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STABILITY OF A NONAUTONOMOUS DELAYED PERIODIC REACTION-DIFFUSION PREDATOR-PREY MODEL

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Abstract This article focuses on the stability of periodic solution of a delayed nonautonomous reaction-diffusion predator-prey model. The fine combination of upper and lower solution methods and Lyapunov stability theory is used to transform the study of the stability problem of delayed reaction-diffusion equations into the stability problem of their corresponding delayed ordinary differential equations. Some sufficient conditions are given to ensure the globally asymptotically stability of the periodic solution for this model. Unlike existing results, the stable solution which are obtained in this article is a time-periodic solution rather than a constant periodic solution or a solution for a steady-state system. We extend a stability theorem on predator-prey model introduced by V. Ortega and C. Rebelo in 2023 to nonautonomous delayed reaction-diffusion model. Finally, in order to show the application of the theoretical results, the proposed conditions are numerically validated over a 2-periodic delayed nonautonomous reaction-diffusion predator-prey model.

Keywords Reaction-diffusion predators-prey model, nonautonomous model, periodic solution, stability and permanent property, method of upper and lower solutions.

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1. Introduction

The study of the properties of reaction-diffusion equations (RDEs) can be used to describe many phenomena in the population and epidemic system, which has attracted increasing attention from scholars [3–5, 8–10, 14, 19, 20]. Especially, the research on predator-prey RDEs has recently achieved many excellent results. For example, in 2013, Ko and Ahn [17] studied RDEs with two competing predators and one prey and obtained some sufficient conditions to ensure the persistence and global attractiveness for solutions of the system. In 2015, Yang et al. [34] studied RDEs with Leslie-Gower functional response and gained sufficient conditions to guarantee the coexistence state and attractor existence of the model by using fixed point index theory. In 2017, Wang [29] studied RDEs with Neumann boundary

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conditions and Holling Type III functional response and obtained sufficient conditions to guarantee the existence of periodic orbits by using coincidence theory and bifurcation methods. In 2020, Wu and Zhao [30] studied RDEs with the Allee effect and threshold hunting and analyzed the asymptotic stability of the equilibrium point of the model by constructing a Jacobian matrix. In 2021, Bentout et al. [2] studied an age-structured predator-prey infection model, and explored the impact of predator maturity on interspecies interactions and infectious disease spread. The findings reveal that the minimum maturation time of predators can influence the behavior of the system's solutions. In the same year, Djilali and Cattani [12] analyzed a superdiffusive predator-prey system with a hunting cooperation functional response. The presence of superdiffusion represents the fear effect of the prey and the organized hunting strategy of the predator. The study indicates that superdiffusion leads to complex dynamical behaviors of the system's solutions and can influence the stability of certain equilibria. In 2022, Yan and Zhang [33] studied RDEs with a Beddington-DeAngelis functional response and obtained stability and instability criteria for the positive constant equilibrium point of the model. In 2023, Chen and Wu [6] studied the spatiotemporal behavior of RDEs with a Beddington-DeAngelis functional response function by using the Leray-Schauder degree theory and Poincare inequality. It is worth mentioning that the above models are autonomous RDEs. Due to the fact that the birth rate, the death rate and the interaction between population are not invariable, nonautonomous RDEs can better simulate the interactions among species in predator-prey models. However, the methods used in the previous literature are difficult to study multi-species nonautonomous predator-prey RDEs. More recently, Jia et al. [15] has considered a 3-species nonautonomous predator-prey RDEs and has obtained some judgment criteria to ensure the globally asymptotically stability of strictly positive homogenous periodic solution for the system by using the upper and lower solutions method and Lyapunov stability theory.

In a large amount of the real world, the state of a system is influenced not only by its current state, but also often by their past state. Even in some phenomena, if you ignore the impact of the system's past state on its future state, the entire research is meaningless. Therefore, when describe the impact of the interaction between time delayed feedback and spatial transfer on the system state, scientists have proposed a new type of mathematical model-delayed reaction-diffusion equations (DRDEs). Using these equations, many real natural phenomena are described and well explained. In recent years, the research on DRDEs has attracted more and more attention of scholars. Early research on DRDEs was mostly included in academic works [28, 31]. In recent years, some excellent achievements have been achieved in the study of periodic solutions to DRDEs. For example, in 2016, Chen and Yu [7] considered a DRDEs with nonlocal delay effect and Dirichlet boundary condition and obtained stability criteria for the positive equilibrium point of the model. In 2017, Shi et al. [25] studied a DRDEs with distributed delay and Dirichlet boundary condition and obtained stability conditions of the positive steady state for the model. In 2018, Yuan and Guo [36] studied a nonlocal DRDE and achieved the the existence and stability of solutions for the model with the help of monotone iteration methods. In 2019, Shen and Wei [24] studied a mussel-algae DRDEs with Neumann boundary conditions and obtained the stability conditions for the positive constant steady state. In 2021, Zuo and Shi [38] researched a general DRDEs with spatiotemporal nonlocal delay effect and Dirichlet boundary conditions and obtained some criteria to ensure the existence and stability of positive steady-state solutions for the system. In 2022, Xu et al. [32] analyzed a general DRDEs with predator maturation delay and obtained global asymptotic stability of the positive constant steady state. In 2023, Yuan and Guo [35] studied a class of DRDEs with spatial nonlocality and achieved some criteria to ensure the stability of positive steady-state solutions. In the same year, Djilali et al. [11] studied a class of spatially heterogeneous DRDEs. By employing the Kuratowski measure of noncompactness, the existence of a global compact attractor for the system is demonstrated. Furthermore, sufficient conditions for the uniform persistence of solutions and the asymptotic stability of equilibrium solutions are obtained. In 2024, Kumar [18] investigated a prey-predator DRDEs with Leslie-Gower functional response and Smith growth functions and established sufficient conditions of the global stability for the positive constant steady state. It is worth mentioning that the issues studied in the above literatures are the stability of constant equilibrium solutions or steady-state solutions of autonomous DRDEs. Moreover, the research methods in the previous papers, such as eigenvalues, which cannot be used to study nonautonomous DRDEs. To the best of our knowledge, the results about the stability of periodic solution to nonautonomous DRDEs rarely occurred.

Due to the fact that nonautonomous DRDEs can better simulate the interactions between species in predator-prey models and the study of its dynamic properties has very important practical significance. In this article, we focus on the following nonautonomous periodic DRDEs

$$\begin{cases} \partial u_1(x,t)/\partial t - d_1(t)\Delta u_1(x,t) = u_1(x,t)[r_1(t) - a_{11}(t)u_1(x,t-\tau_1) \\ \qquad\qquad\qquad\qquad\qquad\qquad -a_{12}(t)u_2(x,t)], \\ \partial u_2(x,t)/\partial t - d_2(t)\Delta u_2(x,t) = u_2(x,t)[-r_2(t) - a_{22}(t)u_2(x,t-\tau_2) \\ \qquad\qquad\qquad\qquad\qquad\qquad +a_{21}(t)u_1(x,t-\tau_1)], \end{cases} \tag{1.1}$$

with the Neumman boundary and initial conditions

$$\begin{aligned} &\partial u_i(x,t)/\partial n = 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+, u_i(x,t) = \eta_{i0}(x,t) > 0, \\ &(x,t) \in \Omega \times [-\tau,\ 0],\ i = 1,2. \end{aligned} \tag{1.2}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, Δ is a Laplace operator on Ω, $\partial/\partial n$ denotes the outward normal derivation on $\partial\Omega$, $u_i(x,t)$ represents the density of i-th species at location $x = (x_1, x_2, \cdots, x_n)$ and the time t. τ_1 and τ_2 are two constants representing delay and $\tau = \max\{\tau_1, \tau_2\}$. $d_1(t)$ and $d_2(t)$ denote the diffusion rates of the prey and the predator species at time t respectively. $a_{ii}(t)$ represent the interaction within i-th species, $a_{12}(t)$ is the capturing rate of the predator, and $a_{21}(t)$ is the effective conversion rate of predator. $r_1(t)$ and $r_2(t)$ are the reproduction rate of prey (in the absence of predator) and the natural death rate of predator. All the coefficients of the model (1.1)-(1.2) are continuous and positive ω-periodic functions. The model (1.1)-(1.2) is an extension of the classic Lotka-Volterra model, and its degenerate model has been extensively studied, for example see [1, 22, 23]. Amine and Ortega [1] obtained a stability criterion on non-constant periodic solutions for the following model

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)(a(t) - b(t)u_1(t) - c(t)u_2(t)), \\ \dfrac{du_2(t)}{dt} = u_2(t)(d(t) - e(t)u_2(t) + f(t)u_1(t)), \end{cases} \tag{1.3}$$

in terms of the L^∞ norm of the coefficients of a planar linear system associated to the model (1.3). Ortega [22] gave another stability criteria in terms of the L^1 norm and Ortega and Rebelo [23] obtained a new stability criterion which establishes a bridge between the stability criteria in [1] and [22] in terms of L^p norm.

The stability on time-periodic solution for nonautonomous DRDEs has not been studied before. In this article, we intend to study the time-periodic solutions for the predator-prey Lotka-Volterra models governed by nonautonomous DRDEs and generalize the stability result on (1.3) obtained in [23]. Meanwhile, the methods obtained in this article can also be used to extend the permanent result obtained in [21] to nonautonomous cooperative DRDEs.

The article organization are showed as follows. In Section 2, we will investigate the existence of the time-periodic solution of the nonautonomous predator-prey DRDEs. In Section 3, we pay more attention to the globally asymptotically stability of the time-periodic solution. In Section 4, we will give a numerical example to show the application of the theoretical findings obtained in this article.

Remark 1.1. The innovations and achievements of this article are listed as follows: (1) By introducing the time delays and the variable coefficient into the known population models, a new Lotka-Volterra predator-prey model (nonautonomous predator-prey DRDEs) that can more truly depict the interaction among populations is proposed. (2) By considering of the upper and lower solution methods and Lyapunov stability theory as well as fixed point theory, some new theories and methods have been creatively developed, the existence and stability of the positive time-dependent periodic solution of the new predator-prey DRDEs are obtained only a set of simplify verified conditions are needed. (3) The technique of constructing Lyapunov functions for delayed differential equations step by step can be used to study related problems, which will provide an effective method to study the stability of solutions to delayed partial differential equations. (4) Compared with the existing results, the stable solution obtained in this article is a time-periodic solution rather than a constant periodic solution or a solution for a steady-state system, which will be more in line with the objective law of seasonal cyclical changes in species density.

2. Existence of spatial homogeneous periodic solutions

Set $\varphi(t)$ be a ω -periodic function in $\mathbb{R}_+$. We denote

$$\varphi^m = \sup\{\varphi(t), t \in \mathbb{R}_+\}, \varphi^l = \inf\{\varphi(t), t \in \mathbb{R}_+\}.$$

Next, we study the functional differential equations corresponding to the model

(1.1)

$$\begin{cases} \dfrac{du_1(t)}{dt} = u_1(t)[r_1(t) - a_{11}(t)u_1(t-\tau_1) - a_{12}(t)u_2(t)], \\ \dfrac{du_2(t)}{dt} = u_2(t)[-r_2(t) - a_{22}(t)u_2(t-\tau_2) + a_{21}(t)u_1(t-\tau_1)], \end{cases} \tag{2.1}$$

with the initial conditions

$$u_i(t) = \eta_{i0}(t) > 0,\ t \in [-\tau,\ 0],\ i = 1, 2. \tag{2.2}$$

Theorem 2.1. *For any positive initial conditions, the solution of the models (2.1-(2.2) is positive.*

Proof. Due to the continuity of the functions on the right-hand side of model (2.1) and its satisfaction of the local Lipschitz condition, models (2.1)-(2.2) possesses a unique local solution, denoted as $u_1(t)$ and $u_2(t)$, on a small interval $[0, T)$ according to the existence and uniqueness theorem for solutions of functional differential equations. Next, we prove that for any positive initial values, this local solution remains positive and can be extended to the entire positive time axis.

According to the first equation of model (2.1), when $u_1(t) > 0$ and $u_2(t) > 0$, the sign of the right-hand side of the equation depends on $r_1(t) - a_{11}(t)u_1(t-\tau_1) - a_{12}(t)u_2(t) \triangleq \Lambda_1(t)$. If the initial value results in $\Lambda_1(t) > 0$, then the rate of change of $u_1(t)$ is positive, and since the initial value $\eta_{10}(t)$ is positive, $u_1(t)$ will remain positive. If the initial values result in $\Lambda_1(t) < 0$, then the rate of change of $u_1(t)$ is negative, and due to the positive initial value, the prey population $u_1(t)$ will decrease. Based on the interaction mechanisms among populations in ecosystems and the continuity of population dynamics. Subsequently, the predator population $u_2(t)$ will also decrease due to insufficient food. Since $r_1(t) > 0$, as the populations of $u_1(t)$ and $u_2(t)$ decrease, eventually $\Lambda_1(t)$ will become positive, causing the population of $u_1(t)$ to increase before reaching zero. In summary, regardless of whether the initial values make $\Lambda_1(t) > 0$ or $\Lambda_1(t) < 0$, $u_1(t)$ remains positive.

Similarly, according to the second equation of model (2.1), when $u_1(t) > 0$ and $u_2(t) > 0$, the sign of the right-hand side of the equation depends on $-r_2(t) - a_{22}(t)u_2(t-\tau_2) + a_{21}(t)u_1(t-\tau_1) \triangleq \Lambda_2(t)$. If the initial values result in $\Lambda_2(t) > 0$, then the rate of change of $u_2(t)$ is positive, and since the initial value $\eta_{20}(t)$ is positive, $u_2(t)$ will remain positive. If the initial values result in $\Lambda_2(t) < 0$, then the rate of change of $u_2(t)$ is negative, and due to the positive initial value, the predator population $u_2(t)$ will continuously decrease. Based on the interaction mechanisms among populations in ecosystems and the continuity of population dynamics. Simultaneously, the prey population $u_1(t)$ will increase due to the reduction in predators. As $u_2(t)$ decreases and $u_1(t)$ increases, eventually $\Lambda_2(t)$ will become positive (since $a_{12}(t) > 0$), causing the population of $u_2(t)$ to increase before decreasing to zero. In summary, regardless of whether the initial values make $\Lambda_2(t) > 0$ or $\Lambda_2(t) < 0$, $u_2(t)$ remains positive.

Furthermore, since the local solution is unique and positive, we can utilize the continuous dependence theorem for solutions of functional differential equations to extend the local solution to the entire positive time axis while maintaining its positivity. Therefore, given the initial conditions $\eta_{10}(t) > 0$ and $\eta_{20}(t) > 0$, the solutions of the predator-prey models (2.1)-(2.2) remain positive on the entire positive time axis. □

For the model (2.1), set

$$
\begin{aligned}
M_1 &= \frac{r_1^m}{a_{11}^l}\exp\left\{r_1^m\tau_1\right\}, \ M_2 = \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l}\exp\left\{(a_{21}^m M_1 - r_2^l)\tau_2\right\},\\
m_1 &= \frac{r_1^m - a_{12}^m M_2}{a_{11}^m}\exp\left\{(r_1^l - a_{12}^m M_2 - a_{11}^m M_1)\tau_1\right\},\\
m_2 &= \frac{a_{21}^l m_1 - r_2^m}{a_{22}^m}\exp\left\{(a_{21}^l m_1 - r_2^m - a_{22}^m M_2)\tau_2\right\}.
\end{aligned}
$$

Theorem 2.2. *Assume the following conditions satisfy*

$$
\begin{aligned}
&(H_1)\ a_{21}^l m_1 > r_2^m,\\
&(H_2)\ r_1^l > a_{12}^m M_2.
\end{aligned}
$$

Then the model (2.1)-(2.2) is permanent.

Proof. By the first equation of model (2.1), it follows that

$$
\frac{du_1(t)}{dt} = u_1(t)[r_1(t) - a_{11}(t)u_1(t-\tau_1) - a_{12}(t)u_2(t)] \le u_1(t)[r_1^m - a_{11}^l u_1(t-\tau_1)]. \tag{2.3}
$$

From the Lemma 2.2 in [21], one has

$$
\limsup_{t\to+\infty} u_1(t) \le \frac{r_1^m}{a_{11}^l}\exp\left\{r_1^m\tau_1\right\} = M_1. \tag{2.4}
$$

Moreover, from the second equation of model (2.1), it holds that

$$
\begin{aligned}
\frac{du_2(t)}{dt} &\le u_2(t)[-r_2(t) - a_{22}(t)u_2(t-\tau_2) + a_{21}(t)u_1(t-\tau_1)]\\
&\le u_2(t)[-r_2^l - a_{22}^l u_2(t-\tau_2) + a_{21}^m M_1].
\end{aligned}
$$

By (H_1), we have $a_{21}^m M_1 > r_2^l$. Thus, by Lemma 2.2 in [21],

$$
\limsup_{t\to+\infty} u_2(t) \le \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l}\exp\left\{(a_{21}^m M_1 - r_2^l)\tau_2\right\} = M_2. \tag{2.5}
$$

On the other hand, by (2.1),

$$
\frac{du_1(t)}{dt} \ge u_1(t)[r_1^l - a_{11}^m u_1(t-\tau_1) - a_{12}^m M_2] = u_1(t)[r_1^l - a_{12}^m M_2 - a_{11}^m u_1(t-\tau_1)].
$$

By (H_2) and Lemma 2.3 in [21],

$$
\liminf_{t\to+\infty} u_1(t) \ge \frac{r_1^l - a_{12}^m M_2}{a_{11}^m}\exp[(r_1^l - a_{12}^m M_2 - a_{11}^m M_1)\tau_1] = m_1. \tag{2.6}
$$

Similarly, by the second equation in model (2.1), we have

$$
\frac{du_2(t)}{dt} = u_2(t)[-r_2(t) - a_{22}(t)u_2(t-\tau_2) + a_{21}(t)u_1(t-\tau_1)]
$$

$$\geq u_2(t)[-r_2^m - a_{22}^m u_2(t-\tau_2) + a_{21}^l m_1]$$
$$= u_2(t)[a_{21}^l m_1 - r_2^m - a_{22}^m u_2(t-\tau_2)].$$

By (H_1) and Lemma 2.3 in [21],

$$\liminf_{t\to+\infty} u_2(t) \geq \frac{a_{21}^l m_1 - r_2^m}{a_{22}^m} \exp[(a_{21}^l m_1 - r_2^m - a_{22}^m M_2)\tau_2] = m_2. \tag{2.7}$$

By (2.4)-(2.7), we see that the model (2.1)-(2.2) is permanent, see [Definition 2.1, [27]] for the definition of permanent property. □

Theorem 2.3. *Assume that* $(H_1) - (H_2)$ *hold. Then there is a strictly positive spatial homogeneity* ω*-periodic solution of (1.1)-(1.2).*

Proof. Let $V = C([-\tau, +\infty), \mathbb{R}_+^2)$ be a Banach space consisting of continuous, bounded, ω-periodic and positive functions defined on $[-\tau, +\infty)$, equipped with the infinite norm. Based on the existence and uniqueness theorem of solutions of the functional differential equations, see [Theorem 2.3, page 42 of [13]], we define a Poincaré mapping $\psi:\ V \to V$ in the following form

$$\psi(U_0) = U(t,\ \omega,\ U_0),$$

where $U(t,\omega,U_0) = (u_1(t), u_2(t))$ is a positive solution of the functional differential equations (2.1) subject to the initial conditions $U_0 = (\eta_{10}(t), \eta_{20}(t)), t \in [-\tau,\ 0]$.

It easy to see that ψ is continuous mapping by using the continuity of solution of the functional differential equations (2.1) with regard to the above initial conditions, see [Theorem 4.1, page 46 of [13]]. Assume that K is any bounded set in V. For any $U_0 \in K$, and let $L = \sqrt{M_1^2 + M_2^2}$. From the permanence of solutions to models (2.1)-(2.2), we have $\|\psi(U_0)\| = \|U(t,\omega,U_0)\| = \|(u_1(t), u_2(t))\| = \sqrt{u_1(t)^2 + u_2(t)^2} \leq \sqrt{M_1^2 + M_2^2} = L$. Hence, $\psi(K)$ is uniformly bounded. Furthermore, according to Theorem 2.2, the derivative of the mapping ψ is also bounded, which can then be used to prove that the $\psi(K)$ is equicontinuous. The Arzela-Ascoli theorem implies that ψ is completely continuous.

We define

$$S = \{(u_1(t), u_2(t)) \in V \mid m_i \leqslant u_i(t) \leqslant M_i,\ i = 1,\ 2\}, \tag{2.8}$$

then it is obvious that S is a closed bounded convex subset of the Banach space V. By Theorem 2.2 we have that ψ is a mapping from S to S. Thus, by Schauder fixed-point theorem, see [Lemma 2.4, page 40 of [13]], the mapping ψ has a fixed point $(u_1^*(t), u_2^*(t))$. That is, the equations (2.1)-(2.2) have a positive ω-periodic solution $(u_1^*(t), u_2^*(t))$ which is the spatial homogeneity ω-periodic solution for models (1.1)-(1.2), see [Definition 2.2, [37]]. □

3. Stability of spatial homogeneity periodic solution

In this section, we provide some sufficient conditions to obtain the globally asymptotically stable of spatial homogeneity ω-periodic solution of (1.1) by using the method of upper and lower solutions for the delayed parabolic partial differential equations and Lyapunov stability theory.

Theorem 3.1. *Assume that* $(H_1)-(H_2)$ *and the following assumptions hold.*

(H_3) $A_1 = a_{11}^l - a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2] - M_1(a_{11}^m)^2\tau_1 - a_{21}^m(1 + a_{22}^m M_2\tau_2) > 0,$

(H_4) $A_2 = a_{22}^l - a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1] - M_2(a_{22}^m)^2\tau_2 - (1 + M_1 a_{11}^m\tau_1)a_{12}^m > 0.$

Then there exists a spatial homogeneity strictly positive and globally asymptotically stable ω*-periodic solution* $(u_1^*(t), u_2^*(t))$*, that is, the solution* $(u_1(x,t), u_2(x,t))$ *of (1.1)-(1.2) with any positive initial values fulfills*

$$\lim_{t\to\infty}(u_i(x,t) - u_i^*(t)) = 0, \textit{uniformly for } x \in \bar\Omega, i = 1,2. \tag{3.1}$$

Proof. By Theorem 2.3, (1.1)-(1.2) has a spatial homogeneity strictly positive ω-periodic solution. We prove the stability of the solution. Because the solutions and coefficients of model (1.1) are positive, it is easy to see that the reaction functions of model (1.1) are mixed quasimonotone. Let $l_i = \min_{x\in\bar\Omega,\ t\in[-\tau,\ 0]} \eta_{i0}(x,t)$, $r_i = \max_{x\in\bar\Omega,\ t\in[-\tau,\ 0]} \eta_{i0}(x,t)$. Then $0 < l_i \le \eta_{i0}(x,t) \le r_i$. Let $(\tilde u_1(t), \tilde u_2(t))$ and $(\hat u_1(t), \hat u_2(t))$ be the solutions of (2.1) subject to initial values $(\eta_{10}(t), \eta_{20}(t)) = (r_1, r_2)$ and $(\eta_{10}(t), \eta_{20}(t)) = (l_1, l_2)$ respectively, then there exist upper and lower solutions $(\tilde u_1(t), \tilde u_2(t))$ and $(\hat u_1(t), \hat u_2(t))$ of (1.1)-(1.2). By Theorem 2.1 in [26], (1.1)-(1.2) has a unique solution $(u_1(x,t), u_2(x,t))$, $(x,t) \in \bar\Omega \times [-\tau,\ +\infty)$, which satisfies

$$(\hat u_1(t), \hat u_2(t)) \le (u_1(x,t), u_2(x,t)) \le (\tilde u_1(t), \tilde u_2(t)). \tag{3.2}$$

We prove

$$\lim_{t\to\infty}(\tilde u_i(t) - u_i^*(t)) = \lim_{t\to\infty}(\hat u_i(t) - u_i^*(t)) = 0, (i = 1,2). \tag{3.3}$$

We first prove the solution $(u_1(t), u_2(t))$ for the functional differential equations (2.1) with any positive initial $(u_1(t), u_2(t)) = (\eta_{10}(t), \eta_{20}(t))$ satisfies

$$\lim_{t\to\infty}(u_i(t) - u_i^*(t)) = 0,\ i = 1,2. \tag{3.4}$$

By Theorem 2.2, there exist five positive real numbers M_i, m_i and T such that

$$m_i \le u_i(t) \le M_i \text{ when } t > T.$$

Let

$$V_{11}(t) = |\ln u_1(t) - \ln u_1^*(t)|.$$

We denote by $D^+V_{11}(t)$ the right-side derivative of $V_{11}(t)$, then

$$\begin{aligned} D^+V_{11}(t) &= \operatorname{sgn}(u_1(t) - u_1^*(t))[-a_{11}(t)(u_1(t-\tau_1) - u_1^*(t-\tau_1)) - a_{12}(t)(u_2(t) \\ &\quad - u_2^*(t))] \\ &= \operatorname{sgn}(u_1(t) - u_1^*(t))[-a_{11}(t)(u_1(t) - u_1^*(t)) - a_{12}(t)(u_2(t) - u_2^*(t)) \\ &\quad + a_{11}(t)\int_{t-\tau_1}^{t}(\dot u_1(\theta) - \dot u_1^*(\theta))d\theta] \\ &= \operatorname{sgn}(u_1(t) - u_1^*(t))[-a_{11}(t)(u_1(t) - u_1^*(t)) - a_{12}(t)(u_2(t) - u_2^*(t)) \end{aligned}$$

$$\begin{aligned}
&+ a_{11}(t)\int_{t-\tau_1}^{t}\{u_1(\theta)[r_1(\theta) - a_{11}(\theta)u_1(\theta-\tau_1) - a_{12}(\theta)u_2(\theta)] \\
&- u_1^*(\theta)[r_1(\theta) - a_{11}(\theta)u_1^*(\theta-\tau_1) - a_{12}(\theta)u_2^*(\theta)]\}d\theta] \\
={}& \operatorname{sgn}(u_1(t)-u_1^*(t))[-a_{11}(t)(u_1(t)-u_1^*(t)) - a_{12}(t)(u_2(t)-u_2^*(t)) \\
&+ a_{11}(t)\int_{t-\tau_1}^{t}\{(u_1(\theta)-u_1^*(\theta))[r_1(\theta) - a_{11}(\theta)u_1^*(\theta-\tau_1) - a_{12}(\theta)u_2^*(\theta)] \\
&- u_1(\theta)[a_{11}(\theta)(u_1(\theta-\tau_1)-u_1^*(\theta-\tau_1)) + a_{12}(\theta)(u_2(\theta)-u_2^*(\theta))]\}d\theta] \\
\le{}& -a_{11}(t)\left|u_1(t)-u_1^*(t)\right| + a_{12}(t)\left|u_2(t)-u_2^*(t)\right| \\
&+ a_{11}(t)\int_{t-\tau_1}^{t}([r_1(\theta) + a_{11}(\theta)u_1^*(\theta-\tau_1) + a_{12}(\theta)u_2^*(\theta)]\left|u_1(\theta)-u_1^*(\theta)\right| \\
&+ u_1(\theta)[a_{11}(\theta)\left|u_1(\theta-\tau_1)-u_1^*(\theta-\tau_1)\right| + a_{12}(\theta)\left|u_2(\theta)-u_2^*(\theta)\right|])d\theta.
\end{aligned} \tag{3.5}$$

Let

$$\begin{aligned}
V_{12}(t) = {}&\int_{t-\tau_1}^{t}\int_{s}^{t} a_{11}(s+\tau_1)([r_1(\theta) + a_{11}(\theta)u_1^*(\theta-\tau_1) + a_{12}(\theta)u_2^*(\theta)] \\
&\times\left|u_1(\theta)-u_1^*(\theta)\right| + u_1(\theta)[a_{11}(\theta)\left|u_1(\theta-\tau_1)-u_1^*(\theta-\tau_1)\right| \\
&+a_{12}(\theta)\left|u_2(\theta)-u_2^*(\theta)\right|])d\theta ds.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6),

$$\begin{aligned}
D^+\sum_{i=1}^{2}V_{1i}(t) \le{}& -a_{11}(t)\left|u_1(t)-u_1^*(t)\right| + a_{12}(t)\left|u_2(t)-u_2^*(t)\right| \\
&+ \int_{t-\tau_1}^{t} a_{11}(s+\tau_1)ds([r_1(t) + a_{11}(t)u_1^*(t-\tau_1) \\
&+ a_{12}(t)u_2^*(t)]\left|u_1(t)-u_1^*(t)\right| + u_1(t)[a_{11}(t)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right| \\
&+ a_{12}(t)\left|u_2(t)-u_2^*(t)\right|]) \\
\le{}& -a_{11}(t)\left|u_1(t)-u_1^*(t)\right| + a_{12}(t)\left|u_2(t)-u_2^*(t)\right| \\
&+ \int_{t-\tau_1}^{t} a_{11}(s+\tau_1)ds[r_1(t) + a_{11}(t)M_1 + a_{12}(t)M_2]\left|u_1(t)-u_1^*(t)\right| \\
&+ M_1\int_{t-\tau_1}^{t} a_{11}(s+\tau_1)ds[a_{11}(t)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right| \\
&+ a_{12}(t)\left|u_2(t)-u_2^*(t)\right|] \\
\le{}& (-a_{11}^l + a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2])\left|u_1(t)-u_1^*(t)\right| \\
&+ M_1(a_{11}^m)^2\tau_1\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right| \\
&+ a_{12}^m(1 + a_{11}^m M_1\tau_1)\left|u_2(t)-u_2^*(t)\right|.
\end{aligned} \tag{3.7}$$

Let

$$V_{13}(t) = M_1(a_{11}^m)^2\tau_1\int_{t-\tau_1}^{t}\left|(u_1(w)-u_1^*(w)\right|dw, \tag{3.8}$$

and

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t). \tag{3.9}$$

By (3.7) and (3.8), we have

$$\begin{aligned} D^+V_1(t) \leq{} & (-a_{11}^l + a_{11}^m\tau_1[r_1^m + a_{11}^m M_1 + a_{12}^m M_2] + M_1(a_{11}^m)^2\tau_1)\left|u_1(t) - u_1^*(t)\right| \\ & +(1 + M_1 a_{11}^m \tau_1)a_{12}^m \left|u_2(t) - u_2^*(t)\right|. \end{aligned} \tag{3.10}$$

Similarly, we define $V_{21}(t) = |\ln u_2(t) - \ln u_2^*(t)|$, and we have

$$\begin{aligned} D^+V_{21}(t) &= \operatorname{sgn}(u_2(t) - u_2^*(t))[-a_{22}(t)(u_2(t-\tau_2) - u_2^*(t-\tau_2)) + a_{21}(t)(u_1(t-\tau_1) \\ &\quad - u_1^*(t-\tau_1))] \\ &= \operatorname{sgn}(u_2(t) - u_2^*(t))[-a_{22}(t)(u_2(t) - u_2^*(t)) + a_{21}(t)(u_1(t-\tau_1) \\ &\quad - u_1^*(t-\tau_1)) + a_{22}(t)\int_{t-\tau_2}^{t} (\dot{u}_2(\theta) - \dot{u}_2^*(\theta))d\theta] \\ &= \operatorname{sgn}(u_2(t) - u_2^*(t))[-a_{22}(t)(u_2(t) - u_2^*(t)) + a_{21}(t)(u_1(t-\tau_1) \\ &\quad - u_1^*(t-\tau_1)) + a_{22}(t)\int_{t-\tau_2}^{t} \{u_2(\theta)[-r_2(\theta) - a_{22}(\theta)u_2(\theta-\tau_2) \\ &\quad + a_{21}(\theta)u_1(\theta-\tau_1)] - u_2^*(\theta)[-r_2(\theta) - a_{22}(\theta)u_2^*(\theta-\tau_2) \\ &\quad + a_{21}(\theta)u_1^*(\theta-\tau_1)]\}d\theta] \\ &= \operatorname{sgn}(u_2(t) - u_2^*(t))[-a_{22}(t)(u_2(t) - u_2^*(t)) + a_{21}(t)(u_1(t-\tau_1) \\ &\quad - u_1^*(t-\tau_1)) + a_{22}(t)\int_{t-\tau_2}^{t} \{(u_2(\theta) - u_{2^*}(\theta))[-r_2(\theta) - a_{22}(\theta)u_2^*(\theta-\tau_2) \\ &\quad + a_{21}(\theta)u_{1^*}(\theta-\tau_1)] - u_2(\theta)[a_{22}(\theta)(u_2(\theta-\tau_2) - u_2^*(\theta-\tau_2)) \\ &\quad - a_{21}(\theta)(u_1(\theta-\tau_1) - u_1^*(\theta-\tau_1))]\}d\theta] \\ &\leq -a_{22}(t)\left|u_2(t) - u_2^*(t)\right| + a_{21}(t)\left|u_1(t-\tau_1) - u_1^*(t-\tau_1)\right| \\ &\quad + a_{22}(t)\int_{t-\tau_2}^{t} ([r_2(\theta) + a_{22}(\theta)u_2^*(\theta-\tau_2) \\ &\quad + a_{21}(\theta)u_1^*(\theta-\tau_1)]\left|u_2(\theta) - u_2^*(\theta)\right| \\ &\quad + u_2(\theta)[a_{22}(\theta)\left|u_2(\theta-\tau_2) - u_2^*(\theta-\tau_2)\right| \\ &\quad + a_{21}(\theta)\left|u_1(\theta-\tau_1) - u_1^*(\theta-\tau_1)\right|])d\theta. \end{aligned} \tag{3.11}$$

Let

$$\begin{aligned} V_{22}(t) = \int_{t-\tau_2}^{t}\int_{s}^{t} & a_{22}(s+\tau_2)([r_2(\theta) + a_{22}(\theta)u_2^*(\theta-\tau_2) \\ & + a_{21}(\theta)u_1^*(\theta-\tau_1)]\left|u_2(\theta) - u_2^*(\theta)\right| + u_2(\theta)[a_{22}(\theta)\left|u_2(\theta-\tau_2) - u_2^*(\theta-\tau_2)\right| \\ & + a_{21}(\theta)\left|u_1(\theta-\tau_1) - u_1^*(\theta-\tau_1)\right|])d\theta ds. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12),

$$\begin{aligned} D^+\sum_{i=1}^{2} V_{2i}(t) \leq{} & -a_{22}(t)\left|u_2(t) - u_2^*(t)\right| + a_{21}(t)\left|u_1(t-\tau_1) - u_1^*(t-\tau_1)\right| \\ & + \int_{t-\tau_2}^{t} a_{22}(s+\tau_2)ds([r_2(t) + a_{22}(t)u_2^*(t-\tau_2) \end{aligned}$$

$$\begin{aligned}
&\quad + a_{21}(t)u_1^*(t-\tau_1)]\left|u_2(t)-u_2^*(t)\right| \\
&\quad + u_2(t)[a_{22}(t)\left|u_2(t-\tau_2)-u_2^*(t-\tau_2)\right| \\
&\quad + a_{21}(t)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right|]) \\
&\le -a_{22}(t)\left|u_2(t)-u_2^*(t)\right| + a_{21}(t)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right| \\
&\quad + \int_{t-\tau_2}^{t} a_{22}(s+\tau_2)ds[r_2(t)+a_{22}(t)M_2+a_{21}(t)M_1]\left|u_2(t)-u_2^*(t)\right| \\
&\quad + M_2\int_{t-\tau_2}^{t} a_{22}(s+\tau_2)ds[a_{22}(t)\left|u_2(t-\tau_2)-u_2^*(t-\tau_2)\right| \\
&\quad + a_{21}(t)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right|] \\
&\le (-a_{22}^l + a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1])\left|u_2(t)-u_2^*(t)\right| \\
&\quad + M_2(a_{22}^m)^2\tau_2\left|u_2(t-\tau_2)-u_2^*(t-\tau_2)\right| \\
&\quad + a_{21}^m(1+a_{22}^m M_2\tau_2)\left|u_1(t-\tau_1)-u_1^*(t-\tau_1)\right|.
\end{aligned} \tag{3.13}$$

Let

$$\begin{aligned}
V_{23}(t) = M_2(a_{22}^m)^2\tau_2 &\int_{t-\tau_2}^{t} \left|(u_2(w)-u_2^*(w)\right| dw \\
&+ a_{21}^m(1+a_{22}^m M_2\tau_2)\int_{t-\tau_1}^{t} \left|(u_1(w)-u_1^*(w)\right| dw,
\end{aligned} \tag{3.14}$$

and

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t). \tag{3.15}$$

By (3.13) and (3.14),

$$\begin{aligned}
D^+V_2(t) \le\ & (-a_{22}^l + a_{22}^m\tau_2[r_2^m + a_{22}^m M_2 + a_{21}^m M_1] + M_2(a_{22}^m)^2\tau_2)\left|u_2(t)-u_2^*(t)\right| \\
&+ a_{21}^m(1+a_{22}^m M_2\tau_2)\left|u_1(t)-u_1^*(t)\right|.
\end{aligned} \tag{3.16}$$

We define a Lyapunov function as follows

$$V(t) = V_1(t) + V_2(t).$$

By (3.10) and (3.16), we get

$$D^+V(t) \le -A_1\left|u_1(t)-u_1^*(t)\right| - A_2\left|u_2(t)-u_2^*(t)\right|. \tag{3.17}$$

Integrating from ω to t on both sides of (3.17), we have

$$V(t) + \alpha\int_{\omega}^{t}(\left|u_1(s)-u_1^*(s)\right| + \left|u_2(s)-u_2^*(s)\right|])ds \le V(\omega) < +\infty, \tag{3.18}$$

where $\alpha = \min\{A_1, A_2\} > 0$. Therefore, $V(t)$ is bounded on $[\omega, +\infty)$, and

$$\int_{\omega}^{t}(\left|u_1(s)-u_1^*(s)\right| + \left|u_2(s)-u_2^*(s)\right|])ds \le \frac{V(\omega)}{\alpha} < +\infty. \tag{3.19}$$

By (3.19), we have

$$(\left|u_1(t)-u_1^*(t)\right| + \left|u_2(t)-u_2^*(t)\right|) \in L^1(T, +\infty). \tag{3.20}$$

From the uniform permanence of model (2.1), we have that $|u_1(t)-u_1^*(t)|+|u_2(t)-u_2^*(t)|$ and its derivative are bounded. Thus, $|u_1(t)-u_1^*(t)|+|u_2(t)-u_2^*(t)|$ is uniformly continuous on $[\omega,+\infty)$. By Lemma 8.2 in [16], we get

$$\lim_{t\to+\infty}|u_i(t)-u_i^*(t)|=0,(i=1,2).$$

From (3.2) and the squeeze theorem, (3.1) holds true. That is, (1.1)-(1.2) have a spatial homogeneity strictly positive and globally asymptotically stable ω-periodic solution $(u_1^*(t),u_2^*(t))$, see [Definition 2.3, [37]]. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Suppose that the ω-periodic model (1.1) satisfies assumptions (H_1)–(H_4), then the model (1.1) is permanent, i.e., the solution $(u_1(x,t),u_2(x,t))$ of models (1.1)-(1.2) with any initial values fulfills*

$$m_i\le u_i(x,t)\le M_i,\ \textit{uniformly for}\ (x,t)\in\bar{\Omega}\times[T,+\infty),i=1,2.\tag{3.21}$$

Proof. By means of Theorem 2.3, there exist four positive real numbers m_i,M_i, $(i=1,2)$ such that

$$m_i\le u_i^*(t)=u_i^*(t+\omega)\le M_i,t\in[-\tau,+\infty).\tag{3.22}$$

Moreover, from Theorem 3.1, one has

$$\lim_{t\to+\infty}u_i(x,t)=u_i^*(t),\text{uniformly for }x\in\bar{\Omega},i=1,2.\tag{3.23}$$

Therefore, from (3.22) and (3.23), the model (1.1) is permanent. □

4. Numerical simulations

In this section, we provide a numerical example to show the application of Theorem 3.1. For the convenience of calculation and numerical simulation, we choose 2-period functions as the coefficients for the nonautonomous ω-periodic DRDEs (1.1)-(1.2).

Example 4.1. Consider the following 2-species DRDEs. In view of the conditions $(H_1)-(H_4)$ of Theorem 3.1, with the help of some calculations we choose some special values of parameters shown in models (4.1)-(4.2). It should be noted that, the selection of above parameters is not unique.

$$\begin{cases}\dfrac{\partial u_1(x,t)}{\partial t}-\Delta u_1(x,t)=u_1(x,t)[(24+\cos\pi t)-(6+\sin\pi t)u_1(x,t-0.001)\\ \qquad\qquad\qquad -(0.75+0.25\sin\pi t)u_2(x,t)],\ 2\pi>x>0,t>0,\\ \dfrac{\partial u_2(x,t)}{\partial t}-\Delta u_2(x,t)=u_2(x,t)[-(2+\cos\pi t)-(5+\sin\pi t)u_2(x,t-0.002)\\ \qquad\qquad\qquad +(1.2+0.2\sin\pi t)u_1(x,t-0.001)],\ 2\pi>x>0,t>0,\end{cases}\tag{4.1}$$

with the Neumman boundary and initial conditions

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial n} = \dfrac{\partial u_2(x,t)}{\partial n} = 0, \quad t > 0, x = 0, 2\pi, \\ u_1(x,t) = (4+3t)[1-\sin(x+0.5\pi)], \\ u_2(x,t) = (0.6+5t)[1+\cos(x+\pi)], \\ (x,t) \in (0,2\pi) \times [-0.002,\ 0]. \end{cases} \tag{4.2}$$

By calculating, we have

$$\begin{aligned} M_1 &= \frac{r_1^m}{a_{11}^l} \exp\ \{r_1^m \tau_1\} \approx 5.1266, \\ M_2 &= \frac{a_{21}^m M_1 - r_2^l}{a_{22}^l} \exp\ \left\{(a_{21}^m M_1 - r_2^l)\tau_2\right\} \approx 1.5635, \\ m_1 &= \frac{r_1^m - a_{12}^m M_2}{a_{11}^m} \exp\ \left\{(r_1^l - a_{12}^m M_2 - a_{11}^m M_1)\tau_1\right\} \approx 3.0185, \\ m_2 &= \frac{a_{21}^l m_1 - r_2^m}{a_{22}^m} \exp\ \left\{(a_{21}^l m_1 - r_2^m - a_{22}^m M_2)\tau_2\right\} \approx 0.0030, \\ a_{21}^l m_1 - r_2^m &\approx 0.0185 > 0, \\ r_1^l - a_{12}^m M_2 &\approx 21.4365 > 0, \\ A_1 &= a_{11}^l - a_{11}^m \tau_1 [r_1^m + a_{11}^m M_1 + a_{12}^m M_2] - M_1 (a_{11}^m)^2 \tau_1 - a_{21}^m (1 + a_{22}^m M_2 \tau_2) q \\ &\approx 2.8854 \\ &> 0, \\ A_2 &= a_{22}^l - a_{22}^m \tau_2 [r_2^m + a_{22}^m M_2 + a_{21}^m M_1] - M_2 (a_{22}^m)^2 \tau_2 - (1 + M_1 a_{11}^m \tau_1) a_{12}^m \\ &\approx 2.6166 \\ &> 0. \end{aligned}$$

Based on the above calculation results, it is easy to see that that systems (4.1)-(4.2) satisfy the conditions of Theorem 3.1. From Theorem 3.1 it is easy to know that the systems (4.1)-(4.2) has a strictly positive spatial homogeneity 2-periodic solution $(u_1(x,t), u_2(x,t))$ which satisfies

$$\lim_{t\to+\infty} |u_i(t) - u_i^*(t)| = 0,\ (i = 1,2),\ \text{uniformly for } x \in (0, 2\pi).$$

By employing the software package MATLAB 7.1 and the finite differences method, we can obtain some numerical solutions of the model (4.1) with the boundary conditions and initial conditions (4.2) which are shown in Figure 1 to Figure 2. From Figures 1-2, it is not difficult to find that the model (4.1)-(4.2) have a strictly positive globally asymptotically stable spatial homogeneity 2-periodic solution. In model (4.1)-(4.2), the densities of prey and predator will oscillate periodically with a period of 2 and distribute homogeneously in space when the time is long enough. In order to verify that the periodic solution of the model (4.1)-(4.2) is globally asymptotically stable, we selected different initial values and conducted extensive numerical simulations. The results showed that the 2-periodic solution of the model (4.1)-(4.2) is asymptotically stable for any positive initial value. Please refer to Figure 3 for details.

From the theoretical research in this article, we can clearly see the dynamic stability mechanism of the predator-prey system. When the birth rate of prey species and the post-predation nutrient absorption rates of predator species reach sufficient levels, predator and prey species can maintain a long-term stable survival state, effectively avoiding the risk of population extinction (see Theorem 2.1 and 3.2). More interestingly, under the basic conditions mentioned above, if key factors such as population diffusion rate, interaction strength, and predator natural mortality rate can also meet specific criteria, then under minor time delays, the density of species in the predator-prey system will exhibit periodic changes. This further reveals the beauty of dynamic equilibrium within ecosystems (see Theorem 3.1). This theoretical result has been thoroughly validated through precise numerical simulations, providing a solid theoretical foundation for our understanding, evaluation, and maintenance of ecosystem balance. It is worth noting that the sufficient conditions established in this article are both concise and easy to verify. Specifically, the conditions proposed here are formulated as a series of inequalities rather than strict equations, providing great convenience and flexibility for the application of these theoretical results in practical ecosystem management.

Remark 4.1. The method obtained in this article can not only be used to study the dynamic properties of solutions for various delayed reaction-diffusion predator-prey models, but also to study corresponding cooperative and competition models. In addition, this method may also be used to study the dynamic properties of solutions for time-varying delayed reaction-diffusion population models and fractional-order delayed reaction-diffusion population models.

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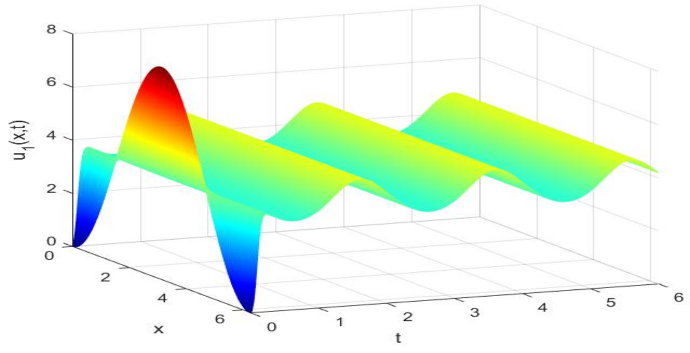

Figure 1. Evolution process of the density for the species $u_1(x,t)$ of model (4.1)-(4.2).

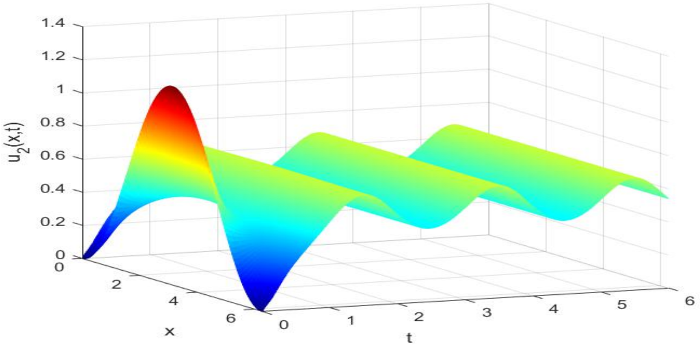

Figure 2. Evolution process of the density for the species $u_2(x,t)$ of model (4.1)-(4.2).

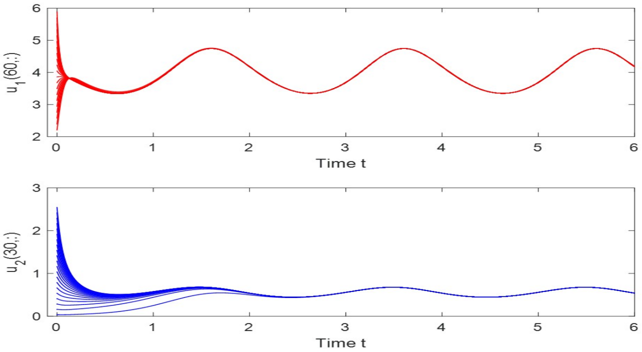

Figure 3. Evolution process of the densities for the species $u_1(x,t)$ and $u_2(x,t)$ of model (4.1)-(4.2) with different positive initial values.

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