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SIMULTANEOUS INVERSION OF THE SOURCE TERM AND INITIAL VALUE OF THE MULTI-TERM TIME FRACTIONAL SLOW DIFFUSION EQUATION*

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Abstract In this paper, the inverse problem of simultaneously identifying the source term and initial value for the multi-term time fractional diffusion equation is studied. We prove this problem is ill-posed, i.e. the solution (if it exists) does not continuous depend on measurement data. A standard Tikhonov regularization method is proposed to solve the inverse problem. In the case of a-priori and a-posteriori, we derive the error estimates between the exact solution and the regularized solution. Finally, we provide two examples to show the validity of the proposed method.

Keywords Simultaneous inversion, multi-term time fractional diffusion equation, ill-posed problem, regularization method, error estimate.

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1. Introduction

Let Ω be the bounded region on R^d $(d = 1, 2, \cdots)$ with fully smooth boundary $\partial\Omega$, and $T > 0$ be fixed. A time fractional diffusion equation with a power-law memory kernel is considered in this paper

$$\begin{cases} {}_0^{GL}D_t^{\alpha}u(x,t) + \displaystyle\int_0^t ker(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau = \Delta u(x,t) + F(x,t), & x \in \Omega, t \in (0,T], \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T], \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t_0) = \psi(x), & x \in \Omega, t_0 \in (0,T], \\ u(x,T) = g(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $ker(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ is the power-law memory kernel. The definition of the one-dimension Laplacian Δ is $\Delta u(x,t) = \frac{\partial^2 u}{\partial x^2}$, the notations ${}_0^{GL}D_t^{\alpha}$ is Grünwald-

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Letnikov fractional derivative of the order α $(0 < \alpha < 1)$ [13]

$$ {}_{a}^{GL}D_t^{\alpha}u(x,t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} u(x, t - jh), $$

where the $\binom{\alpha}{j}$ are the binomial coefficients

$$ \binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}. $$

It can be easily concluded that $\int_0^t ker(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau$ represents a time fractional derivative of order α $(0 < \alpha < 1)$ in the Caputo sense defined as [14]

$$ {}_{0}^{C}D_t^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}d\tau, $$

where $\Gamma(\cdot)$ represents the Gamma function. We assume the unknown source term $F(x,t) = f(x)q(t)$, here $q(t)$ is continuous for $t \in (0,T]$.

In problem (1.1), $f(x)$ and $\varphi(x)$ are unknown. The measurement data $u(x,t_0) = \psi(x)$ and $u(x,T) = g(x)$ are given to identify the unknown source $f(x)$ and initial value $\varphi(x)$. In practical problem, $\psi(x)$ and $g(x)$ are obtained through measurement, $\psi^\delta(x)$ and $g^\delta(x)$ are used to represent measured data, which satisfy

$$ \|\psi(\cdot) - \psi^\delta(\cdot)\| \leqslant \delta, \ \|g(\cdot) - g^\delta(\cdot)\| \leqslant \delta, \tag{1.2} $$

where $\|\cdot\|$ is the $L^2(\Omega)$ norm and $\delta > 0$ is the error level.

In recent decades, time-fractional diffusion equations have attracted widespread attention. From a physical perspective, this generalized diffusion equation is derived from fractional Fick's law, which describes the transfer process with long memory. Scholars have conducted extensive research on such issues [1,2,6,9,12,33]. However, the classical fractional diffusion model is not sufficient to simulate some anomalous diffusion. As a natural extension, the multi-term time fractional diffusion equation (MTTFDE) is proposed, which is expected to improve the modeling accuracy in depicting the anomalous diffusion. At present, MTTFDE have been widely applied in fields such as physics and viscoelastic material mechanics [3]. Apparently, the research of such models have become a new field.

Note that, in (1.1), the second term of the main equation has a kernel function, which has various choices, such as Mittag-Leffler type memory kernel, power-law type memory kernel, etc [24,25]. In this problem, we take power-law type memory kernel $ker(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, the problem becomes MTTFDE. At present, researchers are actively paying attention to such issues. In [21], the existence, uniqueness, and regularity of solutions for nonlinear two-term time fractional diffusion wave problems are given. In [18], Ren et al. studied the L1 approximation of multi-term time Caputo fractional derivatives and provided some efficient numerical schemes. Dehghan et al. [5] applied high order difference and Galerkin spectral method for the numerical solution of multi-term time fractional mixed diffusion-wave equations. In [20], Shen et al. derived the analytical solution for a two-dimensional multi-term time-fractional diffusion and diffusion-wave equation using the method of separation of variables and properties of the multivariate Mittag-Leffler function.

Currently, extensive research has been conducted on numerical method to obtain the solution for MTTFDE, but there is relatively little research on the inverse problem of such equation. Chang et al. [4] investigated an inverse source problem for MTTFDE. Sun et al. [23] studied the recover time-dependent potential function in a MTTFDE. Actually, much inverse problems are ill-posed and needs the regularization method to solve them. Thus several regularization methods are becoming more mature, such as Tikhonov method, Landweber iterative method, Quasi-boundary method and other regularization methods [28–30, 32, 34].

At present, most of the inverse problems of fractional diffusion equations only identify a single unknown parameter. The research on identifying two unknown parameters simultaneously is very limited. Compared to identifying single unknown term, it is more difficult to estimate the error of simultaneously identifying the source term and initial value. In recent decade, several Savants are gradually starting to identify several unknown parameters simultaneously for the fractional diffusion equation. Ruan et al. [19] considered simultaneously identifying the time-independent source terms and initial values of time fractional diffusion equations, and used Tikhonov method to solve the inverse problem. In [17], Ruan et al. studied the simultaneous inversion of fractional order and spatial source terms in time fractional diffusion equations, and transformed inverse problems into optimization problem, then proposed an algorithm to solve it. In [22], Sun et al. studied inversion of the fractional orders and the source term simultaneously of the MTTFDE. This inverse problem is nonlinear and author proposed a numerical method to obtain an approximate solution. Other inverse problem of identifying multiple unknown parameters can refer to [10, 26, 27]. At present, there is little research on simultaneously identifying the unknown source terms and initial values of MTTFDE.

For this reason, in our paper, we aim to identify the unknown source term and initial value simultaneously for MTTFDE. This inverse problem is ill-posed, and the Tikhonov regularization method is used to solve this inverse problem.

The remaining parts of this paper are as follows. Sections 2 provides several important definition and lemmas. In Section 3, we provide the solution to the problem and give the ill-posedness analysis of inverse problem. In Section 4, we use the Tikhonov regularization method to deal with this inverse problem and provide convergent estimates for both a-priori and a-posteriori cases, respectively. Several numerical examples are presented in Section 5 to illustrate the usefulness of the proposed method. Finally, we give some concluding remarks in Section 6.

2. Preliminary

In this section, we present some important Definition and Lemmas.

Definition 2.1. Let λ_n and X_n be the Dirichlet eigenvalues and eigenfunctions of $-\Delta$ on the domain Ω, satisfies

$$\begin{cases} \Delta X_n(x) = -\lambda_n X_n(x), & x \in \Omega, \\ X_n(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$, $\lim_{n\to\infty} \lambda_n = +\infty$ and $X_n(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, then $\{X_n\}_{n=1}^{\infty}$ can be normalized as the orthonormal basis in space $L^2(\Omega)$.

Definition 2.2. For arbitrary $p > 0$, we define the space as follows

$$D((-\Delta)^p) := \Big\{\phi \in L^2(\Omega)\Big| \sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2 < \infty\Big\}, \tag{2.2}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and $D((-\Delta)^p)$ is a Hilbert space with the norm

$$\|\phi\|_{D((-\Delta)^p)} := \Big(\sum_{n=1}^{\infty} \lambda_n^p |(\phi, X_n)|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

Definition 2.3. The Mittag-leffler function is defined as follows [15]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C}, \tag{2.4}$$

where $\alpha > 0$ and $\beta \in R$ are arbitrary constants.

Lemma 2.1. *Let ${}_0^{GL}D_t^{\alpha} g(t)$ be Grünwald-Letnikov fractional derivative of the order α $(0 < \alpha < 1)$, then* [13]

$$\mathcal{L}[{}_0^{GL}D_t^{\alpha} g(t)] = p^{\alpha} G(p),\ \ Re(p) > 0, \tag{2.5}$$

where $\mathcal{L}$ denotes the Laplace transform operator.

Lemma 2.2. *If $p > 0$, then the following equation holds* [15]*:*

$$\int_0^{\infty} e^{-pt} t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(\pm a t^{\alpha}) dt = \frac{m! p^{\alpha-\beta}}{(p^{\alpha} \mp a)^{m+1}},\ \ Re(p) > |a|^{\frac{1}{\alpha}}, \tag{2.6}$$

where $E_{\alpha,\beta}^{(m)}(z) := \frac{d^m}{dz^m} E_{\alpha,\beta}(z)$.

Lemma 2.3. *Assume $\lambda > 0$, $t > 0$ and $0 < \alpha < 1$, then* [15]

$$\partial_t^{\alpha} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda E_{\alpha,1}(-\lambda t^{\alpha}),\ \ \frac{d}{dt} E_{\alpha,1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}). \tag{2.7}$$

Moreover $0 < E_{\alpha,1}(-t) < 1$ and $E_{\alpha,1}(-t)$ is completely monotonic function, i.e.,

$$(-1)^n \frac{d^n}{dt^n} E_{\alpha,1}(-t) \geqslant 0, \forall n \in \mathbb{N}. \tag{2.8}$$

Lemma 2.4. *For the Mittag-leffler function, the following formula holds* [7]*:*

$$E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, z \in \mathbb{C}, \tag{2.9}$$

where $\alpha > 0$, $\beta > 0$.

Lemma 2.5. *Assume $0<\alpha_0<\alpha_1<1$. Then there exist constants $C_-, C^->0$ depending only on α_0, α_1 such that for all $\alpha \in [\alpha_0, \alpha_1]$, then* [11]

$$\frac{C_-}{\Gamma(1-\alpha)} \frac{1}{1-z} \leq E_{\alpha,1}(z) \leq \frac{C^-}{\Gamma(1-\alpha)} \frac{1}{1-z}, z \leq 0. \tag{2.10}$$

Lemma 2.6. *For all λ_n that satisfies $0<\lambda_1\le\cdots\le\lambda_n$, there exist positive constant dependent on $\alpha, t_0, T, \lambda_1$, and we have*

$$\frac{C_1}{\lambda_n}\le E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\le\frac{C_2}{\lambda_n},\ \frac{C_3}{\lambda_n}\le E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\le\frac{C_4}{\lambda_n} \tag{2.11}$$

where $C_1:=\frac{C_-}{\Gamma(1-\alpha)(\frac{1}{\lambda_1}+\frac{1}{2}t_0^\alpha)}$, $C_2:=\frac{2C^-}{\Gamma(1-\alpha)t_0^\alpha}$, $C_3:=\frac{C_-}{\Gamma(1-\alpha)(\frac{1}{\lambda_1}+\frac{1}{2}T^\alpha)}$, $C_4:=\frac{2C^-}{\Gamma(1-\alpha)T^\alpha}$.

Proof. From Lemma 2.5 , then

$$\begin{aligned}E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)&\le\frac{C^-}{\Gamma(1-\alpha)}\frac{1}{1+\frac{\lambda_n}{2}t_0^\alpha}\le\frac{C^-}{\Gamma(1-\alpha)}\frac{2}{t_0^\alpha}\frac{1}{\lambda_n}=\frac{C_2}{\lambda_n},\\E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)&\ge\frac{C_-}{\Gamma(1-\alpha)}\frac{1}{1+\frac{\lambda_n}{2}t_0^\alpha}\ge\frac{C_-}{\Gamma(1-\alpha)}\frac{1}{\frac{\lambda_n}{\lambda_1}+\frac{\lambda_n}{2}t_0^\alpha}=\frac{C_1}{\lambda_n}.\end{aligned}$$

The proof of $E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)$ is similar to the above, so it is omitted. □

Lemma 2.7. *For any λ_n that satisfies $0<\lambda_1\le\cdots\le\lambda_n$, there exist positive constant dependent on $\alpha, t_0, T, \lambda_1$, and we have*

$$\frac{C_5}{\lambda_n}\le t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)\le\frac{2}{\lambda_n},\ \frac{C_6}{\lambda_n}\le T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)\le\frac{2}{\lambda_n}, \tag{2.12}$$

where $C_5:=2(1-E_{\alpha,1}(-\frac{\lambda_1}{2}t_0^\alpha)), C_6:=2(1-E_{\alpha,1}(-\frac{\lambda_1}{2}T^\alpha))$.

Proof. From Lemma 2.4, then

$$\begin{aligned}E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)&=\frac{1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{\frac{\lambda_n}{2}t_0^\alpha}\le\frac{2}{t_0^\alpha}\cdot\frac{1}{\lambda_n},\\E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)&\ge\frac{1-E_{\alpha,1}(-\frac{\lambda_1}{2}t_0^\alpha)}{\frac{\lambda_n}{2}t_0^\alpha}=\frac{2(1-E_{\alpha,1}(-\frac{\lambda_1}{2}t_0^\alpha))}{t_0^\alpha}\cdot\frac{1}{\lambda_n}.\end{aligned}$$

Therefore

$$\frac{C_5}{\lambda_n}\le t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)\le\frac{2}{\lambda_n}.$$

Similarly, it can be inferred that

$$\frac{C_6}{\lambda_n}\le T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)\le\frac{2}{\lambda_n}.$$

□

Lemma 2.8. *Assume $q(\xi(T))>q(\xi(t_0))$, denote $M:=\max_{t\in[0,T]}|q(t)|$, for all λ_n that satisfies $0<\lambda_1\le\cdots\le\lambda_n$, then there exist positive constant dependent on $\alpha, t_0, T, \lambda_1, M$, such that*

$$\begin{aligned}\frac{C_7}{\lambda_n}\le&q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\\&-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\\\le&\frac{C_8}{\lambda_n},\end{aligned} \tag{2.13}$$

where $C_7:=\frac{C_3C_6(q(\xi(T))-q(\xi(t_0)))}{2}$, $C_8:=C_2M$.

Proof. Based on Lemma 2.3, Lemma 2.4, Lemma 2.6 and Lemma 2.7, we can derive

$$
\begin{aligned}
&q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\\
&\quad-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\\
\geq&q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\\
&\quad-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\\
=&E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\Big(q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))\Big)\\
\geq&E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\big(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\big)\big(q(\xi(T))-q(\xi(t_0))\big)\\
=&E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\frac{\lambda_n}{2}T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)\big(q(\xi(T))-q(\xi(t_0))\big)\\
\geq&E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\frac{\lambda_n}{2}\frac{C_6}{\lambda_n}\big(q(\xi(T))-q(\xi(t_0))\big)\\
\geq&\frac{C_3}{\lambda_n}\frac{C_6}{2}\big(q(\xi(T))-q(\xi(t_0))\big)\\
=&\frac{C_7}{\lambda_n}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\\
&\quad-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\\
\leq&\ q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\\
\leq&\ q(\xi(T))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\\
\leq&\ q(\xi(T))\frac{C_2}{\lambda_n}\\
\leq&\ \frac{C_2M}{\lambda_n}\\
:=&\ \frac{C_8}{\lambda_n}.
\end{aligned}
$$

□

Lemma 2.9. *For all $p>0$, $0<\mu<1$, and $0<\lambda_1\leqslant s$, $m>0$ is a constant, then:*

$$
F(s)=\frac{\mu s^{2-\frac{p}{2}}}{m+\mu s^2}\leqslant\begin{cases}C_9\mu^{\frac{p}{4}},\ 0<p<4,\\ C_{10}\mu,\ \ p\geqslant 4,\end{cases}\tag{2.14}
$$

where $C_9:=\frac{1}{4}(4-p)^{1-\frac{p}{4}}p^{\frac{p}{4}}m^{-\frac{p}{4}}$, $C_{10}:=m^{-1}\lambda_1^{2-\frac{p}{2}}$ and $s=\lambda_n$.

Proof. When $0 < p < 4$, due to $\lim_{s\to 0} F(s) = 0$ and $\lim_{s\to\infty} F(s) = 0$, On the other hand, $F(s)>0$, for any s, then we obtain

$$F(s) \leqslant F(s^*).$$

Where s^* satisfies equation $F^{'}(s) = 0$ and its value is $s^* = (\frac{(4-p)m}{p\mu})^{\frac{1}{2}}$.

Therefore,

$$F(s) \leqslant F(s^*) = \mu\big(\frac{(4-p)m}{p\mu}\big)^{1-\frac{p}{4}} \frac{1}{m+\mu\frac{(4-p)m}{p\mu}} := C_9\mu^{\frac{p}{4}}. \tag{2.15}$$

When $p \geqslant 4$,

$$F(s) = \frac{\mu s^{2-\frac{p}{2}}}{m+\mu s^2} = \frac{\mu}{s^{\frac{p}{2}-2}(m+\mu s^2)} \leq \frac{\mu}{s^{\frac{p}{2}-2}m} \leq \frac{\mu}{\lambda_1^{\frac{p}{2}-2}m} := C_{10}\mu. \tag{2.16}$$

□

Lemma 2.10. *For all $p > 0$, $0 < \mu < 1$, $0 < \lambda_1 \leqslant s$ and $b > 0$ is a constant, then:*

$$G(s) = \frac{\mu s^{1-\frac{p}{2}}}{b+\mu s^2} \leqslant \begin{cases} C_{11}\mu^{\frac{p+2}{4}}, & 0 < p < 2, \\ C_{12}\mu, & p \geqslant 2, \end{cases} \tag{2.17}$$

where $C_{11} := \frac{1}{2}(2-p)^{\frac{2-p}{4}}(2+p)^{\frac{2+p}{4}}b^{-\frac{2+p}{4}}$, $C_{12} := b^{-1}\lambda_1^{1-\frac{p}{2}}$ *and* $s = \lambda_n$.

Proof. It is similar to Lemma 2.9, we omit it. □

3. Solution of the problem and ill-posed analysis

In Section 3, we obtain the solution of problem (1.1) and briefly analyze the ill-posedness of the inverse problem. Using the separated variable method, Laplace transform and Lemma 2.2, we have

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{2}\Big(E_{\alpha,1}(-\frac{\lambda_n}{2}t^\alpha)\varphi_n + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda_n}{2}(t-\tau)^\alpha)q(\tau)d\tau f_n\Big)X_n(x), \tag{3.1}$$

where $\varphi_n = (\varphi(x), X_n(x))$, $f_n = (f(x), X_n(x))$ are the Fourier coefficients. By **Lemma 2.3**, we have $\frac{dE_{\alpha,1}(-\frac{\lambda_n}{2}(t-\tau)^\alpha)}{dt} = -\frac{\lambda_n}{2}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda_n}{2}(t-\tau)^\alpha) \leq 0$. With the mean value theorem for integrals and Lemma 2.3, we can get

$$\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda_n}{2}(t-\tau)^\alpha)q(\tau)d\tau = \frac{2q(\xi(t))}{\lambda_n}\big(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t^\alpha)\big), \tag{3.2}$$

where $0 \leq \xi(t) \leq t$. According to (3.2), $u(x,t)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty}\Big(\frac{1}{2}E_{\alpha,1}(-\frac{\lambda_n}{2}t^\alpha)\varphi_n + \frac{q(\xi(t))}{\lambda_n}\big(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t^\alpha)\big)f_n\Big)X_n(x). \tag{3.3}$$

By using $u(x,t_0) = \psi(x)$, $u(x,T) = g(x)$ and (3.3), we can obtain

$$f(x) = \sum_{n=1}^{\infty} \frac{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)g_n - \lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\psi_n}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha) - q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}X_n(x), \tag{3.4}$$

$$\varphi(x)=\sum_{n=1}^{\infty}\frac{q(\xi(T))T^{\alpha}\lambda_nE_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^{\alpha})\psi_n-q(\xi(t_0))t_0^{\alpha}\lambda_nE_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^{\alpha})g_n}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})}X_n(x), \tag{3.5}$$

where $g_n=(g(x),X_n(x))$ and $\psi_n=(\psi(x),X_n(x))$.

Now, we denote formula (3.4) as

$$f=K_1^{-1}g+K_2^{-1}\psi, \tag{3.6}$$

K_1, K_2 are self-adjoint operators and the singular values of operators K_1^{-1}, K_2^{-1}

$$k_1^{-1}=\frac{\lambda_nE_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})},$$

$$k_2^{-1}=\frac{-\lambda_nE_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})}.$$

For (3.5), we have the same representation as above:

$$\varphi=K_3^{-1}\psi+K_4^{-1}g, \tag{3.7}$$

similarly,

$$k_3^{-1}=\frac{q(\xi(T))T^{\alpha}\lambda_nE_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^{\alpha})}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})},$$
$$k_4^{-1}=\frac{-q(\xi(t_0))t_0^{\alpha}\lambda_nE_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^{\alpha})}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha}))E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})}.$$

Due to $n\to\infty$, $\lambda_n\to\infty$, then $k_1^{-1}\to\infty$, $k_2^{-1}\to\infty$. Therefore, according to (3.4), small disturbances of $g(x)$ and $\psi(x)$ can cause significant changes of $f(x)$. Similarly, small disturbances of $g(x)$ and $\psi(x)$ can also cause significant changes of $\varphi(x)$, then this problem is ill-posed. Next, we will use the Tikhonov method to solve the ill-posed problem. In order to obtain the a priori and a posteriori convergent results, we assume that both $f(x)$ and $\varphi(x)$ satisfy the following a-priori bound condition:

$$\max\left\{\|f(\cdot)\|_{D((-\Delta)^p)},\|\varphi(\cdot)\|_{D((-\Delta)^p)}\right\}\le E, \tag{3.8}$$

where $E>0$ and $p>0$, $\|f(\cdot)\|_{D((-\Delta)^p)}=\left(\sum_{n=1}^{\infty}\lambda_n^p|(f,X_n)|^2\right)^{\frac{1}{2}}$, $\|\varphi(\cdot)\|_{D((-\Delta)^p)}=\left(\sum_{n=1}^{\infty}\lambda_n^p|(\varphi,X_n)|^2\right)^{\frac{1}{2}}$.

4. The Tikhonov regularization method and convergent estimation

In Section 4, we will use the Tikhonov regularization method to solve ill-posed problems. Meanwhile, regularization parameter choice rules based on a-priori and a-posteriori, we derive the convergent estimates.

4.1. The Tikhonov regularization method

Consider the following operator equation:

$$Kx = y, \tag{4.1}$$

where $K : X \to Y$ is a bounded linear operator, $x \in X$, $y \in Y$, X, Y are Hibert spaces. The Tikhonov regularization method can be expressed as solving the following minimization functional

$$J_\mu(x) = \|Kx - y\|_Y^2 + \mu\|x\|_X^2, \tag{4.2}$$

here $0 < \mu < 1$ is the Tikhonov regularization parameter. Following [8], we have a result as follows:

$$(K^*K + \mu I)x^\mu = K^*y, \tag{4.3}$$

$K^* : Y \to X$ is the self-adjoint operator of K. In the Section 3, we denote (3.4) as:

$$f = K_1^{-1}g + K_2^{-1}\psi,$$

where $K_1, K_2 : L^2(\Omega) \to L^2(\Omega)$ are self-adjoint operators. For (3.6), we have

$$K_1(K_2f - \psi) = K_2g. \tag{4.4}$$

The Tikhonov functional of operator equation (4.4) is

$$\|K_1(K_2f - \psi) - K_2g\|^2 + \mu\|(K_2f - \psi)\|^2. \tag{4.5}$$

According to (4.2) and (4.3), we can obtain

$$(K_1^*K_1 + \mu I)(K_2f - \psi) = K_1^*(K_2g), \tag{4.6}$$

rewrite (4.6) as

$$K_2(f - (K_1^*K_1 + \mu I)^{-1}K_1^*g) = \psi. \tag{4.7}$$

The Tikhonov functional of operator Equation (4.7) is

$$\|K_2(f - (K_1^*K_1 + \mu I)^{-1}K_1^*g) - \psi\|^2 + \mu\|(f - (K_1^*K_1 + \mu I)^{-1}K_1^*g)\|^2. \tag{4.8}$$

According to (4.2) and (4.3), we can obtain

$$(K_2^*K_2 + \mu I)(f_\mu - (K_1^*K_1 + \mu I)^{-1}K_1^*g) = K_2^*\psi. \tag{4.9}$$

Thus

$$f_\mu = (K_1^*K_1 + \mu I)^{-1}K_1^*g + (K_2^*K_2 + \mu I)^{-1}K_2^*\psi. \tag{4.10}$$

Similarly, we can get the regularization solution of the initial value

$$\varphi_\mu = (K_3^*K_3 + \mu I)^{-1}K_3^*\psi + (K_4^*K_4 + \mu I)^{-1}K_4^*g. \tag{4.11}$$

Using measurement data with errors, singular value of the operator and (4.10), we have

$$\begin{aligned} f_\mu^\delta(x) =& \sum_{n=1}^{\infty}\Bigg(\frac{\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}}{\left(\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}\right)^2 + \mu} g_n^\delta \\ &- \frac{\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}}{\left(\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\right)^2 + \mu}\psi_n^\delta\Bigg)X_n(x). \end{aligned} \tag{4.12}$$

Similarly, we can obtain

$$\varphi_\mu^\delta(x)=\sum_{n=1}^{\infty}\left(\frac{\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}}{\left(\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}\right)^2+\mu}\psi_n^\delta\right.$$
$$\left.-\frac{\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}}{\left(\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}\right)^2+\mu}g_n^\delta\right)X_n(x). \tag{4.13}$$

In order to simplify the calculation process, we introduce some notations as follow

$$\sigma_1=\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)},$$
$$\sigma_2=\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)},$$
$$\sigma_3=\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)},$$
$$\sigma_4=\frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}.$$

Then the formula (3.4), (3.5), (4.12) and (4.13) can be rewritten as

$$f(x)=\sum_{n=1}^{\infty}(\sigma_1^{-1}g_n-\sigma_2^{-1}\psi_n)X_n(x),\quad \varphi(x)=\sum_{n=1}^{\infty}(\sigma_3^{-1}\psi_n-\sigma_4^{-1}g_n)X_n(x), \tag{4.14}$$

$$f_\mu^\delta(x)=\sum_{n=1}^{\infty}(\frac{\sigma_1}{\sigma_1^2+\mu}g_n^\delta-\frac{\sigma_2}{\sigma_2^2+\mu}\psi_n^\delta)X_n(x),$$
$$\varphi_\mu^\delta(x)=\sum_{n=1}^{\infty}(\frac{\sigma_3}{\sigma_3^2+\mu}\psi_n^\delta-\frac{\sigma_4}{\sigma_4^2+\mu}g_n^\delta)X_n(x). \tag{4.15}$$

4.2. The error estimate with a-priori parameter choice

Based on the a-priori regularization parameter choice rule, the error estimates can be obtained respectively.

Theorem 4.1. *Assume exact solution $f(x)$ satisfies* (3.8) *and assumptions* (1.2) *hold.*
(1) If $0<p<4$, and $\mu=(\frac{\delta}{E})^{\frac{4}{p+2}}$, then

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\le\widetilde{C_1}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}; \tag{4.16}$$

(2) If $p\geqslant 4$, and $\mu=(\frac{\delta}{E})^{\frac{2}{3}}$, then

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\le\widetilde{C_2}\delta^{\frac{2}{3}}E^{\frac{1}{3}}, \tag{4.17}$$

where $\widetilde{C_1}$ and $\widetilde{C_2}$ are positive constants independent of μ, δ, E.

Proof. By means of a triangular inequality, we have

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\leqslant\|f_\mu^\delta(\cdot)-f_\mu(\cdot)\|+\|f_\mu(\cdot)-f(\cdot)\|. \tag{4.18}$$

Now we give an estimate of the first term. Through (4.14), (4.15) and (1.2), we obtain

$$\begin{aligned}\|f_\mu^\delta(\cdot)-f_\mu(\cdot)\| &= \Big\|\sum_{n=1}^{\infty}\big(\frac{\sigma_1}{\sigma_1^2+\mu}(g_n^\delta-g_n)-\frac{\sigma_2}{\sigma_2^2+\mu}(\psi_n^\delta-\psi_n)\big)X_n(x)\Big\| \\ &\le \frac{\delta}{2\sqrt{\mu}}+\frac{\delta}{2\sqrt{\mu}} \\ &= \frac{\delta}{\sqrt{\mu}}.\end{aligned} \tag{4.19}$$

Then we give an estimate the second term of (4.18). Through (3.8), (4.10), (4.14) and Lemma 2.9, we can deduce

$$\begin{aligned}&\|f_\mu(\cdot)-f(\cdot)\|^2 \\ =&\Big\|\sum_{n=1}^{\infty}(\frac{\sigma_1}{\sigma_1^2+\mu}g_n-\frac{\sigma_2}{\sigma_2^2+\mu}\psi_n)X_n(x)-\sum_{n=1}^{\infty}(\sigma_1^{-1}g_n-\sigma_2^{-1}\psi_n)X_n(x)\Big\|^2 \\ =&\Big\|\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_1(\sigma_1^2+\mu)}g_n-\frac{\mu}{\sigma_2(\sigma_2^2+\mu)})X_n(x)\Big\|^2 \\ \le&2\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\lambda_n g_n)^2 \\ &+2\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\lambda_n \psi_n)^2 \\ \le&2(\frac{C_2}{C_7})^2\sup_{n\ge1}|A_1(n)|^2\Big(\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n g_n)^2+\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n\psi_n)^2\Big) \\ \le&2(\frac{C_2}{C_7})^2\sup_{n\ge1}|A_1(n)|^2\Big(\sum_{n=1}^{\infty}\lambda_n^p\Big(\lambda_n\big(\frac{1}{2}E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)\varphi_n \\ &+\frac{q(\xi(T))}{2}T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)f_n\big)\Big)^2 \\ &+\sum_{n=1}^{\infty}\lambda_n^p\Big(\lambda_n\big(\frac{1}{2}E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)\varphi_n+\frac{q(\xi(t_0))}{2}t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)f_n\big)\Big)^2\Big) \\ \le&2(\frac{C_2}{C_7})^2\sup_{n\ge1}|A_1(n)|^2(C_2^2\sum_{n=1}^{\infty}\lambda_n^p\varphi_n^2+4M^2\sum_{n=1}^{\infty}\lambda_n^p f_n^2) \\ \le&2(\frac{C_2}{C_7})^2\sup_{n\ge1}|A_1(n)|^2(C_2^2+4M^2)E^2 \\ =&\frac{2C_2^2(C_2^2+4M^2)}{C_7^2}\sup_{n\ge1}\big|A_1(n)\big|^2E^2,\end{aligned}$$

where $A_1(n)=\frac{\mu\lambda_n^{2-\frac{p}{2}}}{(\frac{C_7}{C_2})^2+\mu\lambda_n^2}$. Then $\lambda_n:=s$, according to Lemma 2.9, we have

$$A_1(n)\le\begin{cases}C_9\mu^{\frac{p}{4}},\ 0<p<4,\\ C_{10}\mu,\ \ p\geqslant 4,\end{cases}$$

where $C_9 = \frac{1}{4}(4-p)^{1-\frac{p}{4}} p^{\frac{p}{4}} (\frac{C_7}{C_2})^{-\frac{p}{2}}$, $C_{10} = (\frac{C_7}{C_2})^{-2} \lambda_1^{2-\frac{p}{2}}$, consequently

$$\|f_\mu(\cdot) - f(\cdot)\| \leq \begin{cases} \dfrac{2^{\frac{1}{2}} C_2 (C_2^2 + 4M^2)^{\frac{1}{2}}}{C_7} C_9 \mu^{\frac{p}{4}} E, \ 0 < p < 4, \\ \dfrac{2^{\frac{1}{2}} C_2 (C_2^2 + 4M^2)^{\frac{1}{2}}}{C_7} C_{10} \mu E, \ \ p \geqslant 4. \end{cases} \tag{4.20}$$

Combining (4.19) with (4.20), we can obtain a-priori regularization parameter μ

$$\mu = \begin{cases} (\dfrac{\delta}{E})^{\frac{4}{p+2}}, \ 0 < p < 4, \\ (\dfrac{\delta}{E})^{\frac{2}{3}}, \quad p \geqslant 4, \end{cases}$$

then, we have

$$\|f_\mu^\delta(\cdot) - f(\cdot)\| \leq \begin{cases} \widetilde{C_1} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, \ 0 < p < 4, \\ \widetilde{C_2} \delta^{\frac{2}{3}} E^{\frac{1}{3}}, \qquad p \geqslant 4. \end{cases} \tag{4.21}$$

Where $\widetilde{C_1} = \big(1 + \frac{2^{\frac{1}{2}} C_2 (C_2^2 + 4M^2)^{\frac{1}{2}}}{C_7} C_9\big)$, $\widetilde{C_2} = \big(1 + \frac{2^{\frac{1}{2}} C_2 (C_2^2 + 4M^2)^{\frac{1}{2}}}{C_7} C_{10}\big)$. Now the proof of Theorem 4.1 has been completed. □

Theorem 4.2. *Assume exact solution* $\varphi(x)$ *satisfies* (3.8) *and assumptions* (1.2) *hold.*
(1) If $0 < p < 4$, *and* $\mu = (\frac{\delta}{E})^{\frac{4}{p+2}}$, *then*

$$\|\varphi_\mu^\delta(\cdot) - \varphi(\cdot)\| \leq \widetilde{C_3} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}; \tag{4.22}$$

(2) If $p \geqslant 4$, *and* $\mu = (\frac{\delta}{E})^{\frac{2}{3}}$, *then*

$$\|\varphi_\mu^\delta(\cdot) - \varphi(\cdot)\| \leq \widetilde{C_4} \delta^{\frac{2}{3}} E^{\frac{1}{3}}. \tag{4.23}$$

where $\widetilde{C_3}$ *and* $\widetilde{C_4}$ *are positive constants independent of* μ, δ, E.

Proof. The proof of Theorem 4.2 is similar to Theorem 4.1, so it is omitted. □

4.3. The error estimate with a-posteriori parameter choice

In Section 4.2, we know that the a-priori parameter choice is related to the a-priori bound E. However, the a-priori bound E is an additional assumption. In most practical applications, the condition is usually not readily available. Thus an a-posteriori parameter selection rule for choosing the regularization parameter is necessary. In this subsection, Morozov's discrepancy principle is considered to choose a-posteriori parameter μ.

4.3.1. The a-posteriori convergence error estimates of unknown source term

The a-posteriori rule satisfies the following equation:

$$\|K_2 f_\mu^\delta(x) - K_2 K_1^{-1} g^\delta(x) - \psi^\delta(x)\| = \tau_1 \delta, \tag{4.24}$$

where $\tau_1 \geq \frac{C_2}{C_3} + 1$ and $\|K_2 K_1^{-1} g^\delta + \psi^\delta\| > \tau_1 \delta$.

Lemma 4.1. *Let* $\rho(\mu) = \|K_2 f_\mu^\delta(x) - K_2 K_1^{-1} g^\delta(x) - \psi^\delta(x)\|$, *the following results hold*
(a) $\rho(\mu)$ *is continuous;*
(b) $\lim_{\mu\to 0}\rho(\mu) = 0$*;*
(c) $\lim_{\mu\to\infty}\rho(\mu) = \|K_2 K_1^{-1} g^\delta + \psi^\delta\|$*;*
(d) $\rho(\mu)$ *is a strictly monotone function for any* $\mu \in (0,\infty)$*.*

Proof. According to the expression of $\rho(\mu)$ as follows, the lemma clearly holds

$$\rho(\mu) = \|\sum_{n=1}^{\infty}\big(\frac{\mu I}{K_1^* K_1 + \mu I} K_2 K_1^{-1} g_n^\delta + \frac{\mu I}{K_2^* K_2 + \mu I}\psi_n^\delta\big) X_n(x)\|. \tag{4.25}$$

□

Theorem 4.3. *Assume exact solution* $f(x)$ *satisfies* (3.8) *and assumptions* (1.2) *hold. The a-posteriori regularization parameter* μ *is chosen by* (4.24)*, then*
(1) If $0 < p < 2$*, we can obtain*

$$\|f_\mu^\delta(\cdot) - f(\cdot)\| \le \widetilde{C_5}\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}; \tag{4.26}$$

(2) If $p \geqslant 2$*, we can obtain*

$$\|f_\mu^\delta(\cdot) - f(\cdot)\| \le \widetilde{C_6}\delta^{\frac{1}{2}} E^{\frac{1}{2}}, \tag{4.27}$$

where $\widetilde{C_5}$ *and* $\widetilde{C_6}$ *are positive constants independent of* μ*,* δ*,* E*.*

Proof. From the basic inequality, we have

$$\|f_\mu^\delta(\cdot) - f(\cdot)\| \leqslant \|f_\mu^\delta(\cdot) - f_\mu(\cdot)\| + \|f_\mu(\cdot) - f(\cdot)\|. \tag{4.28}$$

For the first term, according to (4.19), we have

$$\|f_\mu^\delta(\cdot) - f_\mu(\cdot)\| \le \frac{\delta}{\sqrt{\mu}}. \tag{4.29}$$

Using (4.24) and (1.2), we have

$$\begin{aligned}
\tau_1\delta &= \|\sum_{n=1}^{\infty}\big(\frac{\mu}{\sigma_1^2+\mu}\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} g_n^\delta - \frac{\mu}{\sigma_2^2+\mu}\psi_n^\delta\big) X_n(x)\| \\
&\le \|\sum_{n=1}^{\infty}\frac{\mu}{\sigma_1^2+\mu}\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}(g_n^\delta - g_n) X_n(x)\| + \|\sum_{n=1}^{\infty}\frac{\mu}{\sigma_2^2+\mu}(\psi_n^\delta - \psi_n) X_n(x)\| \\
&\quad + \|\sum_{n=1}^{\infty}\big(\frac{\mu}{\sigma_1^2+\mu}\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} g_n - \frac{\mu}{\sigma_2^2+\mu}\psi_n\big) X_n(x)\| \\
&\le \sup_{n\ge 1}\Big|\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\Big|\delta + \delta + J_1 \\
&\le \big(\frac{C_2}{C_3} + 1\big)\delta + J_1.
\end{aligned}$$

Combining a-priori bound (3.8), we can obtain

$$J_1^2 = \sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_1^2+\mu}\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} g_n - \frac{\mu}{\sigma_2^2+\mu}\psi_n\Big)^2$$

$$
\begin{aligned}
&\leq 2\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_1^2+\mu}\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})}g_n\Big)^2+2\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_2^2+\mu}\psi_n\Big)^2\\
&\leq 2\sup_{n\geq 1}\big|\big((\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^{\alpha})}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^{\alpha})})^2+1\big)^{\frac{1}{2}}\frac{\mu\lambda_n^{-\frac{p}{2}-1}}{\sigma_1^2+\mu}\big|^2\\
&\quad\times\Big(\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n g_n)^2+\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n\psi_n)^2\Big)\\
&\leq 2\sup_{n\geq 1}\big|\big((\frac{C_2}{C_3})^2+1\big)^{\frac{1}{2}}\frac{\mu\lambda_n^{1-\frac{p}{2}}}{(\frac{C_7}{C_2})^2+\mu\lambda_n^2}\big|^2(C_2^2+4M^2)E^2\\
&\leq 2\big((\frac{C_2}{C_3})^2+1\big)\big(C_2^2+4M^2\big)\sup_{n\geq 1}\big|B_1(n)\big|^2E^2\\
&=C_{13}^2\sup_{n\geq 1}\big|B_1(n)\big|^2E^2,
\end{aligned}
$$

where $B_1(n)=\frac{\mu\lambda_n^{1-\frac{p}{2}}}{(\frac{C_7}{C_2})^2+\mu\lambda_n^2}$. Then $\lambda_n := s$, from Lemma 2.10, we have

$$
B_1(n)\leqslant\begin{cases} C_{11}\mu^{\frac{p+2}{4}},\ 0<p<2,\\ C_{12}\mu, \qquad p\geqslant 2,\end{cases} \tag{4.30}
$$

where $C_{11}=\frac{1}{4}(2-p)^{\frac{2-p}{4}}(2+p)^{\frac{2+p}{4}}(\frac{C_7}{C_2})^{-\frac{2+p}{2}}$, $C_{12}=(\frac{C_7}{C_2})^{-2}\lambda_1^{1-\frac{p}{2}}$, therefore

$$
(\tau_1-\frac{C_2}{C_3}-1)\delta\leq\begin{cases} C_{11}C_{13}\mu^{\frac{p+2}{4}}E,\ 0<p<2,\\ C_{12}C_{13}\mu E, \qquad p\geqslant 2.\end{cases} \tag{4.31}
$$

Then

$$
\frac{1}{\mu}\leq\begin{cases} (\frac{C_{11}C_{13}}{\tau_1-\frac{C_2}{C_3}-1})^{\frac{4}{p+2}}\delta^{-\frac{4}{p+2}}E^{\frac{4}{p+2}},\ 0<p<2,\\ \frac{C_{12}C_{13}}{\tau_1-\frac{C_2}{C_3}-1}\delta^{-1}E, \qquad p\geqslant 2.\end{cases} \tag{4.32}
$$

According to (4.29) and (4.32), we have

$$
\|f_\mu^\delta(\cdot)-f_\mu(\cdot)\|\leq\begin{cases} (\frac{C_{11}C_{13}}{\tau_1-\frac{C_2}{C_3}-1})^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}},\ 0<p<2,\\ (\frac{C_{12}C_{13}}{\tau_1-\frac{C_2}{C_3}-1})^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, \qquad p\geqslant 2.\end{cases} \tag{4.33}
$$

Next, we consider the second term and apply the Hölder inequality

$$
\begin{aligned}
&\|f_\mu(\cdot)-f(\cdot)\|^2\\
=&\big\|\sum_{n=1}^{\infty}(\frac{\sigma_1}{\sigma_1^2+\mu}g_n-\frac{\sigma_2}{\sigma_2^2+\mu}\psi_n)X_n(x)-\sum_{n=1}^{\infty}(\sigma_1^{-1}g_n-\sigma_2^{-1}\psi_n)X_n(x)\big\|^2\\
=&\big\|\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_1(\sigma_1^2+\mu)}g_n-\frac{\mu}{\sigma_2(\sigma_2^2+\mu)})X_n(x)\big\|^2
\end{aligned}
$$

$$=\sum_{n=1}^{\infty}\Big|\Big(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n$$
$$-\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n\Big)\lambda_n^{\frac{p+2}{2}}\Big|^{\frac{4}{p+2}}$$
$$\times\Big|\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n$$
$$-\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n\Big|^{\frac{2p}{p+2}}$$
$$\le I_1I_2.$$

Now we give the estimate of I_1

$$I_1=\Big(\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n$$
$$-\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n\Big)^2\lambda_n^{p+2}\Big)^{\frac{2}{p+2}}$$
$$\le\Big(2(\frac{C_2}{C_7})^2\Big(\sum_{n=1}^{\infty}\lambda_n^p(\lambda_ng_n)^2+\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n\psi_n)^2\Big)\Big)^{\frac{2}{p+2}}$$
$$\le\big(2(\frac{C_2}{C_7})^2(C_2^2+4M^2)E^2\big)^{\frac{2}{p+2}}.$$

Then we consider I_2

$$I_2=\Big(\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n$$
$$-\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n\Big)^2\Big)^{\frac{p}{p+2}}$$
$$=\Big(\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}$$
$$\times(g_n-g_n^\delta+g_n^\delta)$$
$$-\frac{\mu}{\sigma_2^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}$$
$$\times(\psi_n-\psi_n^\delta+\psi_n^\delta)\Big)^2\Big)^{\frac{p}{p+2}}$$
$$\le\Big(4(\frac{C_2}{C_7})^2\sum_{n=1}^{\infty}(g_n-g_n^\delta)^2+4(\frac{C_4}{C_7})^2\sum_{n=1}^{\infty}(\psi_n-\psi_n^\delta)^2$$
$$+2\sum_{n=1}^{\infty}\Big(\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\Big)^2$$
$$\times\Big(\frac{\mu}{\sigma_1^2+\mu}\cdot\frac{E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)}{E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n^\delta-\frac{\mu}{\sigma_2^2+\mu}\psi_n^\delta\Big)^2\Big)^{\frac{p}{p+2}}$$
$$\le\big(4(\frac{C_2}{C_7})^2\delta^2+4(\frac{C_4}{C_7})^2\delta^2+2(\frac{C_4}{C_7})^2(\tau_1\delta)^2\big)^{\frac{p}{p+2}}.$$

Therefore, we have

$$\|f_\mu(\cdot)-f(\cdot)\|\le C_{14}E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}. \tag{4.34}$$

Combining (4.28), (4.33) and (4.34), we obtain

$$\|f_\mu^\delta(\cdot)-f(\cdot)\|\le\begin{cases}\widetilde{C_5}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, & 0<p<2,\\ \widetilde{C_6}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, & p\geqslant 2,\end{cases} \tag{4.35}$$

where $\widetilde{C_5} = C_{14} + (\frac{C_{11}C_{13}}{\tau_1 - \frac{C_2}{C_3} - 1})^{\frac{2}{p+2}}$, $\widetilde{C_6} = C_{14} + (\frac{C_{12}C_{13}}{\tau_1 - \frac{C_2}{C_3} - 1})^{\frac{1}{2}}$. The proof of Theorem 4.3 has been completed. □

4.3.2. The a-posteriori error estimates of initial value

The a-posteriori rule satisfies the following equation:

$$\|K_3\varphi_\mu^\delta(x) - K_3K_4^{-1}g^\delta(x) - \psi^\delta(x)\| = \tau_2\delta, \tag{4.36}$$

where $\tau_2 \geq \frac{2}{C_6} + 1$ and $\|K_3K_4^{-1}g^\delta + \psi^\delta\| > \tau_2\delta$.

Lemma 4.2. *Let $\rho(\mu) = \|K_3\varphi_\mu^\delta(x) - K_3K_4^{-1}g^\delta(x) - \psi^\delta(x)\|$, the following results hold*
(a) $\rho(\mu)$ is continuous;
(b) $\lim_{\mu\to 0}\rho(\mu) = 0$;
(c) $\lim_{\mu\to\infty}\rho(\mu) = \|K_3K_4^{-1}g^\delta + \psi^\delta\|$;
(d) $\rho(\mu)$ is a strictly monotone function for any $\mu \in (0, \infty)$.

Proof. According to the expression of $\rho(\mu)$ as follows, the lemma clearly holds

$$\rho(\mu) = \|\sum_{n=1}^{\infty}(\frac{\mu I}{K_4^*K_4 + \mu I}K_3K_4^{-1}g_n^\delta + \frac{\mu I}{K_3^*K_3 + \mu I}\psi_n^\delta)X_n(x)\|. \tag{4.37}$$

□

Theorem 4.4. *Assume the exact solution $\varphi(x)$ satisfies* (3.8) *and assumptions* (1.2) *hold. The a-posteriori regularization parameter μ is selected through* (4.36), *then*
(1) If $0 < p < 2$, we can obtain

$$\|\varphi_\mu^\delta(\cdot) - \varphi(\cdot)\| \leq \widetilde{C_7}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}; \tag{4.38}$$

(2) If $p \geqslant 2$, we can obtain

$$\|\varphi_\mu^\delta(\cdot) - \varphi(\cdot)\| \leq \widetilde{C_8}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, \tag{4.39}$$

where $\widetilde{C_7}$ and $\widetilde{C_8}$ are positive constants independent of μ, δ, E.

Proof. According to basic inequality, we have

$$\|\varphi_\mu^\delta(\cdot) - \varphi(\cdot)\| \leq \|\varphi_\mu^\delta - \varphi_\mu\| + \|\varphi_\mu(\cdot) - \varphi(\cdot)\|. \tag{4.40}$$

Now, we consider the first term, similarly

$$\|\varphi_\mu^\delta(\cdot) - \varphi_\mu(\cdot)\| \leq \frac{\delta}{\sqrt{\mu}}. \tag{4.41}$$

Using (4.36) and (1.2), we have

$$\begin{aligned}\tau_2\delta &= \|\sum_{n=1}^{\infty}\Big(\frac{\mu}{\sigma_4^2 + \mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}g_n^\delta - \frac{\mu}{\sigma_3^2 + \mu}\psi_n^\delta\Big)X_n(x)\| \\ &\leq \|\sum_{n=1}^{\infty}\frac{\mu}{\sigma_4^2 + \mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}(g_n^\delta - g_n)X_n(x)\|\end{aligned}$$

$$
\begin{aligned}
&\quad + \Big\| \sum_{n=1}^{\infty} \frac{\mu}{\sigma_3^2+\mu}(\psi_n^\delta - \psi_n) X_n(x) \Big\| \\
&\quad + \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu}{\sigma_4^2+\mu} \cdot \frac{q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} t_0^\alpha)}{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} T^\alpha)} g_n - \frac{\mu}{\sigma_3^2+\mu}\psi_n \Big) X_n(x) \Big\| \\
&\le \sup_{n\ge 1} \Big| \frac{q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} t_0^\alpha)}{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} T^\alpha)} \Big| \delta + \delta + J \\
&\le (\frac{2}{C_6} + 1)\delta + J_2,
\end{aligned}
$$

$$
\begin{aligned}
J_2^2 &= \Big\| \sum_{n=1}^{\infty} \Big(\frac{\mu}{\sigma_4^2+\mu} \cdot \frac{q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} t_0^\alpha)}{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} T^\alpha)} g_n - \frac{\mu}{\sigma_3^2+\mu}\psi_n \Big) X_n(x) \Big\|^2 \\
&= \sum_{n=1}^{\infty} \Big(\frac{\mu}{\sigma_4^2+\mu} \cdot \frac{q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} t_0^\alpha)}{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} T^\alpha)} g_n - \frac{\mu}{\sigma_3^2+\mu}\psi_n \Big)^2 \\
&\le \sum_{n=1}^{\infty} \Big(\frac{\mu}{\sigma_3^2+\mu} \cdot (\frac{q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} t_0^\alpha)}{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2} T^\alpha)} g_n - \psi_n) \Big)^2 \\
&\le \sum_{n=1}^{\infty} \Big(\frac{\mu}{\sigma_3^2+\mu} \cdot \frac{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha) - q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}{\lambda_n q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)} \Big)^2 \\
&\quad \times \Big| \lambda_n \frac{q(\xi(T)) T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)\psi_n - q(\xi(t_0)) t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha) g_n}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha) - q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} \Big|^2 \\
&\le (\frac{C_2}{C_6})^2 \sup_{n\ge 1} \Big| \frac{\mu \lambda^{-1-\frac{p}{2}}}{\sigma_3^2+\mu} \Big|^2 \cdot \sum_{n=1}^{\infty} \lambda_n^p \varphi_n^2 \\
&\le (\frac{C_2}{C_6})^2 \sup_{n\ge 1} \Big| \frac{\mu \lambda_n^{1-\frac{p}{2}}}{(\frac{C_7}{2M})^2 + \mu\lambda_n^2} \Big|^2 E^2 \\
&\le (\frac{C_2}{C_6})^2 \sup_{n\ge 1} |B_1(n)|^2 E^2,
\end{aligned}
$$

where $B_2(n) = \frac{\mu \lambda_n^{1-\frac{p}{2}}}{(\frac{C_7}{2M})^2 + \mu\lambda_n^2}$, then $\lambda_n := s$, according to Lemma 2.10, we have

$$
B_2(n) \leqslant \begin{cases} C_{11}' \mu^{\frac{p+2}{4}},\ 0 < p < 2, \\ C_{12}' \mu, \qquad p \geqslant 2, \end{cases} \tag{4.42}
$$

where $C_{11}' = \frac{1}{2}(2-p)^{\frac{2-p}{4}}(2+p)^{\frac{2+p}{4}}(\frac{C_7}{2M})^{-\frac{2+p}{2}}, C_{12}' = (\frac{C_7}{2M})^{-2}\lambda_1^{1-\frac{p}{2}}$, therefore

$$
(\tau_2 - \frac{2}{C_6} - 1)\delta \le \begin{cases} \frac{C_2 C_{11}'}{C_6} \mu^{\frac{p+2}{4}} E,\ 0 < p < 2, \\ \frac{C_2 C_{12}'}{C_6} \mu E, \qquad p \geqslant 2. \end{cases} \tag{4.43}
$$

Then

$$
\frac{1}{\mu} \le \begin{cases} (\frac{C_2 C_{11}'}{C_6(\tau_2 - \frac{2}{C_6} - 1)})^{\frac{4}{p+2}} \delta^{-\frac{4}{p+2}} E^{\frac{4}{p+2}},\ 0 < p < 2, \\ \frac{C_2 C_{12}'}{C_6(\tau_2 - \frac{2}{C_6} - 1)} \delta^{-1} E, \qquad p \geqslant 2. \end{cases} \tag{4.44}
$$

According to (4.41) and (4.44), we obtain

$$\|\varphi_\mu^\delta(\cdot)-\varphi_\mu(\cdot)\| \le \begin{cases} (\frac{C_2C_{11}'}{C_6(\tau_2-\frac{2}{C_6}-1)})^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, \ 0<p<2, \\ (\frac{C_2C_{12}'}{C_6(\tau_2-\frac{2}{C_6}-1)})^{\frac{1}{2}}\delta^{\frac{1}{2}}E^{\frac{1}{2}}, \qquad p\geqslant 2. \end{cases} \tag{4.45}$$

Next, we consider the second term and apply the Hölder inequality

$$\begin{aligned}
&\|\varphi_\mu(\cdot)-\varphi(\cdot)\|^2 \\
=&\|\sum_{n=1}^{\infty}(\frac{\sigma_3}{\sigma_3^2+\mu}\psi_n^\delta-\frac{\sigma_4}{\sigma_4^2+\mu}g_n^\delta)X_n(x)-\sum_{n=1}^{\infty}(\sigma_3^{-1}\psi_n-\sigma_4^{-1}g_n)X_n(x)\|^2 \\
=&\|\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_3(\sigma_3^2+\mu)}\psi_n^\delta-\frac{\mu}{\sigma_4(\sigma_4^2+\mu)}g_n^\delta)X_n(x)\|^2 \\
=&\sum_{n=1}^{\infty}|(\frac{\mu}{\sigma_3^2+\mu}\cdot\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n \\
&-\frac{\mu}{\sigma_4^2+\mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n)\lambda_n^{\frac{p+2}{2}}|^{\frac{4}{p+2}} \\
&\times|\frac{\mu}{\sigma_3^2+\mu}\cdot\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n \\
&-\frac{\mu}{\sigma_4^2+\mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n|^{\frac{2p}{p+2}} \\
\le& I_3I_4.
\end{aligned}$$

We first estimate I_3, from (3.8)

$$\begin{aligned}
I_3=&\Big(\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_3^2+\mu}\cdot\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n \\
&-\frac{\mu}{\sigma_4^2+\mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n\big)^2\lambda_n^{p+2}\Big)^{\frac{2}{p+2}} \\
\le&\Big(2\big(\frac{2M}{C_7}\big)^2\big(\sum_{n=1}^{\infty}\lambda_n^p(\lambda_n\psi_n)^2+\sum_{n=1}^{\infty}\lambda_n^p(\lambda_ng_n)^2\big)\Big)^{\frac{2}{p+2}} \\
\le&\big(2(\frac{2M}{C_7})^2(C_2^2+4M^2)E^2\big)^{\frac{2}{p+2}}.
\end{aligned}$$

Next we give the estimate of I_4

$$\begin{aligned}
I_4=&\Big(\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_3^2+\mu}\cdot\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}\psi_n \\
&-\frac{\mu}{\sigma_4^2+\mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)}g_n\big)^2\Big)^{\frac{p}{p+2}} \\
=&\Big(\sum_{n=1}^{\infty}(\frac{\mu}{\sigma_3^2+\mu}\cdot\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} \\
&\times(\psi_n-\psi_n^\delta+\psi_n^\delta) \\
&-\frac{\mu}{\sigma_4^2+\mu}\cdot\frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha)-q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} \\
&\times(g_n-g_n^\delta+g_n^\delta))^2\Big)^{\frac{p}{p+2}}
\end{aligned}$$

$$\leq \Big(4(\frac{2M}{C_7})^2 \sum_{n=1}^{\infty} (\psi_n - \psi_n^\delta)^2 + 4(\frac{2M}{C_7})^2 \sum_{n=1}^{\infty} (g_n - g_n^\delta)^2$$
$$+2 \sum_{n=1}^{\infty} \big(\frac{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)}{q(\xi(T))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha) - q(\xi(t_0))(1-E_{\alpha,1}(-\frac{\lambda_n}{2}t_0^\alpha))E_{\alpha,1}(-\frac{\lambda_n}{2}T^\alpha)} \big)^2$$
$$\times \big(\frac{\mu}{\sigma_4^2+\mu} \frac{q(\xi(t_0))t_0^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}t_0^\alpha)}{q(\xi(T))T^\alpha E_{\alpha,\alpha+1}(-\frac{\lambda_n}{2}T^\alpha)} g_n^\delta - \frac{\mu}{\sigma_3^2+\mu}\psi_n^\delta \big)^2 \Big)^{\frac{p}{p+2}}$$
$$\leq \big(4(\frac{2M}{C_7})^2\delta^2 + 4(\frac{2M}{C_7})^2\delta^2 + 2(\frac{2M}{C_7})^2(\tau_2\delta)^2 \big)^{\frac{p}{p+2}}.$$

Therefore, we have

$$\|\varphi_\mu(\cdot) - \varphi(\cdot)\| \leq C_{15}\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}. \tag{4.46}$$

Combining (4.40), (4.45) and (4.46), we obtain

$$\|\varphi_\mu^\delta(\cdot) - \varphi(x)\| \leq \begin{cases} \widetilde{C_7}\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, \ 0<p<2, \\ \widetilde{C_8}\delta^{\frac{1}{2}} E^{\frac{1}{2}}, \qquad p \geqslant 2. \end{cases} \tag{4.47}$$

Where $\widetilde{C_7} = C_{15} + (\frac{C_2 C_{11}'}{C_6(\tau_2 - \frac{2}{C_6} - 1)})^{\frac{2}{p+2}}$, $\widetilde{C_8} = C_{15} + (\frac{C_2 C_{12}'}{C_6(\tau_2 - \frac{2}{C_6} - 1)})^{\frac{1}{2}}$. The proof of Theorem 4.4 is completed. □

5. Numerical implementation

In Section 5, we provide two examples to demonstrate the effectiveness and feasibility of using the Tikhonov method to simultaneously identify initial values and source terms. We take $\Omega = (0,1)$ for the one-dimensional situation and $\Delta u = \frac{d^2u}{dx^2}$. Let $\Omega = (0,1)\times(0,1)$ for the two-dimensional situation and $\Delta u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}$. In both cases, $t_0 = \frac{T}{2}, T = 1$ is fixed. First of all, we consider following one-dimensional forward problem

$$\begin{cases} {}_0^{GL}D_t^\alpha u(x,t) + \int_0^t ker(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau & \\ = \Delta u(x,t) + F(x,t), & x \in (0,1), t\in(0,1], \\ u(0,t) = u(1,t) = 0, & t \in (0,1], \\ u(x,0) = \varphi(x), & x\in(0,1), \\ u(x,0.5) = \psi(x), & x \in (0,1), t \in (0,1], \\ u(x,1) = g(x), & x \in (0,1). \end{cases} \tag{5.1}$$

We define $h = \frac{1}{M}, \tau = \frac{1}{N}$, then

$$x_i = ih(i=0,1,...,M),\ t_k = k\tau(k=0,1,...,N). \tag{5.2}$$

The finite difference method is used to solve the problem. Grünwald-Letnikov discretization method is given by [31]

$${}_0^{GL}D_t^\alpha U(x_i,t_n) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{[\frac{t-a}{h}]}(-1)^j\binom{\alpha}{j}U(x_i,t_n - jh) \approx \frac{1}{h^\alpha}\sum_{j=0}^{[\frac{t-a}{h}]}\omega_j^{(\alpha)}U(x_i,t_n - jh), \tag{5.3}$$

where $\omega_j^{(\alpha)} = (-1)^j \binom{\alpha}{j}$ is the polynomial coefficients of $(1-z)^\alpha$, and can be calculate by the following the recurrence formula:

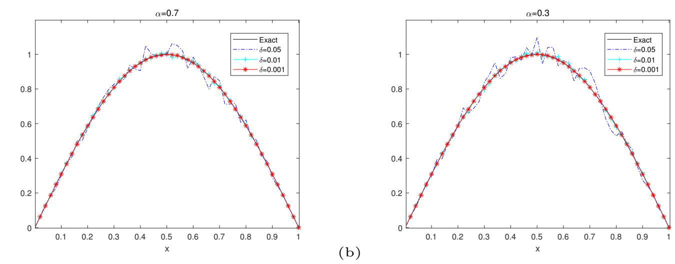
$$\omega_0^{(\alpha)} = 1, \omega_j^{(\alpha)} = \Big(1 - \frac{\alpha+1}{j}\Big)\omega_{j-1}^{(\alpha)}, j = 1, 2, ... \tag{5.4}$$

The discretization method for the second term of equation

$$\begin{aligned}&\int_0^t ker(t-\tau)\frac{\partial u(x,\tau)}{\partial \tau}d\tau \\ =&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0^{(\alpha)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)}U_i^0\Big].\end{aligned} \tag{5.5}$$

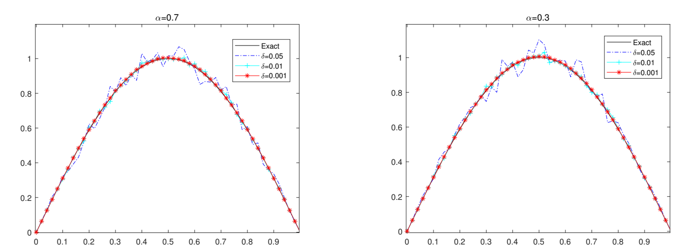
The finite difference method for Laplace operator Δ is as follows

$$\Delta U(x_i, t_n) = \frac{1}{h^2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n). \tag{5.6}$$



(a) (b)

Figure 1. The exact solution and regularized solution of the source term.



(a) (b)

Figure 2. The exact solution and regularized solution of the initial value.

Consider Equation (5.1) at node (x_i, t_n) and combine it with (5.3), (5.5), (5.6), we can obtain following difference scheme

$$\begin{cases} \dfrac{1}{h^\alpha}\displaystyle\sum_{j=0}^{[\frac{t-a}{h}]}\omega_j^{(\alpha)}U_i^{n-jh} + \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[a_0^{(\alpha)}U_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)})U_i^k - a_{n-1}^{(\alpha)}U_i^0\Big] \\ = \dfrac{1}{h^2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) + f_i^n, \quad 1 \leqslant i \leqslant M-1, 1 \leqslant n \leqslant N, \\ U_i^0 = \varphi(x_i), \quad 1 \leqslant i \leqslant M-1, \\ U_0^n = 0, U_M^n = 0, \ 0 \leqslant n \leqslant N. \end{cases} \tag{5.7}$$

Similarly, we provide a two-dimensional example. By using Matlab to implement the difference scheme above, we can obtain $\psi(x)$ and $g(x)$. In practical situations, there is a certain degree of error in the measurement data, so we add random noise to $\psi(x)$ and $g(x)$

$$\psi^{\delta}=\psi(1+\delta*(2R_d-1)),\ g^{\delta}=g(1+\delta*(2R_d-1)), \tag{5.8}$$

where R_d is generated by the Matlab function *rand.m*. We define the relative error

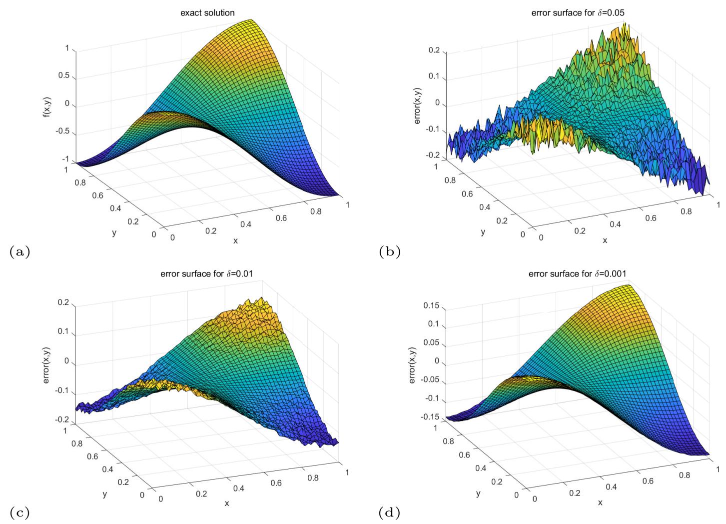


Figure 3. The exact solution and error surface of the source term.

as

$$e_f=\frac{\|f-f_{\mu}^{\delta}\|}{\|f\|},\ e_{\varphi}=\frac{\|\varphi-\varphi_{\mu}^{\delta}\|}{\|\varphi\|}. \tag{5.9}$$

The regularization solution is given by the following equation

$$\begin{aligned}f_{\mu}^{\delta}&=(K_1^*K_1+\mu I)^{-1}K_1^*g^{\delta}+(K_2^*K_2+\mu I)^{-1}K_2^*\psi^{\delta},\\ \varphi_{\mu}^{\delta}&=(K_3^*K_3+\mu I)^{-1}K_3^*\psi^{\delta}+(K_4^*K_4+\mu I)^{-1}K_4^*g^{\delta}.\end{aligned}$$

We use a-posteriori regularization parameter choice rule to select the regularization parameter μ and take $\tau_1=2.1,\tau_2=2.1$. Use the method of Igor Podlubny [16] to obtain f_{μ}^{δ} and φ_{μ}^{δ}. Let $M=50, N=50$, we give the following two examples.

Example 5.1. Let exact source term $f(x)=\sin(\pi x)$ and initial value $\varphi(x)=\sin(\pi x)$. We take $q(t)=\pi^2(\frac{2t^{1-\alpha}}{\Gamma(2-\alpha)}+\pi^2 t)$, consider two scenarios: $\alpha=0.7,\ \alpha=0.3$.

Example 5.2. Let exact source term $f(x,y)=\cos(\pi x)\cos(\pi y)$ and initial value $\varphi(x,y)=2\cos(\pi x)\cos(\pi y)$. We take $q(t)=\frac{1}{2}\pi^2 t+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$, consider $\alpha=0.6$.

Figure 1 and Figure 2 show the curves of exact solution and the Tikhonov approximation solution in one-dimensional case. Figure 1 shows the curves of the exact solution $f(x)$ and the regularized solution $f_{\mu}^{\delta}(x)$ of Example 5.1 for different errors $\delta=0.05,0.01,0.001$, when the fractional order is $\alpha=0.7,0.3$. Figure 2

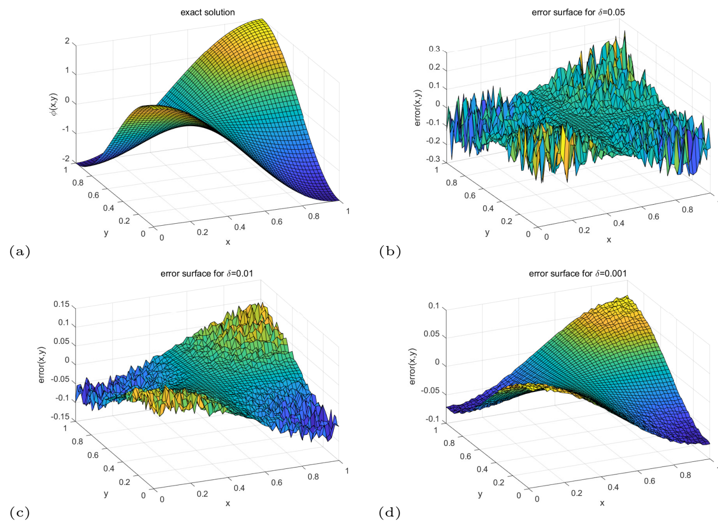


Figure 4. The exact solution and error surface of the initial value.

Table 1. Numerical results of the two examples.

Example	δ	e_f	e_φ
Example 5.2($\alpha = 0.7$)	0.05	0.0355	0.0432
	0.01	0.0133	0.0214
	0.001	0.0028	0.0043
Example 5.1($\alpha = 0.3$)	0.05	0.0274	0.0316
	0.01	0.0124	0.0184
	0.001	0.0012	0.0054
Example 5.2($\alpha = 0.6$)	0.05	0.1298	0.0927
	0.01	0.0628	0.0494
	0.001	0.0139	0.0358

shows the curves of the exact solution $\varphi(x)$ and the regularized solution $\varphi_\mu^\delta(x)$ of Example 5.1 for different errors $\delta = 0.05, 0.01, 0.001$, when the fractional order is $\alpha = 0.7, 0.3$. In addition, from Figure 1 and Figure 2, we can see that the trend of the regularized solution and exact solution is roughly the same. As the error decreases, the the fitting effect of the Tikhonov regularization approximation solution and exact solution is improved.

Figure 3 and Figure 4 show exact solution and the error surface in two-dimensional case. Figure 3 shows source term $f(x, y)$ of Example 5.2, and the error surfaces between the exact solution $f(x, y)$ and the Tikhonov regularization approximation solution $f_\mu^\delta(x, y)$ with errors $\delta = 0.05, 0.01, 0.001$. Figure 4 shows initial value $\varphi(x, y)$ of Example 5.2, and the error surfaces between the exact solution $\varphi(x, y)$ and the Tikhonov regularization approximation solution $\varphi_\mu^\delta(x, y)$ with errors $\delta = 0.05, 0.01, 0.001$.

Table 1 presents the relative error results of Example 5.1 and Example 5.2 for

$\delta = 0.05, 0.01, 0.001$. From numerical simulation results, we can see that the relative error is small. As the error decreases, the fitting effect of the exact solution and the regularized solution is improved. Based on the above two examples, our method is reasonable.

6. Conclusion

In this paper, we investigate the simultaneous identification of source terms and initial values for the multi-term time fractional diffusion equation. This is an inverse problem and it is ill-posed. Therefore, we use the Tikhonov regularization method. Based on a-priori bound assumption and a-priori, a-posteriori regularization parameter choice rules, we derive the convergence estimates. The convergence estimates and several numerical examples demonstrate the efficiency and feasibility of the proposed regularization method for simultaneously identifying initial values and source terms.

Disclosure statement

The authors declare no conflict of interest.

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