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A NEW SMOOTHING FUNCTION TECHNIQUE FOR SOLVING MINIMAX PROBLEMS

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Abstract In this study, we consider non-smooth finite minimax problems. A new approach for solving minimax problems is developed, employing indicator functions and smoothing functions. First, the formulation of minimax problems is revised using indicator functions. Then, a new generation smoothing technique is used for the revised formulation. An algorithm is developed to solve the revised and smoothed problems numerically. The efficiency of the algorithm is demonstrated on several test problems, and a comparison is conducted between the numerical results achieved and those of alternative approaches. Finally, the portfolio planning problem is considered as a real-life application, and satisfactory results are obtained.

Keywords Minimax problem, non-smooth optimization, smoothing technique.

MSC(2010) 90C30, 65K05, 65D10.

1. Introduction

We consider the following minimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1.1}$$

where

$$f(x)=\max_{j\in J} f_j(x) \tag{1.2}$$

and $f_j:\mathbb{R}^n\to\mathbb{R}$, $j\in J=\{1,2,\dots,m\}$ are continuously differentiable. The different versions of the problem (1.1) have been considered for many papers [1,13,22] and appear in many application areas such as engineering design [31], vehicle routing [2], resource-allocation [12], portfolio selection [26], the problem of multi-model regulatory networks under polyhedral uncertainty [27] and etc. [11,14,19,25].

The problem (1.1) is difficult to solve since the objective function defined in (1.2) may be non-differentiable [9]. Many algorithms have been developed in order to solve the problem (1.1) such as sub-gradient based methods [17], bundle-methods [16], homotopy methods [46] and smoothing methods [32,37,47].

In particular, we concentrate on smoothing techniques for non-smooth functions. Smoothing techniques provide an opportunity to use the existing gradient-based

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methods in solving finite minimax problems [8]. The smoothing techniques have been considered for min-max type problems [20,21] The idea of the smoothing approaches is based on approximating the original, non-smooth functions by using smooth functions [7,18,33]. The approximation is controlled by adjustable parameters. There are two important classes of smoothing techniques. The first technique is called local smoothing, which is based on smoothing out the original function in a suitable neighborhood of the kink points. The second technique, known as global smoothing, relies on building smooth functions that approximate the original function across the entire domain.

Developing a smoothing function for the mathematical function $f(x)$ described in equation (1.2) is a difficult task since it has many kink points. To address these challenges, alternative formulations have been suggested. Chronologically, we list some of them. In [5], the function $f(x)$ is restated as follows:

$$\begin{aligned} f(x) = f_1(x) + \max\{f_2(x) - f_1(x) + \max\{\dots \max\{f_{m-1}(x) - f_{m-2}(x) \\ + \max\{f_m(x) - f_{m-1}(x), 0\}, 0\} \dots, 0\}, 0\}, \end{aligned} \tag{1.3}$$

and for the first time, one of the global smoothing approaches is proposed for solving minimax problems. One of the first local smoothing techniques is proposed in [45] for solving minimax problems by considering the form (1.3). However, the above formula is useful, but coding it using computer programs is again complicated when m is large. An alternative penalty form with a smooth approximation is stated in [40] as

$$F(x, \varepsilon) = \beta \ln \sum_{j=1}^{m} \exp\left(\frac{f_j(x)}{\varepsilon}\right), \tag{1.4}$$

where $\varepsilon > 0$ is a smoothing parameter. The formula (1.4) is efficiently used with many gradient-based algorithms [44]. However, when ε is too small ($\varepsilon \to 0$), the numerical stabilization is uncontrolled because of an exponential term. Another interesting formulation of $f(x)$ is given as

$$F(x, r) = r + \sum_{j=1}^{m} \max\{f_j(x) - r, 0\}, \tag{1.5}$$

by adding a new variable r and the relation

$$f(x) = \min_{r \in \mathbb{R}} F(x, r)$$

is proved by [3, 4, 15]. Moreover, the hyperbolic smoothing technique proposed by [35,39] is applied to solve minimax problems in [3,4] by considering the formula (1.5). In recent years, there has been considerable attention on smoothing methods, and new generation smoothing techniques have been proposed and successfully applied for many non-smooth problems [34, 38, 42, 43]. However, minimax problems have not been studied with these new generation smoothing techniques. In this study, we consider the formula (1.5) and reformulate it in order to make it possible to apply the new generation smoothing techniques to solve the problem (1.1). We modify the smoothing technique for minimax problems inspired by the paper [42] and introduce the useful properties of this smoothing technique. We propose a new algorithm to numerically solve the reformulated and smoothed problem. In order to show the efficiency of the algorithm, some numerical examples are considered.

The next section focuses on providing some preliminary knowledge about smoothing approaches. In Section 3, the formulation of the minimax problem is adapted for the new generation smoothing technique, and the convergence properties of the smoothing technique are investigated. In Section 4, we present the minimization algorithm in order to find an approximate solution for the problem (1.1). In Section 5, we apply the algorithms to the important test problems and a portfolio planning problem in order to evaluate the numerical performance of the proposed algorithm. The final section presents concluding remarks.

2. Preliminaries

Throughout the paper, $\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$ is used to denote the Euclidean norm in $\mathbb{R}^n$. The $L^1[a,b]-$norm is defined as

$$\|f\|_{L^1[a,b]} = \int_a^b |f(t)|dt,$$

where f is an integrable function. Moreover, x_k^* denotes the $k-$th local minimizer of f and x^* denotes the global minimizer.

The sub-differential of the function f at the point x_0 is defined as $\partial f(x_0) = \operatorname{conv}\{\nabla f_j(x_0) : j \in \{j \in \mathbb{N} : f_j(x_0) = f(x_0)\}\}$ where conv is a convex hull of a set. A point $x_0 \in \mathbb{R}^n$ is called a stationary point of f if $0 \in \partial f(x_0)$.

Definition 2.1. [8] Let h be a continuous function defined on $\mathbb{R}^n$ to R. The function $\tilde{h} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $h(x)$, if $\tilde{h}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed β, and for any $x \in \mathbb{R}^n$,

$$\lim_{y \to x, \varepsilon \to 0} \tilde{h}(y, \varepsilon) = h(x).$$

3. A new formulation of minimax problems and smoothing approach

In this section, we revise the formula of minimax problems given in (1.5) and we apply the smoothing technique to this new formulation.

By considering the technique in [41], let us re-define the function (1.5) as follows:

$$F(x) = r + \sum_{j=1}^{m} (f_j(x) - r)\chi_{A_j}(x), \tag{3.1}$$

where $\chi_{A_j}(x)$ function is the indicator function of the set A_j defined by

$$\chi_{A_j}(x) = \begin{cases} 0, \, x \notin A_j, \\ 1, \, x \in A_j, \end{cases}$$

where $A_j = \{x \in \mathbb{R}^n : f_j(x) - r \geq 0\}$ for $j = 1, 2, \ldots, m$. It is evident that the function $F(x)$ may exhibit a non-smooth structure. Indeed, the non-smoothness of $F(x)$

is originated from the existence of $\chi_{A_j}(x)$ since $f_j(x)$ is continuously differentiable for $j = 1, \dots, m$. The idea for eliminating this lack is that if the indicator function $\chi_{B_{ji}}(x)$ is smoothed, then the function $F(x)$ becomes smooth. First, we define the smoothing function for indicator functions.

Definition 3.1. Let h be a semi-continuous function (upper or lower) defined on $\mathbb{R}$ to R. The function $\tilde{g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $g(t)$, if $\tilde{h}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed ε, and for any $t \in \mathbb{R}$,

$$\lim_{z \to t, \beta \to 0} \tilde{g}(z, \varepsilon) = h(t).$$

For $t_j = f_j(x) - r$, we re-define the indicator functions as

$$\chi_{A_j}(t) = \begin{cases} 0, \, t_j < 0, \\ 1, \, t_j \geq 0. \end{cases}$$

In the following we give the smoothing function of indicator function as

$$\tilde{\chi}_{A_j}(t, \varepsilon) = \begin{cases} 0, & t \leq -\varepsilon, \\ Q(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ 1, & t \geq \varepsilon, \end{cases} \tag{3.2}$$

where $Q(t_j, \varepsilon) = \frac{3}{16\varepsilon^5} t_j^5 - \frac{10}{16\varepsilon^3} t_j^3 + \frac{15}{16\varepsilon} t_j + \frac{1}{2}$ and $\varepsilon > 0$. The function $Q(t_j, \varepsilon)$ is called smooth transition function. It is designed in order to supply twice continuously differentiability between the pieces of the indicator function. Therefore, $\chi_{A_j}(t_j, \varepsilon)$ is second-order continuously differentiable. We have

$$\tilde{\chi}'_{A_j}(j, \varepsilon) = \begin{cases} 0, & t_j \leq -\varepsilon, \\ Q'(t_j, \varepsilon), & -\varepsilon \leq t_j \leq \varepsilon, \\ 0, & t_j \geq \varepsilon, \end{cases} \tag{3.3}$$

where $Q'(t_j, \varepsilon) = \frac{15}{16\varepsilon^5} t_j^4 - \frac{30}{16\varepsilon^3} t_j^2 + \frac{15}{16\varepsilon}$ and

$$\tilde{\chi}''_{A_j}(t_j, \varepsilon) = \begin{cases} 0, & t_j \leq -\varepsilon, \\ Q''(t_j, \varepsilon), & -\varepsilon \leq t_j \leq \varepsilon, \\ 0, & t_j \geq \varepsilon, \end{cases} \tag{3.4}$$

where $Q''(t_j, \varepsilon) = \frac{15}{4\varepsilon^5} t_j^3 - \frac{15}{4\varepsilon^3} t_j$.

In the following lemmas, we investigate the relation between $\chi_{A_j}(t)$ and its smoothing function $\tilde{\chi}_{A_j}(t, \varepsilon)$.

Lemma 3.1. *Assume that $\chi_{A_j}(t_j)$ is an indicator function of the set $A_j \subset \mathbb{R}^n$ and $\tilde{\chi}_{A_j}(t_j, \varepsilon)$ is a smoothing function of $\chi_{A_j}(t_j)$. Then, we have*

$$|\tilde{\chi}_{A_j}(t_j, \varepsilon) - \chi_{A_j}(t_j)| \leq \frac{1}{2},$$

for any $\varepsilon > 0$.

Proof. Since we have $\tilde{\chi}_{A_j}(t_j,\varepsilon) = \chi_{A_j}(t_j)$ for $t_j \leq -\varepsilon$ and $t_j \geq \varepsilon$, we discuss the cases $-\varepsilon \leq t_j \leq 0$ and $0 \leq t_j \leq \varepsilon$. For $-\varepsilon \leq t_j \leq 0$, we obtain

$$\left|\tilde{\chi}_{A_j}(t_j,\varepsilon) - \chi_{A_j}(t_j)\right| = |Q(t_j,\varepsilon)| \leq \frac{1}{2},$$

and for $0 \leq t_j \leq \varepsilon$

$$\left|\tilde{\chi}_{A_j}(t_j,\varepsilon) - \chi_{A_j}(t_j)\right| = |Q(t_j,\varepsilon) - 1| \leq \frac{1}{2}.$$

Therefore, the proof is completed. □

Lemma 3.2. *Assume that $\chi_{A_j}(t_j)$ is an indicator function of the set $A_j \subset \mathbb{R}^n$ and $\tilde{\chi}_{A_j}(t_j,\varepsilon)$ is the smoothing function. Then, we have*

$$\|\tilde{\chi}_{A_j}(t_j,\varepsilon) - \chi_{A_j}(t_j)\|_{L^1(\mathbb{R})} \leq \frac{\varepsilon}{2},$$

for any $\varepsilon > 0$.

Proof. Since we have $\tilde{\chi}_{A_j}(t_j,\varepsilon) = \chi_{A_j}(t_j)$ for $t_j \leq -\varepsilon$ and $t_j \geq \varepsilon$, we deal with the case $-\varepsilon \leq t_j \leq \varepsilon$. For $-\varepsilon \leq t_j \leq \varepsilon$,

$$\begin{aligned}
\left\|\tilde{\chi}_{A_j}(t_j,\varepsilon) - \chi_A(t_j)\right\|_{L^1(\mathbb{R})} &= \int_{-\varepsilon}^{\varepsilon} \left|\tilde{\chi}_{A_j}(t_j,\varepsilon) - \chi_{A_j}(t_j)\right| dt \\
&= \int_{-\varepsilon}^{0} |Q(t_j,\varepsilon)|\, dt + \int_{0}^{\varepsilon} |Q(t_j,\varepsilon) - 1|\, dt \\
&= \frac{5\varepsilon}{32} + \frac{5\varepsilon}{32} \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Therefore, the proof is completed. □

Based on the new formulation and smoothing technique we define the smoothing function of the objective function $F(x)$ as

$$\tilde{F}(x,\varepsilon) = r + \sum_{j=1}^{m} t_j \chi_{A_j}(t_j,\varepsilon), \tag{3.5}$$

where $t_j = f_j(x) - r$ and the problem given in (1.1) is re-defined as

$$\min_{x\in\mathbb{R}^n} \tilde{F}\left(x,\varepsilon\right), \tag{3.6}$$

for $\varepsilon > 0$. First, we introduce the case $m = 2$ and obtain the following results.

Theorem 3.1. *Let $x \in \mathbb{R}^n$, $\varepsilon > 0$*

$$|F(x) - \tilde{F}(x,\varepsilon)| \leq \varepsilon.$$

Proof. Since $\tilde{\chi}_{A_1}(t_1,\varepsilon) = \chi_{A_1}(t_1)$ for $t_1 \leq -\varepsilon$ and $t_1 \geq \varepsilon$ and $\tilde{\chi}_{A_2}(t_2,\varepsilon) = \chi_{A_2}(t_2)$ for $t_2 \leq -\varepsilon$ and $t_2 \geq \varepsilon$, we concern with the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$ for $\varepsilon > 0$. Let us consider the case $t_1 \in [-\varepsilon,\varepsilon]$ and $t_2 \notin [-\varepsilon,\varepsilon]$

$$|F(x) - \tilde{F}(x,\varepsilon)|$$

$$
\begin{aligned}
&= \left|r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2) - \left(r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| \\
&= \left|t_1\chi_{A_1}(t_1) - t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)\right| \\
&\leq \frac{\varepsilon}{2}.
\end{aligned}
$$

Similar result is obtained for the case $t_1 \notin [-\varepsilon, \varepsilon]$ and $t_2 \in [-\varepsilon, \varepsilon]$. Now, we consider the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$. By considering the Lemma 3.1, we obtain

$$
\begin{aligned}
&\left|F(x) - \tilde{F}(x,\varepsilon)\right| \\
&= \left|r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2) - \left(r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| \\
&= \left|t_1\left(\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)\right) + t_2\left(\chi_{A_2}(t_2) - t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| \\
&\leq |t_1||\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)| + |t_2||\chi_{A_2}(t_2) - \tilde{\chi}_{A_2}(t_2,\varepsilon)| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

Thus, the proof is completed. □

Theorem 3.2. *Let $\varepsilon > 0$ and $x \in \mathbb{R}^n$*

$$
\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \leq 2\varepsilon^2.
$$

Proof. We start the proof similar to the Lemma 3.1. Since $\tilde{\chi}_{A_1}(t_1,\varepsilon) = \chi_{A_1}(t_1)$ for $t_1 \leq -\varepsilon$ and $t_1 \geq \varepsilon$ and $\tilde{\chi}_{A_2}(t_2,\varepsilon) = \chi_{A_2}(t_2)$ for $t_2 \leq -\varepsilon$ and $t_2 \geq \varepsilon$, we concern with the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$ for $\varepsilon > 0$. Let us consider the case $t_1 \in [-\varepsilon, \varepsilon]$ and $t_2 \notin [-\varepsilon, \varepsilon]$

$$
\begin{aligned}
&\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \\
&= \int_{-\varepsilon}^{\varepsilon} \left|r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2)\right. \\
&\quad \left. - \left(r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| dt \\
&= \int_{-\varepsilon}^{\varepsilon} \left|t_1\chi_{A_1}(t_1) - t_1\tilde{\chi}_{A_1}(t_1,\varepsilon)\right| dt \\
&= \int_{-\varepsilon}^{\varepsilon} |t_1| \left|\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)\right| dt.
\end{aligned}
$$

Since $|t_1| \leq \varepsilon$ and from Lemma 3.2, we have

$$
\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \leq \varepsilon^2.
$$

Similar result is obtained for the case $t_1 \notin [-\varepsilon, \varepsilon]$ and $t_2 \in [-\varepsilon, \varepsilon]$. Now, we consider the case $-\varepsilon \leq t_1, t_2 \leq \varepsilon$. By considering the Lemma 3.2, we obtain

$$
\begin{aligned}
&\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \\
&= \int_{-\varepsilon}^{\varepsilon} \left|r + t_1\chi_{A_1}(t_1) + t_2\chi_{A_2}(t_2) - \left(r + t_1\tilde{\chi}_{A_1}(t_1,\varepsilon) + t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| dt \\
&= \int_{-\varepsilon}^{\varepsilon} \left|t_1\left(\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)\right) + t_2\left(\chi_{A_2}(t_2) - t_2\tilde{\chi}_{A_2}(t_2,\varepsilon)\right)\right| dt \\
&\leq \int_{-\varepsilon}^{\varepsilon} |t_1||\chi_{A_1}(t_1) - \tilde{\chi}_{A_1}(t_1,\varepsilon)| dt + \int_{-\varepsilon}^{\varepsilon} |t_2||\chi_{A_2}(t_2) - \tilde{\chi}_{A_2}(t_2,\varepsilon)| dt
\end{aligned}
$$

$$\leq \varepsilon^2 + \varepsilon^2$$
$$= 2\varepsilon^2.$$

Thus, the proof is completed. □

These two theorems, namely 3.1 and 3.2, provide theoretical verification that the approach that has been proposed is a smoothing approach. To make it better understandable, the smoothing process, we will illustrate it with the following example:

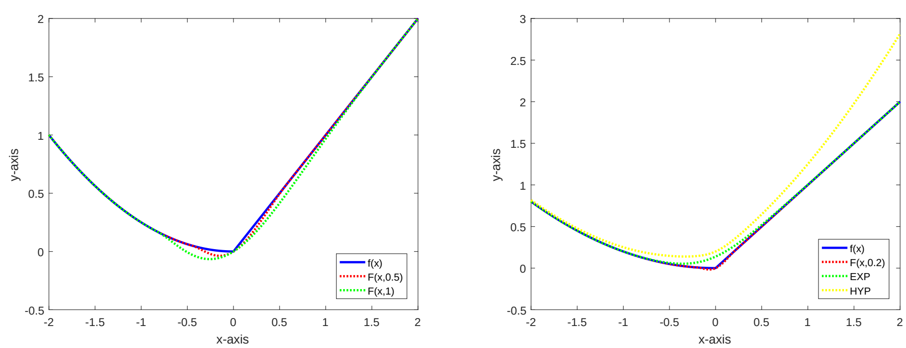
Example 3.1. Let the function f be defined as

$$f(x) = \max\{f_1(x), f_2(x)\},$$

where $f_1(x) = \frac{1}{5}x^2$ and $f_2(x) = x$. It can be observed from the definition that the function f is continuous but non-differentiable and $\partial f(0) = [0, 1]$. The point $x_0 = 0$ is considered to be the stationary point in accordance with the conceptual framework of the sub-differential. It is possible to see the graph of the function f by taking into consideration the max function of f_1 and f_2, which is depicted in Figure 1 (blue and solid). By utilizing the smoothing technique described above, the smoothing function $\tilde{F}(x, \varepsilon)$ of f may be obtained as follows:

$$\tilde{F}(x, \varepsilon) = r + (f_1(x) - r)\tilde{\chi}_{A_1}(t_1, \varepsilon) + (f_2(x) - r)\tilde{\chi}_{A_2}(t_2, \varepsilon),$$

where $A_1 = \{x \in \mathbb{R} : f_1(x) - r \geq 0\}$, $A_2 = \{x \in \mathbb{R} : f_2(x) - r \geq 0\}$ for $x \in \mathbb{R}$. The graph of the function $F(x, \varepsilon)$ is depicted in Fig. 1 (a) (red and dotted) when the value of r is set to zero. In fact, with the assistance of the smoothing technique described above, we are able to acquire an outer approximation to the original function. By deducing that the inequality $f(x) = F(x) \geq \tilde{F}(x, \varepsilon)$ holds for every function $f(x) = \max\{f_1(x), f_2(x)\}$, we are able to establish that the inequality holds.



(a) (b)

Figure 1. (a) The blue graph is the graph of $f(x)$, the red and dotted one is the graph of $\tilde{F}(x, 0.5)$ and the green and dotted one is the graph of $\tilde{F}(x, 1)$, and (b) The blue graph is the graph of $f(x)$, the red one is the graph of $\tilde{F}(x, 0.2)$, the green one is the graph of exponential smoothing with $\varepsilon = 0.2$ and the yellow one is the graph of hyperbolic smoothing function with $\varepsilon = 0.2$.

In accordance with the findings presented in Figure 1 (a), selecting smaller values for ε results in more accurate approximations to the original function. In order to visually compare the smoothing functions that we discussed in the introduction, we have included them in a single framework and illustrated them in Figure 1(b). Our smoothing strategy yields the best approximation when all approaches use the same value of $\varepsilon = 0.2$.

Let us move forward with the presentation of the results concerning the degree of approximation of smoothing approach. Now, we will provide the results of convergence for any finite value of m.

Theorem 3.3. *Let* $x \in \mathbb{R}^n$, $\varepsilon > 0$

$$\left|F(x) - \tilde{F}(x,\varepsilon)\right| \leq \frac{m}{2}\varepsilon.$$

Proof. For any $x \in \mathbb{R}^n$, we have

$$\left|F(x) - \tilde{F}(x,\varepsilon)\right| = \left|r + \sum_{j=1}^{m} t_j \chi_{A_j}(t_j) - \left(r + \sum_{j=1}^{m} t_j \tilde{\chi}_{A_j}(t_j,\varepsilon)\right)\right|.$$

By considering the similar way of the proof of Theorem 3.1, we obtain

$$\left|F(x) - \tilde{F}(x,\varepsilon)\right| \leq \sum_{j=1}^{m} |t_j| \left|\chi_{A_j}(t_j) - \tilde{\chi}_{A_j}(t_j,\varepsilon)\right| \leq \frac{m\varepsilon}{2}.$$

Thus, the proof is completed. □

Theorem 3.4. *Let* $x \in \mathbb{R}^n$, $\varepsilon > 0$

$$\|\tilde{F}(x,\varepsilon) - F(x)\|_{L^1} \leq m\varepsilon^2.$$

Proof. The proof is obtained by following similar ways as Theorems 3.2 and 3.3. □

Theorem 3.5. *Suppose that the point x^* is an optimal solution for the problem* (1.1) *and $\overline{x}$ is an optimal solution for the problem* (3.5)*. Then,*

$$|F(x^*) - \tilde{F}(\overline{x},\varepsilon)| \leq \varepsilon.$$

Proof. Since $F(\bar{x}) \geq F(x^*) \geq \tilde{F}(\bar{x},\varepsilon)$ we have

$$|F(x^*) - \tilde{F}(\bar{x},\varepsilon)| \leq |F(\bar{x}) - \tilde{F}(\bar{x},\varepsilon)|.$$

With the help of Theorem 3.1 and 3.3, we obtain

$$|F(\bar{x}) - \tilde{F}(\bar{x},\varepsilon)| \leq \varepsilon.$$

It completes the proof. □

Theorem 3.6. *Let* $\{\varepsilon_j\} \to 0$ *and x^k be a solution of* (3.5)*. Assume that $\overline{x}$ is an accumulation point of* $\left\{x^k\right\}$*. Then $\overline{x}$ is an optimal solution for* (1.1)*.*

Proof. By considering the Theorem 3.5, the proof is obtained. □

4. Algorithm and minimization procedure

A new method that takes its inspiration from [3] is presented in this section for the purpose of solving the minimax problem that is defined in (1.1). For the purpose of solving the problem presented in equation (1.1), we suggest utilizing the smoothed form of the problem (3.6).

Algorithm I

Step 1 Choose an initial point x^0 and set $r_0 = f(x_0)$. Determine $\varepsilon_0 > 0$, $0 < q < 1$ and $\tau = 10^{-4}$ let $k = 0$ and go to Step 2.

Step 2 Consider x^k as an initial point to solve the problem (3.5) by using smooth optimization solver. Let x^{k+1} be the solution.

Step 3 If $\|\nabla \tilde{F}(x^k, \varepsilon_k)\| \leq \tau$ then stop and x^{k+1} is the optimal solution otherwise; determine $\varepsilon_{k+1} = q\varepsilon_k$, $r_{k+1} = f(x^{k+1})$ and $k = k + 1$, then go to Step 2.

We need the following assumption for convergence of the Algorithm I.

Assumption 4.1. *For a point x^0 consider the level set*

$$\mathcal{L}(x^0) = \left\{ x \in \mathbb{R}^n : f(x) \leq f(x^0) \right\}$$

is bounded.

The convergence of Algorithm I is stated by the following theorem:

Theorem 4.1. *Let Assumption 4.1 hold. Suppose the set*

$$\underset{x \in \mathbb{R}^n}{\mathit{argmin}}\ \tilde{F}(x, \varepsilon) \neq \emptyset,$$

for $\varepsilon \in (0, \varepsilon_0]$. Let x^k be generated by Algorithm I. If $\{x^k\}$ has an accumulation point, then the accumulation point of $\{x^k\}$ is the solution for (1.1).

Proof. Let us define the set $\mathcal{L}(x^0) = \left\{ x \in \mathbb{R}^n : f(x) \leq f(x^0) \right\}$ for starting point x_0. Since $\mathcal{L}(x^0)$ is bounded, the sequence $\{x^k\}$ has at least one accumulation point. Let $\overline{x}$ be an accumulation point of $\{x^k\}$. We first show that $\overline{x} \in \mathcal{L}(x_0)$. Since

$$\tilde{F}(x_0, \varepsilon) \geq \tilde{F}(x^k, \varepsilon),$$

and according to Theorem 3.4, we have $f(x_0) \geq f(x^k)$ and $x^k \in \mathcal{L}(x_0)$. Since $\mathcal{L}(x_0)$ is bounded we obtain $\overline{x} \in \mathcal{L}(x_0)$. By the Theorem 3.6, $\overline{x}$ is the solution for (1.1). $\square$

5. Numerical examples

This section is devoted to presenting the numerical results of *Algorithm I* with the smoothing approach on finite minimax problems. Moreover, the obtained results are compared with *Algorithm I* with exponential smoothing used in [23] and *Algorithm I* with hyperbolic smoothing used in [3, 39]. We consider the BFGS method a local search for *Algorithm I.* We apply the *Algorithm I* by using MATLAB on a PC with the configuration of an Intel Core i3 with 8GB of RAM. In this algorithm, the parameters are selected as $\varepsilon = 10^{-1}$ and $q = 10^{-1}$. It is accepted that the problem is solved if the accuracy of 10^{-4} with respect to the function value is obtained.

5.1. Test problems

We first consider the well known test problems given in [10, 24] and the obtained results are compared with the competing algorithm declared at above. The explicit formulas of test problems are presented as follows:

Problem 5.1. [24] $\min f(x) = \max_{1\le j\le 2} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$f_1(x) = x_1^2 + (x_2-1)^2 + x_2 - 1, \quad f_2(x) = -x_1^2 - (x_2-1)^2 + x_2 + 1,$$

the global minimum value of the objective function f *is* $f^* = 0$.

Problem 5.2. [24] $\min f(x) = \max_{1\le j\le 3} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$f_1(x) = x_1^2 + x_2^4, \quad f_2(x) = (2-x_1)^2 + (2-x_2)^2, \quad f_3(x) = 2\exp(x_2 - x_1),$$

the global minimum value of the objective function f *is* $f^* = 1.9522245$.

Problem 5.3. [24] $\min f(x) = \max_{1\le j\le 3} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$ *and*

$$f_1(x) = 5x_1 + x_2, \quad f_2(x) = -5x_1 + x_2, \quad f_3(x) = x_1^2 + x_2^2 + 4x_2,$$

the global minimum value of the objective function f *is* $f^* = -3$.

Problem 5.4. [24] $\min f(x) = \max_{1\le j\le 6} f_j(x)$ *where* $f : \mathbb{R}^3 \to \mathbb{R}$ *and*

$$\begin{aligned}
f_1(x) &= x_1^2 + x_2^2 + x_3^2 - 1,\\
f_2(x) &= x_1^2 + x_2^2 + (x_3-2)^2,\\
f_3(x) &= x_1 + x_2 + x_3 - 1,\\
f_4(x) &= x_1 + x_2 - x_3 + 1,\\
f_5(x) &= 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2,\\
f_6(x) &= x_1^2 - 9x_3,
\end{aligned}$$

the global minimum value of the objective function f *is* $f^* = 3.5997$.

Problem 5.5. [24] $\min f(x) = \max_{1\le j\le 4} f_j(x)$ *where* $f : \mathbb{R}^4 \to \mathbb{R}$ *and*

$$\begin{aligned}
f_1(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4,\\
f_2(x) &= f_1(x) + 10\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8\right),\\
f_3(x) &= f_1(x) + 10\left(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10\right),\\
f_4(x) &= f_1(x) + 10\left(2x_1^2 + 2x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5\right),
\end{aligned}$$

the global minimum value of the objective function f *is* $f^* = -44$.

Problem 5.6. [24] $\min f(x) = \max_{1\le j\le 5} f_j(x)$ *where* $f : \mathbb{R}^7 \to \mathbb{R}$ *and*

$$\begin{aligned}
f_1(x) = {} & (x_1-10)^2 + 5(x_2-12)^2 + x_3^4 + 3(x_4-11)^2 + 10x_5^6\\
& + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 + 8x_7,\\
f_2(x) = {} & f_1(x) + 10\left(2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127\right),
\end{aligned}$$

$$f_3(x) = f_1(x) + 10\left(7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282\right),$$
$$f_4(x) = f_1(x) + 10\left(23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196\right),$$
$$f_5(x) = f_1(x) + 10\left(4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7\right),$$

the global minimum value of the objective function f *is* $f^* = 680.63006$.

Problem 5.7. [10] $\min f(x) = \max\limits_{1\le j<m} f_j(x)$ *where* $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f_j(x) = x_1^2 + 2x_1t_j^2 + \exp(x_1 + x_2) - \exp(t_j),$$

and $t_j = \frac{j}{(q-1)}$, $j = 0, 1, \dots, m-1$. *The global minimum value of objective function* f *is* $f^* = -1$.

Problem 5.8. [10] $\min f(x) = \max\limits_{1\le i,j<m} f_{i,j}(x)$ *where* $f : \mathbb{R}^4 \to \mathbb{R}$,

$$f_{i,j}(x) = \frac{(t_i - x_i)^2}{x_3^2} + \frac{(r_j - x_2)^2}{x_4^2} - 4,$$

and $t_i = \frac{i}{\sqrt{m}-1}$, $t_j = \frac{j}{\sqrt{m}-1}$, $i, j = 0, 1, \dots, m-1$. *The global minimum value of the objective function* f *is* $f^* = -4$.

Table 1. The numerical results comparison with smoothing.

			ISA				ESA				HSA			
Problem No.	n	m	*iter*	*feval*	*f.val*	Time	*iter*	*f.eval*	*f.val*	Time	*iter*	*f.eval*	*f.val*	Time
1	2	2	10	93	−0.0000	**0.0131**	65	338	0.0000	0.0947	38	210	0.0000	0.0656
2	2	3	2	21	1.9522	**0.0233**	57	262	1.9582	0.0550	61	252	1.9523	0.0644
3	2	3	17	252	−3.0000	**0.0382**	17	153	−2.9940	0.0601	18	144	−3	0.0423
4	3	6	33	256	3.5997	**0.0563**	99	578	3.6030	0.0685	97	1220	3.5998	0.2136
5	4	4	36	640	−44.0000	**0.0864**	115	705	−43.832	0.1069	114	945	−43.99	0.1270
6	7	5	98	1832	680.3800	**0.1230**	236	2174	678.9000	0.1941	106	2528	693.02	0.3226
7	2	5	47	156	−1.0000	**0.0263**	42	193	−0.9999	0.0354	52	186	−0.9999	0.0362
7	2	10	57	195	−1.0000	**0.0424**	11	99	−1.0000	0.0471	56	222	−0.9999	0.0632
7	2	50	51	186	−1.0000	0.1062	16	105	−1.0000	**0.0672**	79	321	−1	0.2010
7	2	100	43	144	−1.0000	**0.4833**	35	186	−0.9888	0.6020	64	246	−0.9999	0.2951
7	2	500	36	156	−1.0000	**0.8912**	45	201	−0.9876	0.9571	61	216	−0.9999	0.9096
7	2	1000	50	228	−1.0000	7.2473	61	267	−0.9982	**1.1384**	58	231	−0.9999	1.7948
8	4	5	22	115	−4.0000	**0.0342**	24	175	−3.9868	0.0594	33	210	−4	0.0483
8	4	10	70	120	−4.0000	**0.0101**	18	145	−3.9650	0.0495	16	110	−3.9999	0.0427
8	4	50	13	184	−4.0000	0.2096	6	105	−3.9679	**0.0629**	36	255	−4	0.1703
8	4	100	54	432	−4.0000	0.7779	26	195	−3.9731	0.6396	31	210	−3.9999	**0.2575**
8	4	500	69	647	−4.0000	**1.9314**	44	385	−3.9660	1.8309	28	200	−4	1.9911
8	4	1000	96	859	−4.0000	5.7579	21	155	−3.8949	4.7227	17	145	−4	**3.6680**

The numerical results are reported in Table 1. In the table, the number dimension "n" and the number of functions "m" for each of the problems are presented. We illustrate the results on total iteration numbers "iter", total function evaluations "feval", function values "f.val", and the CPU time in seconds "Time" obtained by using Algorithm I with our formulation and smoothing approach (ISA) in the left side of the Table 1. In the rest of Table 1, we show the results in the same groups as our method for exponential smoothing (in [23]) with Algorithm I (ESA) and hyperbolic smoothing (in [3]) with Algorithm I (HSA). We consider the same starting points for both algorithms, selecting them randomly.

It can be seen from Table 1 that ISA presents better results than ESA and HSA at the rate of 50% considering all test problems in terms of the total number of iterations. In terms of total function evaluations, ISA presents better results than the ESA and HSA at the rate of 56% considering all test problems. Moreover, ISA and HSA, with the exception of Problem 6, yield correct solutions for all test problems, while ESA fails to achieve the desired results, with the exception of Problem 5. Moreover, if anyone compares ISA with the ESA and HSA in terms of CPU time, it is seen that ISA is faster than the ESA and HSA at the rate of 72% considering all test problems. However, the exponential term makes the ESA difficult to use. When the smoothing parameter $\varepsilon \to 0^+$ again, the exponential function $\exp(\frac{f_j(x)}{\varepsilon})$ reaches huge values. Therefore, the function "fminunc" gives an error and cannot continue. While the HSA is easy to control and can yield results with desired precision, it is slower than the ISA. We can conclude that the ISA is very easy to use and has no drawbacks, unlike the ESA.

5.2. Application of *Algorithm I* in portfolio planning problem

In this section, a minimax portfolio planning model is considered. The problem is first defined by [30] and it is reformulated by Cai et al. in [6] and Teo et al. in [36]. The final of this problem is given in [26]. The problem is mathematically formulated as follows:

$$
\begin{aligned}
& \min f(x) = \frac{1}{12}\sum_{t=1}^{12} y_t, \qquad (5.1)\\
& s.t.,\\
& Ax \leq y,\\
& 0.0207x_1 + 0.0316x_2 + 0.0323x_3 + 0.0337x_4 + 0.0376x_5 \geq 0.03,\\
& \sum_{j=1}^{5} x_j = 1,\\
& 0 \leq x_j \leq 0.75 \quad j = 1, 2, \ldots, 5,\\
& y_i \geq 0 \quad i = 1, 2, \ldots, 12,
\end{aligned}
$$

where $x = (x_1, x_2, \ldots, x_5)^T$ is the decision variable, $y = (y_1, y_2, \ldots, y_{12})^T$ and A is a 12×5 matrix given as

$$
A = \begin{pmatrix}
0.0333 & 0.0004 & 0.0083 & 0.0043 & 0.0114\\
0.0243 & 0.0234 & 0.0203 & 0.0283 & 0.0294\\
0.0507 & 0.0676 & 0.0123 & 0.0707 & 0.0766\\
0.0387 & 0.0204 & 0.0087 & 0.0163 & 0.0134\\
0.0223 & 0.0154 & 0.0173 & 0.0313 & 0.0114\\
0.0263 & 0.0024 & 0.0003 & 0.0767 & 0.0006\\
0.0334 & 0.0314 & 0.0013 & 0.0283 & 0.0174\\
0.0153 & 0.0164 & 0.0113 & 0.0003 & 0.0126\\
0.0597 & 0.0066 & 0.0747 & 0.0013 & 0.0144\\
0.0637 & 0.0084 & 0.0113 & 0.0223 & 0.0176\\
0.0253 & 0.0044 & 0.0043 & 0.0233 & 0.0074\\
0.0313 & 0.0486 & 0.0037 & 0.0087 & 0.0024
\end{pmatrix}. \qquad (5.2)
$$

For more details of the problem, we refer to [26]. We consider the equivalent formulation of the problem (5.2), defined as

$$\min f(x) = \frac{1}{12}\sum_{t=1}^{12} \max_j A(t,j)x_j, \tag{5.3}$$

$$\begin{aligned}
&s.t.,\\
&0.0207x_1 + 0.0316x_2 + 0.0323x_3 + 0.0337x_4 + 0.0376x_5 \geq 0.03,\\
&\sum_{j=1}^{5} x_j = 1,\\
&0 \leq x_j \leq 0.75 \quad j = 1, 2, \ldots, 5,\\
&y_i \geq 0 \quad i = 1, 2, \ldots, 12.
\end{aligned}$$

By considering *Algorithm I*, the numerical solution of the problem is obtained as $x^* = (0.0000, 0.0000, 0.7500, 0.0000, 0.2500)^T$ with the corresponding function value $f(x^*) = 0.015$ which verifies the solution given in [26].

6. Conclusion

In this study, new generation smoothing techniques are successfully applied to the finite minimax problems. The formulation of minimax problems is revised based on the indicator functions. The error estimates are presented, and the relations between the original and smoothed problems are investigated in detail. This reformulation and suggested smoothing technique not only simplify the formulation of minimax problems but also provide a smooth approximation for such non-smooth problems.

A new algorithm for solving reformulated and smoothed finite minimax problems is presented, and the efficiency of our algorithm on some numerical examples is illustrated. According to the comparison of the results with the other methods, it is shown that our approach is competitive with well-known prestigious approaches.

For future studies, indicator functions can be used to derive effective formulations of minimax problems. The concept of employing new generation smoothing techniques utilizing indicator functions can also be extended to address various non-smooth problems, including complementarity, exact penalty, l_1 signal reconstruction, and so on. On the other hand, the methodology proposed in this article can be considered to solve the minimax part of the problem of the optimization of desirability functions under model uncertainty [27–29].

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