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EXACT SOLUTIONS OF TWO HIGH ORDER DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS: DYNAMICAL SYSTEM METHOD*

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Abstract For two generalization models of the first type derivative nonlinear Schrödinger (DNLSI) equation and the second type derivative nonlinear Schrödinger (DNLSII) equation, by using the method of dynamical systems to investigate the existence of exact explicit solutions with the form $q(x,t) = \phi(\xi)\exp[i(\kappa x - \omega t + \theta(\xi))], \xi = x - ct$. This paper show that in some given parameter conditions, explicit exact parametric representations of $\phi(\xi)$ and $\theta(\xi)$ can be given.

Keywords Bifurcation, exact solution, planar integrable system, generalization models of Kaup-Newell equation and Chen-Lee-Liu equation.

MSC(2010) 34C23, 35Q51-53, 58J55.

1. Introduction

It is well known that the nonlinear Schrödinger (NLS) equation is one of the most generic soliton equations, and arises from a wide variety of fields, such as quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied. Among them, there are three celebrated equations with derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger (DNLS) equations. One is the Kaup-Newell equation [7, 19, 37, 38, 46]:

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0, \tag{1.1}$$

which is usually called DNLSI equation. The second type is the Chen-Lee-Liu equation [2, 15, 22, 28]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0, \tag{1.2}$$

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which is called DNLSII equation. The last one takes the form [9]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 (q^*)^2 = 0, \tag{1.3}$$

which is called the Gerjikov-Ivanov (GI) equation or DNLSIII equation. In equation (1.3), q^* denotes the complex conjugation of q.

These equations have been studied by many authors [1, 5–8, 10–14, 16–18, 21, 24, 26, 27, 29–32, 35, 36, 39, 41–44]. As an important generalization of the Kaup-Newell model given in equation (1.1), in 2018, Triki and Biswas [34] introduced a novel class of DNLS equations given by

$$iq_t + aq_{xx} + ib(|q|^{2n} q)_x = 0, \tag{1.4}$$

which incorporates the non-Kerr dispersion term $(|q|^n q)_x$ for the case $n \geq 2$. This new model can be used as a basis for the description of the pulse propagation in highly nonlinear optical fibers beyond the Kerr limit. By considering the representation of the complex field $q(x,t)$ in the form: $q(x,t) = \rho(\xi) e^{i[\theta(\xi) - \omega t]}$, where $\rho(\xi)$ and $\theta(\xi)$ are real functions of the traveling coordinate $\xi = x - vt$. Here v is the wave velocity, and ω is the frequency of the wave oscillation. The authors of [34] demonstrated that this nonlinear wave equation offers a very rich model that supports envelope soliton solutions of different waveforms and shapes. In Saima [33], the author again studied Triki-Biswas equation (1.4) by using two strategic and efficient integration schemes. In [23], two first integrals were discovered for the Triki-Biswas equation, demonstrating its transformation into a nonlinear first-order ordinary differential equation. We shall show that their obtained exact solutions in [33, 34] and [23] are not complete.

As an generalization of the Chen-Lee-Liu equation (1.2), we introduce the equation

$$iq_t + aq_{xx} + ib|q|^{2n} q_x = 0, \tag{1.5}$$

where $n \neq 1$ and $n = 2, 3, 4, \cdots$.

We seek the exact explicit solutions of the above two equations with the form:

$$q(x,t) = \phi(\xi) \exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \quad \xi = x - ct, \tag{1.6}$$

where c is the wave velocity and $\phi(\xi), \theta(\xi)$ are two functions with variable ξ, the parameters κ and ω are constant.

The purpose of this paper is to study whether or not existence exact solutions in explicit form (1.6) for the above two equations (1.4) and (1.5) by using the method of bifurcation theory of dynamical systems.

(i) Substituting (1.6) into equation (1.4) and separating the real and imaginary parts, respectively, we have

$$a\phi'' = b\theta'\phi^{2n+1} + (2a\kappa - c)\theta'\phi + a(\theta')^2\phi + (a\kappa^2 - \omega)\phi + b\kappa\phi^{2n+1}, \tag*{$(1.7)_r$}$$

$$a\theta''\phi + 2a\theta'\phi' + (2a\kappa - c)\phi' + b(2n+1)\phi'\phi^{2n} = 0. \tag*{$(1.7)_i$}$$

Integrating $(1.7)_i$, it follows that $(2a\kappa - c)\phi + b\phi^{2n+1} + a\theta'\phi + a\int \theta' d\phi = C_1$, where C_1 is an integral constant. Taking $C_1 = 0$, we obtain

$$\theta' = \left(\frac{c}{2a} - \kappa\right) - \frac{b(2n+1)}{2a(n+1)}\phi^{2n}, \quad \theta(\xi) = \left(\frac{c}{2a} - \kappa\right)\xi - \frac{b(2n+1)}{2a(n+1)} \int \phi^{2n}(\xi) d\xi. \tag{1.8}$$

Substituting (1.8) into $(1.7)_r$, we obtain the following two order equation:

$$\phi'' + \frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right)\phi - \frac{bc}{2a^2}\phi^{2n+1} + \frac{(2n+1)b^2}{4a^2(n+1)^2}\phi^{4n+1} = 0. \tag{1.9}$$

Write that $\alpha_1 = -\frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right), \alpha_2 = \frac{bc}{2a^2}, \alpha_3 = -\frac{(2n+1)b^2}{4a^2(n+1)^2}$. Then, equation (1.9) is equivalent to the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \alpha_1\phi + \alpha_2\phi^{2n+1} + \alpha_3\phi^{4n+1} \tag{1.10}$$

with the first integral:

$$H(\phi, y) = \frac{1}{2}y^2 - \frac{1}{2}\alpha_1\phi^2 - \frac{\alpha_2}{2n+2}\phi^{2n+2} - \frac{\alpha_3}{4n+2}\phi^{4n+2}. \tag{1.11}$$

(ii) Substituting (1.6) into equation (1.5) and separating the real and imaginary parts, respectively, we obtain

$$a\phi'' - b\theta'\phi^{2n+1} - (2a\kappa - c)\theta'\phi - a(\theta')^2\phi - (a\kappa^2 - \omega)\phi - b\kappa\phi^{2n+1} = 0, \quad (1.12)_r$$

$$a\theta''\phi + 2a\theta'\phi' + (2a\kappa - c)\phi' + b\phi'\phi^{2n} = 0. \quad (1.12)_i$$

Integrating $(1.12)_i$, it follows that $(2a\kappa - c)\phi + \frac{b}{2n+1}\phi^{2n+1} + a\theta'\phi + a\int\theta' d\phi = C_2$, where C_2 is an integral constant. Taking $C_2 = 0$, we obtain

$$\theta' = \left(\frac{c - 2a\kappa}{2a}\right) - \frac{b}{2a(n+1)}\phi^{2n}, \quad \theta(\xi) = \left(\frac{c - 2a\kappa}{2a}\right)\xi - \frac{b}{2a(n+1)}\int\phi^{2n}(\xi)d\xi. \tag{1.13}$$

Substituting (1.13) into $(1.12)_r$, we obtain the following two order equation:

$$\phi'' + \frac{1}{a}\left(\frac{c^2}{4a} - c\kappa + \omega\right)\phi - \frac{bc}{2a^2}\phi^{2n+1} + \frac{(2n+1)b^2}{4a^2(n+1)^2}\phi^{4n+1} = 0. \tag{1.14}$$

Equation (1.9) and (1.14) have the same form, so we only need to further discuss system 10.

Let $\phi = \psi^{\frac{1}{2n}}$. Notice that $\phi' = \frac{1}{2n}\psi^{\frac{1}{2n}-1}\psi'$, $\phi'' = \frac{1}{2n}\left(\frac{1}{2n} - 1\right)\psi^{\frac{1}{2n}-2}(\psi')^2 + \frac{1}{2n}\psi^{\frac{1}{2n}-1}\psi''$. The substitution of the above expression into equation (1.9) yields the subsequent equation:

$$2n\psi\psi'' - (2n-1)(\psi')^2 - 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1) = 0. \tag{1.15}$$

Then, system (1.10) becomes that

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{(2n-1)y^2 + 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1)}{2n\psi}. \tag{1.16}$$

System (1.16) has the first integral

$$H_n(\psi, y) = y^2\psi^{-2+\frac{1}{n}} - 4n^2\psi^{\frac{1}{n}}\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right] = h. \tag{1.17}$$

Remark 1.1. The authors of [33,34] and [23] did not obtain system (1.16) and integral (1.17). They used the first integral (1.11) to make transformation. Therefore, obtained exact solutions of equation (1.4) by [33,34] and [23] are not complete.

Remark 1.2. We see from (1.17) that

$$y^2 = h\psi^{2-\frac{1}{n}} + 4n^2\psi^2\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right]. \tag{1.18}$$

By using the first equation of (1.16), we have

$$\xi = \int_{\psi_0}^{\psi} \frac{d\psi}{\sqrt{h\psi^{2-\frac{1}{n}} + 4n^2\psi^2\left[\frac{\alpha_3}{2n+1}\psi^2 + \frac{\alpha_2}{n+1}\psi + \alpha_1\right]}}. \tag{1.19}$$

It is evident that the integral (1.19) can only be integral when $h = 0$ for all $n \geq 2$. The case of $n = 1$ has already been addressed in [47].

Systems (1.16) is a four-parameter planar dynamical system depending on the parameter group $(n, \alpha_1, \alpha_2, \alpha_3)$. Since the parametric representations of the phase orbits defined by the vector fields of systems (1.16) give rise to all exact solutions with the form (1.6) of equation (1.4) and (1.5), we need to investigate the bifurcations of phase portraits for system (1.16) in the (ψ, y)-phase plane as the parameters are changed [3, 4, 20, 40, 45].

The rest of this paper is organized as follows. In section 2, the bifurcations of the phase portraits of systems (1.16) are studied. In section 3, in given parameter regions, corresponding to the bounded level curves defined by $H_n(\psi, y) = 0$, exact explicit parameter representations of system (1.16) are given. In section 4, we state the main conclusion of this paper.

2. Bifurcations of phase portraits of system (1.16)

Consider the associated regular system of system (1.16):

$$\frac{d\psi}{d\zeta} = 2n\psi y, \quad \frac{dy}{d\zeta} = (2n-1)y^2 + 4n^2\psi^2(\alpha_3\psi^2 + \alpha_2\psi + \alpha_1), \tag{2.1}$$

where $d\xi = 2n\psi d\zeta$, for $\zeta \neq 0$.

To find the equilibrium points of system (2.1), write that $f(\psi) = \alpha_1 + \alpha_2\psi + \alpha_3\psi^2$. Clearly, if $\Delta = \alpha_2^2 - 4\alpha_1\alpha_3 > 0$, then, $f(\psi)$ has zeros $\psi_1 = \frac{1}{2\alpha_3}(-\alpha_2 - \sqrt{\Delta})$, $\psi_2 = \frac{1}{2\alpha_3}(-\alpha_2 + \sqrt{\Delta})$. It is easy to see that if $\Delta > 0$, system (2.1) has three equilibrium points $O(0,0)$ and $E_j(\psi_j, 0), j = 1, 2$.

The assumption is made that at least one of ψ_1 and ψ_2 assumes a non-negative real value. Additionally, it can be observed that in equation (1.9), $\alpha_3 = -\frac{(2n+1)b^2}{4a^2(n+1)^2}$ is a real number which is less than or equal to zero.

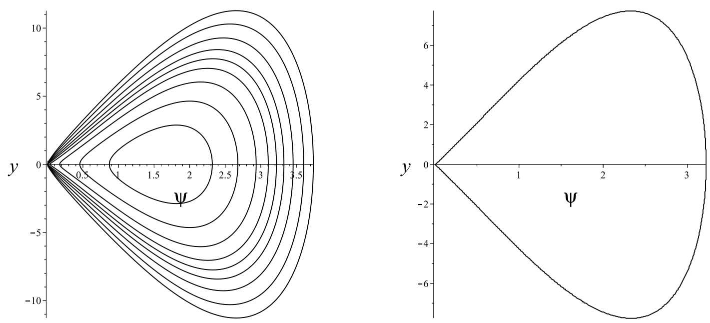
Let $M(\psi_j, 0)$ be the coefficient matrix of the linearized system of system (2.1) at the equilibrium point E_j. Let $J(\psi_j, 0)$ be its Jacobin determinant. Then, one has

$$J(0,0) = 0, \quad J(\psi_j, 0) = -8n^3\psi_j^3 f'(\psi_j).$$

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $(\mathbf{Trace}(M(\psi_j, 0)))^2 - 4J(\psi_j, 0) < 0$, then it is a center point; if $J > 0$ and $(\mathbf{Trace}(M(\psi_j, 0)))^2 - 4J(\psi_j, 0) > 0$, then it is a node; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp [25].

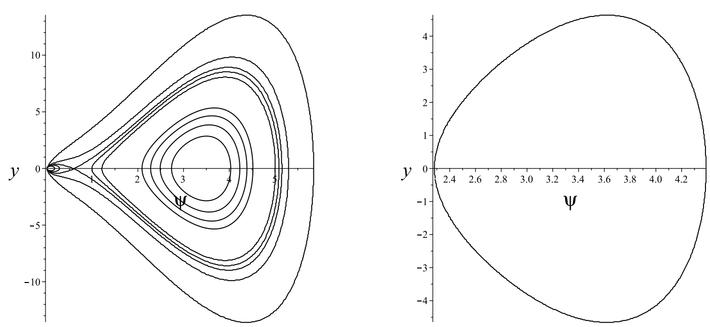
Now, write that $h_1 = H_n(\psi_1, 0) = \frac{2n^3\psi_1^{\frac{1}{n}}[\Delta+\alpha_2\sqrt{\Delta}-4n\alpha_1\alpha_3]}{(n+1)(2n+1)\alpha_3}$, $h_2 = H_n(\psi_2, 0) = \frac{2^{1-\frac{1}{n}}n^3\psi_2^{\frac{1}{n}}[-\Delta+\alpha_2\sqrt{\Delta}+4n\alpha_1\alpha_3]}{(n+1)(2n+1)\alpha_3}$. For a given pair (α_1, α_2), when $\alpha_3 = \frac{(2n+1)\alpha_2^2}{4(n+1)^2\alpha_1}$, we have $h_2 = 0$ or $h_1 = 0$.

Based on the above results, we obtain the bifurcations of the phase portraits of system (1.16) which are shown in Figure 1, Figure 2 and Figure 3.



(a) (b) The level curves $H_n(\psi, y) = 0$.

Figure 1. Bifurcations of phase portraits of system (1.16) when $\alpha_1 > 0, \alpha_2 \in R, \alpha_3 < 0$.



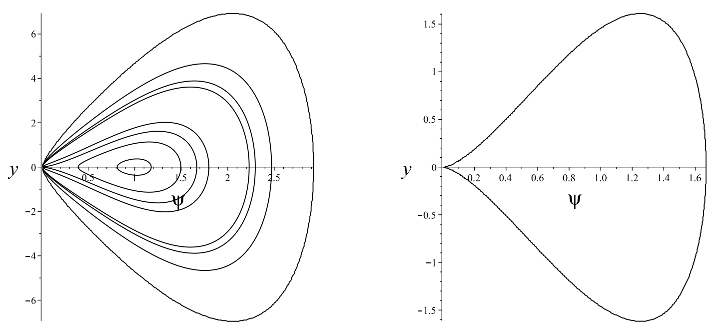
(a) $h_1 < 0 < h_2$ (b) The level curves $H_n(\psi, y) = 0$.

Figure 2. Bifurcations of phase portraits of (1.16) when $\alpha_1 < 0, \alpha_2 > 0$, $\frac{(2n+1)\alpha_2^2}{4(n+1)^2\alpha_1} < \alpha_3 < 0$.

3. Exact parametric representations of the level curves defined by $H_n(\psi, y) = 0$ of system (1.16)

3.1. Exact homoclinic solution of system (1.16) when $\alpha_1 > 0$, $\alpha_2 \in R$(R is any real number), $\alpha_3 < 0$ in Figure 1 (b).

Corresponding the level curves defined by $H_n(\psi, y) = 0$, there exist a homoclinic orbits of system (1.16) to the origin $O(0, 0)$, enclosing the singular point $E_1(\psi_1, 0)$. Now, (1.19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi = \int_{\psi}^{\psi_a} \frac{d\psi}{\psi\sqrt{(\psi_a-\psi)(\psi-\psi_b)}}$, $\psi_a > 0 > \psi_b$. It



(a) $\psi_2 = 0, h_1 = 0$

(b) The level curves defined by $H_n(\psi, y) = 0$.

Figure 3. Bifurcations of phase portraits of system (1.16) when $\alpha_1 = 0, \alpha_2 > 0, \alpha_3 < 0$.

gives rise to the following exact parametric representations of the homoclinic orbit of system (1.16):

$$\psi(\xi) = \frac{2\psi_a\psi_b}{(\psi_a + \psi_b) - (\psi_a - \psi_b)\cosh(\omega_1\xi)}, \tag{3.1}$$

where $\omega_1 = 2n\sqrt{\frac{|\alpha_3|\cdot\psi_a\cdot|\psi_b|}{2n+1}}$.

We see from (3.1) that for $\theta(\xi)$ in (8) and (1.13), we have

$$\begin{aligned} \int \phi^{2n}(\xi)d\xi &= \int \psi(\xi)d\xi \\ &= \int \left(\frac{2\psi_a\psi_b}{(\psi_a + \psi_b) - (\psi_a - \psi_b)\cosh(\omega_1\xi)}\right) d\xi \\ &= -\frac{\sqrt{2n+1}}{n\sqrt{|\alpha_3|}} \arctan\left(\sqrt{\frac{|\psi_b|}{\psi_a}} \tan\left(\frac{1}{2}\omega_1\xi\right)\right). \end{aligned} \tag{3.2}$$

3.2. Exact periodic solution of system (1.16) when $\alpha_1 < 0, \alpha_2 > 0, \frac{\alpha_2^2(2n+1)}{4\alpha_1(n+1)^2} < \alpha_3 < 0$ in Figure 2 (b).

Corresponding the level curves defined by $H_n(\psi, y) = 0$, there exists an periodic orbit of system (1.16), enclosing the singular point $E_1(\psi_1, 0)$. Now, (1.19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi = \int_{\psi_b}^{\psi} \frac{d\psi}{\psi\sqrt{(\psi_a-\psi)(\psi-\psi_b)}}$, $\psi_a > \psi_b > 0$. It gives rise to the following exact parametric representations of the periodic orbit of system (1.16):

$$\psi(\xi) = \frac{2\psi_a\psi_b}{(\psi_a + \psi_b) + (\psi_a - \psi_b)\cos(\omega_2\xi)}, \tag{3.3}$$

where $\omega_2 = 2n\sqrt{\frac{|\alpha_3|\psi_a\psi_b}{2n+1}}$.

We see from (3.3) that for $\theta(\xi)$ in (8) and (1.13), we have

$$\begin{aligned}\int \phi^{2n}(\xi)d\xi &= \int \psi(\xi)d\xi \\ &= \int \left(\frac{2\psi_a\psi_b}{(\psi_a+\psi_b)+(\psi_a-\psi_b)\cos(\omega_2\xi)}\right)d\xi \\ &= \frac{\sqrt{2n+1}}{n\sqrt{|\alpha_3|}}\arctan\left(\sqrt{\frac{\psi_b}{\psi_a}}\tan\left(\frac{1}{2}\omega_2\xi\right)\right).\end{aligned} \tag{3.4}$$

3.3. Exact homoclinic solution of system (1.16) when $\alpha_1 = 0, \alpha_2 > 0, \alpha_3 < 0$ in Figure 3 (b).

Corresponding the level curves defined by $H_n(\psi, y) = 0$, there exists a homocinic orbit of system (1.16), enclosing the singular points $E_1(\psi_1, 0)$. Now, (1.19)can be written as $2n\sqrt{\frac{|\alpha_3|}{2n+1}}\xi = \int_{\psi}^{\psi_M}\frac{d\psi}{\psi\sqrt{(\psi_M-\psi)\psi}}$. It follows the exact parametric representations of the homoclinic orbit of system(1.16):

$$\psi(\xi) = \frac{4\psi_M}{4+\omega_2^2\xi^2}, \tag{3.5}$$

where $\omega_2 = 2n\psi_M\sqrt{\frac{|\alpha_3|}{2n+1}}$.

We see from (3.5) that for $\theta(\xi)$ in (8) and (1.13), we have

$$\begin{aligned}\int \phi^{2n}(\xi)d\xi &= \int \psi(\xi)d\xi \\ &= \int \left(\frac{4\psi_M}{4+\omega_2^2\xi^2}\right)d\xi \\ &= \frac{2\psi_M}{\omega_2}\arctan\left(\frac{1}{2}\omega_2\xi\right).\end{aligned} \tag{3.6}$$

4. Conclusion

The main results in the present paper are summarized as follows.

Theorem 4.1. *Consider the solutions of equations (1.4) and (1.5) with the form* $q(x,t) = \phi(\xi)\exp[i(\kappa x - \omega t + \theta(\xi))]$. *Then, the following conclusions hold.*

(i) The function $\phi(\xi)$ *is the solutions of the planar Hamiltonian systems (1.10). The function* $\theta(\xi)$ *is given by (1.8) and (1.13), respectively.*

(ii) In order to find the exact solutions of (1.10), by making the transformation $\phi(\xi) = \psi^{\frac{1}{2n}}(\xi)$, *system (1.10) has evolved into system (1.16). System (1.16) has the bifurcations of phase portraits which are shown in Figure 1, Figure 2 and Figure 3.*

(iii) Corresponding to the bounded level curves defined by $H_n(\psi, y) = 0$, *system (1.16) has exact explicit solutions* $\psi(\xi)$ *given by (3.1), (3.3) and (3.5). The formulas (3.2), (3.4) and (3.6) give rise to the exact solutions for* $\theta(\xi)$ *in (1.8) and (1.13).*

(iv) Equation (1.4) and (1.5) have exact explicit envelope soliton solutions and envelope periodic solution.

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