb{Q}_h)\|_\infty\|[(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\\
&+(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0]\|_\infty+\|\mathbb{L}_h\mathbb{Q}_h[(\mathbb{K}_h-\mathbb{K})\eta_0+(\mathbb{L}_h-\mathbb{L})\eta_0]\|_\infty. \qquad (3.37)
\end{aligned}$$

From the Lemma 3.2, we have

$$\begin{aligned}
&\|(\mathbb{K}_h\mathbb{Q}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h\mathbb{Q}_h-\mathbb{L})\eta_0\|_\infty\\
=&\mathbb{O}(h^{\min(d+1,\ m+m_1)})+\mathbb{O}(h^{\min(d+1,\ m+m_2)})\\
=&\mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}). \qquad (3.38)
\end{aligned}$$

And by Proposition 2.1, we have

$$\begin{aligned}
\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty+\|(\mathbb{L}_h-\mathbb{L})\eta_0\|_\infty&=\mathbb{O}(h^{d+1})+\mathbb{O}(h^{d+1})\\
&=\mathbb{O}(h^{d+1}). \qquad (3.39)
\end{aligned}$$

Next using the estimates (3.38)-(3.39) and Lemma 2.1 in Eq. (3.37), we obtain

$$\|\tilde{\eta}_h^M-\eta_0\|_\infty=\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}). \qquad (3.40)$$

Now from the Eqs. (2.8) and (2.40), we get

$$\begin{aligned}
\|\tilde{z}_h^M-z_0\|_\infty&=\|\mathbb{K}_h\tilde{\eta}_h^M-\mathbb{K}\eta_0\|_\infty\\
&=\|(\mathbb{K}_h\tilde{\eta}_h^M-\mathbb{K}_h\eta_0+\mathbb{K}_h\eta_0-\mathbb{K}\eta_0\|_\infty\\
&\leq\|(\mathbb{K}_h(\tilde{\eta}_h^M-\eta_0)+(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty\\
&\leq\|(\mathbb{K}_h(\tilde{\eta}_h^M-\eta_0)\|_\infty+\|(\mathbb{K}_h-\mathbb{K})\eta_0\|_\infty\\
&=\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)})+\mathbb{O}(h^{d+1})\\
&=\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}).
\end{aligned}$$

Hence proved the Theorem. □

Remark 3.1. According to Theorems 3.5 and 3.6, it is clear that by selecting a quadrature rule with $d+1 \geq \max\{m+2m_1, m+2m_2\}$, then we achieve the convergence rates in discrete multi-Galerkin and its iterated version as form:

$$\|z_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(\ m+m_1,\ m+m_2)}),$$
$$\|\tilde{z}_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(\ m+2m_1,\ m+2m_2)}).$$

The iterated discrete multi-Galerkin method exhibits superior convergence rates than the discrete multi-Galerkin method.

Remark 3.2. When employing linear piecewise ($r = 1$) polynomials for approximation, the choice of the Gauss two-point quadrature rule becomes crucial, as it ensures a precision degree $d = 4$.

4. Numerical results

Here, some numerical aspects are given to validate our theoretical outcomes. To tackle this problem using discrete multi-Galerkin methods, initially, we choose linear ($r = 1$) piecewise polynomials as approximation subspace $\mathbb{X}_h$. For mathematical simulation, we use a PC equipped with an Intel(R), 8.00 GB-RAM, Core $i5-3470$ CPU@2.10 GHz processor, 64-bit operating system, and Matlab (R2015a). The solutions acquired through the discrete multi-Galerkin method and its iterated version are given in the infinity norm, along with their associated errors and convergence rates. If we consider the approximation in the discrete multi-Galerkin method and its iterated version as z_h^M and $\tilde{z}_h^M$, respectively, then we assume that $\|z_h^M - z_0\|_\infty = \mathcal{O}(h^\alpha)$ and $\|\tilde{z}_h^M - z_0\|_\infty = \mathcal{O}(h^\beta)$, where z_0 is the unique solution.

Example 4.1. [40] Let us consider the given derivative-dependent BVP

$$(z'(t))' = -t^{\lambda-2}(\lambda t e^z z' + \lambda(\lambda-1)e^z),$$

with boundary conditions

$$z(0) = \ln(\frac{1}{4}) \ \text{ and } z(1) = \ln(\frac{1}{5}), \quad \lambda > 0,$$

gives the corresponding Hammerstein integral equation as

$$z(t) = f(t) + \int_0^1 \mathcal{K}(t,s)\psi(s, z(s), z'(s))\, ds, \quad 0 \leq t \leq 1,$$

with right side function $f(t) = \ln(\frac{4}{5})t + \ln(\frac{1}{4})$, $\psi(s, z(s), z'(s)) = -t^{\lambda-2}(\lambda t e^z z' + \lambda(\lambda-1)e^z)$, $z(t) = \ln(\frac{1}{4+t^2})$ and kernel

$$\mathcal{K}(t,s) = \begin{cases} t(s-1), \forall\ t \in [0,s], \\ s(t-1), \forall\ t \in [s,1]. \end{cases}$$

Since we are assuming $(n+1)$ dimensional linear ($r = 1$) piecewise polynomials as approximation subspaces. Then, for the discrete multi-Galerkin method, we get the expected convergence rates as

$$\|z_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}),$$

and the iterated discrete multi-Galerkin method, we get the expected convergence rates as

$$\|\tilde{z}_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}),$$

for $\mu = 0$ and $k_1 \geq 2$, we get $m = 2$, $m_1 = 2$ and $m_2 = 1$. Since the degree of precision $d = 4$. Hence for $r = 1$, then the obtained convergence rates are $\alpha = 3$ and $\beta = 4$, shown in Table 1.

Table 1. Error bounds and corresponding expected order of convergence in z_n^M and $\tilde{z}_n^M$, when $\lambda = 2$.

n	$\|z_h^M - z_0\|_\infty$	α	$\|\tilde{z}_h^M - z_0\|_\infty$	β
2	1.32×10^{-4}	-	3.11×10^{-4}	-
4	1.59×10^{-5}	3.05	1.92×10^{-5}	4.01
8	1.81×10^{-6}	3.13	1.22×10^{-6}	3.97
16	2.21×10^{-7}	3.03	7.18×10^{-8}	4.08
32	2.89×10^{-8}	2.93	4.47×10^{-9}	4.00
64	4.10×10^{-9}	2.81	2.69×10^{-10}	4.05

In Example 4.1, we will compare our proposed methods with the discrete Galerkin method and its iterated version presented by Kant et al. [22]. Here, we use the quadrature rule with the precision degree of $d = 4$. As a result, for $r = 1$ the expected convergence rates are $\alpha = 2$ and $\beta = 3$, respectively (see Table 2). The comparison of the proposed methods with the existing methods shown in Tables 1 and 2 is illustrated in Figure 1.

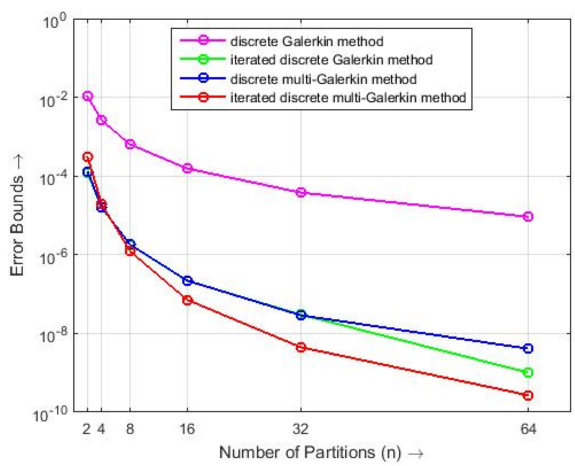

Figure 1. Comparison of errors among proposed methods with the discrete Galerkin and iterated discrete Galerkin methods.

Example 4.2. [22] Let us consider the given derivative-dependent BVP

$$(z'(t))' = -z'e^z,$$

Table 2. [22] Error bounds and corresponding expected order of convergence in z_n and $\tilde{z}_n$, when $\lambda = 2$.

n	$\|z_h - z_0\|_\infty$	α	$\|\tilde{z}_h - z_0\|_\infty$	β
2	1.06×10^{-2}	-	1.21×10^{-4}	-
4	2.60×10^{-3}	2.03	1.56×10^{-5}	2.95
8	6.34×10^{-4}	2.03	1.79×10^{-6}	3.12
16	1.55×10^{-4}	2.02	2.18×10^{-7}	3.03
32	3.79×10^{-5}	2.03	3.01×10^{-8}	2.85
64	9.22×10^{-6}	2.04	1.01×10^{-9}	2.81

with boundary conditions

$$z(0) = \ln \frac{1}{2} \text{ and } z(1) = \ln \frac{1}{3},$$

gives the corresponding Hammerstein integral equation as

$$z(t) = g(t) + \int_0^1 \mathcal{K}(t,s)\psi(s, z(s), z'(s))\, ds, \quad 0 \le t \le 1,$$

with right side function $g(t) = t \ln \frac{2}{3} + \ln \frac{1}{2}$, $\psi(s, z(s), z'(s)) = -z' e^z$, $z(t) = \ln(\frac{1}{2+t})$ and kernel

$$\mathcal{K}(t,s) = \begin{cases} t(s-1), \forall\ t \in [0,s], \\ s(t-1), \forall\ t \in [s,1]. \end{cases}$$

Since we are assuming $(n+1)$ dimensional linear $(r=1)$ piecewise polynomials as approximation subspaces. Then, for the discrete multi-Galerkin method, we get the expected convergence rates as

$$\|z_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+m_1,\ m+m_2)}),$$

and the iterated discrete multi-Galerkin method, we get the expected convergence rates as

$$\|\tilde{z}_h^M - z_0\|_\infty = \mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)}),$$

for $\mu = 0$ and $k_1 \ge 2$, we get $m = 2$, $m_1 = 2$ and $m_2 = 1$. Since the degree of precision $d = 4$. Hence for $r = 1$, then the obtained convergence rates are $\alpha = 3$ and $\beta = 4$, shown in Table 3.

Table 3. Error bounds and corresponding expected order of convergence in z_n^M and $\tilde{z}_n^M$, when $\lambda = 2$.

n	$\|z_h^M - z_0\|_\infty$	α	$\|\tilde{z}_h^M - z_0\|_\infty$	β
2	1.13×10^{-4}	-	1.42×10^{-4}	-
4	1.50×10^{-5}	2.92	1.05×10^{-5}	3.75
8	1.92×10^{-6}	2.96	7.70×10^{-7}	3.76
16	2.42×10^{-7}	2.98	5.42×10^{-8}	3.82
32	3.01×10^{-8}	3.00	3.52×10^{-9}	3.94
64	4.10×10^{-9}	2.87	2.24×10^{-10}	3.97

Similar for Example 4.2, we will compare our proposed methods with the discrete Galerkin method and its iterated version [22]. We use the quadrature rule with precision degree of $d = 4$ and obtain corresponding expected convergence rates $\alpha = 2$ and $\beta = 3$ for $r = 1$ (see Table 4), with comparison shown in Figure 2.

Table 4. [22] Error bounds and corresponding expected order of convergence in z_n and $\tilde{z}_n$, when $\lambda = 2$.

n	$\|z_h - z_0\|_\infty$	α	$\|\tilde{z}_h - z_0\|_\infty$	β
2	4.50×10^{-3}	-	1.04×10^{-4}	-
4	1.20×10^{-3}	1.90	1.41×10^{-5}	2.87
8	3.08×10^{-4}	1.96	1.84×10^{-6}	2.93
16	7.68×10^{-5}	2.00	2.34×10^{-7}	2.96
32	1.88×10^{-5}	2.02	2.83×10^{-8}	3.06
64	4.61×10^{-6}	2.03	3.50×10^{-9}	3.01

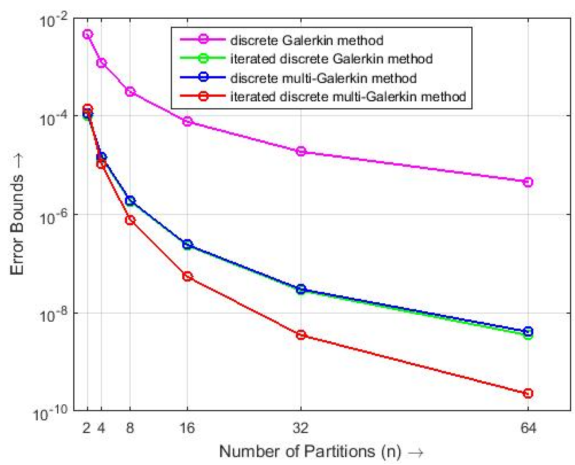

Figure 2. Comparison of errors among proposed methods with the discrete Galerkin and iterated discrete Galerkin methods.

5. Conclusion

In this article, we have studied the discrete multi-Galerkin method and its iterated version for solving the derivative-dependent nonlinear Hammerstein type Fredholm integral equations with Green's kernels. The main advantage of these methods is their ability to achieve superconvergence and high accuracy efficiently. We obtained superconvergence result for the iterated discrete multi-Galerkin method with the rate of convergence $\mathbb{O}(h^{\min(d+1,\ m+2m_1,\ m+2m_2)})$. The comparison of errors among proposed methods with the discrete Galerkin method and its iterated version presented by Kant et al. [22], as illustrated in Figures 1 and 2. These Figures show that the proposed discrete multi-Galerkin method, along with its iterated version, achieves higher accuracy compared to discrete Galerkin and iterated discrete Galerkin methods. Finally, numerical results confirm our theoretical findings and demonstrate that the proposed approach is computationally more efficient than previous methods.

These methods can be extended to solve derivative-dependent nonlinear systems, such as Hammerstein and Urysohn integral equations with Green's kernels through certain modifications. Moreover, these methods can be further improved by employing approximation using Jacobi or Legendre polynomials. In the future, collocation and multi-collocation methods could be studied for solving derivative-dependent nonlinear Hammerstein-type Fredholm integral equations with Green's kernels.

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