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INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH BOTH JUMP CONDITIONS DEPENDENT ON THE SPECTRAL PARAMETER*

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Abstract The inverse spectral problem of Sturm-Liouville operator with both of the jump conditions dependent on the spectral parameter is investigated. Firstly, by theoretical operator formulation the self-adjointness of the problem is proven and then some of the eigenvalue properties, especially the asymptotic formulas of eigenvalues and eigenfunctions are given. Finally, the uniqueness theorems of the corresponding inverse problems are given by Weyl function theory and inverse spectral data approach.

Keywords Inverse problem, Sturm-Liouville operator, Weyl function, eigenparameter-dependent jump conditions.

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1. Introduction

Inverse spectral problems are motivated to recovering operators from their spectral characteristics. These problems often appear in mathematics, physics, mechanics, electronics, and some other branches of science and engineering problems, and, hence, are very important to understanding the real world. A significant progress has been made in the inverse problem theory for regular self-adjoint Sturm-Liouville (S-L) operators and nonself-adjoint Sturm-Liouville operators and even third order differential equations [6, 11, 21, 23, 32].

Historically speaking, the inverse problem of Sturm-Liouville operator was initiated by Ambarzumian [2] and Borg [8], after that, there are various generalizations on the inverse problems of Sturm-Liouville operators. Beside the classical regular Sturm-Liouville operators [11, 23], recent years there are a lot of inverse problems for Sturm-Liouville operators with eigenparameter-dependent boundary conditions and Sturm-Liouville operators with transmission conditions duo to the wide applications of such problems [5, 7, 12–15, 17, 19, 29–31, 34]. Binding et al. have discussed boundary conditions depend nonlinearly on the spectral parameter [5]. Hald has studied the discontinuous S-L problem and shown the direct and inverse spectral theory

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on the S-L problem with internal discontinuous point conditions [17]. The corresponding direct problems of boundary value problems with transmission conditions and/or eigenparameter-dependent boundary conditions, we refer to [18,20,24–26,33] and the references therein.

As an organic combination of the above mentioned two problems, the boundary value problems with eigenparameter-dependent transmission conditions have drown scholars' much attention and have achieved significant progress recently, including direct and inverse spectral theory [1, 3, 4, 9, 10, 16, 22, 27, 28, 35]. In 2005, Akdogan et al. investigated the discontinuous Sturm-Liouville problems, where the spectral parameter not only appear in differential equations, but also in boundary conditions and one of the jump conditions. They got the asymptotic approximation of fundamental solutions and the asymptotic formulae for eigenvalues of such problems [1]. In 2012, Ozkan et al. considered the inverse spectral problems for Sturm-Liouville operator with both boundary and one of the jump conditions linearly dependent on the eigenparameter [22]. In 2014, Guo et al. investigated the inverse spectral problem of Sturm-Liouville operator with finite number of jump conditions dependent on the eigenparameter [16]. In 2016, Wei et al. investigated the inverse spectral problem for Dirac operator with boundary and jump conditions dependent on the spectral parameter [28]. Through inducting the generalized normal constants they have proved the uniqueness theorem. In 2018 and 2021, Bartels et al. presented Sturm-Liouville problems with transfer condition Herglotz dependent on the eigenparameter, and showed the Hilbert space formulation of the problem and calculated out the asymptotic formulas of eigenvalue and eigenfunction on this problem [3, 4]. Zhang et al. studied the finite spectrum of Sturm-Liouville problems with both jump conditions dependent on the spectral parameter [35].

The Sturm-Liouville problems with jump conditions containing the spectral parameter have been widely studied, however, for the problems with both jump conditions containing the spectral parameter attach less attention, which often appear in heat transfer, electronic signal amplifiers and other issues of sciences, hence have high research significance. It is also a good complement to the study of spectral and inverse spectral problems of boundary value problems for differential equations.

In this article, we mainly investigate the inverse spectral problem of Sturm-Liouville operator in which the spectral parameter not only appears in the differential equation, but also appears in both jump conditions. To show the inverse spectral theory of this problem, the operator formulation of this problem is constructed and then some spectral properties are given, next the asymptotic behavior of the solutions and eigenvalues is provided, finaly several uniqueness results for this inverse spectral problem are given by using the Weyl function theory.

2. Notation and basic properties

Consider the following boundary value problem (denoted by L) consisting of the following Sturm-Liouville equation

$$l(y) := -y'' + q(x)y = \lambda y, \quad x \in J = [0, c) \cup (c, \pi], \tag{2.1}$$

together with boundary conditions (BCs)

$$l_1(y) := y'(0) - hy(0) = 0, \tag{2.2}$$

$$l_2(y) := y'(\pi) + Hy(\pi) = 0, \tag{2.3}$$

and jump conditions with spectral parameter

$$y(c^-) + (\lambda\eta_1 - \xi_1)y'(c^-) + y'(c^+) = 0, \tag{2.4}$$

$$y'(c^-) - y(c^+) + (\lambda\eta_2 - \xi_2)y'(c^+) = 0, \tag{2.5}$$

where $q(x) \in L_2(J)$ is real valued, $0 < c < \pi$, h, H, η_k, $\xi_k \in \mathbb{R}$, $\eta_k > 0$, k=1, 2. Here λ is a spectral parameter.

In order to describe the self-adjointness of the operator corresponding to the problem L, we will introduce an inner product in the Hilbert space $\mathcal{H} := L^2(J) \oplus \mathbb{C}^2$ as

$$(F, G) = \int_0^c f\bar{g}dx + \int_c^\pi f\bar{g}dx + \eta_1 f_1\bar{g}_1 + \eta_2 f_2\bar{g}_2, \tag{2.6}$$

for

$$F = (f, f_1, f_2)^T, \ G = (g, g_1, g_2)^T \in \mathcal{H}.$$

To facilitate the description, the following notation need to be listed. Let

$$\tilde{M}_1(y) = \frac{1}{\eta_1}[\xi_1 y'(c^-) - y(c^-) - y'(c^+)], \ M_1(y) = y'(c^-),$$

$$\tilde{M}_2(y) = \frac{1}{\eta_2}[\xi_2 y'(c^+) + y(c^+) - y'(c^-)], \ M_2(y) = y'(c^+),$$

then the eigenparameter-dependent jump conditions (2.4) and (2.5) can be written as

$$\tilde{M}_1(y) = \lambda M_1(y), \qquad \tilde{M}_2(y) = \lambda M_2(y).$$

In the Hilbert space $\mathcal{H}$ we define a linear operator $\mathcal{A}$: $\mathcal{H} \to \mathcal{H}$ as

$$\mathcal{A}F = \begin{pmatrix} l(f) \\ \tilde{M}_1(f) \\ \tilde{M}_2(f) \end{pmatrix} = \begin{pmatrix} -f'' + qf \\ \frac{1}{\eta_1}[\xi_1 f'(c^-) - f(c^-) - f'(c^+)] \\ \frac{1}{\eta_2}[\xi_2 f'(c^+) + f(c^+) - f'(c^-)] \end{pmatrix}, \tag{2.7}$$

and the domain of the operator $\mathcal{A}$ as

$$\begin{aligned} D(\mathcal{A}) :=\{ & F = (f(x), f_1, f_2)^T \in \mathcal{H} : f, \ f' \in AC(J), \text{ and have finite limits} \\ & f(c^\pm) = \lim_{x \to c\pm 0} f(x), \ f'(c^\pm) = \lim_{x \to c\pm 0} f'(x), \\ & l(f) \in L^2(J), \ f_1 = M_1(f), \ f_2 = M_2(f), \ l_1(f) = 0, \ l_2(f) = 0\}. \end{aligned}$$

Thus, the problem L can be ruled as the following form

$$\mathcal{A}F = \lambda F,$$

where $F = (f, f_1, f_2)^T \in D(\mathcal{A})$.

Let $W(f, g; x) := f(x)g'(x) - f'(x)g(x)$ be the Wronskian for any f, $g \in L^2(J)$, then the following theorem holds.

Theorem 2.1. *The linear operator $\mathcal{A}$ is self-adjoint in the Hilbert space $\mathcal{H}$.*

Proof. This theorem can be proved by the following three steps.

(i) $D(\mathcal{A})$ is dense in $\mathcal{H}$. Since this is routine here we omit the details.

(ii) The operator $\mathcal{A}$ is symmetric.

$$\begin{aligned}
&(\mathcal{A}F,G)-(F,\mathcal{A}G)\\
=&\int_0^c[-f''+qf]\bar{g}dx+\int_c^\pi[-f''+qf]\bar{g}dx+\eta_1\tilde{M}_1(f)M_1(\bar{g})+\eta_2\tilde{M}_2(f)M_2(\bar{g})\\
&-\int_0^c f[-\bar{g}''+q\bar{g}]dx-\int_c^\pi f[-\bar{g}''+q\bar{g}]dx-\eta_1M_1(f)\tilde{M}_1(\bar{g})-\eta_2M_2(f)\tilde{M}_2(\bar{g})\\
=&W(f,\bar{g};c^-)-W(f,\bar{g};0)+W(f,\bar{g};\pi)-W(f,\bar{g};c^+)\\
&+\eta_1[\tilde{M}_1(f)M_1(\bar{g})-M_1(f)\tilde{M}_1(\bar{g})]+\eta_2[\tilde{M}_2(f)M_2(\bar{g})-M_2(f)\tilde{M}_2(\bar{g})].
\end{aligned}\tag{2.8}$$

Since $F,G\in D(\mathcal{A})$, from (2.2) and (2.3) we get

$$W(f,\bar{g};0)=0,\ W(f,\bar{g};\pi)=0.\tag{2.9}$$

It is easy to show that

$$\begin{aligned}
&\eta_1(\tilde{M}_1(f)M_1(\bar{g})-M_1(f)\tilde{M}_1(\bar{g}))+\eta_2(\tilde{M}_2(f)M_2(\bar{g})-M_2(f)\tilde{M}_2(\bar{g}))\\
=&-W(f,\bar{g};c^-)+W(f,\bar{g};c^+).
\end{aligned}\tag{2.10}$$

Substituting into (2.8) we arrive at

$$(\mathcal{A}F,G)=(F,\mathcal{A}G),$$

namely that $\mathcal{A}$ is symmetric.

(iii) The symmetric operator $\mathcal{A}$ is self-adjoint.

This means we need to show that: for any $F=(f,f_1,f_2)^T\in D(\mathcal{A})$ and some $Z\in D(\mathcal{A}^*)$, satisfying $(\mathcal{A}F,Z)=(F,V)$, then $Z\in D(\mathcal{A})$ and $\mathcal{A}Z=V$, where $Z=(z,z_1,z_2)^T,V=(v,v_1,v_2)^T$, that is the following conditions must be satisfied.

(1) $z,\ z'\in AC(J),\ l(z)\in L^2(J)$;

(2) $z_1=M_1(z)=z'(c^-),\ z_2=M_2(z)=z'(c^+)$;

(3) $v_1=\tilde{M}_1(z)=\frac{1}{\eta_1}[\xi_1z'(c^-)-z(c^-)-z'(c^+)]$,

$v_2=\tilde{M}_2(z)=\frac{1}{\eta_2}[\xi_2z'(c^+)+z(c^+)-z'(c^-)]$;

(4) $v=l(z)$;

(5) $l_1(z)=0,\ l_2(z)=0$.

Firstly, as $q\in L(J,\mathbb{R})$, it follows that $C_0^\infty(J)\oplus\{0\}\oplus\{0\}\subset D(\mathcal{A})$. For any $F\in C_0^\infty(J)\oplus\{0\}\oplus\{0\}\subset D(\mathcal{A})$, $(\mathcal{A}F,Z)=(F,V)$, then

$$\int_0^c l(f)\bar{z}dx+\int_c^\pi l(f)\bar{z}dx=\int_0^c f\bar{v}dx+\int_c^\pi f\bar{v}dx,\tag{2.11}$$

according to standard Sturm-Liouville theory, we have $z\in D(\mathcal{A})$, so (1) is established.

Next, because $\mathcal{A}$ is symmetric, then $(\mathcal{A}F,Z)=(F,\mathcal{A}Z)$, it can be obtained from the above F

$$\int_0^c l(f)\bar{z}dx+\int_c^\pi l(f)\bar{z}dx=\int_0^c f\overline{l(z)}dx+\int_c^\pi f\overline{l(z)}dx,$$

and combining with (2.11), then $\bar{v} = \overline{l(z)}$, *i.e.* $v = l(z)$, this means (4) is true.

Now according to (4), for any $F \in D(\mathcal{A})$, $(\mathcal{A}F, Z) = (F, V)$, then

$$\begin{aligned}&\int_0^c l(f)\bar{z}dx + \int_c^\pi l(f)\bar{z}dx + \eta_1\tilde{M}_1(f)\bar{z}_1 + \eta_2\tilde{M}_2(f)\bar{z}_2\\ =&\int_0^c f\bar{v}dx + \int_c^\pi f\bar{v}dx + \eta_1 M_1(f)\bar{v}_1 + \eta_2 M_2(f)\bar{v}_2\\ =&\int_0^c f\overline{l(z)}dx + \int_c^\pi f\overline{l(z)}dx + \eta_1 M_1(f)\bar{v}_1 + \eta_2 M_2(f)\bar{v}_2,\end{aligned}$$

i.e.

$$\begin{aligned}&\int_0^c (l(f)\bar{z} - f\overline{l(z)})dx + \int_c^\pi (l(f)\bar{z} - f\overline{l(z)})dx\\ =&\eta_1(M_1(f)\bar{v}_1 - \tilde{M}_1(f)\bar{z}_1) + \eta_2(M_2(f)\bar{v}_2 - \tilde{M}_2(f)\bar{z}_2).\end{aligned}$$

Using (2.8), (2.9) to simplify the above we obtain

$$\begin{aligned}&W(f,\bar{z},c^-) - W(f,\bar{z},c^+)\\ =&\eta_1(M_1(f)\bar{v}_1 - \tilde{M}_1(f)\bar{z}_1) + \eta_2(M_2(f)\bar{v}_2 - \tilde{M}_2(f)\bar{z}_2).\end{aligned} \tag{2.12}$$

According to Naimark's Patching Lemma, there exists a function $F \in D(\mathcal{A})$, such that

$$f(c^-) = 1,\ f'(c^-) = 0,\ f(c^+) = 0,\ f'(c^+) = 0.$$

For such an F,

$$M_1(f) = M_2(f) = \tilde{M}_2(f) = 0,\ \tilde{M}_1(f) = -\frac{1}{\eta_1},$$

then from (2.12), we obtain

$$z_1 = M_1(z) = z'(c^-).$$

Similarly, we can prove that $z_2 = M_2(z)$. That is, there exists a function $F \in D(\mathcal{A})$, such that

$$f(c^-) = 0,\ f'(c^-) = 0,\ f(c^+) = 1,\ f'(c^+) = 0.$$

For such an F,

$$M_1(f) = M_2(f) = \tilde{M}_1(f) = 0,\ \tilde{M}_2(f) = -\frac{1}{\eta_2}.$$

So from (2.12), we get $z_2 = M_2(z) = z'(c^+)$. Thus (2) is true.

(3) and (5) can be proved similarly, here we omit the details.

Hence, the linear operator $\mathcal{A}$ is self-adjoint in the Hilbert space $\mathcal{H}$. □

As the eigenvalues of problem L coincide with the eigenvalues of the operator $\mathcal{A}$, and the eigenfunctions of problem L are the first components of the eigenvectors of the operator $\mathcal{A}$, we can directly deduce that

Corollary 2.1. *All eigenvalues of the problem L are real.*

Corollary 2.2. *Two eigenfunctions $\varphi(x,\lambda_1),\ \varphi(x,\lambda_2)$ corresponding to different eigenvalues $\lambda_1,\ \lambda_2$, are orthogonal, i.e.*

$$\int_0^c \varphi(x,\lambda_1)\varphi(x,\lambda_2)dx+\int_c^\pi \varphi(x,\lambda_1)\varphi(x,\lambda_2)dx \\ +\eta_1\varphi'(c^-,\lambda_1)\varphi'(c^-,\lambda_2)+\eta_2\varphi'(c^+,\lambda_1)\varphi'(c^+,\lambda_2)=0.$$

Now define two fundamental solutions $\varphi(x,\lambda),\ \chi(x,\lambda)$ of equation (2.1) on whole $[0,c)\cup(c,\pi]$ satisfying the jump conditions (2.4), (2.5) and the following initial conditions, respectively

$$\begin{pmatrix}\varphi(0,\lambda)\\ \varphi'(0,\lambda)\end{pmatrix}=\begin{pmatrix}1\\ h\end{pmatrix},\quad \begin{pmatrix}\chi(\pi,\lambda)\\ \chi'(\pi,\lambda)\end{pmatrix}=\begin{pmatrix}1\\ -H\end{pmatrix}.$$

Since these solutions $\varphi(x,\lambda)$ and $\chi(x,\lambda)$ satisfy the jump conditions (2.4), (2.5), the following relations

$$\begin{pmatrix}\varphi(c^+,\lambda)\\ \varphi'(c^+,\lambda)\end{pmatrix}=\begin{pmatrix}(1-a_\lambda b_\lambda)\varphi'(c^-,\lambda)-b_\lambda\varphi(c^-,\lambda)\\ -a_\lambda\varphi'(c^-,\lambda)-\varphi(c^-,\lambda)\end{pmatrix},$$
$$\begin{pmatrix}\chi(c^-,\lambda)\\ \chi'(c^-,\lambda)\end{pmatrix}=\begin{pmatrix}-a_\lambda\chi(c^+,\lambda)+(a_\lambda b_\lambda-1)\chi'(c^+,\lambda)\\ \chi(c^+,\lambda)-b_\lambda\chi'(c^+,\lambda)\end{pmatrix}$$

hold, where $a_\lambda=\lambda\eta_1-\xi_1, b_\lambda=\lambda\eta_2-\xi_2$.

For each x, these solutions satisfy the relation $l_1(\varphi)=l_2(\chi)=0$. Then the characteristic function can be introduced as

$$\Delta(\lambda)=\langle\varphi(x,\lambda),\chi(x,\lambda)\rangle=\varphi(x,\lambda)\chi'(x,\lambda)-\varphi'(x,\lambda)\chi(x,\lambda), \tag{2.13}$$

according to the Liouville formula, the Wronskian $\langle\varphi(x,\lambda),\chi(x,\lambda)\rangle$ is an entire function in λ and the zeros namely $\{\lambda_n\}_{n\geq0}$ of $\Delta(\lambda)$ coincide with the eigenvalues of the problem L. Substituting $x=\pi$ into (2.13) we get

$$\Delta(\lambda)=-(\varphi'(\pi)+H\varphi(\pi)). \tag{2.14}$$

The normal constants α_n of the problem L are as follows

$$\alpha_n=\int_0^c\varphi^2(x,\lambda_n)dx+\int_c^\pi\varphi^2(x,\lambda_n)dx+\eta_1\varphi'^2(c^-,\lambda_n)+\eta_2\varphi'^2(c^+,\lambda_n). \tag{2.15}$$

Lemma 2.1. [11] *If the functions $\varphi(x,\lambda_n)$ and $\chi(x,\lambda_n)$ are the eigenfunctions of the problem L, then there exists a sequence $\{\beta_n\}$ such that*

$$\chi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n),\quad \beta_n\neq0. \tag{2.16}$$

Theorem 2.2. *Let $\{\lambda_n\}$ be the zeros of the function $\Delta(\lambda)$, then*

$$\dot{\Delta}(\lambda_n)=\beta_n\alpha_n, \tag{2.17}$$

where $\dot{\Delta}(\lambda)=\frac{d\Delta}{d\lambda}$, $\alpha_n,\ \beta_n$ are defined by (2.15) and (2.16), respectively.

Proof. Let us write the following equations,

$$-\chi''(x,\lambda)+q(x)\chi(x,\lambda)=\lambda\chi(x,\lambda), \tag{2.18}$$
$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n\varphi(x,\lambda_n). \tag{2.19}$$

Let (2.18), (2.19) multiplied by $\varphi(x,\lambda_n)$ and $\chi(x,\lambda)$, respectively, and subtracting them, then the equality

$$\frac{d}{dx}\left\langle\varphi(x,\lambda_n),\chi(x,\lambda)\right\rangle=(\lambda_n-\lambda)\varphi(x,\lambda_n)\chi(x,\lambda) \tag{2.20}$$

is obtained. Integrating over the interval J

$$\begin{aligned}&(\lambda_n-\lambda)(\int_0^c\chi(x,\lambda)\varphi(x,\lambda_n)dx+\int_c^\pi\chi(x,\lambda)\varphi(x,\lambda_n)dx)\\
=&-\Delta(\lambda)-(\lambda_n-\lambda)\eta_1\varphi(c^+,\lambda_n)\chi(c^+,\lambda)\\
&+(\lambda_n-\lambda)\eta_1(\lambda\eta_2-\xi_2)\varphi(c^+,\lambda_n)\chi'(c^+,\lambda)\\
&+(\lambda_n-\lambda)\eta_1(\lambda_n\eta_2-\xi_2)\varphi'(c^+,\lambda_n)\chi(c^+,\lambda)\\
&-(\lambda_n-\lambda)(\eta_1(\lambda_n\eta_2-\xi_2)(\lambda\eta_2-\xi_2)+\eta_2)\varphi'(c^+,\lambda_n)\chi'(c^+,\lambda).\end{aligned}$$

Dividing both sides of the above equality by $\lambda_n-\lambda$, and let $\lambda\to\lambda_n$, then we have

$$\begin{aligned}-\dot{\Delta}(\lambda_n)=&-\int_0^c\chi(x,\lambda_n)\varphi(x,\lambda_n)dx-\int_c^\pi\chi(x,\lambda_n)\varphi(x,\lambda_n)dx\\
&-\eta_1\varphi(c^+,\lambda_n)\chi(c^+,\lambda_n)\\
&+[\eta_1(\lambda_n\eta_2-\xi_2)]\varphi(c^+,\lambda_n)\chi'(c^+,\lambda_n)\\
&+[\eta_1(\lambda_n\eta_2-\xi_2)]\varphi'(c^+,\lambda_n)\chi(c^+,\lambda_n)\\
&-[\eta_1(\lambda_n\eta_2-\xi_2)^2+\eta_2]\varphi'(c^+,\lambda_n)\chi'(c^+,\lambda_n).\end{aligned}$$

Using (2.16)

$$\begin{aligned}\dot{\Delta}(\lambda_n)=&\beta_n[\int_0^c\varphi^2(x,\lambda_n)dx+\int_c^\pi\varphi^2(x,\lambda_n)dx+\eta_1\varphi^2(c^+,\lambda_n)\\
&-2\eta_1(\lambda_n\eta_2-\xi_2)\varphi(c^+,\lambda_n)\varphi'(c^+,\lambda_n)+((\lambda_n\eta_2-\xi_2)^2\eta_1)\varphi'^2(c^+,\lambda_n)\\
&+\eta_2\varphi'^2(c^+,\lambda_n)]\\
=&\beta_n[\int_0^c\varphi^2(x,\lambda_n)dx+\int_c^\pi\varphi^2(x,\lambda_n)dx+\eta_1\varphi'^2(c^-,\lambda_n)+\eta_2\varphi'^2(c^+,\lambda_n)]\\
=&\beta_n\alpha_n.\end{aligned}$$

Thus the equality (2.17) holds. □

Theorem 2.2 means that the zeros of $\Delta(\lambda)$, thus the eigenvalues of problem L are simple.

3. Asymptotic approximation of fundamental solutions and eigenvalues

In this section, we will obtain the asymptotic approximation of fundamental solutions and eigenvalues of the problem L.

Lemma 3.1. *Let $s=\sqrt{\lambda}=\sigma+i\tau$. Then the following asymptotics hold.*

$$\varphi(x,\lambda)=\begin{cases}\cos sx+O(\frac{1}{s}exp(|\tau|x)), \quad x\in[0,c),\\ s^6\eta_1\eta_2[\frac{\sin sx}{2s}-\frac{\sin s(x-2c)}{2s}]+O(s^4exp(|\tau|x)),\ x\in(c,\pi],\end{cases} \tag{3.1}$$

$$\chi(x,\lambda)=\begin{cases}s^6\eta_1\eta_2[\frac{\sin s(\pi-x)}{2s}-\frac{\sin s(2c-x-\pi)}{2s}]+O(s^4exp(|\tau|x)),\ x\in[0,c),\\ \cos s(\pi-x)+O(\frac{1}{s}exp(|\tau|x)),\ x\in(c,\pi].\end{cases} \tag{3.2}$$

Proof. From method of variation of parameters, it can be calculated out that when $x\in[0,c)$

$$\varphi(x,\lambda)=c_1\cos(sx)+c_2\frac{\sin(sx)}{s}+\int_0^x\frac{\sin[s(x-t)]}{s}q(t)\varphi(t,\lambda)dt. \tag{3.3}$$

Since

$$\varphi(0,\lambda)=1,\quad \varphi'(0,\lambda)=h,$$

substituting into (3.3), we get

$$\varphi(x,\lambda)=\cos sx+h\frac{\sin sx}{s}+\int_0^x\frac{\sin s(x-t)}{s}q(t)\varphi(t,\lambda)dt.$$

Using method of variation of parameters again, for $x\in(c,\pi]$ we can calculate out

$$\varphi(x,\lambda)=c_3\cos s(x-c)+c_4\frac{\sin s(x-c)}{s}+\int_c^x\frac{\sin s(x-t)}{s}q(t)\varphi(t,\lambda)dt, \tag{3.4}$$

$$\varphi'(x,\lambda)=-sc_3\sin s(x-c)+c_4\cos s(x-c)+\int_c^x\cos s(x-t)q(t)\varphi(t,\lambda)dt. \tag{3.5}$$

Since

$$\varphi(c^+,\lambda)=(1-a_\lambda b_\lambda)\varphi'(c^-,\lambda)-b_\lambda\varphi(c^-,\lambda),$$
$$\varphi'(c^+,\lambda)=-a_\lambda\varphi'(c^-,\lambda)-\varphi(c^-,\lambda),$$

substituting into (3.4) and (3.5), we get

$$c_3=(1-a_\lambda b_\lambda)\varphi'(c^-,\lambda)-b_\lambda\varphi(c^-,\lambda),$$
$$c_4=-a_\lambda\varphi'(c^-,\lambda)-\varphi(c^-,\lambda).$$

So, if $x\in(c,\pi]$ the equation for $\varphi(x,\lambda)$ is

$$\begin{aligned}\varphi(x,\lambda)=&[(1-a_\lambda b_\lambda)\varphi'(c^-,\lambda)-b_\lambda\varphi(c^-,\lambda)]\cos s(x-c)-[a_\lambda\varphi'(c^-,\lambda)\\ &+\varphi(c^-,\lambda)]\frac{\sin s(x-c)}{s}+\int_c^x\frac{\sin s(x-t)}{s}q(t)\varphi(t,\lambda)dt\\ =&\frac{s^2-s^2(s^2\eta_1-\xi_1)(s^2\eta_2-\xi_2)+(s^2\eta_2-\xi_2)h}{2s}(\sin s(x-2c)-\sin(sx))\\ &-\frac{1+h(s^2\eta_1-\xi_1)}{2s}(\sin s(x-2c)+\sin(sx))\end{aligned}$$

$$
\begin{aligned}
&+\frac{s^2(s^2\eta_1-\xi_1)-h}{2s^2}(\cos s(x-2c)-\cos(sx))\\
&+\frac{h-(s^2\eta_1-\xi_1)(s^2\eta_2-\xi_2)h-s^2\eta_2+\xi_2}{2}(\cos s(x-2c)+\cos(sx))\\
&+\frac{1-(s^2\eta_1-\xi_1)(s^2\eta_2-\xi_2)}{2}\\
&\times\int_0^c[\cos s(x-t)+\cos s(x-2c+t)]q(t)\varphi(t,\lambda)dt\\
&+\frac{1}{2s^2}\int_0^c[\cos s(x-t)-\cos s(x-2c+t)]q(t)\varphi(t,\lambda)dt\\
&-\frac{s^2\eta_2-\xi_2}{2s}\int_0^c[\sin s(x-t)-\sin s(x-2c+t)]q(t)\varphi(t,\lambda)dt\\
&-\frac{s^2\eta_1-\xi_1}{2s}\int_0^c[\sin s(x-t)+\sin s(x-2c+t)]q(t)\varphi(t,\lambda)dt\\
&+\int_c^x\frac{\sin s(x-t)}{s}q(t)\varphi(t,\lambda)dt.
\end{aligned}
$$

Similarly, the equation for $\chi(x,\lambda)$ can be drawn

When $s\to\infty$, then the asymptotic representations (3.1) and (3.2) are established. □

According to (2.14) and (3.1), the characteristic function $\Delta(\lambda)$ as $s\to\infty$ is

$$\Delta(\lambda)=\frac{s^6}{2}\eta_1\eta_2(\cos s(\pi-2c)-\cos s\pi)+O(s^5(exp(|\tau|\pi)). \tag{3.6}$$

Let $\Delta(\lambda)=\Delta_1(\lambda)+\Delta_2(\lambda)$, where

$$
\begin{aligned}
\Delta_1(\lambda)&=\frac{s^6}{2}\eta_1\eta_2(\cos s(\pi-2c)-\cos s\pi)=-s^6\eta_1\eta_2\sin s(\pi-c)\sin sc,\\
\Delta_2(\lambda)&=O(s^5exp(|\tau|\pi)).
\end{aligned}
$$

We now can establish the asymptotic approximation formulas for the eigenvalues and characteristic function by similar arguments as those in [16].

Lemma 3.2. *Let λ_n be the eigenvalues of the problem L, $\lambda=s^2$. Let $\{r_k\}_{k=0}^{\infty}$ be the zeros (counting with multiplicities if any) of the entire function*

$$\Delta_1(\lambda)=-s^6\eta_1\eta_2\sin s(\pi-c)\sin sc.$$

If $r_{n'}$ is the closest point to λ_n, then as $n\to\infty$

$$s_n=\sqrt{r_{n'}}+O(n^{-\frac{1}{2}}). \tag{3.7}$$

In addition, if there exists a positive number C_0 such that $\{r_k\}_{k=0}^{\infty}$ satisfy

$$|\sqrt{r_k}-\sqrt{r_{k'}}|>C_0,\quad as\quad r_k\neq r_{k'}, \tag{3.8}$$

then (3.7) holds for $n=n'$.

Lemma 3.3. *Fix $\delta>0$. Let $G_\delta=\{s:|s-\sqrt{r_n}|\geq\delta,\quad n\geq0\}$, then for sufficient large $s^*>0$ there exists a constant C_δ such that*

$$|\Delta(\lambda)|>C_\delta|s^6|exp(|\tau|\pi),\quad s\in G_\delta,\quad |s|>s^*. \tag{3.9}$$

4. Inverse problems

In this section, we mainly consider the reconstruction of the problem L, from the Weyl function, from the spectral data $\{\lambda_n, \alpha_n\}$, and from two spectra $\{\lambda_n\} \cup \{\mu_n\}$.

Denote

$$M(\lambda) = \frac{\chi(0,\lambda)}{\Delta(\lambda)}. \tag{4.1}$$

Let $S(x,\lambda)$ be the solution of (2.1), satisfying the following initial conditions and jump conditions (2.4) and (2.5)

$$S(0,\lambda) = 0, \quad S'(0,\lambda) = 1.$$

Because of $W[\varphi, S; x] = 1$, we have

$$\chi(x,\lambda) = \Delta(\lambda) S(x,\lambda) + \chi(0,\lambda)\varphi(x,\lambda),$$

or

$$\frac{\chi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda) + M(\lambda)\varphi(x,\lambda). \tag{4.2}$$

Denote

$$\Phi(x,\lambda) = \frac{\chi(x,\lambda)}{\Delta(\lambda)}. \tag{4.3}$$

Thus $\Phi(x,\lambda)$ is the solution of (2.1) that satisfies the conditions $l_1(\Phi) = 1$, $l_2(\Phi) = 0$ and the jump conditions (2.4) and (2.5), where $\Delta(\lambda)$ is defined in (2.13).

The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the boundary value problem L.

Next, the uniqueness theorem for problem L will be given by the Weyl function. For studying the inverse problem we agree that together with L consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{H}$, $\tilde{\eta}_i$, $\tilde{\xi}_i$, $i = 1, 2$.

Theorem 4.1. *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$, i.e. $q(x) = \tilde{q}(x)$, a.e. and $h = \tilde{h}$, $H = \tilde{H}$, $\eta_i = \tilde{\eta}_i$, $\xi_i = \tilde{\xi}_i$, $i = 1, 2$.*

Proof. Let us define the matrix $P(x,\lambda) = [p_{j,k}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix} \tilde{\varphi}(x,\lambda) & \tilde{\Phi}(x,\lambda) \\ \tilde{\varphi}'(x,\lambda) & \tilde{\Phi}'(x,\lambda) \end{pmatrix} = \begin{pmatrix} \varphi(x,\lambda) & \Phi(x,\lambda) \\ \varphi'(x,\lambda) & \Phi'(x,\lambda) \end{pmatrix},$$

then we can calculate that

$$\begin{cases} p_{j,1}(x,\lambda) = \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}'(x,\lambda), \\ p_{j,2}(x,\lambda) = \varphi^{(j-1)}(x,\lambda)\tilde{\Phi}(x,\lambda) - \Phi^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \varphi(x,\lambda) = p_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + p_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda), \\ \Phi(x,\lambda) = p_{11}(x,\lambda)\tilde{\Phi}(x,\lambda) + p_{12}(x,\lambda)\tilde{\Phi}'(x,\lambda). \end{cases} \tag{4.5}$$

From (4.2)-(4.4) it can be obtained that

$$p_{11}(x,\lambda) = 1 + \frac{1}{\Delta(\lambda)}(\chi(x,\lambda)(\tilde{\varphi}'(x,\lambda) - \varphi'(x,\lambda)) - \varphi(x,\lambda)(\tilde{\chi}'(x,\lambda) - \chi'(x,\lambda))),$$

$$p_{12}(x,\lambda) = \frac{1}{\Delta(\lambda)}(\varphi(x,\lambda)\tilde{\chi}(x,\lambda) - \tilde{\varphi}(x,\lambda)\chi(x,\lambda)).$$

By virtue of (3.1), (3.2) and (3.9), for sufficiently large s^*, there exists a constant $C_\delta > 0$ such that

$$|p_{11}(x,\lambda) - 1| \le \frac{C_\delta}{|s|},\ |p_{12}(x,\lambda)| \le \frac{C_\delta}{|s|},\ s \in G_\delta,\ |s| \ge s^*. \tag{4.6}$$

According to (4.2) and (4.4), the following equations can be obtained

$$p_{11}(x,\lambda) = S(x,\lambda)\tilde{\varphi}'(x,\lambda) - \varphi(x,\lambda)\tilde{S}'(x,\lambda) + (M(\lambda) - \tilde{M}(\lambda))\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda),$$

$$p_{12}(x,\lambda) = \varphi(x,\lambda)\tilde{S}(x,\lambda) - S(x,\lambda)\tilde{\varphi}(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).$$

Thus, if $M(\lambda) = \tilde{M}(\lambda)$, then for each fixed x, the functions $p_{11}(x,\lambda)$ and $p_{12}(x,\lambda)$ are entire in λ. Combined with (4.6), and according to Liouville's theorem, we can get

$$p_{11}(x,\lambda) = 1,\quad p_{12}(x,\lambda) = 0. \tag{4.7}$$

Substituting (4.7) into (4.5), then for each $x \in J$ and $\lambda \in \mathbb{C}$ it has

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda),\quad \Phi(x,\lambda) = \tilde{\Phi}(x,\lambda). \tag{4.8}$$

Thus if $M(\lambda) = \tilde{M}(\lambda)$ holds, then we can conclude $q(x) = \tilde{q}(x)$, a.e. and $h = \tilde{h},\ H = \tilde{H},\ \eta_i = \tilde{\eta}_i,\ \xi_i = \tilde{\xi}_i,\ i = 1,2$. So consequently, $L = \tilde{L}$. □

The following lemma can be established similarly as those in [11,16].

Lemma 4.1. *The function $M(\lambda)$ is defined by (4.1), then the following expression can be established*

$$M(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\alpha_n(\lambda - \lambda_n)}. \tag{4.9}$$

Theorem 4.2. *If $\lambda_n = \tilde{\lambda}_n$ and $\alpha_n = \tilde{\alpha}_n,\ n \in \mathbb{N}_0$, then $q(x) = \tilde{q}(x)$ a.e., $h = \tilde{h},\ H = \tilde{H},\ \eta_i = \tilde{\eta}_i,\ \xi_i = \tilde{\xi}_i,\ i = 1,2$. Thus $L = \tilde{L}$.*

Proof. From lemma 4.1, if $\lambda_n = \tilde{\lambda}_n$ and $\alpha_n = \tilde{\alpha}_n$, then $M(\lambda) = \tilde{M}(\lambda)$. According to Theorem 4.1, this theorem can be proved. □

Lastly, through the two spectra $\{\lambda_n\} \cup \{\mu_n\}$, let us prove the uniqueness theorem. Let $\{\mu_n\}_{n=0}^{\infty}$ be the spectra of the problem L_1 consisting of the equation (2.1) with the boundary conditions $y(0) = 0,\ y'(\pi) + Hy(\pi) = 0$ and jump conditions (2.4) and (2.5). It is obvious that μ_n are the zeros of $\Delta_0(\lambda) = \chi(0,\lambda)$.

Lemma 4.2. *The spectra $\{\lambda_n\}_{n\ge 0}$ of L uniquely determine the characteristic function $\Delta(\lambda)$ as the formula*

$$\Delta(\lambda) = -C_0 \prod_{n=0}^{3}(\lambda_n - \lambda) \prod_{n=4}^{\infty} \frac{\lambda_n - \lambda}{\lambda_n^0}, \tag{4.10}$$

where $C_0 = -c(\pi - c)\eta_1\eta_2$, λ_n are the zeros of $\Delta(\lambda)$, λ_n^0 are the zeros of $\Delta_1(\lambda)$.

Proof. Since $\Delta(\lambda)$ and $\Delta_1(\lambda)$ are all entire in λ of order $\frac{1}{2}$, without of generality assume that $\lambda_n > 0$, so according to Hadamard's factroization theorem, $\Delta(\lambda)$ and $\Delta_1(\lambda)$ can only be represented by following equations

$$\Delta(\lambda) = C\prod_{n=0}^{\infty}(1-\frac{\lambda}{\lambda_n}), \quad \Delta_1(\lambda) = C_0\lambda^4\prod_{n=4}^{\infty}(1-\frac{\lambda}{\lambda_n^0}), \tag{4.11}$$

where $C_0 = -c(\pi - c)\eta_1\eta_2$. Then

$$\frac{\Delta(\lambda)}{\Delta_1(\lambda)} = \frac{C}{C_0}\prod_{n=0}^{3}(\frac{1}{\lambda}-\frac{1}{\lambda_n})\prod_{n=4}^{\infty}(\frac{\lambda_n^0}{\lambda_n})\prod_{n=4}^{\infty}(1+\frac{\lambda_n-\lambda_n^0}{\lambda_n^0-\lambda}). \tag{4.12}$$

According to (3.6) and from [11], it has

$$\lim_{\lambda\to-\infty}\frac{\Delta(\lambda)}{\Delta_1(\lambda)} = 1, \quad \lim_{\lambda\to-\infty}\prod_{n=4}^{\infty}(1+\frac{\lambda_n-\lambda_n^0}{\lambda_n^0-\lambda}) = 1, \tag{4.13}$$

and

$$\lim_{\lambda\to-\infty}\prod_{n=0}^{3}(\frac{1}{\lambda}-\frac{1}{\lambda_n}) = -\prod_{n=0}^{3}\frac{1}{\lambda_n}.$$

Hence, from (4.12), it can be obtained that

$$C = -C_0\prod_{n=0}^{3}\lambda_n\prod_{n=4}^{\infty}\frac{\lambda_n}{\lambda_n^0}, \tag{4.14}$$

substituting (4.14) into (4.11), one can calculate out

$$\Delta(\lambda) = -C_0\prod_{n=0}^{3}(\lambda_n-\lambda)\prod_{n=4}^{\infty}\frac{\lambda_n-\lambda}{\lambda_n^0}.$$

□

Theorem 4.3. *If $\lambda_n = \tilde{\lambda}_n,\ \mu_n = \tilde{\mu}_n,\ n \geq 0$, then $q(x) = \tilde{q}(x)$ a.e., $H = \tilde{H},\ \eta_i = \tilde{\eta}_i,\ \xi_i = \tilde{\xi}_i,\ i = 1, 2$. Thus $L = \tilde{L}$.*

Proof. From Lemma 4.2, we know that each λ uniquely determines its characteristic function $\Delta(\lambda)$, similarly μ_n also uniquely determines $\Delta_0(\lambda)$. Therefore, one has $\Delta(\lambda) = \tilde{\Delta}(\lambda),\ \Delta_0(\lambda) = \tilde{\Delta}_0(\lambda)$, i.e. $\chi(0,\lambda) = \tilde{\chi}(0,\lambda)$ when $\lambda_n = \tilde{\lambda}_n,\ \mu_n = \tilde{\mu}_n$. Consequently $M(\lambda) = \tilde{M}(\lambda)$, according to Theorem 4.1, the proof is completed. □

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