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STABILITY AND HOPF BIFURCATION ANALYSIS OF A NETWORKED SIR EPIDEMIC MODEL WITH TWO DELAYS AND DELAY DEPENDENT PARAMETERS*

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Abstract The spread of infectious diseases is generally influenced by the random contact of different individuals in uneven spatial structure. To describe this contact effect, network is introduced into a two delays SIR epidemic model with incubation period delay and temporary immunity delay. Due to the existence of the temporary immunity term, the characteristic equation of epidemic model has two delays and the parameters depend on one of them. We prove the stability of the disease-free equilibrium and the endemic equilibrium. We additionally obtain the stability switching curves to study the stability switching properties of the endemic equilibrium on the two delays plane when two delays change simultaneously, and further discuss the existence of Hopf bifurcation. The stability and the direction of the Hopf bifurcation are investigated with the normal form method and center manifold theorem. To illustrate our theoretical conclusions visually, we performed numerical simulation on a small-world Watts-Strogatz graph.

Keywords Graph network, two delays, delay dependent parameters, stability switching curves, Hopf bifurcation.

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1. Introduction

Mathematical analysis and modeling of infectious diseases play an important role in studying the transmission dynamics of infectious diseases. Due to the complexity of the propagation, the classic SIR epidemic model proposed by Kermack and McKendric [22] can not completely simulate the real situation. Much recent research works have focused on incorporating various factors into epidemic models like incidence rate, treatment rate, disease incubation period, immunity, etc. Wang and Jiang [35] studied the stability of equilibria of SIS epidemic model with saturated incidence rate and treatment. Enatsu et al. [11] established global stability analysis of equilibrium of an SIRS epidemic model with a class of nonlinear incidence rates

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and distributed delays. Huang et al. [19] incorporated delays and nonlinear incidence rate into SIR and SEIR epidemic models, and proved respectively the global stability of endemic equilibrium and the disease-free equilibrium.

In an epidemic model, the incidence rate is the rate of susceptible individuals becoming infected through contact with infective agent. Many researchers have conducted extensive studies by using various incidence rates, such as bilinear incidence rate $\beta SI/N$ in [17], β is the disease contact rate; nonlinear incidence rate $\beta S^p I^q$ in [18], p and q represent the index of the number of susceptible and infected persons, respectively; nonlinear incidence rate $\frac{\beta I^p S}{(1+kI^q)}$ in [27], $k > 0$ is the basic propagation coefficient; nonlinear incidence rate $kS\ln(1 + vP/k)$ in [5], P is the density of pathogen particles; Crowley-Martin incidence rate $\frac{\beta SI}{(1+\alpha S)(1+\gamma I)}$ in [7]; Beddington-DeAngelis incidence rate $\frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha S(t-\tau)+\gamma I(t-\tau)}$ in [3], α represents the inhibitory effect of preventive measures on the spread of diseases in susceptible individuals, γ represents the inhibitory effect of treatment on the spread of diseases in infected individuals.

Treatment rate plays an important role in preventing and controlling the spread of infectious diseases. When the number of infected person is small and treatment resources are sufficient, treatment rate was considered to be either constant [37] or proportional to number of infected individuals [36]. However, in order to control the disease, many researchers pay more attention to the nonlinear treatment rate. Holling II type, Holling III type and Holling IV type treatment rates are respectively introduced by Dubey et al. [8–10] to their models and studied the dynamics of models. Saturated treatment rate $\frac{aI^2}{bI^2+cI+1}$ is considered by Goel and Nilam [15] to study the dynamics of SIR epidemic model, where $a > 0$ represents the treatment rate of infected individuals, $b > 0$ is the limitation rate in treatment availability, and $c \geq 0$ is the saturation constant in absence of inhibitory effect.

The transmission dynamics of infectious diseases are influenced by both the current and historical states of infection. For instance, influenza does not manifest in an individual at the exact moment of infection but rather has a time delay. The time delay for COVID-19 is between roughly 2 and 14 days after exposure. During the incubation period, the pathogen is actively replicating within the host, but the individual does not yet exhibit any outward symptoms of the illness. The temporary immunity delay is the time needed for the immune system to effectively respond after encountering the antigen through vaccination or infection. During this time, the individual may not yet have full immunity and could still be susceptible to infection. In infectious diseases, the incubation period and the temporary immunity delay are crucial for understanding epidemiological phenomena and for designing appropriate prevention and control policies. Consequently, it is imperative to incorporate time delays into our models to accurately reflect the impact of historical infection states on the current situation. Xu and Ma [39], Kumar and Nilam [13,23] considered the influence of incubation delay on SIRS, SIR and SVIRS epidemic models, respectively. They also detailed the necessary conditions that guarantee the stability of equilibria of their proposed models. In [15], Goel and Nilam discussed the stability of the following SIR epidemic model with a delay

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \pi - \mu S(t) - \frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha S(t-\tau)+\gamma I(t-\tau)},\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha S(t-\tau)+\gamma I(t-\tau)} - (\mu+d+\theta)I(t) - \frac{aI^2(t)}{bI^2(t)+cI(t)+1},
\end{aligned}
$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \theta I(t) + \frac{aI^2(t)}{bI^2(t) + cI(t) + 1} - \mu R(t), \tag{1.1}$$

where susceptible individuals moved into the total population with a constant rate π. $S(t), I(t)$ and $R(t)$ are the numbers of susceptible, infected, and recovered persons respectively. μ, d, θ represent the natural death rate, disease-induced death rate and recovery rate, respectively. $\frac{\beta S(t-\tau)I(t-\tau)}{1+\alpha S(t-\tau)+\gamma I(t-\tau)}$ is Beddington-DeAngelis type incidence rate. $\frac{aI^2(t)}{bI^2(t)+cI(t)+1}$ is the saturated treatment rate, τ is incubation period delay of disease, which characterizes the delay that the susceptible individual becomes infected but is not yet infectious.

The recovered individual has temporary immunity $\theta I(t-\tau_2)e^{-\mu\tau_2}$ to a disease [12], the term reflects that an individual has survived from natural death in a recovery pool before becoming susceptible again, τ_2 is temporary immunity period, which reflects the time delay between a recovered individual recovering and becoming susceptible again. This biological process can be captured by a model with delay-dependent parameters. Kyrychko and Blyuss [24] proposed an SIR epidemic model with temporary immunity and nonlinear incidence rate to study the stability of equilibria. A two delays SIR epidemic model was proposed by Goel et al. [14] to discuss the stability of equilibria. Xu and Wei [40] recently considered a diffusive budworm model with delay-dependent coefficients to discuss the existence of Hopf bifurcation when space is heterogeneous. The local asymptotical stability and the existence of Hopf bifurcation at the equilibrium of an SIR epidemic model with latent period and temporary immunity were established by Jiang and Ma [21].

Considering the inherently non-uniform nature of environments, individuals engage with others in a random manner, which could help spread diseases over time. Incorporating complex networks into infectious disease models can effectively represent these contact dynamics. Individuals are regarded as nodes, the connection between people is the edge. Many epidemic models based on network have been proposed to control disease outbreaks. Cheng et al. [6] introduced a network-based SIQS epidemic model with nonmonotone incidence rate and proved the stability of disease-free equilibrium and the permanence of the disease. The result indicated that quarantine strategy was an effective measure in preventing epidemic spreading. Lv et al. [29] investigated a robust optimal control problem of a network-based SIVS system with an uncertain time delay, discussed the effects of different weight factors on disease control. An epidemic model about the effect of awareness programs on complex networks was proposed by Chen et al. [41], the stability of disease-free equilibrium and endemic equilibrium was proved. Tian et al. [32] introduced the continuous Laplacian operator to network-based SIR epidemic model, and demonstrated the global stability of the model. Barman and Mishra [2] explored the existence and the property of Hopf bifurcation of a delayed nonlinear SEIR epidemic model on weighted network. In [28], weighted network was incorporated to SIRS epidemic model and the asymptotical behavior of the model was discussed. The stability of equilibrium of SEIR epidemic model and transient dynamics of model were studied on the network by Tian et al. [31]. A new methodology based on the construction of epidemic networks is introduced by Herrera-Diestra [16], used symbolic networks to analyse dynamical properties of disease outbreaks. The impact of population size on epidemic spreading in a bipartite metapopulation network with recurrent mobility is studied [34].

In this paper, considering the influence of temporary immunity and network on disease transmission, on the basis of model (1.1), we construct the following model with graph Laplacian operator

$$\begin{cases}\dfrac{\partial S}{\partial t}-\varepsilon_1\Delta S(x,t)=\pi-\mu S(x,t)-\dfrac{\beta S(x,t-\tau_1)I(x,t-\tau_1)}{1+\alpha S(x,t-\tau_1)+\gamma I(x,t-\tau_1)}\\ \qquad\qquad +\theta I(x,t-\tau_2)e^{-\mu\tau_2},\\ \dfrac{\partial I}{\partial t}-\varepsilon_2\Delta I(x,t)=\dfrac{\beta S(x,t-\tau_1)I(x,t-\tau_1)}{1+\alpha S(x,t-\tau_1)+\gamma I(x,t-\tau_1)}-(\mu+d+\theta)I(x,t)\\ \qquad\qquad -\dfrac{aI^2(x,t)}{bI^2(x,t)+cI(x,t)+1},\\ \dfrac{\partial R}{\partial t}-\varepsilon_3\Delta R(x,t)=\theta I(x,t)-\theta I(x,t-\tau_2)e^{-\mu\tau_2}+\dfrac{aI^2(x,t)}{bI^2(x,t)+cI(x,t)+1}\\ \qquad\qquad -\mu R(x,t),\\ S_0(x,q)=\Gamma_1(x,q),I_0(x,q)=\Gamma_2(x,q),R_0(x,q)=\Gamma_3(x,q),\\ q\in[-\tau,0],\Gamma_i(x,0)>0,i=1,2,3,\end{cases}\tag{1.2}$$

where $(x,t)\in\mathcal{V}\times(0,\infty)$, $(\Gamma_1(x,q),\Gamma_2(x,q),\Gamma_3(x,q))\in\mathcal{C}\left([-\tau,0],\mathbb{R}^3_+\right)$, $\mathcal{V}$ is set of nodes, $\mathcal{C}$ denotes the Banach space $\mathcal{C}([-\tau,0],\mathbb{R}^3_+)$ of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^3_+$, $\mathbb{R}^3_+=\{y\in\mathbb{R}^3: y\geq 0\}$, $\tau=\max\{\tau_1,\tau_2\}$. The term $\theta I(t-\tau_2)e^{-\mu\tau_2}$ is temporary immunity, τ_2 is immunity delay. $\varepsilon_1,\varepsilon_2$ and ε_3 are the diffusion rates, Δ is the graph Laplacian operator, other parameters have the same biological significance as model (1.1). From an epidemiological point of view, we hypothesize all parameters are positive.

Due to $R(t)$ does not exist in the first two equations of system (1.2), we can just consider the following equation

$$\begin{cases}\dfrac{\partial S}{\partial t}-\varepsilon_1\Delta S(x,t)=\pi-\mu S(x,t)-\dfrac{\beta S(x,t-\tau_1)I(x,t-\tau_1)}{1+\alpha S(x,t-\tau_1)+\gamma I(x,t-\tau_1)}\\ \qquad\qquad +\theta I(x,t-\tau_2)e^{-\mu\tau_2}\\ \qquad\qquad \triangleq F_1,\\ \dfrac{\partial I}{\partial t}-\varepsilon_2\Delta I(x,t)=\dfrac{\beta S(x,t-\tau_1)I(x,t-\tau_1)}{1+\alpha S(x,t-\tau_1)+\gamma I(x,t-\tau_1)}-(\mu+d+\theta)I(x,t)\\ \qquad\qquad -\dfrac{aI^2(x,t)}{bI^2(x,t)+cI(x,t)+1}\\ \qquad\qquad \triangleq F_2,\\ S_0(x,q)=\Gamma_1(x,q),\ I_0(x,q)=\Gamma_2(x,q),\\ q\in[-\tau,0],\ \Gamma_i(x,0)>0,\ i=1,2,\end{cases}\tag{1.3}$$

where $(x,t)\in\mathcal{V}\times(0,\infty)$, $(\Gamma_1(x,q),\Gamma_2(x,q))\in\mathcal{C}\left([-\tau,0],\mathbb{R}^2_+\right),\tau=\max\{\tau_1,\tau_2\}$.

Due to the existence of temporary immunity item $e^{-\mu\tau_2}$, the characteristic equation depend on the delay τ_2. When two delays change simultaneously, and characteristic equation depend on the delay τ_2, An et al. [1] and Wang et al. [33] used the method of stability switching curves to analyze the existence of Hopf bifurcation. They determined the stability switching curves in the two delays plane, analyzed the direction of the characteristic roots crossing the imaginary axis, and obtained the direction of bifurcation in stability switching curves. This method is also applied to examine the impact of two delays on the stability of planktonic resource-consumer model [20], fractional-order ring-hub structure neural network [30] and diffusive predator-prey model [25]. In this paper, we use the method of stability switching curves proposed by [1] to study the existence of Hopf bifurcation of model (1.3).

The following content is organized as follows: preliminary knowledge are presented in Section 2. In Section 3, the existence analysis and local stability analysis of disease-free equilibrium and endemic equilibrium are performed. The method of stability switching curves is applied to prove the existence of Hopf bifurcation. In Section 4, the calculation of the normal form regarding Hopf bifurcation is completed and the property of Hopf bifurcation is obtained. Numerical simulations for supporting results have been carried out in Section 5. We give conclusion in Section 6.

2. Preliminaries

Model (1.3) is relevant to a undirected weighted network with m nodes, that is, human migration is inhomogeneous. The flow of people is represented by a network. From the mathematical viewpoint a network is a graph $G := \langle \mathcal{V}, \mathcal{E} \rangle$, $\mathcal{V} = \{1, 2, \ldots, m\}$ represents a set of nodes and $\mathcal{E}$ stands for a collection of edges, $|\mathcal{V}| = m$ is total number of nodes. The motion speeds of susceptible and infected people are positive constants ε_1 and ε_2. If a node y is adjacent to a node x, we write $y \sim x$. The graph Laplacian operator Δ acts on the function $f : \mathcal{V} \to \mathbb{R}$ from continuous space to a finite standard weight graph [28] is : $\Delta f(x) = \sum_{y \in \mathcal{V}, y \sim x} [f(y) - f(x)] \omega_{xy}$, where $\omega_{xy} > 0$ is the weight of each edge and satisfies $\omega_{xy} = \omega_{yx}$.

Lemma 2.1. (see Lemma 2.2 of [26]) *Consider the eigenvalue problem*

$$\begin{cases} -\Delta \Psi(x) = \alpha_j \Psi(x), \ x \in \mathcal{V}, \\ \displaystyle\int_{\mathcal{V}} \Psi^2(x) = 1, \end{cases}$$

there exists a series of eigenvalues $\{\alpha_j\}_{j=1}^m$ such that it satisfies the following

$$0 = \alpha_1 < \alpha_2 \leq \cdots \leq \alpha_{m-1} \leq \alpha_m, \tag{2.1}$$

whose associated eigenfunctions are $\{\Psi_j\}_{j=1}^m$.

Lemma 2.2. (see Lemma 2.3 of [2]) *Suppose $\mathbf{u}(x,t) = \left(u^{(1)}(x,t), u^{(2)}(x,t)\right)^{\mathrm{T}}$ be the solution of*

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t}(x,t) - \epsilon \Delta \mathbf{u}(x,t) = F(\mathbf{u}(x,t), \mathbf{u}(x,t-\tau_1), \mathbf{u}(x,t-\tau_2)), \\ &\mathbf{u}(x,0) = \mathbf{u}_0(x) \geq 0 (\not\equiv 0), \ x \in \mathcal{V}, \ t \in [0, \mathbf{T}), \end{aligned}$$

where $F = (F_1, F_2)^T$ is a function in $(x,t) \in \mathcal{V} \times [0, \mathbf{T})$, $\mathbf{T}$ is a given positive real number or ∞. Consider $\overline{\mathbf{u}} := \left(\bar{u}^{(1)}, \bar{u}^{(2)}\right)^{\mathrm{T}}$ be an equilibrium, $\delta \mathbf{u} := \left(\delta u^{(1)}, \delta u^{(2)}\right)^{\mathrm{T}}$ be a perturbation near the equilibrium $\overline{\mathbf{u}}$. Hence the linearized system of (1.3) is *expressed as follows:*

$$\delta\dot{\mathbf{u}} = J|_{\overline{\mathbf{u}}}\,\delta\mathbf{u} + J_{\tau_1}|_{\overline{\mathbf{u}}}\,\delta\mathbf{u}_{\tau_1} + J_{\tau_2}|_{\overline{\mathbf{u}}}\,\delta\mathbf{u}_{\tau_2} + \epsilon\Delta\delta\mathbf{u}, \quad \mathbf{u}_{\tau_1} = \mathbf{u}(t-\tau_1), \mathbf{u}_{\tau_2} = \mathbf{u}(t-\tau_2), \tag{2.2}$$

where

$$J|_{\overline{\mathbf{u}}} = \left.\frac{\partial\left(F_1, F_2\right)}{\partial\left(u^{(1)}, u^{(2)}\right)}\right|_{\mathbf{u}=\overline{\mathbf{u}}}, \qquad J_{\tau_1}|_{\overline{\mathbf{u}}} = \left.\frac{\partial\left(F_1,\ F_2\right)}{\partial\left(u^{(1)}_{\tau_1}, u^{(2)}_{\tau_1}\right)}\right|_{\mathbf{u}_{\tau_1}=\overline{\mathbf{u}}},$$
$$J_{\tau_2}|_{\overline{\mathbf{u}}} = \left.\frac{\partial\left(F_1, F_2\right)}{\partial\left(u^{(1)}_{\tau_2}, u^{(2)}_{\tau_2}\right)}\right|_{\mathbf{u}_{\tau_2}=\overline{\mathbf{u}}}.$$

In accordance with Lemma 2, the solution of (2.2) is considered in the following form: $\delta\mathbf{u}(x,t) = \sum_{j=1}^{m}\begin{pmatrix} a_j(t) \\ b_j(t)\end{pmatrix}\Psi_j(x) = \sum_{j=1}^{m}\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix} e^{\lambda_j t}\Psi_j(x),\ c_j^{(1)}, c_j^{(2)} \in \mathbb{R}, j = 1, 2, ..., m$. From Lemma 1, we can derive

$$\begin{aligned}&\lambda_j\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix}e^{\lambda_j t} \\ =&J\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix}e^{\lambda_j t} + J_{\tau_1}\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix}e^{\lambda_j(t-\tau_1)} + J_{\tau_2}\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix}e^{\lambda_j(t-\tau_2)} - \epsilon\alpha_j\begin{pmatrix} c_j^{(1)} \\ c_j^{(2)}\end{pmatrix}e^{\lambda_j t},\end{aligned} \tag{2.3}$$

which leads us the following expression for the characteristic equation

$$P(\lambda_j, \tau_1, \tau_2) = det(J + e^{-\lambda_j\tau_1}J_{\tau_1} + e^{-\lambda_j\tau_2}J_{\tau_2} - \epsilon\alpha_j - \lambda_j I) = 0. \tag{2.4}$$

3. Stability and Hopf bifurcation of model (1.3)

The model (1.3) has two equilibria: disease-free equilibrium $E_1 = (\frac{\pi}{\mu}, 0)$ and endemic equilibrium $E_2 = (\hat{S}, \hat{I})$. The value of $\hat{S}$ in terms of $\hat{I}$ is as follows:

$$\hat{S} = \frac{(1+\gamma\hat{I})(p(b\hat{I}^2 + c\hat{I} + 1) + a\hat{I})}{(\beta - p\alpha)(b\hat{I}^2 + 1) + \hat{I}(c(\beta - p\alpha) - a\alpha)},$$

where $p = \mu + d + \theta$, $\hat{S} > 0$ if $c(\beta - p\alpha) - a\alpha > 0$. This condition implies that $\beta - p\alpha > 0$.

The value of $\hat{I}$ satisfies following equation

$$P(\hat{I}) := K_1\hat{I}^5 + K_2\hat{I}^4 + K_3\hat{I}^3 + K_4\hat{I}^2 + K_5\hat{I} + K_6 = 0,$$

where the coefficients are as follows

$$
\begin{aligned}
K_1 =& pb^2(\beta-\alpha p+\gamma\mu)+\theta b e^{-\mu\tau_2}(\beta-p\alpha),\\
K_2 =& ab(\beta-2p\alpha+\gamma\mu)+b(\beta-p\alpha)(2pc-b\pi)+bp\mu(b+2c\gamma)\\
&-\theta b e^{-\mu\tau_2}(c\beta-cp\alpha-a\alpha)-\theta bce^{-\mu\tau_2}(\beta-p\alpha),\\
K_3 =& ab(\mu+\pi\alpha)+pc^2\gamma\mu+2bp\mu(c+\gamma)+(\beta-p\alpha)(c(a+pc)+2b(p-c\pi))\\
&-a^2\alpha-ac(p\alpha-\gamma\mu)-\theta b e^{-\mu\tau_2}(\beta-p\alpha)-\theta c e^{-\mu\tau_2}(c\beta-cp\alpha-a\alpha)\\
&-\theta e^{-\mu\tau_2}(\beta-p\alpha),\\
K_4 =& ac(\mu+\pi\alpha)+2pc(\beta-p\alpha+\gamma\mu)+p\mu\left(2b+c^2\right)+a(\beta+\gamma\mu)\\
&-\left(2\alpha pa+\pi\left(2b+c^2\right)(\beta-p\alpha)\right)-\theta c e^{-\mu\tau_2}(\beta-p\alpha)\\
&-\theta e^{-\mu\tau_2}(c\beta-cp\alpha-a\alpha),\\
K_5 =& a(\mu+\pi\alpha)+(\beta-p\alpha)(p-2c\pi)+p\mu(2c+\gamma)-\theta e^{-\mu\tau_2}(\beta-p\alpha),\\
K_6 =& p(\mu+\pi\alpha)(1-R_0),
\end{aligned}
$$

and the basic reproduction number R_0 is expressed as $R_0=\frac{\beta\pi}{(\mu+\alpha\pi)(\mu+\theta+d)}$.

If the condition (H_1): $R_0>1$, $\beta-p\alpha>0$ holds, then $P(0)=K_6<0$ and the coefficient K_1 is always positive. Hence $\lim_{\hat{I}\to\infty}F(\hat{I})=+\infty$. We obtain that the polynomial $P(\hat{I})$ has at least a positive real root. After obtaining the value of $\hat{I}$, the value of $\hat{S}$ can be obtained accordingly. Therefore the positive endemic equilibrium $E_2=(\hat{S},\hat{I})$ exists.

Next, in order to get insight into the local stability of two equilibria, we introduce a small perturbation $S=\hat{S}+\delta S, I=\hat{I}+\delta I$ to the endemic equilibrium $E_2=(\hat{S},\hat{I})$. According to Section 2, the characteristic equation at the endemic equilibrium E_2 is:

$$
\begin{aligned}
P(\lambda_j,\tau_1,\tau_2) &= det(J+e^{-\lambda_j\tau_1}J_{\tau_1}+e^{-\lambda_j\tau_2}J_{\tau_2}-\epsilon\alpha_j-\lambda_j\hat{I})\\
&= P_0(\lambda_j,\tau_2)+P_1(\lambda_j,\tau_2)e^{-\lambda_j\tau_1}+P_2(\lambda_j,\tau_2)e^{-\lambda_j(\tau_1+\tau_2)}\\
&=0,
\end{aligned}
\tag{3.1}
$$

where

$$
J=\begin{pmatrix}-\mu & 0\\ 0 & -(\mu+\theta+d)-\dfrac{2a\hat{I}+ac\hat{I}^2}{(b\hat{I}^2+c\hat{I}+1)^2}\end{pmatrix},
$$

$$
J_{\tau_1}=\begin{pmatrix}-\dfrac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & -\dfrac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2}\\ \dfrac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & \dfrac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2}\end{pmatrix},
$$

$$
J_{\tau_2}=\begin{pmatrix}0 & \theta e^{-\mu\tau_2}\\ 0 & 0\end{pmatrix},\ \epsilon=\begin{pmatrix}\varepsilon_1 & 0\\ 0 & \varepsilon_2\end{pmatrix},
$$

$$
P_0(\lambda_j,\tau_2)=\lambda_j^2+(q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j)\lambda_j+(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j),
$$

$$
P_1(\lambda_j,\tau_2)=q_3(q_1+\varepsilon_2\alpha_j+\lambda_j)-q_4(q_2+\varepsilon_1\alpha_j+\lambda_j),
$$

$$
\begin{aligned}
&P_2(\lambda_j, \tau_2) = q_5 e^{-\mu\tau_2},\\
&q_1 = \mu + \theta + d + \frac{2a\hat{I} + ac\hat{I}^2}{(b\hat{I}^2 + c\hat{I} + 1)^2},\\
&q_2 = \mu,\\
&q_3 = \frac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2},\\
&q_4 = \frac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2},\\
&q_5 = -q_3\theta.
\end{aligned}
$$

3.1. Stability analysis of $E_1 = (\frac{\pi}{\mu}, 0)$

In this subsection, we will analyze the local stability of disease-free equilibrium. Through judging the distribution of the roots of the characteristic equation in different cases, we obtain the following result.

Theorem 3.1. *The disease-free equilibrium $E_1 = (\frac{\pi}{\mu}, 0)$ of the model (1.3) is locally asymptotically stable if $R_0 < 1$.*

Proof. At $E_1 = (\frac{\pi}{\mu}, 0)$, we have $q_1 = \mu + \theta + d, q_2 = \mu, q_3 = 0, q_4 = \frac{\beta\pi}{\mu+\alpha\pi}, q_5 = 0$.

(1) When $\tau_1 = \tau_2 = 0$, Eq. (3.1) becomes

$$(\lambda_j + \mu + \varepsilon_1\alpha_j)(\lambda_j + \mu + \theta + d + \varepsilon_2\alpha_j - \frac{\beta\pi}{\mu+\alpha\pi}) = 0.$$

After plugging in R_0, we get

$$\lambda_j = -\mu - \varepsilon_1\alpha_j, \lambda_j = -(\mu+\theta+d)(1-R_0) - \varepsilon_2\alpha_j.$$

If $R_0 < 1$, then $\lambda_j < 0, j = 1, 2, ..., m$, hence the disease-free equilibrium E_1 is locally asymptotically stable.

(2) When $\tau_1 = 0, \tau_2 > 0$, Eq. (3.1) becomes

$$(\lambda_j + \mu + \varepsilon_1\alpha_j)(\lambda_j + \mu + \theta + d + \varepsilon_2\alpha_j - \frac{\beta\pi}{\mu+\alpha\pi}) = 0.$$

This case is similar to case (1), where the procedure is omitted.

(3) When $\tau_1 > 0, \tau_2 = 0$, then above Eq. (3.1) becomes

$$\lambda_j^2 + G_1\lambda_j + G_2 + e^{-\lambda_j\tau_1}(G_3\lambda_j + G_4) = 0, \tag{3.2}$$

where $G_1 = 2\mu + \theta + d + \varepsilon_1\alpha_j + \varepsilon_2\alpha_j, G_2 = (\mu + \varepsilon_1\alpha_j)(\mu + \theta + d + \varepsilon_2\alpha_j), G_3 = -\frac{\beta\pi}{\mu+\alpha\pi} = -q_4, G_4 = -q_4(\mu + \varepsilon_1\alpha_j)$. Let $\lambda_j = i\omega_{j_{11}} (\omega_{j_{11}} > 0)$, after separating real and imaginary parts, Eq. (3.2) becomes

$$
\begin{aligned}
&\omega_{j_{11}}^2 - G_2 = G_4\cos(\tau_1\omega_{j_{11}}) + \omega_{j_{11}}G_3\sin(\tau_1\omega_{j_{11}}),\\
&\omega_{j_{11}}G_1 = G_4\sin(\tau_1\omega_{j_{11}}) - \omega_{j_{11}}G_3\cos(\tau_1\omega_{j_{11}}).
\end{aligned}
$$

Squaring and adding the above two equations, we obtain

$$\omega_{j_{11}}^4 + \left(G_1^2 - 2G_2 - G_3^2\right)\omega_{j_{11}}^2 + \left(G_2^2 - G_4^2\right) = 0, \tag{3.3}$$

where

$$\begin{aligned}G_2^2-G_4^2&=[(\mu+\varepsilon_1\alpha_j)(\mu+\theta+d+\varepsilon_2\alpha_j)]^2-[q_4(\mu+\varepsilon_1\alpha_j)]^2\\&=(\mu+\varepsilon_1\alpha_j)^2[(\mu+\theta+d+\varepsilon_2\alpha_j)^2-q_4^2]\\&=(\mu+\varepsilon_1\alpha_j)^2[(\mu+\theta+d+\varepsilon_2\alpha_j+q_4)(\mu+\theta+d+\varepsilon_2\alpha_j-q_4)]\\&=(\mu+\varepsilon_1\alpha_j)^2(\mu+\theta+d+\varepsilon_2\alpha_j+q_4)[(\mu+\theta+d)(1-R_0)+\varepsilon_2\alpha_j],\end{aligned}$$

$$\begin{aligned}&G_1^2-2G_2-G_3^2\\&=(\mu+\theta+d+\mu+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j)^2-2(\mu+\varepsilon_1\alpha_j)(\mu+\theta+d+\varepsilon_2\alpha_j)-q_4^2\\&=[(\mu+\theta+d+\varepsilon_2\alpha_j)+(\mu+\varepsilon_1\alpha_j)]^2-2(\mu+\varepsilon_1\alpha_j)(\mu+\theta+d+\varepsilon_2\alpha_j)-q_4^2\\&=(\mu+\theta+d+\varepsilon_2\alpha_j)^2+(\mu+\varepsilon_1\alpha_j)^2-q_4^2\\&=(\mu+\varepsilon_1\alpha_j)+(\mu+\theta+d+\varepsilon_2\alpha_j+\frac{\beta\pi}{\mu+\alpha\pi})[(\mu+\theta+d)(1-R_0)+\varepsilon_2\alpha_j].\end{aligned}$$

We obtain $G_2^2-G_4^2>0$ and $G_1^2-2G_2-G_3^2>0$ if $R_0<1$ for all $j=1,2,...,m$. Then all the roots of Eq. (3.3) are negative, that is, Eq. (3.2) has no imaginary root. Hence the disease-free equilibrium E_1 is locally asymptotically stable if $R_0<1$.

(4) When $\tau_1>0,\tau_2>0$, Eq. (3.1) becomes

$$\begin{aligned}&\lambda_j^2+(2\mu+\theta+d+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j)\lambda_j+(\mu+\varepsilon_1\alpha_j)(\mu+\theta+d+\varepsilon_2\alpha_j)\\&+e^{-\lambda_j\tau_1}[-\frac{\beta\pi}{\mu+\alpha\pi}\lambda_j-\frac{\beta\pi}{\mu+\alpha\pi}(\mu+\varepsilon_1\alpha_j)]=0.\end{aligned}$$

Case (4) is similar to case (3) in that the procedure is omitted.

To sum up, when $R_0<1$, the disease-free equilibrium $E_1=(\frac{\pi}{\mu},0)$ of the model (1.3) is locally asymptotically stable. □

3.2. Stability and Hopf bifurcation of $E_2=(\hat{S},\hat{I})$

To investigate the stability of endemic equilibrium, we conduct the study by considering the following three cases.

Case 1. $\tau_1=\tau_2=0$.

In this case, Eq. (3.1) becomes

$$\begin{aligned}&\lambda_j^2+(q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j+q_3-q_4)\lambda_j+(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j)\\&+q_3(q_1+\varepsilon_2\alpha_j)-q_4(q_2+\varepsilon_1\alpha_j)+q_5=0.\end{aligned}$$

Suppose (H_2):

$$\begin{aligned}&q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j+q_3-q_4>0,\\&(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j)+q_3(q_1+\varepsilon_2\alpha_j)-q_4(q_2+\varepsilon_1\alpha_j)+q_5>0.\end{aligned}$$

Then by Routh-Hurwitz criteria, we have the following results.

Theorem 3.2. *When $\tau_1=\tau_2=0$, if (H_1) and (H_2) hold, then the positive equilibrium E_2 of model (1.3) is locally asymptotically stable for all $j=1,2,...,m$.*

Case 2. $\tau_1=\tau_2\neq 0$.

In this case, Eq. (3.1) becomes

$$C_0(\lambda_j,\tau_2)+C_1(\lambda_j,\tau_2)e^{-\lambda_j\tau_2}+C_2(\lambda_j,\tau_2)e^{-2\lambda_j\tau_2}=0, \tag{3.4}$$

where $C_0(\lambda_j,\tau_2)=\lambda_j^2+h_1\lambda_j+h_2$, $C_1(\lambda_j,\tau_2)=h_3\lambda_j+h_4$, $C_2(\lambda_j,\tau_2)=q_5e^{-\mu\tau_2}$, $h_1=q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j$, $h_2=(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j)$, $h_3=q_3-q_4$, $h_4=q_3(q_1+\varepsilon_2\alpha_j)-q_4(q_2+\varepsilon_1\alpha_j)$.

Using the method similar to [4], there are the following conditions:

(i) $deg(C_0(\lambda_j,\tau_2))\geq max\{deg(C_1(\lambda_j,\tau_2)),deg(C_2(\lambda_j,\tau_2))\}$;

(ii) $C_0(0,\tau_2)+C_1(0,\tau_2)+C_2(0,\tau_2)\neq 0$;

(iii) $C_0(i\omega_j,\tau_2)+C_1(i\omega_j,\tau_2)+C_2(i\omega_j,\tau_2)\neq 0$;

(iv) $\lim_{|\lambda_j|\to\infty,\mathrm{Re}\,\lambda_j\geq 0}\sup\{|\frac{C_1(\lambda_j,\tau_2)}{C_0(\lambda_j,\tau_2)}|+|\frac{C_2(\lambda_j,\tau_2)}{C_0(\lambda_j,\tau_2)}|\}<1$;

(v) $\mathbf{F}(\omega_j,\tau_2)$ has finite number of zeros, where $\mathbf{F}(\omega_j,\tau_2)$ is given by (3.7);

(vi) If $\omega_j>0$ satisfying $\mathbf{F}(\omega_j,\tau_2)=0$, then $\omega_j>0$ is continuous and differentiable in τ_2.

For $C_0(i\omega_j,\tau_2),C_1(i\omega_j,\tau_2),C_2(i\omega_j,\tau_2)$, (i) is automatically satisfied. On account of $C_0(0,\tau_2)+C_1(0,\tau_2)+C_2(0,\tau_2)=h_2+h_4+q_5e^{-\mu\tau_2}\neq 0$, (ii) holds. (iii) is established to ensure that $C_0(i\omega_j,\tau_2),C_1(i\omega_j,\tau_2)$ and $C_2(i\omega_j,\tau_2)$ have no common pure imaginary root. If they have common factor $e(\lambda_j,\tau_2)$, then equation (3.4) can be rewritten as the product of another transcendental equation of the full condition (iii) and $e(\lambda_j,\tau_2)$.

$$\lim_{\lambda_j\to\infty}\left(\left|\frac{C_1(\lambda_j,\tau_2)}{C_0(\lambda_j,\tau_2)}+\frac{C_2(\lambda_j,\tau_2)}{C_0(\lambda_j,\tau_2)}\right|\right)=0<1,$$

so (iv) is true. For any τ_2, condition (v) holds, then the number of times the characteristic root crosses the imaginary axis is limited. By the implicit function theorem we know that (vi) is true.

Let $\lambda_j=i\omega_{j_1}(\omega_{j_1}>0)$ in Eq. (3.4), then we get

$$\begin{cases}\cos\omega_{j_1}\tau_2(-\omega_{j_1}^2+h_2+q_5e^{-\mu\tau_2})-(\omega_{j_1}h_1)\sin\omega_{j_1}\tau_2=-h_4,\\(\omega_{j_1}h_1)\cos\omega_{j_1}\tau_2+\sin\omega_{j_1}\tau_2(-\omega_{j_1}^2+h_2-q_5e^{-\mu\tau_2})=-\omega_{j_1}h_3.\end{cases} \tag{3.5}$$

Hence, $\omega_{j_1}=\omega_{j_1}(\tau_2)>0$ needs to satisfy the following equation

$$\begin{cases}\cos\omega_{j_1}\tau_2=\cos(\omega_{j_1}(\tau_2)\tau_2)=\dfrac{-h_4(-\omega_{j_1}^2+h_2+q_5e^{-\mu\tau_2})-\omega_{j_1}^2h_1h_3}{(-\omega_{j_1}^2+h_2)^2-(q_5e^{-\mu\tau_2})^2+(\omega_{j_1}h_1)^2}\triangleq\dfrac{C_3}{C_4},\\\sin\omega_{j_1}\tau_2=\sin(\omega_{j_1}(\tau_2)\tau_2)=\dfrac{-\omega_{j_1}h_3(-\omega_{j_1}^2+h_2+q_5e^{-\mu\tau_2})+h_1h_4\omega_{j_1}}{(-\omega_{j_1}^2+h_2)^2-(q_5e^{-\mu\tau_2})^2+(\omega_{j_1}h_1)^2}\triangleq\dfrac{C_5}{C_4}.\end{cases} \tag{3.6}$$

Define

$$\mathbf{F}(\omega_{j_1},\tau_2)=C_4^2-C_3^2-C_5^2=0. \tag{3.7}$$

Let

$$I_{\tau_2}=\{\tau_2>0:\mathbf{F}(\omega_{j_1},\tau_2)=0,\omega_{j_1}>0\},$$

then ω_{j_1} satisfies (3.7) for all $\tau_2\in I_{\tau_2}$. If $\tau_2\notin I_{\tau_2}$, ω_{j_1} is not definited.

Define functions $S_n(\tau_2)$:

$$S_n(\tau_2):=\tau_2-\tau_n(\tau_2),n\in N_0,\tau_2\in I_{\tau_2},$$

where $\tau_n(\tau_2) = \frac{\theta(\tau_2)+2n\pi}{\omega_{j_1}(\tau_2)}$, $\omega_{j_1}(\tau_2)\tau_2 = \theta(\tau_2) + 2n\pi$, $\theta(\tau_2) \in [0, 2\pi)$ satisfies

$$\begin{cases} \cos\theta(\tau_2) = \dfrac{C_3}{C_4}, \\ \sin\theta(\tau_2) = \dfrac{C_5}{C_4}. \end{cases} \tag{3.8}$$

When $S_n(\tau_2)$ has root $\tau_2 = \tau_2^*$, Eq. (3.4) has a purely imaginary root $i\omega_{j_1}^*$. We get the following theorems.

Theorem 3.3. *The Eq. (3.4) has a pair of simply imaginary roots* $\lambda_j = \pm i\omega_{j_1}(\omega_{j_1} > 0)$ *for any* $\tau_2 = \tau_2^* \in I_{\tau_2}$ *if*

$$S_n(\tau_2^*) = 0, n \in N_0.$$

The corresponding pair eigenvalues cross the imaginary axis from left (right) to right (left) if $\delta(\tau_2^*) > 0(< 0)$. *Here,*

$$\begin{aligned} \delta(\tau_2^*) :=& \operatorname{Sign}\left\{ \frac{\mathrm{d}Re\lambda_j}{\mathrm{d}\tau_2}\bigg|_{\lambda_j = i\omega_{j_1}(\tau_2^*)} \right\} \\ =& \operatorname{Sign}\left\{ \mathbf{F}'_{\omega_{j_1}}\left(\omega_{j_1}\left(\tau_2^*\right), \tau_2^*\right)\right\} \operatorname{Sign}\left\{ \frac{\mathrm{d}S_n(\tau_2)}{\mathrm{d}\tau_2}\bigg|_{\tau_2=\tau_2^*} \right\}. \end{aligned}$$

Theorem 3.4. *(i) If* I_{τ_2} *is empty, or* I_{τ_2} *is nonempty and* $S_n(\tau_2) = 0$ *has no positive root in* I_{τ_2}, *then* E_2 *is locally asymptotically stable for all* $\tau_2 > 0$.

(ii) If I_{τ_2} *is nonempty,* $S_n(\tau_2) = 0$ *has positive roots and* $\delta(\tau_2^*) \neq 0$. *Let* $\tau_2^{min} = min\{\tau_2 : S_n(\tau_2) = 0\}$, *then* E_2 *is locally asymptotically stable for* $\tau_2 \in [0, \tau_2^{min})$, *and model (1.3) undergoes Hopf bifurcation at* E_2 *when* $\tau_2 = \tau_2^{min}$.

Case 3. $\tau_1 = 0, \tau_2 \neq 0$.

In this case, Eq. (3.1) becomes

$$H_1(\lambda_j, \tau_2) + H_2(\lambda_j, \tau_2)e^{-\lambda_j\tau_2} = 0, \tag{3.9}$$

where $H_1(\lambda_j, \tau_2) = \lambda_j^2 + b_1\lambda_j + b_2$, $H_2(\lambda_j, \tau_2) = q_5e^{-\mu\tau_2}$, $b_1 = q_1 + q_2 + q_3 - q_4 + \varepsilon_1\alpha_j + \varepsilon_2\alpha_j$, $b_2 = (-q_1 - \varepsilon_2\alpha_j)(-q_2 - \varepsilon_1\alpha_j) + q_3(q_1 + \varepsilon_2\alpha_j) - q_4(q_2 + \varepsilon_1\alpha_j)$.

Similar to case 2, all of the following conditions are true:

(i) $deg(H_1(\lambda_j, \tau_2)) \geq deg(H_2(\lambda_j, \tau_2))$;

(ii) $H_1(0, \tau_2) + H_2(0, \tau_2) \neq 0$;

(iii) $H_1(i\omega_j, \tau_2) + H_2(i\omega_j, \tau_2) \neq 0$;

(iv) $\lim_{|\lambda_j|\to\infty, \mathrm{Re}\,\lambda_j\geq 0} \sup\{|\frac{H_2(\lambda_j,\tau_2)}{H_1(\lambda_j,\tau_2)}|\} < 1$;

(v) $\mathbf{F}(\omega_j, \tau_2) = |H_1(i\omega_j, \tau_2)|^2 - |H_2(i\omega_j, \tau_2)|^2$ has finite number of zeros;

(vi) If $\omega_j > 0$ satisfying $\mathbf{F}(\omega_j, \tau_2) = 0$, then $\omega_j > 0$ is continuous and differentiable in τ_2.

Let $\lambda_j = i\omega_{j_{12}}(\omega_{j_{12}} > 0)$ in Eq. (3.9), substituting it into Eq. (3.9) and separating the real and imaginary parts yield that

$$\begin{cases} \cos\omega_{j_{12}}\tau_2 = \dfrac{\omega_{j_{12}}^2 - b_2}{q_5e^{-\mu\tau_2}}, \\ \sin\omega_{j_{12}}\tau_2 = \dfrac{\omega_{j_{12}}b_1}{q_5e^{-\mu\tau_2}}. \end{cases} \tag{3.10}$$

Furthermore, we have

$$\mathbf{F}(\omega_{j_{12}},\tau_2)=\omega_{j_{12}}^4+(b_1^2-2b_2)\omega_{j_{12}}^2+(b_2^2-q_5^2e^{-2\mu\tau_2})=0. \tag{3.11}$$

Denote $I_{\tau_2}=\{\tau_2>0:\mathbf{F}(\omega_{j_{12}},\tau_2)=0,\omega_{j_{12}}>0\}$, for $\tau_2\in I_{\tau_2}$, let $\theta(\tau_2)\in[0,2\pi)$ be defined by

$$\begin{cases}\cos\theta(\tau_2)=\dfrac{\omega_{j_{12}}^2(\tau_2)-b_2}{q_5e^{-\mu\tau_2}},\\ \sin\theta(\tau_2)=\dfrac{\omega_{j_{12}}(\tau_2)b_1}{q_5e^{-\mu\tau_2}}.\end{cases} \tag{3.12}$$

Define functions $S_n(\tau_2)$:

$$S_n(\tau_2):=\tau_2-\frac{\theta(\tau_2)+2n\pi}{\omega_{j_{12}}(\tau_2)},\ n\in N_0.$$

One can verify that Eq. (3.9) has a purely imaginary root $i\omega_{j_1}^*$ if and only if τ_2^* is a root of the function $S_n(\tau_2)$ for some $n\in N_0$. We introduce the following theorem [4]:

Theorem 3.5. *Suppose $S_n(\tau_2)=0$ has positive real root $\tau_2=\tau_2^*\in I_{\tau_2}$ for $n\in N_0$, then Eq. (3.9) has a pair of simply imaginary roots $\pm i\omega_{j_{12}}(\omega_{j_{12}}>0)$. The corresponding pair eigenvalues cross the imaginary axis from left (right) to right (left) if $\delta(\tau_2^*)>0(<0)$. Here,*

$$\begin{aligned}\delta(\tau_2^*)&:=\operatorname{Sign}\left\{\left.\frac{\mathrm{d}Re\lambda_j}{\mathrm{d}\tau_2}\right|_{\lambda_j=i\omega_{j_{12}}(\tau_2^*)}\right\}\\&=\operatorname{Sign}\left\{\mathbf{F}'_{\omega_{j_{12}}}\left(\omega_{j_{12}}\left(\tau_2^*\right),\tau_2^*\right)\right\}\operatorname{Sign}\left\{\left.\frac{\mathrm{d}S_n(\tau_2)}{\mathrm{d}\tau_2}\right|_{\tau_2=\tau_2^*}\right\}.\end{aligned}$$

Theorem 3.6. *Suppose (H_1) holds, then*
(i) If $S_0(\tau_2)$ has no positive root in I_{τ_2}, then E_2 is locally asymptotically stable for all $\tau_2>0$. (ii) If $S_0(\tau_2)$ has at least one positive root in I_{τ_2}, then there exists τ_2^* such that $\forall\tau_2\in[0,\tau_2^*)$, E_2 is locally asymptotically stable, and model (1.3) undergoes Hopf bifurcation at E_2 when $\tau_2=\tau_2^*$.

The case of $\tau_1\neq0,\tau_2=0$ is discussed similarly to case 3, with the process omitted.

Case 4. $\tau_1>0,\tau_2>0$ and $\tau_1\neq\tau_2$.

In the following, we will examine the existence of purely imaginary roots of Eq. (3.1) by using the method of stability switching curves [1]. We first verify following conditions (I)-(VI):

(I) $deg(P_0(\lambda_j,\tau_2))\geq max\left\{deg(P_1(\lambda_j,\tau_2)),deg(P_2(\lambda_j,\tau_2))\right\}$.

(II) The polynomials P_0, P_1, P_2 and P_3 are coprime.

(III) $P_0(0,\tau_2)+P_1(0,\tau_2)+P_2(0,\tau_2)\neq0$.

(IV) $\lim_{\lambda_j\to\infty}\left(\left|\frac{P_1(\lambda_j,\tau_2)}{P_0(\lambda_j,\tau_2)}+\frac{P_2(\lambda_j,\tau_2)}{P_0(\lambda_j,\tau_2)}\right|\right)<1$.

(V) $P_l(i\omega_j,\tau_2)\neq0,l=0,1,2$ for any $\tau_2\in I$ and $\omega\in\mathbb{R}_+$.

(VI) For any $\omega_j\in\mathbb{R}_+$, at least one of $|P_l(i\omega_j,\tau_2)|,l=0,1,2$ tend to ∞ as $\tau_2\to-\infty$. If there are multiple such P_l, then these functions tend to infinity at different rates.

Obviously, (I) and (II) are satisfied. For $P_0(0) + P_1(0) + P_2(0) = (-q_1 - \varepsilon_2\alpha_j)(-q_2 - \varepsilon_1\alpha_j) + q_3(q_1 + \varepsilon_2\alpha_j) - q_4(q_2 + \varepsilon_1\alpha_j) + e^{-\mu\tau_2}q_5 \neq 0$, (III) is satisfied. Since

$$\lim_{\lambda_j\to\infty}\left(\left|\frac{q_3(q_1+\varepsilon_2\alpha_j+\lambda_j)-q_4(q_2+\varepsilon_1\alpha_j+\lambda_j)}{\lambda_j^2+(q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j)\lambda_j+(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j)}\right|\right)$$
$$+\lim_{\lambda_j\to\infty}\left(\left|\frac{e^{-\mu\tau_2}q_5}{\lambda_j^2+(q_1+q_2+\varepsilon_1\alpha_j+\varepsilon_2\alpha_j)\lambda_j+(-q_1-\varepsilon_2\alpha_j)(-q_2-\varepsilon_1\alpha_j)}\right|\right)=0<1,$$

so (IV) holds. From the expression for $P_l(\lambda_j, \tau_2), l = 0, 1, 2$, (V) is naturally true. The presentation of assumption (VI) helps to reduce the cases of graph for $S_n^{\pm}(\omega_j, \tau_2)$.

Let $\lambda_j = i\omega_j(\omega_j > 0)$ be a root of Eq. (3.1), we have

$$D(i\omega_j,\tau_1,\tau_2)=1+a_1(\omega_j,\tau_2)e^{-i\omega_j\tau_1}+a_2(\omega_j,\tau_2)e^{-i\omega_j(\tau_1+\tau_2)}=0, \tag{3.13}$$

where $a_l(\omega_j, \tau_2) = P_l(i\omega_j, \tau_2)/P_0(i\omega_j, \tau_2), l = 1, 2$. And the right side of Eq. (3.13) can form a triangle on the complex plane. According to the relationship between the three sides of the triangle, we have

$$\begin{aligned}
&|a_1(\omega_j,\tau_2)|+|a_2(\omega_j,\tau_2)|\geq 1,\\
&|a_1(\omega_j,\tau_2)|-|a_2(\omega_j,\tau_2)|\leq 1,\\
&|a_2(\omega_j,\tau_2)|-|a_1(\omega_j,\tau_2)|\leq 1,
\end{aligned} \tag{3.14}$$

which are equivalent to

$$\begin{aligned}
Q_1(\omega_j,\tau_2)&:=|P_1(i\omega_j,\tau_2)|+|P_2(i\omega_j,\tau_2)|-|P_0(i\omega_j,\tau_2)|\geq 0,\\
Q_2(\omega_j,\tau_2)&:=|P_0(i\omega_j,\tau_2)|+|P_2(i\omega_j,\tau_2)|-|P_1(i\omega_j,\tau_2)|\geq 0,\\
Q_3(\omega_j,\tau_2)&:=|P_0(i\omega_j,\tau_2)|+|P_1(i\omega_j,\tau_2)|-|P_2(i\omega_j,\tau_2)|\geq 0.
\end{aligned} \tag{3.15}$$

Obviously, Eq. (3.1) exists root $\lambda_j = i\omega_j(\omega_j > 0)$ if (ω_j, τ_2) satisfies (3.14) or (3.15). Define feasible region $\Omega = \{(\omega_j, \tau_2) \in I \times \mathbb{R}_+ : Q_i(\omega_j, \tau_2) \geq 0, i = 1, 2, 3\}$, such that the characteristic equation Eq. (3.1) may have solutions for $\tau_2 \in \mathbb{R}_+$. Define $\theta_1(\omega_j, \tau_2), \theta_2(\omega_j, \tau_2)$ as the angle formed by 1 and $a_1(\omega_j, \tau_2)e^{-i\omega_j\tau_1}$, and $a_2(\omega_j, \tau_2)e^{-i\omega_j(\tau_1+\tau_2)}$, respectively. From the law of cosine, we have

$$\theta_1(\omega_j,\tau_2)=\arccos\left(\frac{1+|a_1(\omega_j,\tau_2)|^2-|a_2(\omega_j,\tau_2)|^2}{2|a_1(\omega_j,\tau_2)|}\right),$$

$$\theta_2(\omega_j,\tau_2)=\arccos\left(\frac{1+|a_2(\omega_j,\tau_2)|^2-|a_1(\omega_j,\tau_2)|^2}{2|a_2(\omega_j,\tau_2)|}\right).$$

For each connected region Ω_k of Ω, we further denote the feasible set concerning ω_j and interval regarding τ_2 for $\omega_j \in I_k$ by $I_k = [\omega_k^l, \omega_k^r]$ and $I_{\omega_j}^k = [\tau_{2,\omega}^{k,l}, \tau_{2,\omega}^{k,r}]$, $k = 1, 2, \cdots, N$. Then we consider two possible cases as below:

1) If $Im(a_1(\omega_j, \tau_2)e^{-i\omega_j\tau_1}) > 0$, we have

$$\arg(a_1(\omega_j,\tau_2)e^{-i\omega_j\tau_1})=\pi-\theta_1(\omega_j,\tau_2),\quad \arg(a_2(\omega_j,\tau_2)e^{-i\omega_j(\tau_1+\tau_2)})=\theta_2(\omega_j,\tau_2)-\pi.$$

Hence there is an $n \in \mathbb{Z}$, such that

$$\arg(a_2(\omega_j,\tau_2)) - \arg(a_1(\omega_j,\tau_2)) - [\theta_1(\omega_j,\tau_2)+\theta_2(\omega_j,\tau_2)] + 2n\pi = \omega_j\tau_2, \quad (3.16)$$

and

$$\tau_1 = \frac{1}{\omega_j}[\arg(a_1(\omega_j,\tau_2)) + \theta_1(\omega_j,\tau_2) + (2\iota-1)\pi], \quad \iota \geq \iota_0^+, \quad (3.17)$$

where $\iota \geq \iota_0^+$ is the smallest integer such that $\tau_1 > 0$ and

$$\tau_2 = \frac{1}{\omega_j}[\arg(a_2(\omega_j,\tau_2)) - \arg(a_1(\omega_j,\tau_2)) - (\theta_1(\omega_j,\tau_2)+\theta_2(\omega_j,\tau_2)) + 2n\pi].$$

2) If $Im(a_1(\omega_j,\tau_2)e^{-i\omega_j\tau_1}) < 0$, then

$$\begin{aligned}&\arg(a_1(\omega_j,\tau_2)e^{-i\omega_j\tau_1}) = \pi + \theta_1(\omega_j,\tau_2),\\ &\arg(a_2(\omega_j,\tau_2)e^{-i\omega_j(\tau_1+\tau_2)}) = -\theta_2(\omega_j,\tau_2) - \pi.\end{aligned}$$

Hence

$$\arg(a_2(\omega_j,\tau_2)) - \arg(a_1(\omega_j,\tau_2)) + [\theta_1(\omega_j,\tau_2)+\theta_2(\omega_j,\tau_2)] + 2n\pi = \omega_j\tau_2, \quad n \in \mathbb{Z}, \quad (3.18)$$

and

$$\tau_1 = \frac{1}{\omega_j}[\arg(a_1(\omega_j,\tau_2)) - \theta_1(\omega_j,\tau_2) + (2\iota-1)\pi], \quad \iota \geq \iota_0^-, \quad (3.19)$$

where $\iota \geq \iota_0^-$ is the smallest integer such that $\tau_1 > 0$ and

$$\tau_2 = \frac{1}{\omega_j}[\arg(a_2(\omega_j,\tau_2)) - \arg(a_1(\omega_j,\tau_2)) + (\theta_1(\omega_j,\tau_2)+\theta_2(\omega_j,\tau_2)) + 2n\pi].$$

Then we define function $S_n^{\pm} : \Omega \to \mathbb{R}$

$$S_n^{\pm}(\omega_j,\tau_2) = \tau_2 - \frac{1}{\omega_j}[\arg(a_2(\omega_j,\tau_2)) - \arg(a_1(\omega_j,\tau_2)) \mp (\theta_1(\omega_j,\tau_2)+\theta_2(\omega_j,\tau_2)) + 2n\pi]. \quad (3.20)$$

We denote the zeros of Eq. (3.20) as $\tau_2^{i\pm}, i = 1,2,\cdots$, and the corresponding τ_1 are obtained

$$\tau_{1,i}^{\iota\pm}(\omega_j) = \frac{1}{\omega_j}[\arg(a_1(\omega_j,\tau_2^{i\pm})) \pm \theta_1(\omega_j,\tau_2^{i\pm}) + (2\iota^{\pm}-1)\pi], \quad (3.21)$$

for $\iota = \iota_0^{\pm}, \iota_0^{\pm}+1, \cdots$, where $\iota_0^{\pm}$ is the smallest integer such that $\tau_{1,i}^{\iota\pm}(\omega_j) > 0$.

Define

$$\mathcal{C} := \{(\omega_j, \tau_2^{i\pm}(\omega_j)) : \omega_j \in I_k, S_n^{\pm}(\omega_j,\tau_2^{i\pm}) = 0\},$$

and the stability switching curves can be defined by

$$\mathcal{T} = \{(\tau_2^{i\pm}(\omega_j), \tau_{1,i}^{\iota\pm}(\omega_j)) \in I \times \mathbb{R}_+ | \omega_j \in I_k, k = 1,2,\cdots,N\}.$$

There exists a pair of purely imaginary roots $\pm i\omega_j$ of characteristic equation Eq. (3.1) when $(\tau_1,\tau_2) \in \mathcal{T}$, and the stability of model (1.3) is altered when (τ_1,τ_2) traverses curves $\mathcal{T}$.

Subsequently, the direction of the corresponding roots of Eq. (3.1) pass through the imaginary axis as delays goes through curves $\mathcal{T}$ should be ensured. When

$\frac{\partial D}{\partial \lambda_j}(i\omega_j^*, \tau_1^*, \tau_2^*) \neq 0$, suppose that Eq. (3.1) has complex roots $\lambda_j(\tau_1, \tau_2) = \delta_j(\tau_1, \tau_2) \pm i\omega_j(\tau_1, \tau_2)$ surround $(\tau_1^*, \tau_2^*) \in \mathcal{T}$ with $\delta_j(\tau_1^*, \tau_2^*) = 0$, $\omega_j(\tau_1^*, \tau_2^*) = \omega_j^*$. The positive direction of the curve $\mathcal{T}$ is the increasing direction of ω_j. The right (left) region is the right (left)-hand side when we walking along the positive direction of curve $\mathcal{T}$.

Considering τ_1 and τ_2 be a function of δ_j and ω_j from the implicit function theorem, we have

$$\Delta(\omega_j^*) := \begin{pmatrix} \frac{\partial \tau_1}{\partial \delta_j} & \frac{\partial \tau_1}{\partial \omega_j} \\ \frac{\partial \tau_2}{\partial \delta_j} & \frac{\partial \tau_2}{\partial \omega_j} \end{pmatrix}\Bigg|_{\delta_j=0,\omega_j=\omega_j^*\in\Omega} = -\begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 & -I_0 \\ I_0 & R_0 \end{pmatrix},$$

and

$$\begin{aligned}
R_0 &:= \frac{\partial ReD(\lambda_j;\tau_1,\tau_2)}{\partial \delta_j}\Big|_{\lambda_j=i\omega_j^*} \\
&= Re\Big\{P_0'(i\omega_j^*, \tau_2^*) + (P_1'(i\omega_j^*, \tau_2^*) - \tau_1^* P_1(i\omega_j, \tau_2^*))e^{-i\omega_j^*\tau_1^*} + (P_2'(i\omega_j^*, \tau_2^*) \\
&\quad -(\tau_1^* + \tau_2^*)P_2(i\omega_j^*, \tau_2^*))e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\},\\
I_0 &:= \frac{\partial ImD(\lambda_j;\tau_1,\tau_2)}{\partial \delta_j}\Big|_{\lambda_j=i\omega_j^*} \\
&= Im\Big\{P_0'(i\omega_j^*, \tau_2^*) + (P_1'(i\omega_j^*, \tau_2^*) - \tau_1^* P_1(i\omega_j, \tau_2^*))e^{-i\omega_j^*\tau_1^*} + (P_2'(i\omega_j^*, \tau_2^*) \\
&\quad -(\tau_1^* + \tau_2^*)P_2(i\omega_j^*, \tau_2^*))e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\},\\
&\frac{\partial ReD(\lambda_j;\tau_1,\tau_2)}{\partial \omega_j}\Big|_{\lambda_j=i\omega_j^*} \\
&= Re\Big\{iP_0'(i\omega_j^*, \tau_2^*) + (iP_1'(i\omega_j^*, \tau_2^*) - i\tau_1^* P_1(i\omega_j, \tau_2^*)) \\
&\quad \times e^{-i\omega_j^*\tau_1^*} + (iP_2'(i\omega_j^*, \tau_2^*) - i(\tau_1^* + \tau_2^*)P_2(i\omega_j^*, \tau_2^*))e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\} \\
&= -I_0,\\
&\frac{\partial ImD(\lambda_j;\tau_1,\tau_2)}{\partial \omega_j}\Big|_{\lambda_j=i\omega_j^*} \\
&= Im\Big\{iP_0'(i\omega_j^*, \tau_2^*) + (iP_1'(i\omega_j^*, \tau_2^*) - i\tau_1^* P_1(i\omega_j, \tau_2^*)) \\
&\quad \times e^{-i\omega_j^*\tau_1^*} + (iP_2'(i\omega_j^*, \tau_2^*) - i(\tau_1^* + \tau_2^*)P_2(i\omega_j^*, \tau_2^*))e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\} \\
&= R_0,\\
R_1 &:= \frac{\partial ReD(\lambda_j;\tau_1,\tau_2)}{\partial \tau_1}\Big|_{\lambda_j=i\omega_j^*} \\
&= Re\Big\{-i\omega_j^* P_1(i\omega_j^*, \tau_2^*)e^{-i\omega_j^*\tau_1^*} - i\omega_j^* P_2(i\omega_j^*, \tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\},
\end{aligned}$$

$$
\begin{aligned}
I_1 :=& \frac{\partial ImD(\lambda_j;\tau_1,\tau_2)}{\partial \tau_1}\Big|_{\lambda_j=i\omega_j^*}\\
=& Im\Big\{-i\omega_j^* P_1(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*\tau_1^*} - i\omega_j^* P_2(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\},\\
R_2 :=& \frac{\partial ReD(\lambda_j;\tau_1,\tau_2)}{\partial \tau_2}\Big|_{\lambda_j=i\omega_j^*}\\
=& Re\Big\{P'_{0\tau_2^*}(i\omega_j^*,\tau_2^*) + P'_{1\tau_2^*}(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*\tau_1^*} + P'_{2\tau_2^*}(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\\
&-i\omega_j^* P_2(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\},\\
I_2 :=& \frac{\partial ImD(\lambda_j;\tau_1,\tau_2)}{\partial \tau_2}\Big|_{\lambda_j=i\omega_j^*}\\
=& Im\Big\{P'_{0\tau_2^*}(i\omega_j^*,\tau_2^*) + P'_{1\tau_2^*}(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*\tau_1^*} + P'_{2\tau_2^*}(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\\
&-i\omega_j^* P_2(i\omega_j^*,\tau_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\Big\}.
\end{aligned}
$$

Let

$$
det(\Delta(\omega_j^*)) = det\begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} det\begin{pmatrix} R_0 & -I_0 \\ I_0 & R_0 \end{pmatrix}.
$$

By the implicit function theorem, we know $det\begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix} = R_1I_2 - R_2I_1 \neq 0$, and $R_0^2 + I_0^2 \geq 0$, therefore $sign(det(\Delta(\omega_j^*))) = sign\{R_1I_2 - R_2I_1\}$, where

$$
\begin{aligned}
& R_1(\tau_1^*,\tau_2^*)I_2(\tau_1^*,\tau_2^*) - R_2(\tau_1^*,\tau_2^*)I_1(\tau_1^*,\tau_2^*)\\
=& -Im\Big\{\frac{\partial D}{\partial \tau_2}(i\omega_j^*,\tau_1^*,\tau_2^*)\cdot\overline{\frac{\partial D}{\partial \tau_1}(i\omega_j^*,\tau_1^*,\tau_2^*)}\Big\}\\
=& -Im\Big\{\big[P_{0\tau_2}^* + P_{1\tau_2}^* e^{-i\omega_j^*\tau_1^*} + (P_{2\tau_2}^* - i\omega_j^* P_2^*)e^{-i\omega_j^*(\tau_1^*+\tau_2^*)}\big]\\
&\times\overline{(-i\omega_j^*)(P_1^* e^{-i\omega_j^*\tau_1^*} + P_2^* e^{-i\omega_j^*(\tau_1^*+\tau_2^*)})}\Big\}\\
=& -\omega_j^* Re\Big\{P_{0\tau_2}^*(P_1^* e^{i\omega_j^*\tau_1^*} + P_2^* e^{i\omega_j^*(\tau_1^*+\tau_2^*)}) + P_{1\tau_2}^*(P_1^* + P_2^* e^{i\omega_j^*\tau_2^*})\\
&+ (P_{2\tau_2}^* - i\omega_j^* P_2^*)P_1^* e^{-i\omega_j\tau_2^*} + (P_{2\tau_2}^* - i\omega_j^* P_2^*)P_2^*\Big\},
\end{aligned}
$$

and $P_i^* = P_i(i\omega_j^*,\tau_2^*)$, $P_{i\tau_2}^* = \frac{\partial P_i}{\partial \tau_2}(i\omega_j^*,\tau_2^*)$, $i = 0,1,2$. Thereupon, we obtain the following theorem

Theorem 3.7. *Under conditions* (H_1) *and* (H_2)*, if* $R_1I_2 - R_2I_1 > 0(< 0)$*, a pair of pure imaginary roots of Eq. (3.1) crosses the imaginary axis from left to right as* (τ_1,τ_2) *traverses the stability switching curves to the region on the right (left). When* (τ_1,τ_2) *crosses the curves* $\mathcal{T}$*, then model (1.3) undergoes Hopf bifurcation at equilibrium* E_2*.*

4. Direction and stability of Hopf bifurcation

In this section, we study the direction of Hopf bifurcation of model (1.3) and the stability of bifurcation periodic solutions when $\tau_2 = \tau_2^*$ by taking τ_1 as the bifurcation parameter. It is assumed that the characteristic equation Eq. (3.1) has a pair of pure imaginary root $\pm i\omega_j^*$ when $\tau_1 = \tau_1^*$, and model (1.3) appears Hopf bifurcation at endemic equilibrium E_2.

Let $v_1(x,t) = S(x,t) - \hat{S}, v_2(x,t) = I(x,t) - \hat{I}, t = \frac{t}{\tau_1}$, and $\tau_1 = \tau_1^* + \mu, \mu \in \mathbb{R}$, then Eq. (3.1) can be written as the following model in $\mathbb{C} := \mathbb{C}\left([-1,0], \mathbb{R}^2\right)$,

$$\dot{v}_t = L_\mu(v_t) + f(\mu, v_t), \tag{4.1}$$

where $v(t) = (v_1(t), v_2(t))^T$, $v_t(\Theta) = v(t+\Theta)$, $\Theta \in [0,1]$, $L_\mu : \mathbb{C} \to \mathbb{R}^2$ and $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^2$ are given by

$$\begin{aligned} L_\mu\phi = {} & (\tau_1^* + \mu)\, M_0 \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_1^* + \mu)\, M_1 \begin{pmatrix} \phi_1(-\frac{\tau_2^*}{\tau_1}) \\ \phi_2(-\frac{\tau_2^*}{\tau_1}) \end{pmatrix} \\ & + (\tau_1^* + \mu)\, M_2 \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} f(\mu, \phi) = {} & (\tau_1^* + \mu) \\ & \times \begin{pmatrix} \frac{\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_1^2(-\frac{\tau_2^*}{\tau_1}) + \frac{\beta\gamma\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_2^2(-\frac{\tau_2^*}{\tau_1}) - \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_1(-\frac{\tau_2^*}{\tau_1})\phi_2(-\frac{\tau_2^*}{\tau_1}) \\ \frac{-\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_1^2(-\frac{\tau_2^*}{\tau_1}) - \frac{\beta\gamma\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_2^2(-\frac{\tau_2^*}{\tau_1}) + \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\phi_1(-\frac{\tau_2^*}{\tau_1})\phi_2(-\frac{\tau_2^*}{\tau_1}) \end{pmatrix} \\ = {} & (\tau_1^* + \mu) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \end{aligned} \tag{4.3}$$

where

$$M_0 = \begin{pmatrix} -\mu + \varepsilon_1\alpha_j & 0 \\ 0 & -(\mu+\theta+d) - \dfrac{2a\hat{I} + ac\hat{I}^2}{(1+b\hat{I}^2+c\hat{I})^2} + \varepsilon_2\alpha_j \end{pmatrix},$$

$$M_1 = \begin{pmatrix} -\dfrac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & -\dfrac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} \\ \dfrac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & \dfrac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \theta e^{-\mu\tau_2} \\ 0 & 0 \end{pmatrix},$$

here, $\phi = (\phi_1, \phi_2)^T \in \mathbb{C}$.

Define

$$L_\mu\phi = \int_{-1}^{0} d\eta(\Theta,\mu)\phi(\Theta), \text{ for } \phi \in C,$$

where

$$\eta(\Theta,\mu) = (\tau_1^* + \mu)[M_0\delta(\Theta) + M_1\delta(\Theta + \tfrac{\tau_2^*}{\tau_1}) + M_2\delta(\Theta+1)]. \tag{4.4}$$

Then Eq. (4.1) can be rewritten as

$$\dot{v}_t = A(\mu)v_t + Rv_t, \tag{4.5}$$

where $A(\mu)\phi = \begin{cases} \frac{\mathrm{d}\phi(\Theta)}{\mathrm{d}\Theta}, & \Theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \Theta = 0, \end{cases}$ and $R\phi = \begin{cases} 0, & \Theta \in [-1,0), \\ f_\mu(\phi), & \Theta = 0. \end{cases}$

Define

$$A^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0,1], \\ \displaystyle\int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0, \end{cases}$$

and

$$\langle \psi(s), \phi(\Theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\Theta} \bar{\psi}(\xi-\Theta)d\eta(\Theta,0)\phi(\xi)d\xi,$$

where $\eta(\Theta) = \eta(\Theta,0)$. We know that $\pm i\omega_j^*\tau_1^*$ are eigenvalues for adjoint operators A and A^*, respectively. Assuming that $q(\Theta) = (1,\alpha_1)^T e^{i\omega_j^*\tau_1^*\Theta}$ is the eigenvector of A corresponding to $i\omega_j^*\tau_1^*$. According to the definition of A and (4.4), we have

$$\alpha_1 = \frac{-\mu + \varepsilon_1\alpha_j - i\omega_j^* - \frac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma I)^2}e^{-i\omega_j^*\tau_2^*}}{\frac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma I)^2}e^{-i\omega_j^*\tau_2^*} - \theta e^{-\mu\tau_2}e^{-i\omega_j^*\tau_1^*}}.$$

In the same way, from $A^*q^*(s) = -i\omega_j^*\tau_1^*q^*(s), q^*(s) = D(1,\alpha_1^*)e^{i\omega_j^*\tau_1^*s}$, we obtain

$$\alpha_1^* = \frac{-\mu + \varepsilon_1\alpha_j + i\omega_j^* - \frac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma I)^2}e^{-i\omega_j^*\tau_2^*}}{\frac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma I)^2}e^{-i\omega_j^*\tau_2^*} - \theta e^{-\mu\tau_2}e^{-i\omega_j^*\tau_1^*}}.$$

In order to ensure $\langle q^*(s), q(\Theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\Theta)\rangle = 0$ hold, we calculate

$$\begin{aligned}
&\langle q^*(s), q(\Theta)\rangle \\
=&\bar{D}(1,\bar{\alpha}_1^*)(1,\alpha_1)^T - \int_{-1}^{0}\int_{\xi=0}^{\Theta} \bar{D}(1,\bar{\alpha}_1^*)e^{-i\omega_j^*\tau_1^*(\xi-\Theta)}d\eta(\Theta)(1,\alpha_1)^T e^{i\omega_j^*\tau_1^*\xi}d\xi \\
=&\bar{D}(1,\bar{\alpha}_1^*)(1,\alpha_1)^T - \int_{-1}^{0} \bar{D}(1,\bar{\alpha}_1^*)\Theta e^{i\omega_j^*\tau_1^*\Theta}d\eta(\Theta)(1,\alpha_1)^T \\
=&\bar{D}\Bigg[1 + \alpha_1\bar{\alpha}_1^* - \frac{\beta\hat{I}(1+\gamma\hat{I})\tau_1^*e^{-i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2}(1-\bar{\alpha}_1^*) + \alpha_1\tau_1^*\theta e^{-\mu\tau_2}e^{-i\omega_j^*\tau_1^*} \\
&- \frac{\alpha_1\beta\hat{S}(1+\alpha\hat{S})\tau_2^*e^{-i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2}(1-\bar{\alpha}_1^*)\Bigg],
\end{aligned}$$

hence

$$\begin{aligned}
\bar{D} =&\Bigg[1 + \alpha_1\bar{\alpha}_1^* - \frac{\beta\hat{I}(1+\gamma\hat{I})\tau_2^*e^{-i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2}(1-\bar{\alpha}_1^*) + \alpha_1\tau_1^*\theta e^{-\mu\tau_2}e^{-i\omega_j^*\tau_1^*} \\
&- \frac{\alpha_1\beta\hat{S}(1+\alpha\hat{S})\tau_2^*e^{-i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2}(1-\bar{\alpha}_1^*)\Bigg]^{-1}.
\end{aligned}$$

Let v_t be the solution of Eq. (4.1) to describe the center manifold C_0 at $\mu = 0$, and define

$$z(t) = \langle q^*, v_t \rangle, \quad W(t, \Theta) = v_t(\Theta) - 2Re\{z(t)q(\Theta)\}. \tag{4.6}$$

On the center manifold C_0, we have

$$W(t, \Theta) = W(z(t), \bar{z}(t), \Theta) = W_{20}(0)\tfrac{z^2}{2} + W_{11}(\Theta)z\bar{z} + W_{02}(\Theta)\tfrac{\bar{z}^2}{2} + \cdots,$$

where local coordinates in the direction of q and q^* are expressed as z and $\bar{z}$. Then we have

$$\begin{aligned} \dot{z}(t) &= \langle q^*, \dot{v_t} \rangle \\ &= \langle q^*, A(0)v_t + R(0)v_t \rangle \\ &= \langle q^*, A(0)v_t \rangle + \langle q^*, R(0)v_t \rangle \\ &= i\omega_j^* \tau_1^* z + g(z, \bar{z}), \end{aligned} \tag{4.7}$$

where

$$g(z, \bar{z}) = \bar{q^*}(0)f_0(z, \bar{z}) = g_{20}\tfrac{z^2}{2} + g_{11}z\bar{z} + g_{02}\tfrac{\bar{z}^2}{2} + g_{21}\tfrac{\bar{z}z^2}{2} + \cdots. \tag{4.8}$$

Since $v_t(\Theta) = W(t, \Theta) + zq(\Theta) + \bar{z}\bar{q}(\Theta)$ and $q(\Theta) = (1, \alpha_1)^T e^{i\omega_j^* \tau_1^* \Theta}$, we have

$$\begin{aligned} v_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ v_{2t}(0) &= z\alpha_1 + \bar{z}\bar{\alpha}_1 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ v_{1t}(-\frac{\tau_2^*}{\tau_1^*}) &= ze^{-i\omega_j^*\tau_2^*} + \bar{z}e^{i\omega_j^*\tau_2^*} + W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} + W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} \\ &\quad + W_{02}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ v_{2t}(-\frac{\tau_2^*}{\tau_1^*}) &= z\alpha_1 e^{-i\omega_j^*\tau_2^*} + \bar{z}\bar{\alpha}_1 e^{i\omega_j^*\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} + W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} \\ &\quad + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ v_{1t}(-1) &= ze^{-i\omega_j^*\tau_1^*} + \bar{z}e^{i\omega_j^*\tau_1^*} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} \\ &\quad + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3), \\ v_{2t}(-1) &= z\alpha_1 e^{-i\omega_j^*\tau_1^*} + \bar{z}\bar{\alpha}_1 e^{i\omega_j^*\tau_1^*} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} \\ &\quad + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O(|(z, \bar{z})|^3). \end{aligned}$$

Then

$$\begin{aligned} g(z, \bar{z}) &= \bar{q^*}(0)f_0(z, \bar{z}) = \bar{D}\tau_1^*(1, \bar{\alpha}_1^*)(f_1, f_2)^T \\ &= \bar{D}\tau_1^*(1 - \bar{\alpha}_1^*)\Bigg\{\frac{\alpha\beta\hat{I}(1 + \gamma\hat{I})}{(1 + \alpha\hat{S} + \gamma\hat{I})^3}\Bigg[(ze^{-i\omega_j^*\tau_2^*} + \bar{z}e^{i\omega_j^*\tau_2^*} + W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} \end{aligned}$$

$$
\begin{aligned}
&+ W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})(ze^{-i\omega_j^*\tau_2^*} + \bar{z}e^{i\omega_j^*\tau_2^*} + W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} \\
&+ W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})\Big] + \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\Big[(z\alpha_1 e^{-i\omega_j^*\tau_2^*} \\
&+ \bar{z}\bar{\alpha}_1 e^{i\omega_j^*\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} + W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})(z\alpha_1 e^{-i\omega_j^*\tau_2^*} \\
&+ \bar{z}\bar{\alpha}_1 e^{i\omega_j^*\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} + W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})\Big] \\
&- \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I})+2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\Big[(ze^{-i\omega_j^*\tau_2^*} + \bar{z}e^{i\omega_j^*\tau_2^*} + W_{20}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} \\
&+ W_{11}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(1)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})(z\alpha_1 e^{-i\omega_j^*\tau_2^*} + \bar{z}\bar{\alpha}_1 e^{i\omega_j^*\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{z^2}{2} \\
&+ W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})z\bar{z} + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_1^*})\frac{\bar{z}^2}{2})\Big]\Big\}.
\end{aligned}
$$

Comparing coefficients with Eq. (4.7), we have

$$
\begin{aligned}
g_{20} =& 2\bar{D}\tau_1^*(1-\bar{\alpha}_1^*)\Big[\frac{\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}e^{-2i\omega_j^*\tau_2^*} + \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1^2 e^{-2i\omega_j^*\tau_2^*} \\
&- \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I})+2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1 e^{-2i\omega_j^*\tau_2^*}\Big], \\
g_{11} =& \bar{D}\tau_1^*(1-\bar{\alpha}_1^*)\Big[\frac{2\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3} + \frac{2\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1\bar{\alpha}_1 \\
&- \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I})+2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\bar{\alpha}_1\alpha_1\Big], \\
g_{02} =& 2\bar{D}\tau_1^*(1-\bar{\alpha}_1^*)\Big[\frac{\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}e^{2i\omega_j^*\tau_2^*} + \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\bar{\alpha}_1^2 e^{2i\omega_j^*\tau_2^*} \\
&- \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I})+2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\bar{\alpha}_1 e^{2i\omega_j^*\tau_2^*}\Big], \\
g_{21} =& 2\bar{D}\tau_1^*(1-\bar{\alpha}_1^*)\Big[\frac{\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}(2e^{-i\omega_j^*\tau_2^*}W_{11}^{(1)}(\frac{-\tau_2^*}{\tau_1^*}) + e^{i\omega_j^*\tau_2^*}W_{20}^{(1)}(\frac{-\tau_2^*}{\tau_1^*})) \\
&+ \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}(2\alpha_1 e^{-i\omega_j^*\tau_2^*}W_{11}^{(2)}(\frac{-\tau_2^*}{\tau_1^*}) + \bar{\alpha}_1 e^{i\omega_j^*\tau_2^*}W_{20}^{(2)}(\frac{-\tau_2^*}{\tau_1^*})) \\
&- \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I})+2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}(e^{-i\omega_j^*\tau_2^*}W_{11}^{(2)}(\frac{-\tau_2^*}{\tau_1^*}) + e^{i\omega_j^*\tau_2^*}W_{20}^{(2)}(\frac{-\tau_2^*}{\tau_1^*}) \\
&+ \bar{\alpha}_1 e^{i\omega_j^*\tau_2^*}W_{20}^{(1)}(\frac{-\tau_2^*}{\tau_1^*}) + \alpha_1 e^{-i\omega_j^*\tau_2^*}W_{11}^{(1)}(\frac{-\tau_2^*}{\tau_1^*}))\Big],
\end{aligned}
$$

where

$$
W_{20}(\Theta) = \frac{ig_{20}}{\omega_j^*\tau_1^*}q(0)e^{i\omega_j^*\tau_1^*\Theta} + \frac{i\bar{g}_{02}}{3\omega_j^*\tau_1^*}\bar{q}(0)e^{-i\omega_j^*\tau_1^*\Theta} + E_1 e^{2i\omega_j^*\tau_1^*\theta},
$$

$$W_{11}(\Theta) = -\frac{ig_{11}}{\omega_j^*\tau_1^*}q(0)e^{i\omega_j^*\tau_1^*\Theta} + \frac{i\bar{g}_{11}}{\omega_j^*\tau_1^*}\bar{q}(0)e^{-i\omega_j^*\tau_1^*\Theta} + E_2,$$

and

$$E_1 = 2\begin{pmatrix} 2i\omega_j^* + \mu - \varepsilon_1\alpha_j + \frac{\beta\hat{I}(1+\gamma\hat{I})e^{-2i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & \frac{\beta\hat{S}(1+\alpha\hat{S})e^{-2i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2} - \theta e^{-\mu\tau_2}e^{-2i\omega_j^*\tau_1^*} \\ \frac{-\beta\hat{I}(1+\gamma\hat{I})e^{-2i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & N_1 \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} \mu - \varepsilon_1\alpha_j + \frac{\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & \frac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} - \theta e^{-\mu\tau_2} \\ \frac{-\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} & \mu+\theta+d+\frac{2a\hat{I}+ac\hat{I}^2}{(1+b\hat{I}^2+c\hat{I})^2} - \varepsilon_2\alpha_j - \frac{\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^2} \end{pmatrix}^{-1} \begin{pmatrix} X_3 \\ X_4 \end{pmatrix},$$

where

$$\begin{aligned}
N_1 &= 2i\omega_j^* + \mu + \theta + d + \frac{2a\hat{I}+ac\hat{I}^2}{(1+b\hat{I}^2+c\hat{I})^2} - \varepsilon_2\alpha_j - \frac{\beta\hat{S}(1+\alpha\hat{S})e^{-2i\omega_j^*\tau_2^*}}{(1+\alpha\hat{S}+\gamma\hat{I})^2}, \\
X_1 &= \frac{\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}e^{-2i\omega_j^*\tau_2^*} + \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1^2 e^{-2i\omega_j^*\tau_2^*} \\
&\quad - \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1 e^{-2i\omega_j^*\tau_2^*}, \\
X_2 &= \frac{-\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}e^{-2i\omega_j^*\tau_2^*} - \frac{\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1^2 e^{-2i\omega_j^*\tau_2^*} \\
&\quad + \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1 e^{-2i\omega_j^*\tau_2^*}, \\
X_3 &= \frac{2\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3} + \frac{2\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1\bar{\alpha}_1 - \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\bar{\alpha}_1\alpha_1, \\
X_4 &= \frac{-2\alpha\beta\hat{I}(1+\gamma\hat{I})}{(1+\alpha\hat{S}+\gamma\hat{I})^3} - \frac{2\gamma\beta\hat{S}(1+\alpha\hat{S})}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\alpha_1\bar{\alpha}_1 + \frac{\beta(1+\alpha\hat{S}+\gamma\hat{I}) + 2\beta\alpha\gamma\hat{I}\hat{S}}{(1+\alpha\hat{S}+\gamma\hat{I})^3}\bar{\alpha}_1\alpha_1.
\end{aligned}$$

Finally, we can calculate the values of the following items:

$$\begin{aligned}
C_1(0) &= \frac{i}{2\omega_j^*\tau_2^*}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}, \;\; \mu_2 = -\frac{Re(C_1(0))}{Re(\lambda'(\tau_2^*))}, \\
T_2 &= -\frac{ImC_1(0) + \mu_2 Im\lambda'(\tau_2^*)}{\omega_j^*\tau_2^*}, \;\; \beta_2 = 2Re(C_1(0)),
\end{aligned}$$

then the direction of Hopf bifurcation and the stability of periodic solution are obtained.

5. Numerical simulation

In order to solve the model (1.3) numerically, we obtain the following system of differential equations:

$$\begin{cases} \dfrac{dS_i}{dt} - \varepsilon_1 \displaystyle\sum_{j=1}^{m} \mathbf{L}_{ij} S_j = \pi - \mu S_i - \dfrac{\beta S_{i\tau_1} I_{i\tau_1}}{1 + \alpha S_{i\tau_1} + \gamma I_{i\tau_1}} + \theta I_{i\tau_2} e^{-\mu \tau_2}, \\ \dfrac{dI_i}{dt} - \varepsilon_2 \displaystyle\sum_{j=1}^{m} \mathbf{L}_{ij} I_j = \dfrac{\beta S_{i\tau_1} I_{i\tau_1}}{1 + \alpha S_{i\tau_1} + \gamma I_{i\tau_1}} - (\mu + d + \theta) I_i - \dfrac{a I_i^2}{b I_i^2 + c I_i + 1}, \\ i = 1, 2, \ldots, m, \end{cases} \tag{5.1}$$

where S_i and I_i are the number of susceptible and infected individuals at time t at node i, $S_{i\tau_1} = S_i(t - \tau_1)$, $I_{i\tau_1} = I_i(t - \tau_1)$, $I_{i\tau_2} = I_i(t - \tau_2)$. Define $L = A - D$ to represent the Laplacian matrix with regard to the graph G, then

$$D_{ij} = \begin{cases} \deg(v_i), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad A_{ij} = \begin{cases} 1, & \text{if } i \sim j, \\ 0, & \text{otherwise,} \end{cases}$$

where D_{ij} represents degree matrix, A_{ij} is the adjacent matrix.

We consider that graph G is a Watts-Strogatz network [38] with $m = 80$, $r = 4$, $p_1 = 0.28$, and the initial infected node is node 2 (red-color dot in Figure 1). Let the model (5.1) parameters be as follows:

$$\begin{aligned} &\pi = 2.2,\ \mu = 0.05,\ \beta = 9,\ \alpha = 0.107,\ \gamma = 1.5,\ \theta = 29, \\ &d = 11,\ a = 0.1,\ b = 0.2,\ c = 0.2,\ \varepsilon_1 = 0.1,\ \varepsilon_2 = 0.3. \end{aligned}$$

We consider the case of $j = 1$, $\alpha_1 = 0$, and calculate $R_0 = 1.7322 > 1$, $\beta - p\alpha = 4.7147 > 0$, then (H_1) holds, endemic equilibrium $E_2 = (\hat{S}, \hat{I})$ exists. Model (5.1) is discussed in the following three cases.

(1) $\tau_1 = 0, \tau_2 = 0$

We obtain $H_{21} = 1.5090 > 0, H_{22} = 1.2379 > 0$, then (H_2) holds. The positive equilibrium $E_2(10.4348, 0.1517)$ is locally asymptotically stable from Theorem 3.2 (see Figure 2).

(2) $\tau_1 = \tau_2 \neq 0$

When $\tau_1 = \tau_2 = 15$, positive equilibrium $E_2(9.3356, 0.0658)$ is locally asymptotically stable (see Figure 3). When $\tau_1 = \tau_2 = 19$, Hopf bifurcation occurs near positive equilibrium $E_2(9.2648, 0.0602)$ of model (1.3)(see Figure 4).

(3) $\tau_1 = 0, \tau_2 \neq 0$

When $\tau_1 = 0, \tau_2 = 3$, positive equilibrium $E_2(9.9372, 0.1128)$ is locally asymptotically stable (see Figure 5). In this case, positive equilibrium is always locally asymptotically stable, and Hopf bifurcation does not appear.

(4) $\tau_1 > 0, \tau_2 > 0$ and $\tau_1 \neq \tau_2$

We obtain the image of $Q_i(\omega_j, \tau_2)$, $i = 1, 2, 3$ from Eq. (3.15) (see Figure 6(a)). Then feasible region Ω surrounded by blue lines, green lines and ω_j axis is obtained, and curve C formed by satisfying $S_n^{\pm}(\omega_j, \tau_2) = 0$ is obtained (see Figure

6(b)) (Figure 6(b) shows an image of the curve in the first quadrant in (ω_j, τ_2) plane). In feasible region Ω, the stability switching curves $\mathcal{T}$ of system (5.1) on (τ_1, τ_2) plane is shown as Figure 7(a). We partial enlarge Figure 7(a) to get Figure 7(b) and Figure 8. When (τ_1, τ_2) changes along the direction of the arrow in Figure 8(a,b,c), the direction of characteristic root crosses the imaginary axis is from left to right.

When $(\tau_1, \tau_2) = P_1(1.4, 2.15)$, and P_1 is in stable region, positive equilibrium $E_2(10.0438, 0.1211)$ is locally asymptotically stable (see Figure 9). When $(\tau_1, \tau_2) = P_2(0.4, 2.6)$, and P_2 is in stable region, positive equilibrium $E_2(9.9849, 0.1165)$ is locally asymptotically stable (see Figure 10). When $(\tau_1, \tau_2) = P_3(0.07, 2.45)$, delay passes through the stability switching curves, we observe that the system has the periodic solution at positive equilibrium $E_2(10.0039, 0.1180)$ (see Figure 11). Similarly, when $(\tau_1, \tau_2) = P_4(3.8, 2.3)$ or $(\tau_1, \tau_2) = P_6(7.4, 2.272)$, P_4 and P_6 are in stable region, positive equilibrium $E_2(10.0235, 0.1195)$ or $E_2(10.0272, 0.1198)$ is locally asymptotically stable (see Figure 12, 14). If $(\tau_1, \tau_2) = P_5(3.57, 2.282)$ or $(\tau_1, \tau_2) = P_7(7, 2.272)$, delay crosses the stability switching curves, positive equilibrium $E_2(10.0272, 0.1198)$ or $E_2(10.0271, 0.1197)$ is unstable (see Figure 13, 15).

When $\tau_1 = 3.57$, we obtain $\tau_2^* = 2.282$, $\omega^* = 1.8$, then

$$\begin{aligned} &C_1(0) = -3.4650 + 4.4307i, \quad \mu_2 = 198.7477 > 0, \\ &T_2 = 0.7529 > 0, \quad \beta_2 = -6.9299 < 0. \end{aligned}$$

Hence, Hopf bifurcation is supercritical, bifurcating periodic solutions are stable and period of the periodic solution increases with τ_2.

6. Conclusion

In this paper, we studied a networked SIR epidemic model with two delays and delay dependent parameters. The graph of Laplacian diffusion is incorporated into our model to describe population mobility. The stability change and Hopf bifurcation caused by two delays are mainly discussed. The disease-free equilibrium E_1 is locally asymptotically stable when $R_0 < 1$. When $\tau_1 = \tau_2 = 0$, the endemic equilibrium E_2 is locally asymptotically stable. If both τ_1 and τ_2 are not equal to zero and change at the same time, using the method of stability switching curves, we obtain the stability switching curves on the (τ_1, τ_2) plane such that the stability of equilibrium and the existence of Hopf bifurcation can be further discussed. We calculated normal form of Hopf bifurcation and got the property of Hopf bifurcation.

Numerically, using Watts-Strogatz network as a concrete network, we obtained the stable region of equilibrium, further confirmed that Hopf bifurcating is supercritical, bifurcating periodic solutions are stable and period of the periodic solution increases with τ_2. When the equilibrium is locally asymptotically stable, susceptible and infected individuals can coexist in a stable state. However, when the equilibrium is unstable, the existence of periodic solution implies that the densities of susceptible and infected individuals appears periodic oscillation. This phenomenon reflects the phenomenon of seasonal fluctuation or periodic epidemic of actual infectious diseases in the population.

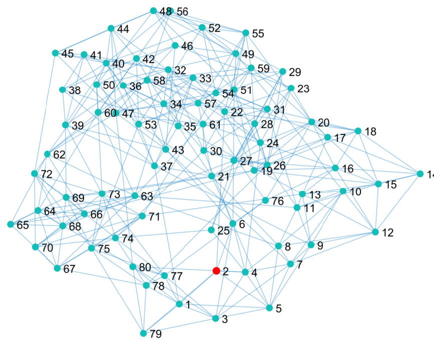


Figure 1. Watts-Strogatz network WS(m, r, p_1) = WS(80, 4, 0.28).

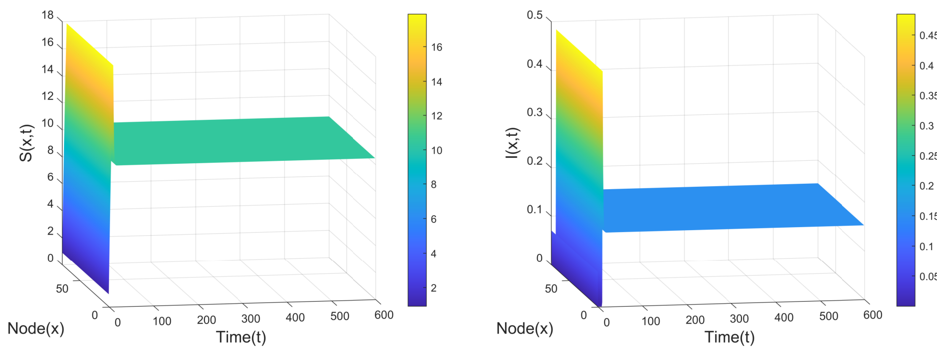


Figure 2. Positive equilibrium $E_2(10.4348, 0.1517)$ is locally asymptotically stable when $\tau_1 = 0, \tau_2 = 0$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

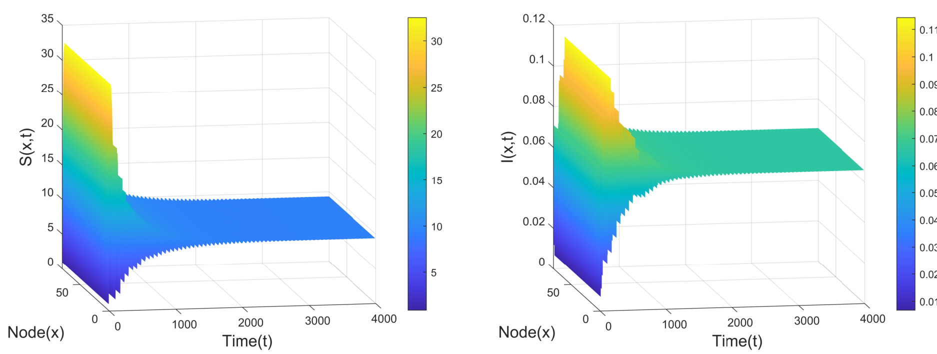


Figure 3. Positive equilibrium $E_2(9.3356, 0.0658)$ is locally asymptotically stable when $\tau_1 = \tau_2 = 15$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

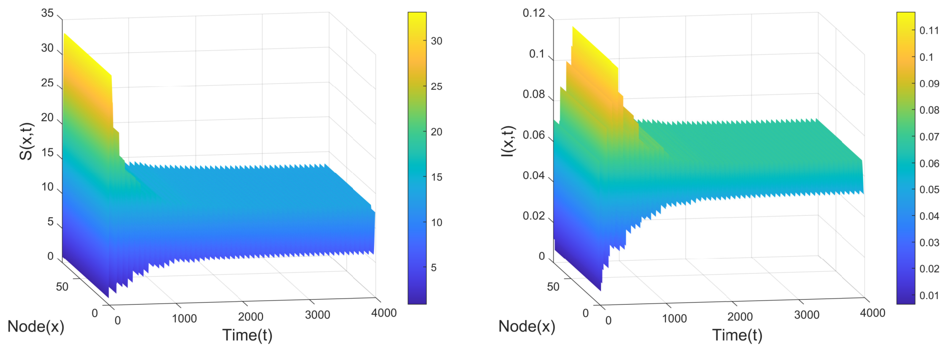


Figure 4. Positive equilibrium $E_2(9.2648, 0.0602)$ is unstable when $\tau_1 = \tau_2 = 19$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

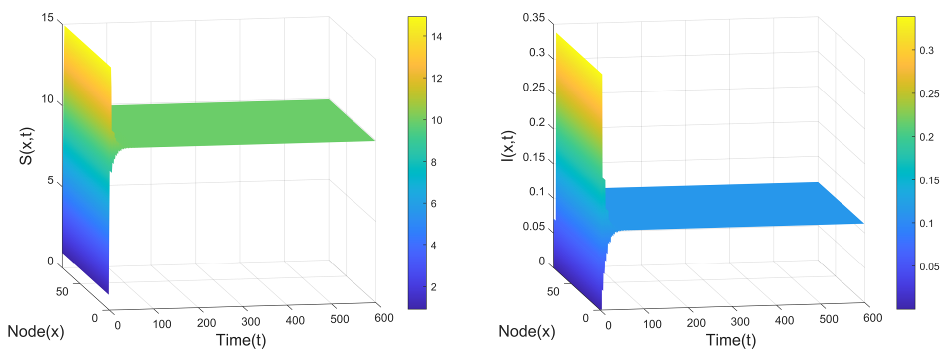
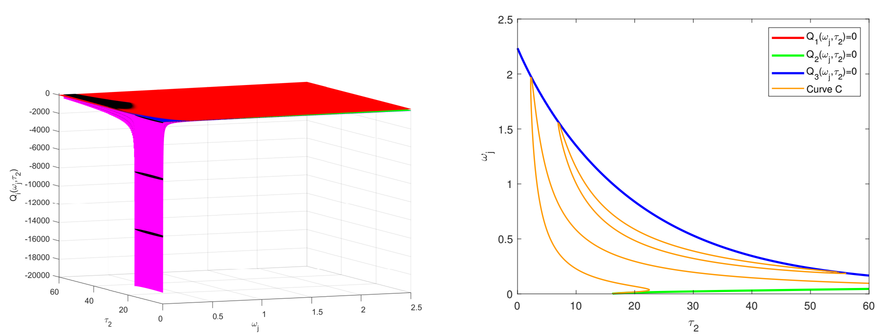
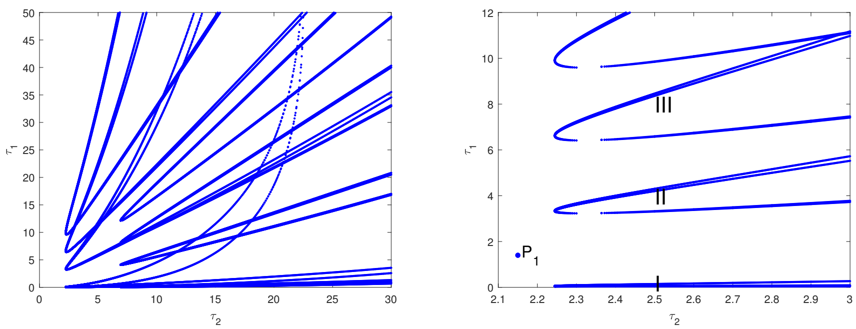


Figure 5. Positive equilibrium $E_2(9.9372, 0.1128)$ is locally asymptotically stable when $\tau_1 = 0, \tau_2 = 3$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.



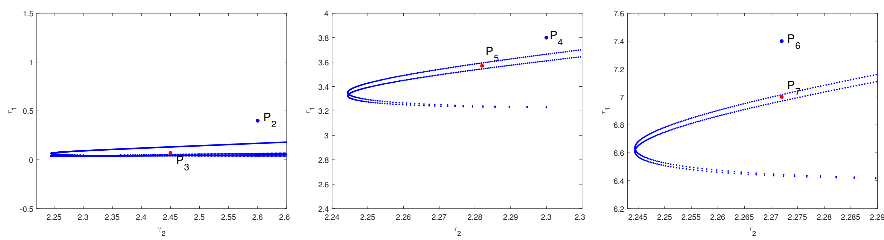
(a) (b)

Figure 6. (a) The image of $Q_i(\omega_j, \tau_2)$, $i = 1, 2, 3$ when $\omega_j \in (0, 2.5]$, $\tau_2 \in [0, 60]$. (b) Feasible region Ω and curve C.



(a) (b)

Figure 7. (a) The stability switching curves $\mathcal{T}$. (b) Partial enlargement.



(a) Partial enlargement of I (b) Partial enlargement of II (c) Partial enlargement of III

Figure 8. Partial enlargements of I, II, III of Figure 6(b) and cross direction.

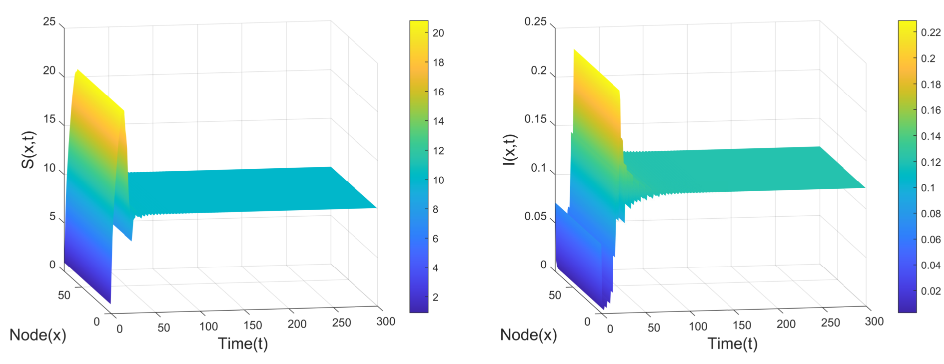


Figure 9. Positive equilibrium $E_2(10.0438, 0.1211)$ is locally asymptotically stable when $(\tau_1, \tau_2) = P_1(1.4, 2.15)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

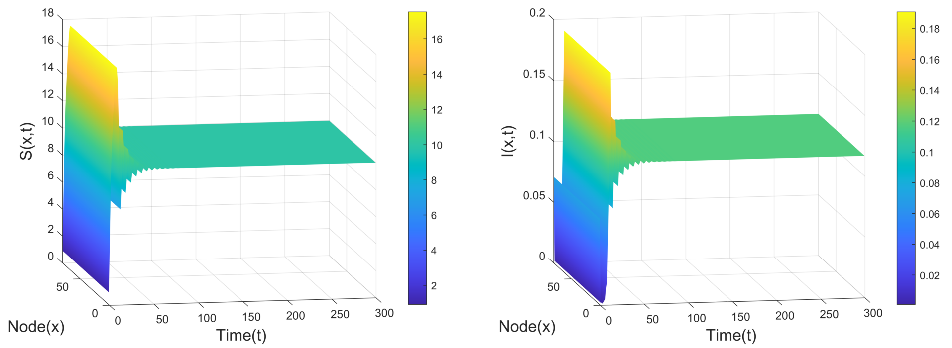


Figure 10. Positive equilibrium $E_2(9.9849, 0.1165)$ is locally asymptotically stable when $(\tau_1, \tau_2) = P_2(0.4, 2.6)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

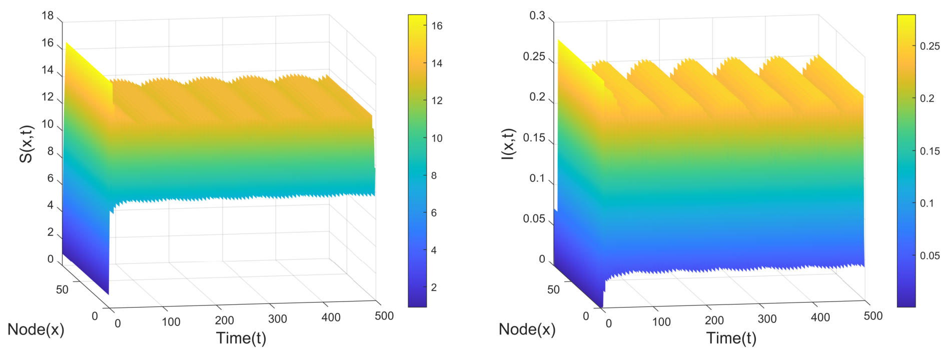


Figure 11. Positive equilibrium $E_2(10.0039, 0.1180)$ is unstable and the periodic solution appears when $(\tau_1, \tau_2) = P_3(0.07, 2.45)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

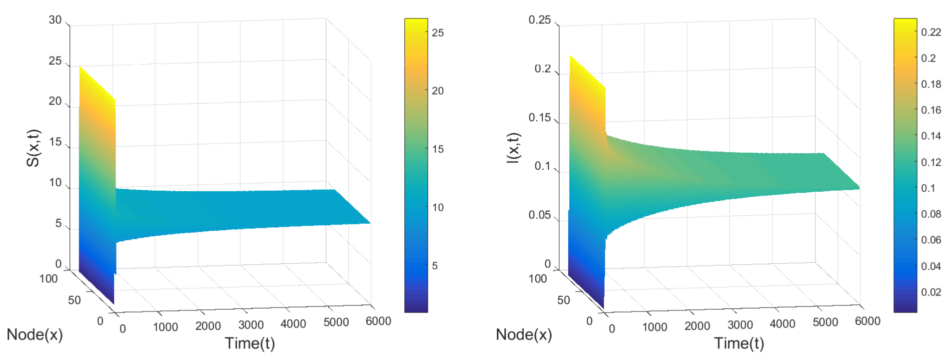


Figure 12. Positive equilibrium $E_2(10.0235, 0.1195)$ is locally asymptotically stable when $(\tau_1, \tau_2) = P_4(3.8, 2.3)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

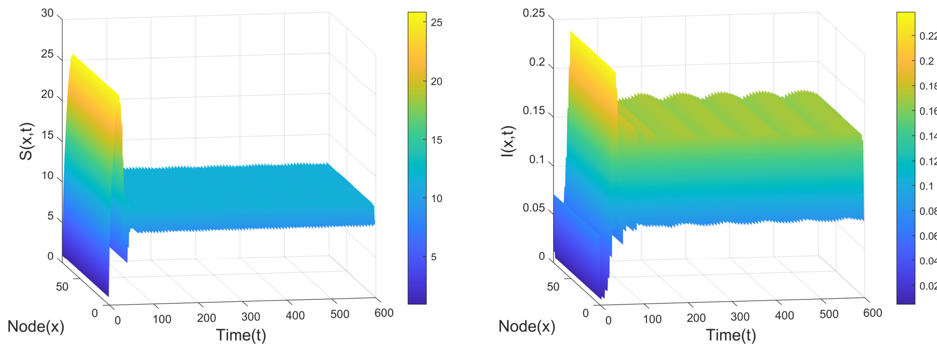


Figure 13. Positive equilibrium $E_2(10.0259, 0.1197)$ is unstable and the periodic solution appears when $(\tau_1, \tau_2) = P_5(3.57, 2.282)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

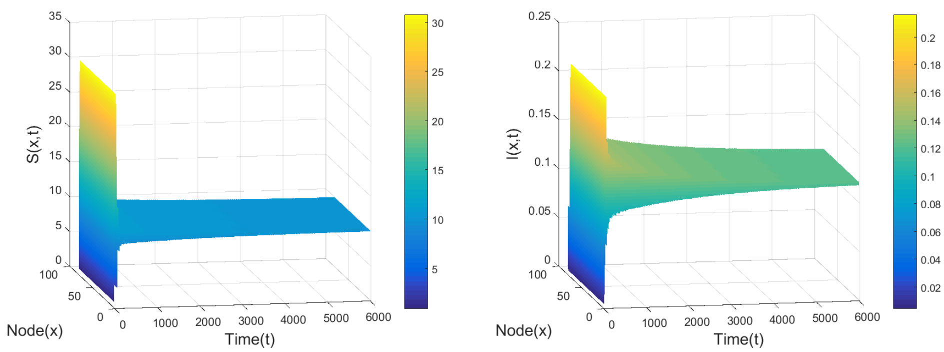


Figure 14. Positive equilibrium $E_2(10.0272, 0.1198)$ is locally asymptotically stable when $(\tau_1, \tau_2) = P_6(7.4, 2.272)$ 1.00,0.00,0.00with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

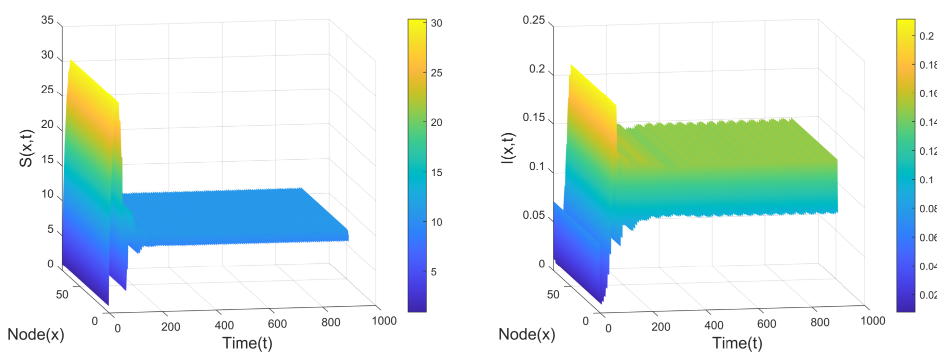


Figure 15. Positive equilibrium $E_2(10.0271, 0.1197)$ is unstable and the periodic solution appears when $(\tau_1, \tau_2) = P_7(7, 2.272)$ 1.00,0.00,0.00 with initial value $(S_0, I_0) = (0.9114, 0.0710)$.

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Declarations.

Conflict of interest. The authors declare that they have no Conflict of interest.

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