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MAXIMUM ERROR ESTIMATES OF DISCONTINUOUS GALERKIN METHODS FOR SOLVING NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract The exact solution to a neutral delay differential equation is generally non-smooth. Some possible loss of accuracy is usually found if certain high-order numerical methods are applied. The discontinuous Galerkin (DG) methods are introduced to numerically solve neutral delay differential equations so as to handle the difficulties. Maximum error estimates of the numerical method is investigated. Theoretical results indicate that the p-degree DG approximate solution has an accuracy of p-th order. Numerical experiments are presented to confirm the effectiveness and performance of the DG methods.

Keywords Neutral delay differential equations, maximum error estimates, discontinuous Galerkin methods.

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1. Introduction

We consider to apply discontinuous Galerkin (DG) methods for solving nonlinear neutral delay differential equations (NDDE) with a constant delay or a proportional delay, having the form

$$\begin{cases} x'(t) = f(t, x(t), x(\varphi(t)), x'(\varphi(t))), & t_0 < t \leq T, \\ x(t) = \phi(t), & t \leq t_0, \end{cases} \tag{1.1}$$

where $\varphi(t) = t - \varsigma$ ($\varsigma > 0$) with ς being the constant delay or $\varphi(t) = mt$ ($0 < m < 1$) with mt being the proportional delay, and $\phi(t)$ is a given initial function.

Many natural phenomena are widely modeled as NDDE in scientific fields such as physics [34,37], biology [3], ecology [2], control systems [30,36], etc. Over the past few decades, a great deal of attention has been devoted to the numerical solution to NDDE by a certain numerical method, such as linear multistep methods [11,20], Runge-Kutta methods [12,40,42], one-leg methods [44] and so on [4,19,46]. The

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major emphasis rests on the stability analysis for specific numerical method. For example, Cong et al. [13] established a sufficient condition of weak delay-dependent stability of multistep Runge-Kutta methods for NDDE. Wang et al. [45] obtained sufficient conditions of delay-dependent stability of a class of Runge-Kutta methods for NDDE. The convergence and stability of finite difference method for nonlinear neutral functional differential equations have been studied intensively in [38]. The convergence of the fully geometric mesh one-leg methods for NDDE with a proportional delay has also been investigated in [39]. To the best of our knowledge, few research has been conducted in convergence analysis of the DG methods for solving problem (1.1). One of the reasons is that the global solutions to NDDE are generally not so smooth (e.g., [2,3,16]). Some possible loss of accuracy is usually found if some high-order numerical methods are applied. As a result, a major concern today is to continue to develop an efficient and effective numerical method for numerically solving the problems.

DG methods have been recognized as efficient and effective methods for numerically solving a diversity of differential equations with discontinuous solutions. For ordinary differential equations (ODE), readers can consult the study by Estep [17], Delfour et al. [15]. In particular, for ODE with non-smooth solutions, Schötzau and Schwab [32,33] investigated the hp-version regarding to the DG methods. The authors derived a priori error estimate explicit in time step and in approximation orders. The DG methods also have been intensively investigated for time-dependent partial differential equations, e.g., nonlinear dispersive equations [25], KdV type equations [50], hyperbolic conservation laws [8], semilinear parabolic problems with time constant delay [29,49], etc. For more details, we refer to [9,10,14] and references therein. So far, the DG methods have been proven to be high-order accurate and stable methods, and generally some superconvergence results are available [1,6,7,48,52]. Consequently, it is reasonable to bring in the DG methods for solving delay differential equations (DDE).

Brunner et al. [5] studied the DG methods for pantograph type DDE in 2010. They obtained some superconvergence results. Furthermore, for the same problem, Huang et al. [21] in 2011 obtained superconvergence result at $t = 0$. They also studied hp DG methods for DDE with nonlinear vanishing delay [22]. At the same time, Li et al. considered nonlinear stability and convergence of DG methods for different kinds of DDE in [26–28]. Recently, Tu et al. [35] introduced a new postprocessing method aimed at improving the numerical errors of DG approximation for DDE. For NDDE, the equations's behavior (e.g., well-posedness, continuity, asymptotic boundedness) are markedly different, see, e.g., [23]. Especially, Brunner et al. [5] showed that the analysis of the convergence of the DG methods for NDDE with proportional delay remained an open problem. Given the aforementioned, the study of the DG methods for NDDE is a far more challenging issue, yet to be adequately addressed.

In the first content of this study, we investigate the DG methods for solving NDDE under a constant delay case. It is proved that the p-degree DG approximate solutions of the NDDE can be p-th order accurate. Therefore, the method can be high-order accuracy. Furthermore, we apply the DG methods for NDDE with proportional delay. By an equivalent transformation, the equations become the NDDE with a constant delay. Then, the convergence analysis follows from the previous mentioned results for NDDE with a constant delay. We would like to mention that the convergence order of the DG methods for NDDE is quite different from that for the DDE investigated in [5, 21, 22, 28, 31]. Because of the effect of neutral delay term, the accuracy order is at least one order lower. Meanwhile, we also test some convergence orders at the nodal points. No superconvergence results are found. Finally, all the conclusions are illustrated by several numerical examples.

We organize the paper as follows. In section 2, a detailed analysis as well as implementation of DG methods for solving NDDE under a constant delay case is presented. Also, we apply the DG methods for solving NDDE under a proportional delay case. Experimental studies are shown in section 3. Finally, conclusions are summarized in the last section.

2. The constant delay case

This section centers around DG methods for solving NDDE under a constant delay case and the method's convergence results.

2.1. DG methods

Take a positive integer k. Let $h = \frac{\varsigma}{k}$. Consider the following uniform mesh:

$$\mathcal{T}^h: \ t_0 < t_1 < \ldots < t_N,$$

where $t_n = nh$, $n = 0, 1, 2, \cdots, N$, and the index N satisfies $t_{N-1} < T \leq t_N$. Let $J_n = (t_{n-1}, t_n)$ and $\mathcal{T} = [t_0, T]$. Define

$$S^h(\mathcal{T}^h, p; \mathbb{R}^d) := \{v \in L^2(\mathcal{T}; \mathbb{R}^d): \ v|_{J_n} \in Q_p(J_n; \mathbb{R}^d), n = 1, 2, \cdots, N\},$$

of which $Q_p(J_n; \mathbb{R}^d)$ is the set of polynomials whose degrees are no more than p. Define $v_n^+ = \lim_{t \to t_n^+} v(t)$ and $v_n^- = \lim_{t \to t_n^-} v(t)$. Define the jump by $[v]_n = v_n^+ - v_n^-$.

Multiplying by $v \in Q_p(J_n; \mathbb{R}^d)$ and integrating over the element J_n, under the constant delay case ($\varphi(t) = t - \varsigma$), then problem (1.1) turns into

$$\int_{J_n} x'(t) v dt = \int_{J_n} f(t, x(t), x(t-\varsigma), x'(t-\varsigma)) v dt, \quad n \geq 1, \ x \in \mathbb{R}^d. \tag{2.1}$$

With this, we could define the p-degree DG solution $X \in S^h(\mathcal{T}^h, p; \mathbb{R}^d)$ as:

$$\begin{aligned} &\sum_{n=1}^{N} \int_{J_n} \Big(X'(t), \ v\Big) dt + \sum_{n=2}^{N} \Big([X]_{n-1}, v_{n-1}^+\Big) + \Big(X_0^+, v_0^+\Big) \qquad (2.2)\\ &= \sum_{n=1}^{N} \int_{J_n} \Big(f(t, X, X(t-\varsigma), X'(t-\varsigma)), \ v\Big) dt + \Big(x(0), v_0^+\Big), \ \forall \ v \in Q_p(J_n; \mathbb{R}^d). \end{aligned}$$

We could also rewrite the above DG methods as a time-stepping scheme, i.e.,

$$\begin{aligned} &\int_{J_n} \Big(X'(t) - f(t, X, X(t-\varsigma), X'(t-\varsigma)), \ v\Big) dt + \Big(X_{n-1}^+, v_{n-1}^+\Big) \\ &= \Big(X_{n-1}^-, v_{n-1}^+\Big), \ \forall \ v \in Q_p(J_n; \mathbb{R}^d). \end{aligned} \tag{2.3}$$

2.2. Error estimates

The main convergence result is revealed in the following theorem under the following assumption

$$\|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)\| \ \leq \ L(\|u_1-u_2\|+\|v_1-v_2\|+\|w_1-w_2\|), \quad (2.4)$$

where $L>0$ is a constant,and $\|\cdot\|$ is the usual norm, which is deduced form the inner product.

Theorem 2.1. *Suppose assumption* (2.4) *holds and the exact solution x to Equation (1.1) is smooth in each interval $((i-1)\varsigma,\ i\varsigma)$, $i=1,2,\ldots$. Then it holds that*

$$\|x-X\|_\infty \ \leq \ Ch^p,$$

where X is the DG approximate solution defined in (2.2), p is the degree of the DG methods and C is a constant independent of h.

In the above theorem,

$$\|x-X\|_\infty := \max_{t\in\mathcal{J}}|x(t)-X(t)|.$$

Remark 2.1. In [51], the authors also introduced the DG methods for solving the NDDEs. In their paper, the error estimates are obtained by assuming the solutions are sufficiently smooth. While in the present paper, it is assumed that the exact solutions to Equation (1.1) are smooth only in each interval $((i-1)\varsigma,\ i\varsigma)$, $i=1,2,\ldots$. It implies that the solutions are discontinuous at the time $i\varsigma$. Therefore, the error estimates are different.

In order to prove Theorem 2.1, we introduce the following several lemmas.

Lemma 2.1 ([28]). *Let $J=(c,d)$. It holds that*

$$\int_c^d\|\psi\|^2dt \leq \frac{1}{d-c}\sum_{i=1}^{d}(\int_c^d\psi_i(t)dt)^2+\frac{1}{2}\int_c^d(d-t)(t-c)\|\psi'(t)\|^2dt,$$

where $\psi(t)=(\psi_1(t),\ldots,\psi_d(t))\in Q_p((c,d);R^d)$, $p\in N_0$ with N_0 being a given positive integer.

Lemma 2.2 ([32]). *For any $p\in N_0$ and $J=(c,d)$, it holds*

$$\|\psi\|^2_{J_n,\infty}\leq C\log(p+1)\int_c^d\|\psi'(t)\|^2(t-c)dt+C\|\psi(d)\|^2$$

for every $\psi(t)=(\psi_1(t),\ldots,\psi_d(t))\in Q_p((c,d);R^d)$, where C denotes a constant and

$$\|\psi\|_{J_n,\infty}:=\max_{t\in J_n}|\psi(t)|.$$

Lemma 2.3 ([28]). *Let $\Pi_h : C(0,1)\longrightarrow S^h(\mathcal{T}^h)$ be an interpolation operator which is the unique function in S^h. Suppose that*

$$\Pi_hx(t_n^-)=x(t_n^-),\ \int_{J_n}\Pi_hxvdx=\int_{J_n}xvdx,\ \forall v\in Q_{p-1}(J_n;\mathbb{R}^d),\ p\geq 1, \quad (2.5)$$

where x denotes a smooth function in a given interval. Then, it holds

$$\|x-\Pi_hx\|_{J_n,\infty}+h\|(x-\Pi_hx)'\|_{J_n,\infty}\leq Ch^{p+1}\|x\|_{J_n,p+1,\infty}. \quad (2.6)$$

Now, let $\epsilon := x - X = x - \Pi_h x + \Pi_h x - X := \xi + \theta$. Then, one can check that for the difference θ, it holds that

$$\begin{aligned}
&\int_{J_n} \Big(\theta'(t),\ v\Big)dt + \Big(\theta^+_{n-1}, v^+_{n-1}\Big) \\
=\ &\Big(\theta^-_{n-1}, v^+_{n-1}\Big) \\
&+ \int_{J_n} \Big(f(t,x,x(t-\varsigma),x'(t-\varsigma)) - f(t,X,X(t-\varsigma),X'(t-\varsigma)),\ v\Big)dt
\end{aligned} \tag{2.7}$$

for all $v \in S^h(\mathcal{T}^h, p; \mathbb{R}^d)$. Equivalently, we have

$$\begin{aligned}
&-\int_{J_n} \Big(\theta(t),\ v'\Big)dt + \Big(\theta^-_{n}, v^-_{n}\Big) \\
=\ &\Big(\theta^-_{n-1}, v^+_{n-1}\Big) \\
&+ \int_{J_n} \Big(f(t,x,x(t-\varsigma),x'(t-\varsigma)) - f(t,X,X(t-\varsigma),X'(t-\varsigma)),\ v\Big)dt.
\end{aligned} \tag{2.8}$$

As a result, we have the following results from formulae (2.7) and (2.8).

Lemma 2.4. *It holds*

$$\begin{aligned}
\|\theta^-_n\|^2 \le \|\theta^-_{n-1}\|^2 + 5L\int_{J_n}\|\theta(t)\|^2dt + L\int_{J_n}\|\xi(t)\|^2dt + L\int_{J_{n-k}}\|\epsilon(t)\|^2dt& \\
+ L\int_{J_{n-k}}\|\epsilon'(t)\|^2dt,&
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\int_{J_n}\|\theta'\|^2(t-t_{n-1})dt \le 6L^2h\int_{J_n}\|\xi(t)\|^2dt + 6L^2h\int_{J_n}\|\theta(t)\|^2dt& \\
+ 3L^2h\int_{J_{n-k}}\|\epsilon(t)\|^2dt + 3L^2h\int_{J_{n-k}}\|\epsilon'(t)\|^2dt,&
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\sum_{i=1}^{d}\Big(\int_{J_n}\theta_i(t)dt\Big)^2 \le 2h^2\|\theta^-_n\|^2 + 4L^2h^3\int_{J_n}\|\xi(t)\|^2dt + 4L^2h^3\int_{J_n}\|\theta(t)\|^2dt& \\
+ 2L^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt + 2L^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt.&
\end{aligned} \tag{2.11}$$

Proof. To show (2.9), we take $v = \theta(t)$ in (2.7) and (2.8). Then, by summing the two equations, we have

$$\begin{aligned}
&\|\theta^-_n\|^2 + \|\theta^+_{n-1}\|^2 \\
=\ &2\int_{J_n}\Big(f(t,x,x(t-\varsigma),x'(t-\varsigma)) - f(t,X,X(t-\varsigma),X'(t-\varsigma)),\ \theta(t)\Big)dt \\
&+2\Big(\theta^-_{n-1}, \theta^+_{n-1}\Big) \\
\le\ &2L\int_{J_n}(\|\epsilon(t-\varsigma)\| + \|\epsilon(t)\| + \|\epsilon'(t-\varsigma)\|)\|\theta(t)\|dt + \|\theta^-_{n-1}\|^2 + \|\theta^+_{n-1}\|^2 \\
\le\ &2L\int_{J_n}(\|\theta(t)\| + \|\xi(t)\| + \|\epsilon(t-\varsigma)\| + \|\epsilon'(t-\varsigma)\|)\|\theta(t)\|dt + \|\theta^-_{n-1}\|^2 + \|\theta^+_{n-1}\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq 5L\int_{J_n}\|\theta(t)\|^2dt+L\int_{J_n}\|\xi(t)\|^2dt+L\int_{J_n}\|\epsilon(t-\varsigma)\|^2dt+L\int_{J_n}\|\epsilon'(t-\varsigma)\|^2dt\\
&\quad+\|\theta_{n-1}^-\|^2+\|\theta_{n-1}^+\|^2\\
&=5L\int_{J_n}\|\theta(t)\|^2dt+L\int_{J_n}\|\xi(t)\|^2dt+L\int_{J_{n-k}}\|\epsilon(t)\|^2dt+L\int_{J_{n-k}}\|\epsilon'(t)\|^2dt\\
&\quad+\|\theta_{n-1}^-\|^2+\|\theta_{n-1}^+\|^2,
\end{aligned}
$$

where we have used the inequality $2\alpha\beta\leq\alpha^2+\beta^2$.

Eliminating the term $\|\theta_{n-1}^+\|^2$, one can get (2.9).

In order to prove (2.10), we let $v=\theta'(t)(t-t_{n-1})$ in (2.7) and then find

$$
\begin{aligned}
&\int_{J_n}\|\theta'(t)\|^2(t-t_{n-1})dt\\
&=\int_{J_n}\Big(f(t,x,x(t-\varsigma),x'(t-\varsigma))-f(t,X,X(t-\varsigma),X'(t-\varsigma)),\ \theta'(t)(t-t_{n-1})\Big)dt\\
&\leq L\int_{J_n}\Big(\|\epsilon(t)\|+\|\epsilon(t-\varsigma)\|+\|\epsilon'(t-\varsigma)\|\Big)\|\theta'(t)\|(t-t_{n-1})dt\\
&=L\int_{J_n}\|\epsilon(t)\|\|\theta'(t)\|(t-t_{n-1})dt+L\int_{J_n}\|\epsilon(t-\varsigma)\|\|\theta'(t)\|(t-t_{n-1})dt\\
&\quad+L\int_{J_n}\|\epsilon'(t-\varsigma)\|\|\theta'(t)\|(t-t_{n-1})dt\\
&\leq L\Big(\int_{J_n}\|\epsilon(t)\|^2(t-t_{n-1})dt\Big)^{\frac12}\Big(\int_{J_n}\|\theta'(t)\|^2(t-t_{n-1})dt\Big)^{\frac12}\\
&\quad+L\Big(\int_{J_n}\|\epsilon(t-\varsigma)\|^2(t-t_{n-1})dt\Big)^{\frac12}\Big(\int_{J_n}\|\theta'(t)\|^2(t-t_{n-1})dt\Big)^{\frac12}\\
&\quad+L\Big(\int_{J_n}\|\epsilon'(t-\varsigma)\|^2(t-t_{n-1})dt\Big)^{\frac12}\Big(\int_{J_n}\|\theta'(t)\|^2(t-t_{n-1})dt\Big)^{\frac12}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{J_n}\|\theta'(t)\|^2(t-t_{n-1})dt\\
&\leq\Big(L(\int_{J_n}\|\epsilon(t)\|^2(t-t_{n-1})dt)^{\frac12}+L(\int_{J_n}\|\epsilon(t-\varsigma)\|^2(t-t_{n-1})dt)^{\frac12}\\
&\quad+L(\int_{J_n}\|\epsilon'(t-\varsigma)\|^2(t-t_{n-1})dt)^{\frac12}\Big)^2\\
&\leq 3L^2\int_{J_n}\|\epsilon(t)\|^2(t-t_{n-1})dt+3L^2\int_{J_n}\|\epsilon(t-\varsigma)\|^2(t-t_{n-1})dt\\
&\quad+3L^2\int_{J_n}\|\epsilon'(t-\varsigma)\|^2(t-t_{n-1})dt\\
&\leq 6L^2h\int_{J_n}(\|\xi(t)\|^2+\|\theta(t)\|^2)dt+3hL^2\int_{J_n}(\|\epsilon(t-\varsigma)\|^2+\|\epsilon'(t-\varsigma)\|^2)dt\\
&=6L^2h\int_{J_n}(\|\xi(t)\|^2+\|\theta(t)\|^2)dt+3hL^2\int_{J_{n-k}}(\|\epsilon(t)\|^2+\|\epsilon'(t)\|^2)dt.
\end{aligned}
$$

In order to obtain (2.11), we take v in (2.7) as $(0\cdots0,t_{n-1}-t,0\cdots0)$, of which

$t_{n-1}-t$ denotes the usual $i-$th component. It holds that

$$\int_{J_n}\theta_i(t)dt-h(\theta_i)_n^-=\int_{J_n}f_i(t_{n-1}-t)dt$$

with f_i being the usual $i-$th component of the nonlinear function $f(t,x,x(t-\varsigma),x'(t-\varsigma))-f(t,X,X(t-\varsigma),X'(t-\varsigma))$.

Then, by the widely used Cauchy-Schwarz inequality, it holds that

$$\begin{aligned}\Big(\int_{J_n}\theta_i(t)dt\Big)^2&\le 2h^2(\theta_i^2)_n^-+2\int_{J_n}f_i^2dt\cdot\int_{J_n}(t_{n-1}-t)^2dt\\&\le 2h^2(\theta_i^2)_n^-+\frac{2h^3}{3}\int_{J_n}f_i^2dt.\end{aligned}$$

Summing over for i from i to d, we have

$$\begin{aligned}&\sum_{i=1}^d\Big(\int_{J_n}\theta_i(t)dt\Big)^2\\&\le 2h^2\|\theta_n^-\|^2+\frac{2h^3L^2}{3}\int_{J_n}\Big(\|\epsilon(t)\|+\|\epsilon(t-\varsigma)\|+\|\epsilon'(t-\varsigma)\|\Big)^2dt\\&\le 2h^2\|\theta_n^-\|^2+2h^3L^2\int_{J_n}\Big(\|\epsilon(t)\|^2+\|\epsilon(t-\varsigma)\|^2+\|\epsilon'(t-\varsigma)\|^2\Big)dt\\&\le 2h^2\|\theta_n^-\|^2+4L^2h^3\int_{J_n}\|\xi(t)\|^2dt+4L^2h^3\int_{J_n}\|\theta(t)\|^2dt\\&\quad+2L^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt+2L^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt.\end{aligned}$$

This finishes the proof of Lemma 2.4. □

Thanks to the above lemmas, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. By mathematical induction, we can prove the main result. First of all, we will show that the required error estimate holds on the interval I_1.

By (2.11) and Lemma 2.1, it holds that

$$\begin{aligned}&\sum_{i=1}^d(\int_{J_1}\theta_i(t)dt)^2\\&\le 2h^2\|\theta_1^-\|^2+4L^2h^3\int_{J_1}\|\xi(t)\|^2dt+4L^2h^3\int_{J_1}\|\theta(t)\|^2dt\\&\quad+2L^2h^3\int_{J_{1-k}}\|\epsilon(t)\|^2dt+2L^2h^3\int_{J_{1-k}}\|\epsilon'(t)\|^2dt\\&\le 2h^2\|\theta_1^-\|^2+4L^2h^3\int_{J_1}\|\xi(t)\|^2dt+4L^2h^2\sum_{i=1}^d(\int_{J_1}\theta_i(t)dt)^2\\&\quad+2L^2h^3\int_{J_1}\|\theta'(t)\|^2(t_1-t)(t-t_0)dt\\&\quad+2L^2h^3\int_{J_{1-k}}\|\epsilon(t)\|^2dt+2L^2h^3\int_{J_{1-k}}\|\epsilon'(t)\|^2dt\end{aligned}$$

$$
\begin{aligned}
&\leq 2h^2\|\theta_1^-\|^2 + 4L^2h^3\int_{J_1}\|\xi(t)\|^2dt + 4L^2h^2\sum_{i=1}^{d}(\int_{J_1}\theta_i(t)dt)^2\\
&\quad +2L^2h^4\int_{J_1}\|\theta'(t)\|^2(t-t_0)dt\\
&\quad +2L^2h^3\int_{J_{1-k}}\|\epsilon(t)\|^2dt + 2L^2h^3\int_{J_{1-k}}\|\epsilon'(t)\|^2dt,
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
&\sum_{i=1}^{d}(\int_{J_1}\theta_i(t)dt)^2\\
&\leq Ch^2\|\theta_1^-\|^2 + CL^2h^3\int_{J_1}\|\xi(t)\|^2dt + CL^2h^4\int_{J_1}\|\theta'(t)\|^2(t-t_0)dt\\
&\quad +CL^2h^3\int_{J_{1-k}}\|\epsilon(t)\|^2dt + CL^2h^3\int_{J_{1-k}}\|\epsilon'(t)\|^2dt.
\end{aligned}
\tag{2.12}
$$

Meanwhile, by (2.9), (2.10) and Lemma 2.1,

$$
\begin{aligned}
&\int_{J_1}\|\theta'\|^2(t-t_0)dt + \|\theta_1^-\|^2\\
&\leq \|\theta_0^-\|^2 + (L+6Lh^2)\int_{J_1}\|\xi(t)\|^2dt + (5L+6Lh^2)\int_{J_1}\|\theta(t)\|^2dt\\
&\quad +(L+3L^2h)\int_{J_{1-k}}\|\epsilon(t)\|^2dt + (L+3L^2h)\int_{J_{1-k}}\|\epsilon'(t)\|^2dt\\
&\leq \|\theta_0^-\|^2 + (L+6Lh^2)\int_{J_1}\|\xi(t)\|^2dt + \frac{5L+6Lh^2}{h}\sum_{i=1}^{d}(\int_{J_1}\theta_i(t)dt)^2\\
&\quad +\frac{5Lh+6Lh^3}{2}\int_{J_1}\|\theta'(t)\|^2(t-t_0)dt\\
&\quad +(L+3L^2h)\int_{J_{1-k}}\|\epsilon(t)\|^2dt + (L+3L^2h)\int_{J_{1-k}}\|\epsilon'(t)\|^2dt.
\end{aligned}
$$

Substituting (2.12) into the above equation, we arrive at

$$
\begin{aligned}
&\int_{J_1}\|\theta'\|^2(t-t_0)dt + \|\theta_1^-\|^2\\
&\leq CL\int_{J_1}\|\xi(t)\|^2dt + CL\int_{J_{1-k}}\|\epsilon(t)\|^2dt\\
&\quad +CL\int_{J_{1-k}}\|\epsilon'(t)\|^2dt + CLh\Big(\int_{J_1}\|\theta'\|^2(t-t_0)dt + \|\theta_1^-\|^2\Big),
\end{aligned}
\tag{2.13}
$$

where we noted that $\theta_0^- = 0$. It further implies that, when h is sufficiently small, it holds

$$
\int_{J_1}\|\theta'\|^2(t-t_0)dt + \|\theta_1^-\|^2
$$

$$\leq CL\int_{J_1}\|\xi(t)\|^2dt+CL\int_{J_{1-k}}\|\epsilon(t)\|^2dt+CL\int_{J_{1-k}}\|\epsilon'(t)\|^2dt. \tag{2.14}$$

Together with lemma 2.2 and (2.14), it holds that

$$\begin{aligned}\|\theta\|_{J_1,\infty} &\leq C\int_{J_1}\|\theta'\|^2(t-t_0)dt+C\|\theta_1^-\|^2 \qquad (2.15)\\ &\leq CL\int_{J_1}\|\xi(t)\|^2dt+CL\int_{J_{1-k}}\|\epsilon(t)\|^2dt+CL\int_{J_{1-k}}\|\epsilon'(t)\|^2dt.\end{aligned}$$

Thus, together with lemma 2.3, we have

$$\|\theta\|_{J_1,\infty}\leq\|\xi\|_{J_1,\infty}+Ch^p.$$

Applying the triangle inequality, we get

$$\|\epsilon\|_{J_1,\infty}\leq Ch^p.$$

Meanwhile,

$$\begin{aligned}\|\epsilon'\|_{J_1,\infty} &= \Big\|f(t,x,x(t-\varsigma),x'(t-\varsigma))-f(t,X,X(t-\varsigma),X'(t-\varsigma))\Big\|_{J_1,\infty} \qquad (2.16)\\ &\leq L\Big(\|\epsilon\|_{J_1,\infty}+\|\epsilon(t-\varsigma)\|_{J_1,\infty}+\|\epsilon'(t-\varsigma)\|_{J_1,\infty}\Big)\\ &\leq Ch^p.\end{aligned}$$

Second, assuming that, for $1\leq\tilde{n}\leq n-1$,

$$\|\epsilon\|_{J_{\tilde{n}},\infty}\leq Ch^p \quad\text{and}\quad \|\epsilon'\|_{J_{\tilde{n}},\infty}\leq Ch^p. \tag{2.17}$$

We are going to show that the convergence results hold for $\tilde{n}=n$.
Combining (2.11) and Lemma 2.1, we have

$$\begin{aligned}&\sum_{i=1}^{d}(\int_{J_n}\theta_i(t)dt)^2\\ &\leq 2h^2\|\theta_n^-\|^2+4L^2h^3\int_{J_n}\|\xi(t)\|^2dt+4L^2h^3\int_{J_n}\|\theta(t)\|^2dt \qquad (2.18)\\ &\quad+2L^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt+4L^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt\\ &\leq 2h^2\|\theta_n^-\|^2+4L^2h^3\int_{J_n}\|\xi(t)\|^2dt+4L^2h^2\sum_{i=1}^{d}(\int_{J_n}\theta_i(t)dt)^2\\ &\quad+2L^2h^3\int_{J_n}\|\theta'(t)\|^2(t_1-t)(t-t_0)dt\\ &\quad+2L^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt+4L^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt\\ &\leq 2h^2\|\theta_n^-\|^2+4L^2h^3\int_{J_n}\|\xi(t)\|^2dt+4L^2h^2\sum_{i=1}^{d}(\int_{J_n}\theta_i(t)dt)^2\\ &\quad+2L^2h^4\int_{J_n}\|\theta'(t)\|^2(t-t_0)dt\end{aligned}$$

$$+2L^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt+4L^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt,$$

which further implies that

$$\begin{aligned}
&\sum_{i=1}^{d}(\int_{J_n}\theta_i(t)dt)^2\\
&\le Ch^2\|\theta_n^-\|^2+CL^2h^3\int_{J_n}\|\xi(t)\|^2dt+CL^2h^4\int_{J_n}\|\theta'(t)\|^2(t-t_0)dt\\
&\quad+CL^2h^3\int_{J_{n-k}}\|\epsilon(t)\|^2dt+CL^2h^3\int_{J_{n-k}}\|\epsilon'(t)\|^2dt. \qquad (2.19)
\end{aligned}$$

Meanwhile, by (2.9), (2.10) and Lemma 2.1, we can find

$$\begin{aligned}
&\int_{J_n}\|\theta'\|^2(t-t_0)dt+\|\theta_n^-\|^2\\
&\le\|\theta_{n-1}^-\|^2+(L+6Lh^2)\int_{J_n}\|\xi(t)\|^2dt+(5L+6Lh^2)\int_{J_n}\|\theta(t)\|^2dt\\
&\quad+(L+3L^2h)\int_{J_{n-k}}\|\epsilon(t)\|^2dt+(L+3L^2h)\int_{J_{n-k}}\|\epsilon'(t)\|^2dt\\
&\le\|\theta_{n-1}^-\|^2+(L+6Lh^2)\int_{J_n}\|\xi(t)\|^2dt+\frac{(5L+6Lh^2)}{h}\sum_{i=1}^{d}(\int_{J_n}\theta_i(t)dt)^2dt\\
&\quad+\frac{5Lh+6Lh^3}{2}\int_{J_n}\|\theta'(t)\|^2(t-t_0)dt\\
&\quad+(L+3L^2h)\int_{J_{n-k}}\|\epsilon(t)\|^2dt+(L+3L^2h)\int_{J_{n-k}}\|\epsilon'(t)\|^2dt.
\end{aligned}$$

Substituting (2.19) into the above equation, we find that

$$\begin{aligned}
&\int_{J_n}\|\theta'\|^2(t-t_0)dt+\|\theta_n^-\|^2 \qquad (2.20)\\
&\le\|\theta_{n-1}^-\|^2+CL\int_{J_n}\|\xi(t)\|^2dt+CLh\Big(\int_{J_n}\|\theta'\|^2(t-t_0)dt+\|\theta_n^-\|^2\Big)\\
&\quad+CL\int_{J_{n-k}}\|\epsilon(t)\|^2dt+CL\int_{J_{n-k}}\|\epsilon'(t)\|^2dt.
\end{aligned}$$

Now, iterating the estimate (2.20) yields

$$\begin{aligned}
&\int_{J_n}\|\theta'\|^2(t-t_0)dt+\|\theta_n^-\|^2 \qquad (2.21)\\
&\le CL\sum_{i=1}^{n}\int_{J_i}\|\xi(t)\|^2dt+CLh\sum_{i=1}^{n}\Big(\int_{J_i}\|\theta'\|^2(t-t_0)dt+\|\theta_i^-\|^2\Big)\\
&\quad+CL\sum_{i=1}^{n}\int_{J_{i-k}}\|\epsilon(t)\|^2dt+CL\sum_{i=1}^{n}\int_{J_{i-k}}\|\epsilon'(t)\|^2dt\\
&\le Ch^{2p}+CLh\sum_{i=1}^{n}\Big(\int_{J_i}\|\theta'\|^2(t-t_0)dt+\|\theta_i^-\|^2\Big),
\end{aligned}$$

where we use the assumptions (2.17).

By the Gronwall's lemma, it holds that

$$\int_{J_n} \|\theta'\|^2 (t-t_0)dt + \|\theta_n^-\|^2 \leq Ch^{2p}.$$

Then, by Lemma 2.3 (i.e., $\psi = \theta$), it holds that

$$\|\theta\|^2_{J_n,\infty} \leq \int_{J_n} \|\theta'\|^2 (t-t_0)dt + \|\theta_n^-\|^2 \leq Ch^{2p}.$$

Moreover, by Lemma 2.4, it holds that

$$\|\xi\|_{J_n,\infty} \leq Ch^{p+1}.$$

Note that $\epsilon := x - X = x - \Pi_h x + \Pi_h x - X := \xi + \theta$, it holds that

$$\|\epsilon\|_{J_n,\infty} \leq \|\theta\|^2_{J_n,\infty} + \|\xi\|^2_{J_n,\infty} \leq Ch^p,$$

which further implies that Theorem 2.1 holds. This completes the proof. □

Remark 2.2. In order to solve the NDDE with proportional delay conveniently, we use the following transformation, which is widely used to investigate the stability of the continuous problems, e.g., [2,24,41,43]. Let $z(t) = x(e^t)$. Then, $z(t)$ satisfies

$$\begin{cases} z'(t) = e^t f(e^t, z(t), z(t-\varsigma), e^{-(t-\varsigma)} z'(t-\varsigma)), & \log t_0 < t \leq \log T, \\ z(t) = x(e^t), & t \leq \log t_0, \end{cases} \tag{2.22}$$

where $\varsigma = -\log m$. Now, we only have to solve the NDDE with a constant delay. Noting that the DG solutions are expressed by the p-degree polynomials, the numerical solution $z(t)$ at every point can be obtained. Therefore, by using the inverse transformation $x(t) = z(\log t)$, the numerical solutions can be easily recovered. Clearly, the stability and convergence analysis follow from the previous mentioned results. And p-degree DG approximate solutions for the NDDE can be p-th order accurate.

3. Numerical examples

We give three numerical experiments to confirm the effectiveness of the DG methods.

Example 3.1. As a first example, we consider the NDDE [31]

$$\begin{cases} x'(t) = -2x(t) + x(t-\varsigma) + 0.5x'(t-\varsigma), & 0 < t \leq 8, \\ x(t) = \sin(\pi t), \quad -\varsigma \leq t \leq 0. \end{cases} \tag{3.1}$$

Take $\varsigma = 2$, then the neutral solution of (3.1) reads

$$x(t) = \begin{cases} \sin(\pi t)/2, & 0 \leq t \leq 2, \\ \sin(\pi t)/4, & 2 < t \leq 4, \\ \sin(\pi t)/8, & 4 < t \leq 6, \\ \sin(\pi t)/16, & 6 < t \leq 8. \end{cases} \tag{3.2}$$

One can check the derivative of $x(t)$ owns a jumping discontinuity for the time $t = 0, 2, 6, 8$. The exact solution of Equation (3.2) is approximated by applying the DG methods under different stepsizes. The L^∞-norm errors are showed in Table 1, where the stepszie $h = 2/N$. The results imply $p-$order convergence result of the DG methods.

We also try to find some possible superconvergence results of DG methods for solving NDDE. Some statistics of the errors and orders at $t = 5$ are listed in Table 2. Although the analytical solutions are smooth in the time interval $[4, 6]$, p-degree DG approximate solution is p-th order accurate. No superconvergence results are found.

Table 1. The numerical error and convergence order for Equation (3.1).

	$p = 1$		$p = 2$		$p = 3$	
N	errors	orders	errors	orders	errors	orders
10	2.79E-2	-	1.75E-3	-	9.42E-5	-
20	1.33E-2	1.07	3.99E-4	2.13	1.03E-5	3.19
40	6.52E-3	1.02	9.46E-5	2.08	1.21E-6	3.09
80	3.25E-3	1.01	2.31E-5	2.03	1.47E-7	3.05
160	1.62E-3	1.00	5.69E-6	2.01	1.83E-8	3.00

Table 2. The numerical error at $t = 5$ and convergence order for Equation (3.1).

	$p = 1$		$p = 2$		$p = 3$	
N	errors	orders	errors	orders	errors	orders
10	6.64E-3	-	2.47E-4	-	1.85E-5	-
20	3.67E-3	0.86	7.13E-5	1.79	2.56E-6	2.85
40	1.93E-3	0.93	1.90E-5	1.90	3.37E-7	2.92
80	9.88E-4	0.96	4.91E-6	1.95	4.34E-8	2.96
160	5.00E-4	0.98	1.25E-6	1.98	5.37E-9	3.01

Example 3.2. Secondly, we solve the following NDDE with a proportional delay

$$x'(t) = -6x(t) + 4x(mt) + 0.1\sin(x'(mt)) + f(t), \tag{3.3}$$

where the initial condition and $f(t)$ is specified so that the exact solution is $x(t) = \sin(t)$. Now, let $z(t) = x(e^t)$. Equation (3.3) becomes

$$z'(t) = -6e^t z(t) + 4e^t z(t-\varsigma) + 0.1m^{-1}\sin(z'(t-\varsigma)) + e^t f(e^t), \tag{3.4}$$

where $\varsigma = -\log m$, the initial condition and exact solution are determined by the exact solution $z(t) = \sin(e^t)$.

We firstly set $m = 0.9$, the time stepsize $h = \varsigma/N$ and solve the problem on the time interval $[0, 10\varsigma]$. The numerical solution $x(t)$ is recovered by $x(t) = z(\log t)$ on the interval $[1, m^{-10}]$. The L^∞-norm of errors and convergence orders are showed in Table 3. The results demonstrate that p-degree DG approximate solution is p-th order accurate.

Table 3. The numerical error and convergence order for Equation (3.3).

	$p=1$		$p=2$		$p=3$	
N	errors	orders	errors	orders	errors	orders
10	1.38E-4	-	4.93E-7	-	2.35E-9	-
20	6.07E-5	1.19	9.36E-8	2.39	1.72E-10	3.77
40	2.82E-5	1.11	1.99E-8	2.24	1.39E-11	3.63
80	1.36E-5	1.05	4.87E-9	2.03	1.25E-12	3.46
160	6.66E-6	1.03	1.21E-9	2.01	1.30E-13	3.27

Example 3.3. To illustrate the stability of the DG methods, we consider the following nonlinear NDDEs with different initial conditions: [18]

$$\begin{cases} x_1'(t) = \sin(t) - ax_1 + by_1 + \dfrac{b_1(x_1(t-1) + b_2 x_1'(t-1))}{1 + (x_1(t-1) + b_2 x_1'(t-1))^2}, & t \geq 0, \\ y_1'(t) = \cos(t) + bx_1 - ay_1 + \dfrac{b_1(y_1(t-1) + b_2 y_1'(t-1))}{1 + (y_1(t-1) + b_2 y_1'(t-1))^2}, & t \geq 0, \\ x_1(t) = \cos(x) \qquad y_1(t) = 2 + \cos(x), \quad t \leq 0, \end{cases} \tag{3.5}$$

$$\begin{cases} x_2'(t) = \sin(t) - ax_2 + by_2 + \dfrac{b_1(x_2(t-1) + b_2 x_2'(t-1))}{1 + (x_2(t-1) + b_2 x_2'(t-1))^2}, & t \geq 0, \\ y_2'(t) = \cos(t) + bx_2 - ay_2 + \dfrac{b_1(y_2(t-1) + b_2 y_2'(t-1))}{1 + (y_2(t-1) + b_2 y_2'(t-1))^2}, & t \geq 0, \\ x_2(t) = 2(\sin(t) + \cos(x)) \qquad y_2(t) = 2(\sin(t) - \cos(x)), \quad t \leq 0. \end{cases} \tag{3.6}$$

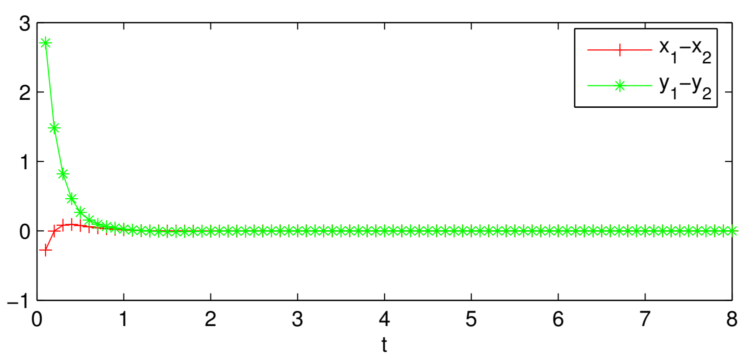

Figure 1. The differences $x_1 - x_2$ and $y_1 - y_2$ to (3.5) and (3.6) computed by 2-degree DG methods.

The equations were used to describe a line array of several mutually coupled lossless transmissionlines which are interconnected by a common resistor [47]. It is also used to investigate the dissipative of numerical methods for NDDEs [18]. Here in order to test effectiveness of the DG methods, we set $a = -6$, $b = 1$, $b_1 = 0.1$ and $b_2 = 1$. We get the numerical approximations by using 2-degree DG methods with stepsize $h = 0.1$. The differences are plotted in Figure 1. Clearly, the differences of the numerical approximations decay in time. These results further confirm the stability results of the DG methods and the DG methods are effective for the kind of the problem.

4. Conclusions

The DG methods are introduced to numerically solve several typical NDDE. The attainable convergence order of the DG methods is quite different from that of DG solutions to the other type of differential equations. We show that, due to the effect of the neutral term and discontinuous of the solutions, the p-degree DG solution of NDDE has p-th order accuracy. No superconvergence results are found.

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References

[1] S. Adjerid, K. Devine, J. Flaherty and L. Krivodonova, *A posteriori error estimation for discontinuous Galerkin solutions of hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 2002, 191, 1097–1112.

[2] A. Bellen and M. Zennaro, *Numerical Methods for Delay Differential Equations*, Oxford University Press, Oxford, 2003.

[3] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004.

[4] H. Brunner and Q. Hu, *Optimal superconvergence results for delay integro-differential equations of pantograph type*, SIAM J. Numer. Anal., 2007, 45, 986–1004.

[5] H. Brunner, Q. Huang and H. Xie, *Discontinuous Galerkin methods for delay differential equations of pantograph type*, SIAM J. Numer. Anal., 2010, 48, 1944–1967.

[6] W. Cao, D. Li and Z. Zhang, *Optimal superconvergence of energy conserving local discontinuous Galerkin methods for wave equations*, Commu. Comput. Phys., 2017, 21, 211–236.

[7] W. Cao, D. Li and Z. Zhang, *Superconvergence of discontinuous Galerkin methods based on upwind-biased fluxes for 1D linear hyperbolic equations*, ESIAM Math. Model. Numer. Anal., 2017, 51(2), 467–486.

[8] Y. Cheng and C. W. Shu, *Superconvergence of discontinuous Galerkin method and local discontinuous Galerkin schemes for linear hyperbolic and convection-diffusion equations in one space dimension*, SIAM J. Numer. Anal., 2010, 47(6), 4044–4072.

[9] B. Cockburn, G. E. Karniadakis and C. W. Shu, *The development of discontinuous Galerkin methods*, in Discontinuous Galerkin Methods. Theory, Computation and Applications, Lecture Notes in Computational Science and Engineering, Springer-Verlag, 2000, 11, 3–50.

[10] B. Cockburn and C. W. Shu, *Foreword for the special issue on discontinuous Galerkin method*, J. Sci. Comp., 2005, 22(23), 1–3.

[11] Y. Cong, *NGP_G-stability of linear multistep methods for systems of generalized neutral delay differential equations*, Appl. Math. Mech., 2001, 22(7), 735–742.

[12] Y. Cong, B. Yang and J. Kuang, *Asymptotic stability and numerical analysis for systems of generalized neutral delay differential equations*, Math. Numer. Sin., 2001, 23(4), 457–468.

[13] Y. Cong, H. Zhao and Y. Zhang, *Delay-dependent stability of multistep Runge-Kutta methods for delay differential systems of neutral type*, J. Numer. Methods Comput. Appl., 2018, 39(4), 301–320.

[14] C. Dawson, *Special issue on discontinuous Galerkin methods*, Comp. Meth. Appl. Mech. Eng., 2006, 195.

[15] M. Delfour, W. Hager and F. Trochu, *Discontinuous Galerkin methods for ordinary differential equations*, Math. Comput., 1981, 36, 455–473.

[16] W. H. Enright and H. Hayashi, *Convergence analysis of the solution of retarded and neutral delay differential equations by continuous numerical emthods*, SIAM. J. Numer. Anal., 1998, 35(2), 572–585.

[17] D. Estep, *A posteriori error bounds and global error control for approximations of ordinary differential equations*, SIMA J. Numer. Anal., 1995, 32, 1–48.

[18] S. Gan, *Dissipativity of θ-methods for nonlinear delay differential equations of neutral type*, Appl. Numer. Math., 2009, 59, 1354–1365.

[19] H. Han and C. Zhang, *Asymptotical-stability-preserving finite element methods in time for 2D neutral delay–reaction–diffusion equations*, Appl. Math. Lett., 2022, 131, p.108082.

[20] P. Hu, C. Huang and S. Wu, *Asymptotic stability of linear multistep methods for nonlinear neutral delay differential equations*, Appl. Math. Comput., 2009, 211, 95–101.

[21] Q. Huang, H. Xie and H. Brunner, *Superconvergence of discontinuous Galerkin solutions for delay differential equations of pantograph type*, SIAM J. Sci. Comput., 2011, 33, 2664–2684.

[22] Q. Huang, H. Xie and H. Brunner, *The hp discontinuous Galerkin method for delay differential equations with nonlinear vanishing delay*, SIAM J. Sci. Comput., 2013, 35, 1614–1620.

[23] A. Iserles, *On the generalized pantograph functional-differential equation*, Euro. J. Appl. Math., 1993, 4(1), 1–38.

[24] T. Koto, *Stability of Runge-Kutta methods for the generalized pantograph equation*, Numer. Math., 1999, 84, 233–247.

[25] D. Levy, C. W. Shu and J. Yan, *Local discontinuous Galerkin methods for nonlinear dispersive equations*, J. Comput. Phys., 2004, 196, 751–772.

[26] D. Li and C. Zhang, *Nonlinear stability of discontinuous Galerkin methods for delay differential equation*, Appl. Math. Lett., 2010, 23, 457–461.

[27] D. Li and C. Zhang, *Superconvergence of a discontinuous Galerkin method for first-order linear delay differential equations*, J. Comp. Math., 2011, 29, 574–588.

[28] D. Li and C. Zhang, *L^∞ error estimates of discontinuous Galerkin methods for delay differential equations*, Appl. Numer. Math., 2014, 82, 1–10.

[29] D. Li, C. Zhang and H. Qin, *LDG method for reaction–diffusion dynamical systems with time delay*, Appl. Math. Comput., 2011, 217(22), 9173–9181.

[30] Y. Li and Y. Li, *Existence and exponential stability of almost periodic solution for neutral delay BAM neural net works with time-varying delays in leak age terms*, J. Fran. Inst., 2013, 350, 2808–2825.

[31] H. Qin, Q. Zhang and S. Wan, *The continuous Galerkin finite element methods for linear neutral delay differential equations*, Appl. Math. Comput., 2019, 346, 76–85.

[32] D. Schötzau and C. Schwab, *An hp a priori error analysis of the DG time-stepping method for initial value problems*, Calcolo, 2000, 37, 207–232.

[33] D. Schötzau and C. Schwab, *Time discretization of parabolic problems by the hp-version of discontinuous Galerkin finite element method*, SIAM J. Numer. Anal., 2000, 38, 837–875.

[34] T. G. Seidel, S. V. Gurevich and J. Javaloyes, *Conservative solitons and reversibility in time delayed systems*, Phys. Rev. Lett., 2022, 128(8), p.083901.

[35] Q. Tu, Z. Li and L. Yi, *Postprocessing technique of the discontinuous Galerkin method for solving delay differential equations*, J. Appl. Math. Comput., 2024, 70, 3603–3630.

[36] J. E. Velázquez and V. L. Kharitonov, *Lyapunov-Krasovskii functionals for scalar neutral type time delay equations*, Sys. Cont. Lett., 2009, 58, 17–25.

[37] A. G. Vladimirov and D. A. Dolinina, *Neutral delay differential equation model of an optically injected Kerr cavity*, Phys. Rev. E., 2024, 109(2), p.024206.

[38] W. S. Wang, *Numerical Analysis of Nonlinear Neutral Functional Differential Equations*, Ph.D. Thesis, 2008.

[39] W. S. Wang, *Optimal convergence orders of fully geometric mesh one-leg methods for neutral differential equations with vanishing variable delay*, Adv. Comput. Math., 2019, 45(3), 1631–1655.

[40] W. S. Wang and D. Li, *Stability analysis of Runge-Kutta methods for nonlinear neutral Volterra delay-integro-differential equations*, Numer. Math. Theo. Meth. Appl., 2011, 4, 537–561.

[41] W. S. Wang and S. F. Li, *Stability analysis of Θ-method for nonlinear neutral functional differential equations*, SIAM J. Sci. Comput., 2008, 30(4), 2181–2205.

[42] W. S. Wang, S. F. Li and K. Su, *Nonlinear stability of Runge-Kutta methods for neutral delay differential equations*, J. Comput. Appl. Math., 2008, 214, 175–185.

[43] W. S. Wang and C. Zhang, *Analytical and numerical dissipativity for nonlinear generalized pantograph equations*, Disc. Cont. Dyna. Syst., 2011, 29(3), 1245–1260.

[44] W. S. Wang, Y. Zhang and S. F. Li, *Nonlinear stability of one-leg methods for delay differential equations of neutral type*, Appl. Numer. Math., 2008, 58, 122–130.

[45] Z. Wang and Y. Cong, *Delay-dependent stability of a class of Runge-Kutta methods for neutral differential equations*, Numer. Algorithms, 2024, 1–24.

[46] F. Wu, D. Li, J. Wen and J. Duan, *Stability and convergence of compact finite difference method for parabolic problems with delay*, Appl. Math. Comput., 2018, 322, 129–139.

[47] S. Wu and T. Huang, *Schwarz waveform relaxation for a neutral functional partial differential equation model of lossless coupled transmission lines*, SIAM J. Sci. Comput., 2013, 35(2), 1161–1191.

[48] Z. Xie and Z. Zhang, *Uniform superconvergence analysis of the discontinuous Galerkin method for a singularity perturbed problem in 1-D*, Math. Comput., 2010, 79, 35–45.

[49] X. Xu and Q. Huang, *Discontinuous Galerkin time stepping for semilinear parabolic problems with time constant delay*, J. Sci. Comput., 2023, 96, 1–21.

[50] J. Yan and C. W. Shu, *A local discontinuous Galerkin method for KdV type equations*, SIAM J. Numer. Anal., 2002, 40, 769–791.

[51] G. Zhang and X. Dai, *Superconvergence of discontinuous Galerkin method for neutral delay differential equations*, Int. J. Comput. Math., 2021, 98(8), 1648–1662.

[52] H. Zhu and Z. Zhang, *Uniform convergence of the LDG method for a singularly perturbed problem with the exponential boundary layer*, Math. Comput., 2014, 83, 635–663.