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ON THE STUDY TO A TYPE OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM WITH TWO DOUBLE ROOTS OF THE DEGENERATE EQUATION*

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Abstract This paper addresses a singularly perturbed boundary value problem where the degenerate equation has three distinct roots: two double roots and one simple root. It is shown that for a sufficiently small parameter, the solution of the problem switches between the two double roots in a neighborhood of the transition point. As a result, the inner layer can be divided into multiple regions. An asymptotic expansion is constructed, and the existence of smooth solutions is established. Additionally, an estimate for the remainder term is provided.

Keywords Singular perturbations, asymptotic theory, multiple roots.

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1. Introduction

This paper discusses the important topic of singularly perturbed problems, specifically contrast structures. The Tikhonov school in the former Soviet Union first introduced the concept of contrast structures in the late 1990s. A contrast structure occurs in singular perturbation problems where the degenerate equation has distinct roots, known as critical manifolds in geometric singular perturbation theory. As the solution switches between these isolated roots, a complex solution structure forms.

Currently, most research focuses on steptype contrast structures. The main challenge in this area is that the position and timing of the switching are unknown. Moreover, the switching happens over a very short time scale. Since the 1970s, Nefedov and Ni [10–13] have conducted numerous studies on singularly perturbed problems involving contrast structures. However, all of their studies assume that the critical manifolds are normal hyperbolic manifolds of saddle type. This raises an important issue: whether contrast structures exist when the critical manifold

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is non-normal hyperbolic, or where the degenerate equation has repeated roots. Butuzov [2] was one of the first to investigate such types of problems

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u, x, \varepsilon), \quad 0 < x < 1, \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(0, \varepsilon) = 0, \quad \dfrac{\mathrm{d}u}{\mathrm{d}x}(1, \varepsilon) = 0, \end{cases} \tag{1.1}$$

where

$$f(u, x, \varepsilon) = (u - \varphi_1(x))^2 (u - \varphi_2(x))(u - \varphi_3(x)) - \varepsilon f_1(u, x, \varepsilon).$$

In [2], authors established a new method for studying singularly perturbed problems with repeated roots, which we call the non-standard boundary layer method. This approach successfully addressed the limitation of Vasil'eva's method [14], which cannot be applied to non-hyperbolic manifolds. As a result, problems involving multiple roots have become a major focus in the study of singularly perturbed problems. Butuzov [3, 4, 7, 8] not only conducted extensive research on double roots, but also expanded his work to include triple roots and elliptic problems [1, 5, 6]. Yang [15–17] studied piecewise-smooth systems based on equation (1.1), where

$$f(u, x, \varepsilon) = \begin{cases} f^{(-)}(u, x, \varepsilon), & 0 \le x < x_0, \\ f^{(+)}(u, x, \varepsilon), & x_0 \le x \le 1. \end{cases}$$

Both $f^{(-)}(u, x, \varepsilon)$ and $f^{(+)}(u, x, \varepsilon)$ contain repeated roots. Yang [18, 19] had also extended the research to reaction-diffusion equation.

In this paper, we consider the problem (1.1), where

$$f(u, x, \varepsilon) = -(u - \varphi_1(x))^2 (u - \varphi_2(x))(u - \varphi_3(x))^2 - \varepsilon f_1(u, x, \varepsilon).$$

In this paper, following Butuzov [2], we study a singularly perturbed boundary value problem where the degenerate equation has three distinct roots: two double roots and one simple root. The key difference between [2] and our study is that we focus on two double roots, whereas Butuzov studied a single root and a double root. The difficulty arises because the degenerate roots can "jump" from one non-hyperbolic manifold to another, making the existence of a smooth solution more uncertain.

This problem can be treated as two separate sub-problems, referred to as the left and right problems. We construct the formal asymptotic solution using the boundary layer method and solve it term by term. Finally, the solutions to the left and right problems are smoothly matched using the seaming method, leading to a smooth solution for the original problem.

We show that when the degenerate equation has repeated roots, i.e., when the critical manifolds are non-normal hyperbolic, there exists a contrast structure between the roots. Unlike the case where the critical manifold is normal hyperbolic, not only is the formal asymptotic solution expanded in terms of fractional powers of the small parameter, but a non-standard method is also used to solve the internal layer functions. These internal layer functions exhibit complex variations, transitioning from exponential decay to power-law decay.

2. Problem statement

We consider singularly perturbed problems with Neumann boundary conditions

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(u,x,\varepsilon), \quad 0 < x < 1, & (2.1) \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(0,\varepsilon) = 0, \quad \dfrac{\mathrm{d}u}{\mathrm{d}x}(1,\varepsilon) = 0, & (2.2) \end{cases}$$

where $\varepsilon > 0$ is a small parameter. Here $f(u,x,\varepsilon)$ has the form

$$f(u,x,\varepsilon) = -(u-\varphi_1(x))^2(u-\varphi_2(x))(u-\varphi_3(x))^2 - \varepsilon f_1(u,x,\varepsilon).$$

We will call a multivariable function smooth if it is infinitely differentiable with respect to all arguments.

Assume that the following conditions are hold.

A1. The functions $\varphi_i(x), i = 1,2,3,$ are smooth and satisfy

$$\varphi_1(x) < \varphi_2(x) < \varphi_3(x)$$

for $0 \leq x \leq 1$.

Condition A1 guarantees that the roots of the degenerate equation are distinct.

A2. The function $f_1(u,x,\varepsilon)$ is smooth, not identically equal to zero for $0 \leq x \leq 1$, and satisfies

$$\bar{f_1}^{(-)}(x) := f_1(\varphi_1(x),x,0) > 0,$$
$$\bar{f_1}^{(+)}(x) := f_1(\varphi_3(x),x,0) < 0.$$

A3. The equation

$$I(\bar{x}_0) := \int_{\varphi_1(x_0)}^{\varphi_3(x_0)} f(u,x_0,0)\mathrm{d}u = 0,$$

has the root $x_0 \in (0,1)$, and $I'(x_0) \neq 0$. The root x_0 is called the transition point.

The boundary layer method is commonly used to solve such singularly perturbed problems. The general process involves expanding the solution as a power series form with respect to small parameters ε, and then finding each coefficient iteratively. This process is known as constructing asymptotic solutions.

3. Construction of asymptotic solutions

To determine the steptype asymptotic solution of the problem (2.1)-(2.2) , we treat the original problem as two separate problems, namely, the left and right problems. The solutions of the left and right problems are then smoothly matched using the seaming method, yielding a smooth solution of the original problem. The asymptotic for function $u(x,\varepsilon)$ has the form

$$u(x,\varepsilon) = \begin{cases} u^{(-)}(x,\varepsilon), & 0 \leq x < x_*, \\ u^{(+)}(x,\varepsilon), & x_* \leq x \leq 1. \end{cases} \tag{3.1}$$

The left and right problems can be written as follows.

Left problem is defined for $0 \le x \le x_*$ as

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u^{(-)}}{\mathrm{d}x^2} = f(u^{(-)}, x, \varepsilon), \\ \dfrac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(0,\varepsilon) = 0, \quad u^{(-)}(x_*, \varepsilon) = \varphi_2(x_*). \end{cases} \tag{3.2}$$

Right problem is defined for $x_* \le x \le 1$ as

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u^{(+)}}{\mathrm{d}x^2} = f(u^{(+)}, x, \varepsilon), \\ u^{(+)}(x_*, \varepsilon) = \varphi_2(x_*), \quad \dfrac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(1,\varepsilon) = 0, \end{cases} \tag{3.3}$$

where

$$u^{(-)}(x,\varepsilon) = \bar{u}^{(-)}(x,\varepsilon) + Q^{(-)}(\tau,\varepsilon) + \Pi^{(-)}(\xi,\varepsilon), \tag{3.4}$$

$$u^{(+)}(x,\varepsilon) = \bar{u}^{(+)}(x,\varepsilon) + Q^{(+)}(\tau,\varepsilon) + \Pi^{(+)}(\tilde{\xi},\varepsilon). \tag{3.5}$$

Here functions $\bar{u}^{(\pm)}(x,\varepsilon)$ are the regular parts of asymptotic, $Q^{(\pm)}(\tau,\varepsilon)$ are the asymptotic inner layer, and $\Pi^{(-)}(\xi,\varepsilon), \Pi^{(+)}(\tilde{\xi},\varepsilon)$ are the boundary layer functions. We represent these functions in power series expansion as follows:

$$\bar{u}^{(\pm)}(x,\varepsilon) = \bar{u}_0^{(\pm)}(x) + \varepsilon^{\frac{1}{2}} \bar{u}_1^{(\pm)}(x) + \cdots + \varepsilon^{\frac{i}{2}} \bar{u}_i^{(\pm)}(x) + \cdots, \tag{3.6}$$

$$Q^{(\pm)}(\tau,\varepsilon) = Q_0^{(\pm)}(\tau) + \varepsilon^{\frac{1}{4}} Q_1^{(\pm)}(\tau) + \cdots + \varepsilon^{\frac{i}{4}} Q_i^{(\pm)}(\tau) + \cdots, \quad \tau = \frac{x - x_*}{\varepsilon}, \tag{3.7}$$

$$\Pi^{(-)}(\xi,\varepsilon) = \varepsilon^{\frac{3}{4}} (\Pi_0^{(-)}(\xi) + \varepsilon^{\frac{1}{4}} \Pi_1^{(-)}(\xi) + \cdots + \varepsilon^{\frac{i}{4}} \Pi_i^{(-)}(\xi) + \cdots), \quad \xi = \frac{x}{\varepsilon^{3/4}}, \tag{3.8}$$

$$\Pi^{(+)}(\xi,\varepsilon) = \varepsilon^{\frac{3}{4}} (\Pi_0^{(+)}(\tilde{\xi}) + \varepsilon^{\frac{1}{4}} \Pi_1^{(+)}(\tilde{\xi}) + \cdots + \varepsilon^{\frac{i}{4}} \Pi_i^{(+)}(\tilde{\xi}) + \cdots), \quad \tilde{\xi} = \frac{1 - x}{\varepsilon^{3/4}}. \tag{3.9}$$

Functions $Q^{(\pm)}(\tau,\varepsilon)$, $\Pi^{(-)}(\xi,\varepsilon)$ and $\Pi^{(+)}(\xi,\varepsilon)$ are expanded in terms of $\varepsilon^{1/4}$, because the degenerate equation contains repeated roots. Then, the refined algorithms for accurate searching coefficients must be used. For $\bar{u}^{(\pm)}(x,\varepsilon)$ the order $\varepsilon^{1/2}$ is enough.

The denominators in series expansions for ξ, $\tilde{\xi}$ and τ are different due to the difference in boundary conditions. For inner layer we have only Neumann boundary conditions. We multiply the power series in equations (3.8), (3.9) by $\varepsilon^{\frac{3}{4}}$ to balance the boundary conditions due to specific choice of ξ. By x_* we denote the transition point, which can be also expressed in the form of power series:

$$x_* = x_0 + \varepsilon^{\frac{1}{4}} x_1 + \cdots + \varepsilon^{\frac{i}{4}} x_i + \cdots. \tag{3.10}$$

Then, we find the coefficients of asymptotic expansions (3.6)-(3.9) for right and left problems. We also prove that there exists $x_* \in (0,1)$ where

$$\frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x_*,\varepsilon) = \frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x_*,\varepsilon). \tag{3.11}$$

The value x_* is used by seaming method for matching solutions of left and right problems.

Finally, we prove that the solution of problem (2.1)-(2.2) is also the solution of (3.2)-(3.3) with contrast structure near $x = x_*$.

4. The asymptotic of the right and left problems solution

Let us consider the left problem (3.2). Substituting series (3.4) to (3.2), we get

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 \bar{u}^{(-)}}{\mathrm{d}x^2} + \dfrac{\mathrm{d}^2 Q^{(-)}}{\mathrm{d}\tau^2} + \varepsilon^{\frac{1}{2}} \dfrac{\mathrm{d}^2 \Pi^{(-)}}{\mathrm{d}\xi^2} = \bar{f}^{(-)} + Q^{(-)} f + \Pi^{(-)} f, \\ \dfrac{\mathrm{d}\bar{u}^{(-)}}{\mathrm{d}x}(0,\varepsilon) + \varepsilon^{\frac{4}{3}} \dfrac{\mathrm{d}\Pi^{(-)}}{\mathrm{d}\xi}(0,\varepsilon) = 0, \quad \bar{u}^{(-)}(x_*,\varepsilon) + Q^{(-)}(0,\varepsilon) = \varphi_2(x_*), \end{cases}$$

where

$$\begin{aligned} &\bar{f}^{(-)} = f(\bar{u}^{(-)}(x,\varepsilon), x, \varepsilon), \\ &\Pi^{(-)} f = f(\bar{u}^{(-)}(\xi\varepsilon^{3/4},\varepsilon) + \Pi^{(-)}(\tau,\varepsilon), \xi\varepsilon^{3/4}, \varepsilon) - f(\bar{u}^{(-)}(\xi\varepsilon^{3/4},\varepsilon), \xi\varepsilon^{3/4}, \varepsilon), \\ &Q^{(-)} f = f(\bar{u}^{(-)}(x_* + \tau\varepsilon, \varepsilon) + Q^{(-)}(\tau,\varepsilon), x_* + \tau\varepsilon, \varepsilon) \\ &\qquad\qquad - f(\bar{u}^{(-)}(x_* + \tau\varepsilon,\varepsilon), x_* + \tau\varepsilon, \varepsilon). \end{aligned}$$

The regular part of asymptotics can be found from the equation

$$\varepsilon^2 \frac{\mathrm{d}^2 \bar{u}^{(-)}}{\mathrm{d}x^2} = \bar{f}^{(-)}, 0 < x < x_*,$$

the inner layer from

$$\frac{\mathrm{d}^2 Q^{(-)}}{\mathrm{d}\tau^2} = Q^{(-)} f, \tau < 0,$$

and boundary layer from

$$\varepsilon^{\frac{1}{2}} \frac{\mathrm{d}^2 \Pi^{(-)}}{\mathrm{d}\xi^2} = \Pi^{(-)} f, \xi > 0.$$

The right problem (3.3) is solved similarly.

4.1. The regular part of asymptotics

Let us consider the regular part $u^{(-)}(x,\varepsilon)$ of asymptotics for left problem solution. It satisfies

$$\varepsilon^2 \frac{\mathrm{d}^2 \bar{u}^{(-)}}{\mathrm{d}x^2} = f(\bar{u}^{(-)}, x, \varepsilon). \tag{4.1}$$

Substituting series (3.6) to equation (4.1), we get

$$\begin{aligned} &\varepsilon^2 \frac{\mathrm{d}^2 \bar{u}_0^{(-)}(x)}{\mathrm{d}x^2} + \varepsilon^{\frac{5}{2}} \frac{\mathrm{d}^2 \bar{u}_1^{(-)}(x)}{\mathrm{d}x^2} + \cdots + \varepsilon^{\frac{k+4}{2}} \frac{\mathrm{d}^2 \bar{u}_k^{(-)}(x)}{\mathrm{d}x^2} + \cdots \\ &= -(\bar{u}^{(-)} - \varphi_1(x))^2 (\bar{u}^{(-)} - \varphi_2(x))(\bar{u}^{(-)} - \varphi_3(x))^2 - \varepsilon f_1(\bar{u}^{(-)}, x, \varepsilon) \\ &= -(\bar{u}_0^{(-)} - \varphi_1(x))^2 (\bar{u}_0^{(-)} - \varphi_2(x))(\bar{u}_0^{(-)} - \varphi_3(x))^2 \\ &\quad + \varepsilon^{\frac{1}{2}} [-(\bar{u}_0^{(-)} - \varphi_1(x))^2 \bar{u}_1^{(-)} (\bar{u}_0^{(-)} - \varphi_3(x))^2] \\ &\quad + \varepsilon [-(\bar{u}_1^{(-)})^2 (\bar{u}_0^{(-)} - \varphi_2(x))(\bar{u}_0^{(-)} - \varphi_3(x))^2 \\ &\quad - (\bar{u}_0^{(-)} - \varphi_1(x))^2 \bar{u}_2^{(-)} (\bar{u}_0^{(-)} - \varphi_3(x))^2 \end{aligned}$$

$$
\begin{aligned}
&-(\bar{u}_0^{(-)}-\varphi_1(x))^2(\bar{u}_0^{(-)}-\varphi_2(x))(\bar{u}_1^{(-)})^2-f_1(\bar{u}_0^{(-)},x,0)]+\cdots \\
&+\varepsilon^{\frac{k}{2}}\sum_{i>0,j,h,l\geq 0}^{2i+2j+2h+2l\leq k}[-\bar{u}_i^{(-)}\bar{u}_l^{(-)}(\bar{u}_{k-2i-2j}^{(-)}-\tilde{\varphi}_2)(\bar{u}_j^{(-)}-\tilde{\varphi}_{3j}) \\
&\times(\bar{u}_h^{(-)}-\tilde{\varphi}_{3h})-\bar{f}_k^{(-)}(x)] \\
&+\cdots
\end{aligned}
$$

where

$$
\tilde{\varphi}_{3j}=\begin{cases}\varphi_3(x), & j=0,\\ 0, & j\neq 0,\end{cases}\qquad \tilde{\varphi}_{3h}=\begin{cases}\varphi_3(x), & h=0,\\ 0, & h\neq 0,\end{cases}
$$

$$
\tilde{\varphi}_2=\begin{cases}\varphi_2(x), & k-2i-2j=0,\\ 0, & k-2i-2j\neq 0.\end{cases}
$$

By equating the coefficients of the same powers of ε on both sides of the equation, for $\bar{u}_0^{(-)}(x)$ we obtain

$$
-(\bar{u}_0^{(-)}-\varphi_1(x))^2(\bar{u}_0^{(-)}-\varphi_2(x))(\bar{u}_0^{(-)}-\varphi_3(x))^2=0.
$$

For solution we can take $\varphi_1(x)$, i.e

$$
\bar{u}_0^{(-)}=\varphi_1(x).
$$

Then $\bar{u}_1^{(-)}$ is determined from the equation

$$
\bar{h}^{(-)}(x)(\bar{u}_1^{(-)}(x))^2-\bar{f}_1^{(-)}(x)=0, \tag{4.2}
$$

where

$$
\begin{aligned}
&\bar{h}^{(-)}(x)=-(\varphi_1(x)-\varphi_3(x))^2(\varphi_1(x)-\varphi_2(x)),\\
&\bar{f}_1^{(-)}(x)=f_1(\varphi_1(x),x,0).
\end{aligned} \tag{4.3}
$$

According to condition A2, the solution of equation (4.2) must exist. Then, we can take a positive root of

$$
\bar{u}_1^{(-)}=\sqrt{\frac{\bar{f}_1^{(-)}(x)}{\bar{h}^{(-)}(x)}} \tag{4.4}
$$

as solution for (4.2).

Similarly, the higher order terms $\bar{u}_k^{(-)}(x)$ of series (3.6) can be obtained from algebraic equations

$$
[2\bar{h}^{(-)}(x)\bar{u}_1^{(-)}(x)]\bar{u}_k^{(-)}(x)-\bar{f}_k^{(-)}(x)=0,\quad k>1, \tag{4.5}
$$

where $\bar{f}_k^{(-)}(x)$ is the known function that depends on $\bar{u}_j^{(-)}(x),(j<k)$. Hence, the solution of (4.4)-(4.5) is unique and equal to $\bar{u}_k^{(-)}(x)$.

4.2. The inner layer of asymptotics

Now let us consider the inner layer term, namely, $Q^{(-)}(\tau,\varepsilon)$. Recall that it satisfies the following equation and boundary conditions

$$\begin{cases} \dfrac{\mathrm{d}^2 Q^{(-)}(\tau,\varepsilon)}{\mathrm{d}\tau^2} = Q^{(-)}f \\ \qquad := f(\bar{u}^{(-)}(x_*+\varepsilon\tau,\varepsilon)+Q^{(-)}(\tau,\varepsilon),x_*+\varepsilon\tau,\varepsilon) \\ \qquad\quad -f(\bar{u}^{(-)}(x_*+\varepsilon\tau,\varepsilon),x_*+\varepsilon\tau,\varepsilon), & (4.6) \\ Q^{(-)}(0,\varepsilon)=\varphi_2(x_*)-\bar{u}^{(-)}(x_*,\varepsilon).\ \ Q^{(-)}(-\infty,\varepsilon)=0. & (4.7)\end{cases}$$

To find the coefficients of power series expansion for $Q^{(-)}(\tau,\varepsilon)$, we follow the Butuzov's nonstandard method from [8]. For convenience, for $Q_i^{(-)}$ we mention only dependance on τ. We have

$$h^{(-)}(u,x)=-(u-\varphi_2(x))(u-\varphi_3(x))^2,\quad \varphi_1(x)=\varphi^{(-)}(x).$$

Substituting series (3.7) into (4.6), we obtain the the following equation for $Q_i^{(-)}(\tau)$:

$$\begin{aligned}
&\frac{\mathrm{d}^2Q_0^{(-)}(\tau)}{\mathrm{d}\tau^2}+\varepsilon^{\frac14}\frac{\mathrm{d}^2Q_1^{(-)}(\tau)}{\mathrm{d}\tau^2}+\cdots+\varepsilon^{\frac{k}{4}}\frac{\mathrm{d}^2Q_k^{(-)}(\tau)}{\mathrm{d}\tau^2}+\cdots\\
=&-(\bar{u}^{(-)}+Q^{(-)}-\varphi_1)^2(\bar{u}^{(-)}+Q^{(-)}-\varphi_2)(\bar{u}^{(-)}+Q^{(-)}\\
&-\varphi_3)^2-\varepsilon f_1(\bar{u}^{(-)}+Q^{(-)},x,\varepsilon)\\
&-[-(\bar{u}^{(-)}-\varphi_1)^2(\bar{u}^{(-)}-\varphi_2)(\bar{u}^{(-)}-\varphi_3)^2-\varepsilon f_1(\bar{u}^{(-)},x,\varepsilon)]\\
=&-(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_2)(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_3)^2(Q_0^{(-)})^2\\
&+\varepsilon^{\frac14}\big\{-2Q_1^{(-)}Q_0^{(-)}(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_2)(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_3)^2\\
&-(Q_0^{(-)})^2[Q_1^{(-)}(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_3)^2\\
&+2Q_1^{(-)}(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_2)(\bar{u}_0^{(-)}+Q_0^{(-)}-\varphi_3)]\big\}+\cdots\\
&+\varepsilon^{\frac{k}{2}}\Bigg\{\sum_{i,j,l,h\ge 0}^{2i+2j+2h+2l\le k}-[(\bar{u}_{i/2}^{(-)}+Q_i^{(-)}-\tilde{\varphi}_1)(\bar{u}_{l/2}^{(-)}+Q_l^{(-)}-\tilde{\varphi}_1)\\
&\times(\bar{u}_{(k-2i-2j-2l-2h)/2}^{(-)}+Q_{k-2i-2j-2l-2h}^{(-)}-\tilde{\varphi}_2)(\bar{u}_{j/2}^{(-)}+Q_j^{(-)}-\tilde{\varphi}_3)\\
&\times(\bar{u}_{h/2}^{(-)}+Q_h^{(-)}-\tilde{\varphi}_3)]\\
&-\sum_{i>0,j,h,l\ge 0}^{2i+2j+2h+2l\le k}-[\bar{u}_i^{(-)}\bar{u}_l^{(-)}(\bar{u}_{k-2i-2j}^{(-)}-\tilde{\varphi}_2)(\bar{u}_j^{(-)}-\tilde{\varphi}_{3j})(\bar{u}_h^{(-)}-\tilde{\varphi}_{3h})]-Qf_k\Bigg\}\\
&+\cdots
\end{aligned}$$

where

$$\tilde{\varphi}_1=\begin{cases}\varphi_1(x_*+\varepsilon\tau), & i=0,\\ 0, & i\ne 0,\end{cases}\qquad \bar{u}_\alpha^{(-)}=\begin{cases}\bar{u}_\alpha^{(-)}, & \text{if } \alpha \text{ is odd},\\ 0, & \text{if } \alpha \text{ is even}.\end{cases}$$

According to Butuzov's nonstandard method, we get the following equation and boundary conditions that determine $(Q_0^{(-)}(\tau))$:

$$\begin{cases} \dfrac{\mathrm{d}^2 Q_0^{(-)}}{\mathrm{d}\tau^2} = h^{(-)}(\varphi^{(-)}(x_*) + Q_0^{(-)}, x_*)[(Q_0^{(-)})^2 + 2\sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)Q_0^{(-)}], & (4.8) \\ Q_0^{(-)}(0) = \varphi_2(x_*) - \varphi^{(-)}(x_*), \quad Q_0^{(-)}(-\infty) = 0. & (4.9) \end{cases}$$

This problem can be reduced to the first-order boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau} = \left[2\displaystyle\int_0^{Q_0^{(-)}} h^{(-)}(\varphi^{(-)}(x_*) + s, x_*)(s^2 + 2\sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)s)\mathrm{d}s\right]^{\frac{1}{2}}, & (4.10) \\ Q_0^{(-)}(0) = \varphi_2(x_*) - \varphi^{(-)}(x_*). & (4.11) \end{cases}$$

It is challenging to find the solution to the above boundary value problem directly. However, it can be estimated using a differential inequality. Since the function $h^{(-)}(\varphi^{(-)}(x_*) + s, x_*)$ is bounded for $0 \le s \le Q_0^{(-)}$, let α_1 denote its minimum value and α_2 its maximum value. Thus, the solution $Q_0^{(-)}(\tau)$ admits the following estimate

$$Q_{\alpha_2}^{(-)}(\tau) \le Q_0^{(-)}(\tau) \le Q_{\alpha_1}^{(-)}(\tau), \tag{4.12}$$

where

$$Q_\alpha^{(-)}(\tau) = \frac{12\sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)(1 + O(\varepsilon^{1/4}))e^{\varepsilon^{1/4}\alpha k_0 \tau}}{\left\{1 - \left[1 - \left(\frac{12\bar{u}_1^{(-)}(x_*)}{\varphi_2(x_*) - \varphi^{(-)}(x_*)}\right)^{1/2}\varepsilon^{1/4} + O(\varepsilon^{1/2})\right]e^{\varepsilon^{1/4}\alpha k_0 \tau}\right\}^2}. \tag{4.13}$$

Here we denote $k_0 = [2\bar{u}_1^{(-)}(x_*)]^{\frac{1}{2}} > 0$.

According to the different decay behaviors of $Q_0^{(-)}(\tau)$, the inner layer can be divided into three regions. Here, we discuss only the left problem, i.e., when $\tau < 0$. Specifically:

1. If $-\frac{1}{\varepsilon^\gamma} \le \tau \le 0$, where $0 \le \gamma \le \frac{1}{4}$, then $Q_0^{(-)}(\tau)$ decays according to a power law as $\tau \to \infty$.
2. If $-\frac{1}{\varepsilon^{1/4}} \le \tau \le -\frac{1}{\varepsilon^\gamma}$, the decay of $Q_0^{(-)}(\tau)$ changes from a power-law decay to an exponential decay.
3. If $\tau \le -\frac{1}{\varepsilon^{1/4}}$, then $Q_0^{(-)}(\tau)$ exhibits exponential decay with respect to the new variable $\theta = \frac{x}{\varepsilon^{3/4}}$.

There are also three analogous regions when $\tau > 0$.

For higher order coefficients of $Q_k^{(-)}(\tau)$ we have boundary value problem

$$\frac{\mathrm{d}^2 Q_k^{(-)}}{\mathrm{d}\tau^2} = \beta^{(-)}(\tau, \varepsilon)Q_k^{(-)} + q_k^{(-)}(\tau, \varepsilon), \tag{4.14}$$

$$Q_k^{(-)}(0) = \begin{cases} -\bar{u}_{\frac{k}{2}}^{(-)}(x_*), & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}, \end{cases} \qquad Q_k^{(-)}(\pm\infty) = 0, \tag{4.15}$$

where

$$\begin{aligned}\beta^{(-)}(\tau,\varepsilon) &= h_u^{(-)}(\tau)[(Q_0^{(-)}(\tau))^2 + 2\sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)Q_0^{(-)}(\tau)]+\\ &\qquad + 2h^{(-)}(\tau)[Q_0^{(-)}(\tau) + \sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)],\\ h_u^{(-)}(\tau) &= \frac{\partial h}{\partial u}(\varphi^{(-)}(x_*) + Q_0^{(-)}(\tau), x_*, 0),\\ h^{(-)}(\tau) &= h^{(-)}(\varphi^{(-)}(x_*) + Q_0^{(-)}(\tau), x_*, 0),\end{aligned} \tag{4.16}$$

and $q_k^{(-)}$ depends on the known functions $Q_j^{(-)}(\tau), j < k$. In particular,

$$q_1^{(-)}(\tau,\varepsilon) = 0,$$

and for $q_2^{(-)}$ we have

$$\begin{aligned}q_2^{(-)}(\tau,\varepsilon) =&\sqrt{\varepsilon}(h^{(-)}(\tau) - \bar{h}^{(-)}(x_*))(\bar{u}_1^{(-)}(x_*))^2 + 2\sqrt{\varepsilon}h^{(-)}(\tau)\bar{u}_2^{(-)}(x_*)Q_0^{(-)}(\tau)\\ &+ h_u^{(-)}(\tau)\bar{u}_1^{(-)}(x_*)[(Q_0^{(-)}(\tau))^2 + 2\sqrt{\varepsilon}\bar{u}_1^{(-)}(x_*)Q_0^{(-)}(\tau)] + \sqrt{\varepsilon}Q_0^{(-)}f_1,\end{aligned}$$

where $Q_0^{(-)}f_1$ is the boundary layer part of $f_1(u,x,\varepsilon)$ power series expansion at x_*. Therefore, for the solution of (4.14) we have

$$Q_k^{(-)}(\tau) = Q_k^{(-)}(0)\Phi^{(-)}(\tau)[\Phi^{(-)}(0)]^{-1} + \Phi^{(-)}(\tau)\int_0^{\tau}[\Phi^{(-)}(\eta)]^{-2}J_k^{(-)}(\eta)\mathrm{d}\eta, \tag{4.17}$$

where

$$J_k^{(-)}(\eta) = \int_{-\infty}^{\eta}\Phi^{(-)}(s)q_k^{(-)}(s)\mathrm{d}s, \qquad \Phi^{(-)}(\tau) = \frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau}(\tau). \tag{4.18}$$

Moreover, for all functions $Q_k^{(-)}(\tau)(k \geq 0)$ we have the following estimate:

$$\|Q_k^{(-)}(\tau)\| \leq cQ_\alpha^{(-)}(\tau), \qquad \tau \in R. \tag{4.19}$$

4.3. The boundary layer of asymptotics

Now we consider the left problem for boundary layer of asymptotics, namely, the functions $\Pi^{(-)}(\xi,\varepsilon)$. For this functions we have the following boundary value problem:

$$\begin{cases}\sqrt{\varepsilon}\dfrac{\mathrm{d}^2\Pi^{(-)}(\xi,\varepsilon)}{\mathrm{d}\xi^2} = \Pi^{(-)}f := f(\bar{u}^{(-)}(\varepsilon^{\frac{3}{4}}\xi,\varepsilon) + \Pi^{(-)}(\xi,\varepsilon), \varepsilon^{\frac{3}{4}}\xi,\varepsilon)\\ \qquad\qquad\qquad\qquad -f(\bar{u}^{(-)}(\varepsilon^{\frac{3}{4}}\xi,\varepsilon), \varepsilon^{\frac{3}{4}}\xi,\varepsilon), & (4.20)\\ \dfrac{\mathrm{d}\bar{u}^{(-)}}{\mathrm{d}x}(0,\varepsilon) + \varepsilon^{\frac{4}{3}}\dfrac{\mathrm{d}\Pi^{(-)}}{\mathrm{d}\xi}(0,\varepsilon) = 0, \quad \Pi^{(-)}(+\infty,\varepsilon) = 0. & (4.21)\end{cases}$$

Substituting (3.8) to (4.20), for $\Pi_k^{(-)}(\xi)$ we get the equations

$$\begin{aligned}
&\varepsilon^{\frac{5}{4}}\frac{\mathrm{d}^2\Pi_0^{(-)}(\xi)}{\mathrm{d}\xi^2}+\varepsilon^{\frac{6}{4}}\frac{\mathrm{d}^2\Pi_1^{(-)}(\xi)}{\mathrm{d}\xi^2}+\cdots+\varepsilon^{\frac{5+k}{4}}\frac{\mathrm{d}^2\Pi_k^{(-)}(\xi)}{\mathrm{d}\xi^2}+\cdots\\
=&-(\bar{u}^{(-)}+\Pi^{(-)}-\varphi_1)^2(\bar{u}^{(-)}+\Pi^{(-)}-\varphi_2)(\bar{u}^{(-)}+\Pi^{(-)}\\
&-\varphi_3)^2-\varepsilon f_1(\bar{u}^{(-)}+\Pi^{(-)},x,\varepsilon)\\
&-[-(\bar{u}^{(-)}-\varphi_1)^2(\bar{u}^{(-)}-\varphi_2)(\bar{u}^{(-)}-\varphi_3)^2-\varepsilon f_1(\bar{u}^{(-)},x,\varepsilon)]\\
=&\varepsilon^{\frac{5}{4}}[-2\Pi_0^{(-)}\bar{u}_1^{(-)}(\bar{u}_0^{(-)}-\varphi_2)(\bar{u}_0^{(-)}-\varphi_3)^2]+\cdots\\
&+\varepsilon^{\frac{5+k}{4}}\Bigg\{\sum_{i,j,l,h\geq 0}^{2i+2j+2h+2l\leq k}-[(\bar{u}_{(i+3)/2}^{(-)}+\Pi_i^{(-)})(\bar{u}_{(l+3)/2}^{(-)}+\Pi_l^{(-)})(\bar{u}_{(k-2i-2j-2l-2h+3)/2}^{(-)}\\
&+\Pi_{k-2i-2j-2l-2h}^{(-)}-\tilde{\varphi}_2)(\bar{u}_{(j+3)/2}^{(-)}+\Pi_j^{(-)}-\tilde{\varphi}_3)(\bar{u}_{(h+3)/2}^{(-)}+\Pi_h^{(-)}-\tilde{\varphi}_3)\\
&-\sum_{i>0,j,h,l\geq 0}^{2i+2j+2h+2l\leq k}-[\bar{u}_i^{(-)}\bar{u}_l^{(-)}(\bar{u}_{k-2i-2j}^{(-)}-\tilde{\varphi}_2)(\bar{u}_j^{(-)}-\tilde{\varphi}_{3j})(\bar{u}_h^{(-)}-\tilde{\varphi}_{3h})]\Pi f_k\Bigg\}\\
&+\cdots.
\end{aligned}$$

By equating the coefficients of the same powers of ε on both sides of the equation, for $\Pi_k^{(-)}, k=0,1,2...$ we get

$$\frac{\mathrm{d}^2\Pi_k^{(-)}}{\mathrm{d}\xi^2}=2\bar{h}^{(-)}(0)\bar{u}_1^{(-)}(0)\Pi_k^{(-)}+\pi_k^{(-)}(\xi),\qquad \xi>0,\qquad k=0,1,...,\tag{4.22}$$

$$\frac{\mathrm{d}\Pi_k^{(-)}}{\mathrm{d}\xi}(0)=\begin{cases}-\dfrac{\mathrm{d}\bar{u}_{k/2}^{(-)}}{\mathrm{d}x}(0), & \text{if } k \text{ is odd},\\ 0, & \text{if } k \text{ is even},\end{cases}\qquad \Pi_k^{(-)}(+\infty)=0,\tag{4.23}$$

where $\pi_k^{(-)}$ depends on known functions $\Pi_j^{(-)}(\xi), j<k$. In particular, $\pi_0^{(-)}\equiv 0$. Since the equations (4.22) are linear, the solution $\Pi_0^{(-)}(\xi)$ can be written as

$$\Pi_0^{(-)}(\xi)=\varphi^{(-)\prime}(0)[2\bar{h}^{(-)}(0)\bar{u}_1^{(-)}(0)]^{-\frac{1}{2}}\exp\big(-(2\bar{h}^{(-)}(0)\bar{u}_1^{(-)}(0))^{\frac{1}{2}}\xi\big).\tag{4.24}$$

We also have the estimate:

$$\|\Pi_k^{(-)}(\xi)\|\leq ce^{-\kappa\xi},\qquad \xi>0,\qquad k=0,1,....\tag{4.25}$$

Currently, each term of the asymptotic expansion for the left problem has been determined and satisfies the exponential decay estimate. The right problem can be analyzed in the same way, so we skip the details here.

4.4. Proof of existence of solution to the problem (2.1)-(2.2)

In this section, we will prove the existence of a smooth solution to the original problem (2.1)-(2.2) using the sewing method.

Theorem 4.1. *Suppose we have the boundary value problem* (2.1)-(2.2) *and the conditions (A1)-(A3) are met. Then for sufficiently small* $\varepsilon>0$, *there exist a* *smooth solution* $u(x,\varepsilon)$ *in the asymptotic form*

$$u(x,\varepsilon)=\begin{cases} U_n^{(-)}(x,\varepsilon)+O(\varepsilon^{\frac{n+1}{2}}), & 0\le x<(x_*)_{2n+5},\\ U_n^{(+)}(x,\varepsilon)+O(\varepsilon^{\frac{n+1}{2}}), & (x_*)_{2n+5}\le x\le 1,\end{cases}$$

where $n\in N$, $(x_*)_{2n+5}=\sum_{k=0}^{2n+5}\varepsilon^{\frac{k}{4}}\bar{x}_k$, *the functions* $U_n^{(\pm)}(x,\varepsilon)$ *can be found from* (4.27) *with* $\tau=[(x-x_*)_{2n+5}]/\varepsilon$.

To prove this theorem, we first introduce two lemmas.

Lemma 4.1. *If (A1) and (A2) are satisfied, then for sufficiently small* $\varepsilon>0$, *the solution* $u^{(-)}(x,\varepsilon)$ *of the left problem* (3.2) *and the solution* $u^{(+)}(x,\varepsilon)$ *of the right problem* (3.3) *are respectively:*

$$\begin{aligned} u^{(-)}(x,\varepsilon)&=U_n^{(-)}(x,\varepsilon)+O(\varepsilon^{\frac{n+1}{2}}), \quad 0\le x\le x_*,\\ u^{(+)}(x,\varepsilon)&=U_n^{(+)}(x,\varepsilon)+O(\varepsilon^{\frac{n+1}{2}}), \quad x_*\le x\le 1,\end{aligned} \tag{4.26}$$

where

$$\begin{aligned} U_n^{(-)}(x,\varepsilon)&=\sum_{k=0}^{n}\varepsilon^{\frac{k}{2}}\bar{u}_k^{(-)}(x)+\sum_{k=0}^{2n+1}\varepsilon^{\frac{k}{4}}Q_k^{(-)}(\tau)+\varepsilon^{\frac{3}{4}}\sum_{k=0}^{2n}\varepsilon^{\frac{k}{4}}\Pi_k^{(-)}(\xi),\\ U_n^{(+)}(x,\varepsilon)&=\sum_{k=0}^{n}\varepsilon^{\frac{k}{2}}\bar{u}_k^{(+)}(x)+\sum_{k=0}^{2n+1}\varepsilon^{\frac{k}{4}}Q_k^{(+)}(\tau)+\varepsilon^{\frac{3}{4}}\sum_{k=0}^{2n}\varepsilon^{\frac{k}{4}}\Pi_k^{(+)}(\tilde{\xi}).\end{aligned} \tag{4.27}$$

The proof is analogous to the one in [2] and skipped here.

Lemma 4.2. *For the derivative* $\frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x,\varepsilon)$, $\frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x,\varepsilon)$, *the asymptotic representations*

$$\begin{aligned} \frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x,\varepsilon)&=\frac{\mathrm{d}U_n^{(-)}}{\mathrm{d}x}(x,\varepsilon)+O(\varepsilon^{\frac{n-1}{2}}), \quad 0\le x<x_*,\\ \frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x,\varepsilon)&=\frac{\mathrm{d}U_n^{(+)}}{\mathrm{d}x}(x,\varepsilon)+O(\varepsilon^{\frac{n-1}{2}}), \quad x_*\le x\le 1,\end{aligned} \tag{4.28}$$

are true.

The proof is also analogous to the one in [2].

Now, we prove the original theorem.

We rewrite the smooth seaming condition (3.11) as

$$I(x_*,\varepsilon)=\varepsilon\frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x_*,\varepsilon)-\varepsilon\frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x_*,\varepsilon)=0. \tag{4.29}$$

Substituting expansion (3.10) for x_* in at $\tau=0$, we get

$$\frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau}(0)=I^{(-)}(x_0)+\sum_{i=1}^{+\infty}\varepsilon^{\frac{i}{4}}g_i^{(-)}(x),$$

where

$$I^{(-)}(x_0)=\left[2\int_{\varphi^{(-)}(x_0)}^{\varphi_2(x_0)}f(u,x_0,0)\mathrm{d}u\right]^{\frac{1}{2}},$$

and $g_i(x)$ is sufficiently smooth functions that depend on $x_k, k \le i$. Similarly for the right problem solution, we have

$$\frac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau}(0) = I^{(+)}(x_0) + \sum_{i=1}^{+\infty} \varepsilon^{\frac{i}{4}} g_i^{(+)}(x),$$

where

$$I^{(+)}(x_0) = \left[2\int_{\varphi^{(+)}(x_0)}^{\varphi_2(x_0)} f(u, x_0, 0)\mathrm{d}u\right]^{\frac{1}{2}}.$$

Substituting power series expansion of (3.4) and (3.5) with respect to $\varepsilon^{\frac{1}{4}}$ to (4.29), we obtain

$$\begin{aligned}
&I(x_*, \varepsilon)\\
=&\varepsilon\left(\frac{\mathrm{d}\varphi^{(-)}}{\mathrm{d}x}(x_*) + \varepsilon^{\frac{1}{2}}\frac{\mathrm{d}\bar{u}_1^{(-)}}{\mathrm{d}x}(x_*) + \cdots\right) + \left(\frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau}(0) + \varepsilon^{\frac{1}{4}}\frac{\mathrm{d}Q_1^{(-)}}{\mathrm{d}\tau}(0) + \cdots\right)\\
&-\varepsilon\left(\frac{\mathrm{d}\varphi^{(+)}}{\mathrm{d}x}(x_*) + \varepsilon^{\frac{1}{2}}\frac{\mathrm{d}\bar{u}_1^{(+)}}{\mathrm{d}x}(x_*) + \cdots\right) - \left(\frac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau}(0) + \varepsilon^{\frac{1}{4}}\frac{\mathrm{d}Q_1^{(+)}}{\mathrm{d}\tau}(0) + \cdots\right)\\
=&\left(\frac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau}(0) - \frac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau}(0)\right) + \varepsilon^{\frac{1}{4}}\left(\frac{\mathrm{d}Q_1^{(-)}}{\mathrm{d}\tau}(0) - \frac{\mathrm{d}Q_1^{(+)}}{\mathrm{d}\tau}(0)\right) + \cdots\\
=&\left[I^{(-)}(x_0) - I^{(+)}(x_0) + \varepsilon^{\frac{1}{4}}(I^{(-)\prime}(x_0)x_1 - I^{(+)\prime}(x_0)x_1) + \cdots\right] + \cdots\\
=&H(x_0) + \varepsilon^{\frac{1}{4}}[H'(x_0)x_1 + m_1] + \cdots + \varepsilon^{\frac{k}{4}}[H'(x_0)x_k + m_k] + \cdots\\
=&0.
\end{aligned} \tag{4.30}$$

Here $H(x) = I^{(-)}(x) - I^{(+)}(x)$. Then, according to (A3), $x = x_0$ is the root of the equation $H(x) = 0$.

Since $H'(x_0) \neq 0$, the higher order coefficients $x_k, k \ge 1$ can be uniquely determined from the following linear algebraic equations

$$H'(x_0)x_k + m_k = 0, \quad k \ge 1, \tag{4.31}$$

where m_k depend on the known numbers $x_{j,j<k}$. Note that

$$m_1 = [H(x_0)^{-1}(J^{(+)} - J^{(-)})]. \tag{4.32}$$

Since $q_1^{(-)} = 0$, we have $m_1 = 0$ and then $x_1 = 0$.

To prove the existence of a solution with contrast structures for the problem (2.1)-(2.2), we reconsider the left (3.2) and right (3.3) problems and modify x_* to the following form:

$$x_* = x_\delta^{(\pm)} := \bar{x}_0 + \varepsilon^{\frac{2}{4}}\bar{x}_1 + \cdots + \varepsilon^{\frac{2m+1}{4}}(\bar{x}_{m+1} \pm \delta), \tag{4.33}$$

where δ is an arbitrary real number, and δ is bounded as $\varepsilon \to 0$. Lemma 4.1 shows that the solutions of the left and right problems exist and have uniform asymptotic expansions. It only requires replacing the variable τ with $\tau = \frac{x - x_\delta^{(\pm)}}{\varepsilon}$, and the estimate in (4.19) still holds.

Rewriting (4.29) and taking $n = \left[\dfrac{2m+1}{4}\right]$ in (4.28), we have

$$\begin{aligned} I(x_\delta^{(\pm)}, \varepsilon) =& \varepsilon \frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x_\delta^{(\pm)}, \varepsilon, \delta) - \varepsilon \frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x_\delta^{(\pm)}, \varepsilon, \delta) \\ =& H(\bar{x}_0) + \sum_{k=2}^{2m} \varepsilon^{\frac{k}{4}} [H'(\bar{x}_0)\bar{x}_k + m_k] + \varepsilon^{\frac{2m+1}{4}} H'(\bar{x}_0)(\pm\delta) + O(\varepsilon^{\frac{2m+2}{4}}), \end{aligned} \tag{4.34}$$

where x_0 and x_k can be determined by (A2) and (4.31). The last two terms on the right-hand side of (4.34) depend on δ and are uniformly small as $\delta \to 0$. Thus,

$$I(x_\delta^{(\pm)}, \varepsilon) = \varepsilon^{\frac{2m+1}{4}} (H'(\bar{x}_0)(\pm\delta) + O(\varepsilon^{\frac{1}{4}})). \tag{4.35}$$

Since $H'(\bar{x}_0) \neq 0$, the sign of (4.35) depends on δ. Therefore, there exists a δ, such that

$$(H'(\bar{x}_0)\delta + O(\varepsilon^{\frac{1}{4}}))(-H'(\bar{x}_0)\delta + O(\varepsilon^{\frac{1}{4}})) < 0.$$

According to the intermediate value theorem, there exists $\delta = \bar{\delta}(\varepsilon) = O(\varepsilon^{1/4})$ such that (4.35) is equal to 0 when ε is sufficiently small. So, we have

$$\varepsilon \frac{\mathrm{d}u^{(-)}}{\mathrm{d}x}(x_{\bar{\delta}}, \varepsilon, \delta) - \varepsilon \frac{\mathrm{d}u^{(+)}}{\mathrm{d}x}(x_{\bar{\delta}}, \varepsilon, \delta) = 0.$$

It means that the function

$$u(x, \varepsilon) = \begin{cases} u^{(-)}(x, \varepsilon), & 0 \le x < x_*, \\ u^{(+)}(x, \varepsilon), & x_* \le x \le 1, \end{cases}$$

is the solution of (2.1)–(2.2) with contrast structure near x_*. This completes the proof of the theorem.

5. Example

In this section we consider the following boundary value problem:

$$\begin{cases} \varepsilon^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = -(u-x)^2(u - \dfrac{6}{5})(u-2)^2 - \varepsilon(\dfrac{6}{5} - u), & 0 < x < 1, \\ \dfrac{\mathrm{d}u}{\mathrm{d}x}(0, \varepsilon) = 0, \quad \dfrac{\mathrm{d}u}{\mathrm{d}x}(1, \varepsilon) = 0. \end{cases} \tag{5.1}$$

Here

$$\begin{aligned} &\varphi_1(x) = \varphi^{(-)} = x, \ \varphi_2(x) = \frac{6}{5}, \ \varphi_3(x) = \varphi^{(+)} = 2, \\ &\bar{h}^{(-)}(x) = -(x-2)^2(x - \frac{6}{5}), \ \bar{h}^{(+)}(x) = -(x-2)^2(2 - \frac{6}{5}), \\ &\bar{f}_1^{(-)}(x) = \frac{6}{5} - x, \ \bar{f}_1^{(+)}(x) = -\frac{4}{5}. \end{aligned}$$

It is easy to verify that the conditions (A1) and (A2) are satisfied. Then $\bar{x}_0 = \dfrac{2}{5}$ is the root of the equation

$$I(\bar{x}_0) := \int_{x_0}^{2} -(u - x_0)^2(u - \frac{6}{5})(u-2)^2 \mathrm{d}u = 0,$$

where $I'(\bar{x}_0) \neq 0$. Therefore, condition (A3) also holds.

From (4.4), we have

$$\bar{u}_1^{(-)} = \frac{1}{2-x}, \quad \bar{u}_1^{(+)} = \frac{-1}{2-x}. \tag{5.2}$$

We have the following equations for determining $Q_0^{(\pm)}(\tau)$ and their boundary conditions:

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}}{\mathrm{d}\tau} = \left[2\displaystyle\int_0^{Q_0^{(-)}} -(x_* + s - \frac{6}{5})(x_* + s - 2)^2(s^2 + \frac{2\sqrt{\varepsilon}}{2-x}s)\mathrm{d}s\right]^{\frac{1}{2}}, \\ Q_0^{(-)}(0) = \dfrac{6}{5} - x_*, \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(+)}}{\mathrm{d}\tau} = \left[2\displaystyle\int_0^{Q_0^{(+)}} -(2 + s - x_*)^2(2 + s - \frac{6}{5})(s^2 - \frac{2\sqrt{\varepsilon}}{2-x}s)\mathrm{d}s\right]^{\frac{1}{2}}, \\ Q_0^{(+)}(0) = -\dfrac{4}{5}. \end{cases}$$

It is difficult to obtain the analytical solutions for these Cauchy problems, so we provide only numerical simulation.

For $\Pi_0^{(\pm)}(\xi)$ we have

$$\begin{cases} \dfrac{\mathrm{d}\Pi_0^{(-)}}{\mathrm{d}\xi} = \dfrac{24}{5}\Pi_0^{(-)}, \\ \dfrac{\mathrm{d}\Pi_0^{(-)}}{\mathrm{d}\xi}(0) = -1, \quad \Pi_0^{(-)}(+\infty) = 0, \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}\Pi_0^{(+)}}{\mathrm{d}\tilde{\xi}} = \dfrac{4}{5}\Pi_0^{(+)}, \\ \dfrac{\mathrm{d}\Pi_0^{(+)}}{\mathrm{d}\tilde{\xi}}(0) = 0, \quad \Pi_0^{(-)}(+\infty) = 0. \end{cases}$$

Here, we can solve these Cauchy problems analytically, and the solution is

$$\Pi_0^{(-)}(\xi) = \sqrt{5/24}\exp(-\sqrt{24/5}\xi), \qquad \Pi_0^{(+)}(\tilde{\xi}) = 0.$$

At Figure 1 you can see the results of numerical simulation for zero approximation $U_0(x, \varepsilon)$ of the solution.

The whole solution can be get from Theorem 4.1 and has the form

$$u(x,\varepsilon) = \begin{cases} x + Q_0^{(-)}(\tau) + \varepsilon^{\frac{3}{4}}\Pi_0^{(-)}(\xi) + O(\varepsilon^{\frac{1}{2}}), & 0 \le x < (x_*)_{2n+5}, \\ 2 + Q_0^{(+)}(\tau) + O(\varepsilon^{\frac{1}{2}}), & (x_*)_{2n+5} \le x \le 1. \end{cases}$$

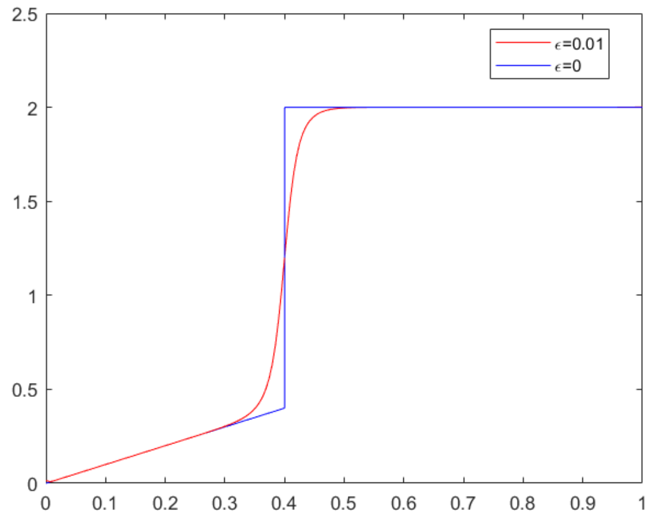

Figure 1. Zero order approximation $U_0(x, \varepsilon)$ of problem (2.1)-(2.2).

6. Conclusion

In this paper, we discuss the singularly perturbed boundary value problem for a degenerate equation with two double roots. The formal asymptotic solution is constructed using the modified boundary layer function method. The existence of multiple inner layers is established, and the existence of smooth solutions to the problem is proven. Moreover, the obtained results are illustrated through numerical simulation for particular right hand part of the equation. The theoretical results can be extended to handle the contrast structure between repeated roots.

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