

DYNAMICAL ANALYSIS OF A DISCRETE AMENSALISM SYSTEM WITH THE BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE AND FEAR EFFECT

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Abstract This paper proposes a discrete amensalism with the Beddington-DeAngelis functional response and fear effect on the first species. By comparison and study of different bifurcations, the introduction of the Beddington-DeAngelis functional response not only increased the dynamical behaviour of the system, including the emergence of pitchfork bifurcation and fold bifurcation, but also reduced the rate of extinction of the first species. Furthermore, we analyze the influence of the fear effect on the system, specifically focusing on the boundary equilibrium E_2 and the positive equilibrium E_1^* . Our findings reveal that when the second species is in a chaotic state, due to the persistence of the second species, the fear effect may have increased the stability of the first species or accelerated the extinction of the first species; when the second species is stable, the fear effect plays an essential part in maintaining the stability of the first species. Moreover, an appropriate fear effect promotes the coexistence of the first and second species. However, if the fear effect becomes excessively large, it directly results in the extinction of the first species. The discovery further enhances the understanding of the influence generated by amensalism through the fear effect.

Keywords Amensalism, Beddington-DeAngelis functional response, fear effect, bifurcation, Chaos control.

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1. Introduction

In nature, it is uncommon for a single species to persist independently of other species, and the survival of two species in the same natural environment inevitably entails interactions; amensalism is one of those interactions. Amensalism is a type of interaction between two species when one species is constrained in some way, while the other species remains unaffected. The presence of *Pseudomonas taetrolens* and *Lactobacillus casei* together results in an antagonistic relationship, where *Lactobacillus casei* suppresses the growth of *Pseudomonas taetrolens*, regardless of the ideal experimental conditions for its growth. In contrast, *Lactobacillus casei* is able

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to survive and remain productive even in unfavorable environmental conditions. Thus, the interaction does not have a detrimental effect on *Lactobacillus casei* but does impede the growth of *Pseudomonas taetrolens* [16]. Another example of amensalism is the relationship between ibex and weevils that feed on the same bush. Although the ibex consumed large amounts of vegetation and may have ingested the weevils, there is no evidence indicating that the presence of weevils impacts the food available to ibex [17]. For further details on the biological background of amensalism, see [27, 41, 44, 48] and the corresponding references.

A lot of scholars have spent the last two decades researching the dynamic behavior of the amensalism system. Sun [38] initially introduced the following amensalism model:

$$\begin{cases} \frac{dx}{dt} = r_1 x \left(\frac{k_1 - x - cy}{k_1} \right), \\ \frac{dy}{dt} = r_2 y \left(\frac{k_2 - y}{k_2} \right), \end{cases} \quad (1.1)$$

where $x(t)$ and $y(t)$ are the densities of the first and second species at time t , respectively; r_1 and r_2 are the intrinsic growth rates of x and y , respectively; k_1 and k_2 are the environmental capacities of x and y , respectively; c reflects the impact of the second species on the first. The author examined the stability of each positive equilibrium point in system (1.1). Subsequently, in order to enhance comprehension of amensalism, scholars have revised system (1.1) by adding features such as the Allee effect [19, 31, 41, 48], non-monotonic functional response [42, 43], Michaelis-Menten type harvesting [27, 29], and refuge [23, 45, 50].

In the study of amensalism ecosystems, Chen and Xie [3, 45] proposed that the impacts of the second species on the first species followed a linear relationship. However, Lin and Wu [26, 43] argued that using a linear functional response to characterise the impact of the second species on the first species may not be suitable and that it would be rational to incorporate a nonlinear functional response to describe amensalism suppression, so they proposed an amensalism ecosystem with a Holling II functional response of amensalism ecosystems and studied its dynamical behavior. In the study of predator-prey systems, to better portray the interspecific relationship between predators and prey, the Beddington-DeAngelis functional response $cx/(1 + mx + ny)$, introduced by Beddington [2] and DeAngelis et al. [7], has been widely studied by scholars, which is comparable to the widely recognized Holling II type functional response $cx/(1 + mx)$, but in the denominator of has an extra additional term ny to describe mutual interference between predators. Many scholars [6, 34, 37, 46], say that predators must look for food to share and compete with each other. This indicates that the functional response in predator-prey models must be contingent upon the predator. The Beddington-DeAngelis functional response more accurately characterizes the interactions between predators and prey during hunting, and due to its theoretical and practical significance, the functional response with Beddington-DeAngelis DeAngelis functional response has been extensively studied. Influenced by their work [6, 34, 37, 46], Guan and Chen [18] proposed an amensalism system with Beddington-DeAngelis functional response and Allee effect, they studied the existence and persistence of equilibrium points and found that bistability occurs in some cases. In addition, they analysed saddle-node bifurcation and transcritical bifurcations. Lou and Wang [30] studied the global dynamics for a Beddington-DeAngelis amensalism model with strong Allee effect on the second species. Subsequently, Zhou [49] converted the model proposed by Guan and

Chen [18] into a corresponding discrete system. By comparing, they found that when the stability of the positive equilibrium point E_2^* changes, the discrete amensalism system will experience flip bifurcation that does not occur in the continuous system. The flip bifurcation will lead to the emergence of chaos phenomena unique to discrete systems, and the results of this study also reveal that discrete systems have more complex dynamical behaviours than continuous systems.

In nature, predator-prey interactions take on a wide variety of forms. Predators have a strong impact on the number and behavior of prey species due to their hunting strategy. A large number of scholars have worked on developing different predator-prey models to study the complex dynamics of their interactions [15,24,39]. In traditional predator-prey models, they directly consider that predators affect prey populations through direct predation. However, the presence of predators causes prey to perceive risks and respond in an anti-predatory manner, such as a decrease in foraging frequency and a decrease in the number of offspring foraging, which can lead to a decrease in the density of prey populations, this phenomenon is known as the fear effect [28]. In an experiment with song sparrows, Zanette et al. [47] observed that simply playing the sound of a predator resulted in a 40% decrease in song sparrow reproduction. In a different independent investigation, Elliott et al. [14] observed that *Drosophila melanogaster* exhibited defensive behavior in response to the scent of praying mantises, and displayed decreased levels of activity during both the breeding and non-breeding periods. The presence of garter snakes reduces the amount of time salamanders spend searching for food due to the perceived threat of being preyed upon [33].

In 2016, Wang et al. [40] first investigated a predator-prey model with fear effects, in which linear functional responses and Holling type II functional responses were used:

$$\begin{cases} \frac{dx}{dt} = \frac{ax}{1+ky} - dx - bx^2 - cxy, \\ \frac{dy}{dt} = \mu cxy - ey, \end{cases} \quad (1.2)$$

$$\begin{cases} \frac{dx}{dt} = \frac{ax}{1+ky} - dx - bx^2 - \frac{cxy}{1+mx}, \\ \frac{dy}{dt} = \frac{\mu cxy}{1+mx} - ey, \end{cases} \quad (1.3)$$

where $x(t)$ and $y(t)$ denote prey and predator populations, respectively, d is the natural mortality rate of prey, b is the mortality rate of intraspecific competition, e denotes the predator's death rate, k represents the fear parameter, c represents the predator's maximal predation rate, and μ represents the rate at which prey biomass is converted to predator biomass. Analyzing system (1.2), they found that the fear effect does not change the dynamic behavior of the system, and if positive equilibria exist, it must be globally asymptotically stable. According to the system (1.3), they found that the stability of the positive equilibrium point of the system is related to the birth rate of prey, i.e., the positive equilibrium point is not always globally asymptotically stable. Furthermore, the scholars discovered that system (1.3) exhibits more intricate dynamic behaviors and phenomena compared to system (1.2). They also confirmed that Hopf bifurcations induced by fear effects can be both supercritical and subcritical, which contrasts with the fact that Hopf bifurcations induced by the classical predator-prey model, which ignores fear

effects, can only be supercritical. This finding suggests that both fear effects and non-monotonic functional response are essential factors to consider in further studies. Subsequently, Din et al. [11] overcome the limitations of previous models by incorporating fear effect into the model and further explore the impact of fear on the stability of predator-prey populations and the possibility of chaotic dynamics. Given the importance of the prey population level, Ishaque et al. [20] introduced fear factors into a discrete-time predator-prey model and systematically studied its impact on the complex dynamic behavior of the model.

Discrete models are more relevant than continuous models because statistics of biological sample data are usually generated based on discrete time intervals rather than continuous time. It has been shown that discrete-time models are not only able to exhibit more complex dynamic behaviour, but also provide more efficient numerical simulation results [8, 10]. Kundu et al. [22] discretized model (1.2) through the segmented constant parameter method and analyzed the discrete model corresponding to model (1.2), their results showed that when predator-prey interactions follow a linear functional response, the fear effect improves the stability of the predator-prey relationship. This is quite different from the continuous model (1.2) in which the fear effect has no effect on dynamical behavior, indicating that the discrete-time model presents a richer dynamical representation than the continuous-time model. Zhou et al. [50] discretized model (1.1) and found that the system has flip bifurcations. Many scholars have studied various discrete-time population models with fear effects. For example, Din and Zulfiqar [12] examined the qualitative dynamics of a discrete predator-prey model influenced by the fear effect. Chen et al. [5] proposed an allelopathic phytoplankton competition ODE model influenced by the fear effect based on natural biological phenomena. Din et al. [9] proposed a predator-prey model that incorporates the effects of fear on prey populations by focusing on non-overlapping generations. Banerjee et al. [1] proposed a model of the impact of prey's fear cost and anti-predation behavior on predation, as well as competition between prey species and competition within predators.

As research deepens, it can be found that the fear effect is not limited to predator-prey interactions, but also exists in amensalism system. Xi et al. [44] studied the population of grassland caterpillars that cocooned, taking into account the presence or absence of grasshoppers. The results revealed that the caterpillars' fear of grasshoppers resulted in a decrease in species diversity. Ogada [35] emphasized that the presence of huge herbivorous animals significantly impacts the behavior and growth of birds. This influence is notably evident in terms of hunting habits, rate of growth, chances of survival, reproductive capacity, and availability of food supplies. Consequently, fear of large herbivores in East African savanna ecosystems has led to a decline in bird birth rates of about 30%. Influenced by the above practical background, Li et al. [25] proposed the following discrete model:

$$\begin{cases} x_{n+1} = x_n \exp \left(\frac{e_1}{1 + ky_n} - e_2 - x_n - cy_n \right), \\ y_{n+1} = y_n \exp(a_2 - y_n), \end{cases} \quad (1.4)$$

they found that the system has at most three boundary equilibrium points and one positive equilibrium point, and will produce transcritical bifurcation and flip bifurcations. Due to the rationality and researchability of non-monotonic functional

response, Zhu and Chen [51] proposed the following continuum model:

$$\begin{cases} \frac{dx}{dt} = x \left(\frac{a_1}{1+k_1y} - b_1 - c_1x_n - \frac{d_1y}{1+mx+ny} \right), \\ \frac{dy}{dt} = y(a_2 - b_2y). \end{cases} \quad (1.5)$$

They observed that under certain parameter conditions, when the intensity of the fear effect is below a particular threshold value, as the fear effect intensifies, it will solely decrease the density of the population of the first species without affecting the extinction or survival of that population. However, if the level of fear becomes too high, it will expedite the extinction of the population of the first species. So far, scholars have not proposed or conducted research on a discrete amensalism model with fear effects and Beddington-DeAngelis functional responses. This work aims to study the model obtained by discretizing model (1.5).

First, with the transform $x = \frac{\bar{x}}{m}$, $y = \frac{\bar{y}}{n}$, $t = \frac{n\bar{t}}{b_2}$, after removing the bars, system (1.5) is obtained

$$\begin{cases} \frac{dx}{dt} = x \left(\frac{\alpha}{1+ky} - \delta - \beta x - \frac{cy}{1+x+y} \right), \\ \frac{dy}{dt} = y(\gamma - y), \end{cases} \quad (1.6)$$

where $\alpha = \frac{a_1n}{b_2}$, $k = \frac{k_1}{n}$, $\delta = \frac{b_1n}{b_2}$, $\beta = \frac{c_1n}{b_2m}$, $\gamma = \frac{a_2n}{b_2}$ and $c = \frac{d_1}{b_2}$. With the help of the piecewise constant argument method introduced by Jiang and Rogers [21], the discrete system can be derived as follows:

$$\begin{cases} x_{n+1} = x_n \exp \left(\frac{\alpha}{1+ky_n} - \delta - \beta x_n - \frac{cy_n}{1+x_n+y_n} \right), \\ y_{n+1} = y_n \exp(\gamma - y_n), \end{cases} \quad (1.7)$$

the initial situation of system (1.7) is $R_+^2 \triangleq \{(x, y) | x > 0, y > 0\}$.

The remainder of this work is organized as follows. In Section 2 and 3, we talk about the stability, presence, and global attractivity of equilibria for system (1.7). Then, Section 4 is the bifurcation analysis. Section 5 is specifically dedicated to the control of chaos. The study is concluded with a summary and discussion.

2. Existence and stability of equilibrium points of system (1.7)

The equation that satisfies the equilibrium $E(x, y)$ of system (1.7) is as follows:

$$x = x \exp \left(\frac{\alpha}{1+ky} - \delta - \beta x - \frac{cy}{1+x+y} \right), \quad y = y \exp(\gamma - y).$$

Obviously, system (1.7) consistently exhibits three boundary fixed points, namely $E_0(0, 0)$, $E_1((\alpha - \delta)/\beta, 0)$, and $E_2(0, \gamma)$. Furthermore, to determine the potential positive equilibrium $E^*(x^*, \gamma)$, we just need to examine the positive solutions of the following equation:

$$F(x) = Ax^2 + Bx + C,$$

where

$$\begin{aligned} A &= \beta\gamma k + \beta, \\ B &= \beta\gamma^2 k + (k\beta + \beta + k\delta)\gamma + \beta - \alpha + \delta, \\ C &= k(c + \delta)\gamma^2 + (k\delta + \delta + c - \alpha)\gamma - \alpha + \delta. \end{aligned}$$

We can observe that $B = 0$ if and only if $k = \frac{\alpha - \beta - \delta - \beta\gamma}{\beta\gamma^2 + \beta\gamma + \delta\gamma} := k^*$ and $\alpha > \beta + \delta + \beta\gamma$. Denote the discriminant of $F(x)$ as $\Delta = B^2 - 4AC$. By calculation, we obtain:

$$\Delta = (\beta\gamma^2 k + k\beta\gamma + \beta\gamma - k\delta\gamma + \beta + \alpha - \delta)^2 - 4c\gamma\beta(k\gamma + 1)^2.$$

Obviously, if

$$0 < c \leq \frac{(\beta\gamma^2 k + k\beta\gamma + \beta\gamma - k\delta\gamma + \beta + \alpha - \delta)^2}{4\gamma\beta(k\gamma + 1)^2} := c^*,$$

then $F(x)$ possesses two roots $x_1^* = \frac{-B + \sqrt{\Delta}}{2A}$ and $x_2^* = \frac{-B - \sqrt{\Delta}}{2A}$. When $(-B)^2 = \Delta$, we can calculate that

$$c = \frac{(\gamma + 1)(\alpha - \delta - \delta\gamma k)}{\gamma(k\gamma + 1)} := c^{**}.$$

In order to ensure that $c^{**} > 0$, k must satisfy $k < \frac{\alpha - \delta}{\delta\gamma} := k^{**}$, $\alpha > \delta$. Hence, if $c < c^{**}$, $(-B)^2 < \Delta$, if $c > c^{**}$, $(-B)^2 > \Delta$. Additionally, we can simply confirm $c^{**} \leq c^*$, $k^* < k^{**}$ and $c^{**} = c^*$ if and only if $(-B) = 0$.

Given all of the above, it may be inferred that:

- (1) When $0 < c < c^{**}$, $k < k^{**}$, $\alpha > \delta$, there is $x_1^* > 0$ and $x_2^* < 0$ holds;
- (2) When $c = c^{**}$,
 - (a) $x_1^* > 0$, $x_2^* = 0$ if $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
 - (b) $x_1^* = 0$, $x_2^* < 0$ if $k^* < k < k^{**}$ and $\alpha > \beta + \delta + \beta\gamma$;
 - (c) $x_1^* = x_2^* = 0$ if $k = k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
- (3) When $c^{**} < c < c^*$,
 - (a) $x_1^* > 0$, $x_2^* > 0$ if $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
 - (b) $x_1^* < 0$, $x_2^* < 0$ if $k^* < k < k^{**}$ and $\alpha > \beta + \delta + \beta\gamma$;
- (4) When $c = c^*$, i.e., $\Delta = 0$. Consequently, the first equation of system (1.7) has a unique positive root x_3^* , and $x_3^* > 0$ to hold is that $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
- (5) When $c > c^*$, i.e., $\Delta < 0$. then the first equation of system (1.7) has no root.

The results of the above discussion are summarised below:

Theorem 2.1. *System (1.7) always has the boundary equilibria $E_0(0, 0)$, $E_1(\frac{\alpha - \delta}{\beta}, 0)$, and $E_2(0, \gamma)$. Regarding positive equilibria, the subsequent outcomes are as follows:*

- (1) When $0 < c < c^{**}$, $k < k^{**}$ and $\alpha > \delta$, system (1.7) contains a positive fixed point $E_1^*(x_1^*, \gamma)$;
- (2) When $c = c^{**}$, system (1.7) contains a positive fixed point $E_1^*(x_1^*, \gamma)$ if $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$, and has no positive equilibrium if $k \geq k^*$ and $\alpha > \beta + \delta + \beta\gamma$;

- (3) When $c^{**} < c < c^*$, system (1.7) contains two positive fixed points $E_1^*(x_1^*, \gamma)$ and $E_2^*(x_2^*, \gamma)$ if $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$, and has no positive equilibrium if $k > k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
- (4) When $c = c^*$, system (1.7) contains a positive fixed point $E_3^*(x_3^*, \gamma)$ if $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$, and has no positive equilibrium if $k \geq k^*$ and $\alpha > \beta + \delta + \beta\gamma$;
- (5) When $c > c^*$, then system (1.7) has no positive equilibrium.

Following that, we investigate the local stability of the fixed points of system (1.7). The Jacobian matrix of system (1.7) at $E(x, y)$ is

$$J(E) = \begin{pmatrix} \left(1 - \beta x + \frac{cxy}{(1+x+y)^2}\right)M & x\left(-\frac{\alpha k}{(1+ky)^2} - \frac{c}{1+x+y} + \frac{cy}{(1+y+x)^2}\right)M \\ 0 & (1-y)\exp(\gamma - y) \end{pmatrix}.$$

With $M = \exp\left(\frac{\alpha}{ky+1} - \delta - \beta x - \frac{cy}{1+x+y}\right)$, let λ_1 and λ_2 are two eigenvalues of $J(E)$.

Theorem 2.2. *The fixed point $E_0(0, 0)$ and $E_1((\alpha - \delta)/\beta, 0)$ of system (1.7) can be summarized as follows:*

- (1) $E_0(0, 0)$ is always a source.
- (2) $E_1((\alpha - \delta)/\beta, 0)$ is always a saddle.

Proof. The Jacobian matrices of $E_0(0, 0)$ and $E_1((\alpha - \delta)/\beta, 0)$ are as follows:

$$J(E_0) = \begin{pmatrix} e^{\alpha-\delta} & 0 \\ 0 & e^\gamma \end{pmatrix}, \quad J(E_1) = \begin{pmatrix} 1 - \alpha + \delta & \frac{(\alpha - \delta)(-\alpha k - \frac{c\beta}{\alpha + \beta - \delta})}{\beta} \\ 0 & e^\gamma \end{pmatrix},$$

first of all, by observing $J(E_0)$, it can be seen that it has two eigenvalues, namely $\lambda_1 = e^{\alpha-\delta} > 1$, $\lambda_2 = e^\gamma > 1$, then, E_0 is always a source. The discussion of E_1 is similar to the above, so the conclusion of Theorem 2.2 holds. $\square$

Theorem 2.3. $E_2(0, \gamma)$ of system (1.7) is

- (a) a sink if $k > \tilde{k}$, $\alpha > \delta + \frac{c\gamma}{1+\gamma}$ and $0 < \gamma < 2$;
- (b) a source if $0 < k < \tilde{k}$, $\alpha > \delta + \frac{c\gamma}{1+\gamma}$ and $\gamma > 2$;
- (c) non-hyperbolic if $k = \tilde{k}$, $\alpha > \delta + \frac{c\gamma}{1+\gamma}$ or $\gamma = 2$;
- (d) a saddle otherwise.

Proof. At E_2 , we have

$$J(E_2) = \begin{pmatrix} e^{\frac{\alpha}{1+k\gamma} - \delta - \frac{c\gamma}{1+\gamma}} & 0 \\ 0 & 1 - \gamma \end{pmatrix},$$

the eigenvalues of $J(E_2)$ are $e^{\frac{\alpha}{1+k\gamma}-\delta-\frac{c\gamma}{1+\gamma}}$ and $1-\gamma$. By calculation, we obtain:

$$e^{\frac{\alpha}{1+k\gamma}-\delta-\frac{c\gamma}{1+\gamma}} \begin{cases} > 1, & \text{if } 0 < k < \tilde{k} := \frac{\alpha - \delta - \frac{c\gamma}{1+\gamma}}{\gamma(\delta + \frac{c\gamma}{1+\gamma})}, \alpha > \delta + \frac{c\gamma}{1+\gamma}, \\ = 1, & \text{if } k = \tilde{k}, \alpha > \delta + \frac{c\gamma}{1+\gamma}, \\ \in (0, 1), & \text{if } k > \tilde{k}, \alpha > \delta + \frac{c\gamma}{1+\gamma}, \end{cases}$$

and

$$1-\gamma \begin{cases} \in (-1, 0), & \text{if } 0 < \gamma < 2, \\ = -1, & \text{if } \gamma = 2, \\ < -1, & \text{if } \gamma > 2, \end{cases} \quad (2.1)$$

thus, Theorem 2.3 is proved by a simple analysis. $\square$

Based on the conclusion of Theorem 2.3, system (1.7) undergoes flip bifurcation when the parameters satisfy the range at

$$F_B := \{(\alpha, \beta, c, \gamma, \delta, k) : \gamma = 2, k \neq \tilde{k}, \alpha > \delta + c\gamma/(1+\gamma), \beta > 0, c > 0, \delta > 0, k > 0\}.$$

Figure 1 displays the bifurcation diagram where γ serves as the bifurcation param-

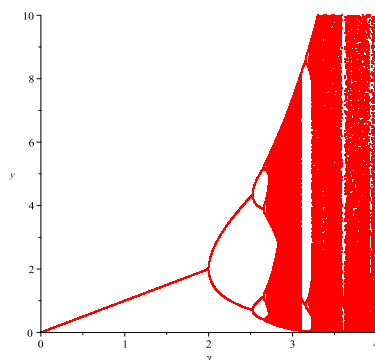


Figure 1. Flip bifurcation image of system (1.7) at $E_2(0, \gamma)$ with $\alpha = 6$, $c = 1$, $\delta = 1$, $\beta = 1$ and $k = 1$.

eter. When $k = 1$, Figure 1 also shows periodic-2, 4, 8 orbits, the second species is stable if $0 < 2 < \gamma$ and it is unstable if $\gamma > 2$. We have observed that when the birth rate is much higher than the death rate, the second population may grow faster than the carrying capacity of the environment, leading to more intense competition between populations, excessive resource consumption, and ultimately instability or chaos in the population. This phenomenon is completely normal in ecology.

Now, we will focus on the positive fixed points of system (1.7). The Jacobian matrix of $E_i^*(x_i^*, \gamma)$ ($i = 1, 2, 3$) is

$$J(E_i^*) = \begin{pmatrix} J_i & x_i \left(-\frac{\alpha k}{(1+k\gamma)^2} - \frac{c}{1+x_i+\gamma} + \frac{cy}{(1+\gamma+x_i)^2} \right) \\ 0 & 1-\gamma \end{pmatrix},$$

where

$$J_i = 1 + \delta - \frac{\alpha}{1+ky} + \frac{c\gamma(1+\gamma+2x_i^*)}{(1+\gamma+x_i^*)^2}.$$

For $J(E_i^*)$, the eigenvalues are J_i and $1 - \gamma$. After computation, $J_3 = 1$. Thus, the stability of E_3^* can be obtained as follows:

Theorem 2.4. *Suppose that $c = c^*$, $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$. Then, the eigenvalues of $J(E_3^*)$ are 1 and $1 - \gamma$. Therefore, E_3^* is always non-hyperbolic.*

It is important to observe that J_i is a function that decreases rigorously as x_i^* increases, as shown by the derivative $\frac{dJ_i}{dx_i^*} = -\frac{2c\gamma x_i^*}{(1+\gamma+x_i^*)^3} < 0$. Furthermore, it can be verified that $x_2^* < x_3^* < x_1^*$. Subsequently, the stability of E_2^* can be obtained as follows:

Theorem 2.5. *Suppose that $c^{**} < c < c^*$, $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$, then E_2^* can be classified as a source, a saddle, non-hyperbolic when $\gamma > 2$, $0 < \gamma < 2$, and $\gamma = 2$, respectively.*

Theorem 2.6. *Suppose the existence of the positive equilibrium E_1^* in system (1.7).*

- (i) *If $A_1 \geq 0$, then E_1^* can be categorised as a sink, a saddle, and non-hyperbolic when $0 < \gamma < 2$, $\gamma > 2$, and $\gamma = 2$, respectively.*
- (ii) *If $A_1 < 0$, then the topology divisions of $E_1^*(x_1^*, \gamma)$ can be found in Table 1.*

Table 1. Topological types of E_1^* in system (1.7).

Conditions		Properties
$0 < c \leq \tilde{c}$	$0 < \gamma < 2$	saddle
	$\gamma = 2$	non-hyperbolic
	$\gamma > 2$	source
$k < k^{**}$	$0 < \beta < \beta^*$	$0 < \gamma < 2$ saddle
		$\gamma = 2$ non-hyperbolic
		$\gamma > 2$ source
	$\beta = \beta^*$ non-hyperbolic	
	$\beta^* < \beta$	$0 < \gamma < 2$ sink
		$\gamma = 2$ non-hyperbolic
		$\gamma > 2$ saddle
$k < k^*$ $\beta_1 > \beta^*$	$0 < \beta < \beta^*$	$0 < \gamma < 2$ saddle
		$\gamma = 2$ non-hyperbolic
		$\gamma > 2$ source
	$\beta = \beta^*$ non-hyperbolic	
	$\beta^* < \beta < \beta_1$	$0 < \gamma < 2$ sink
		$\gamma = 2$ non-hyperbolic
		$\gamma > 2$ saddle
$k < k^*$ $\beta_1 \leq \beta^*$	$0 < \beta < \beta_1 \leq \beta^*$	$0 < \gamma < 2$ saddle
		$\gamma = 2$ non-hyperbolic
		$\gamma > 2$ source

Proof. According to Theorem 2.1, the existence of E_1^* is limited to two cases: when $0 < c < c^{**}$, $k < k^{**}$, and $\alpha > \delta$; or when $c^{**} \leq c < c^*$, $k < k^*$, and

$\alpha > \beta + \delta + \beta\gamma$. Since the relationship between $\lambda_2 = 1 - \gamma$ and -1 is easy to derive, we focus on analyzing the relationship between J_1 and -1 . According to

$$J_1 - (-1) = \frac{H(x_1^*)}{(1 + \gamma + x_1^*)^2},$$

where

$$H(x_1^*) = A_1 x_1^{*2} + 2T x_1^* + (1 + \gamma)T,$$

with $A_1 = 2 + \delta - \frac{\alpha}{1+\gamma k}$, $T = A_1(1 + \gamma) + c\gamma$. The discriminant of H is $\Delta_1 = 4c\gamma T$. To identify the value of H and the correlation between J_1 and -1 , we consider two cases.

Case 1. $A_1 \geq 0$. $H(x_1^*) > 0$ and hence $J_1 \in (-1, 1)$.

Case 2. $A_1 < 0$. At this point, the sign of T needs to be discussed.

- (1) When $T = 0$, i.e., $c = \tilde{c} = (\frac{\alpha}{\gamma k+1} - 2 - \delta)(1 + \gamma)/\gamma$. At this time, $H(x_1^*) < 0$, so $|\lambda_1| > 1$.
- (2) When $T < 0$, i.e., $0 < c < \tilde{c}$. At this time, $H(x_1^*) < 0$, so $|\lambda_1| > 1$.
- (3) When $T > 0$, i.e., $c > \tilde{c}$. Due to the existence of E_1^* , it is only necessary to study the cases in the two ranges of $0 < c < c^{**}$, $k < k^{**}$, $\alpha > \delta$ and ($c^{**} \leq c < c^*$ and $k < k^*$ and $\alpha > \beta + \delta + \beta\gamma$). Then

$$H(x_1^*) \begin{cases} > 0, & \text{if } 0 \leq x_1^* < \tilde{x} \triangleq \frac{T + \sqrt{c\gamma T}}{\frac{\alpha}{\gamma k+1} - 2 - \delta}, \\ = 0, & \text{if } x_1^* = \tilde{x}, \\ < 0, & \text{if } x_1^* > \tilde{x}. \end{cases}$$

(i) If $\tilde{c} < c < c^{**}$, $k < k^{**}$, $\alpha > \delta$, we have:

$$x_1^* \begin{cases} > \tilde{x}, & \text{if } 0 < \beta < \beta^* \triangleq \frac{(\frac{\alpha}{\gamma k+1} - 2 - \delta) \left[(\frac{\alpha}{\gamma k+1} - \delta) \sqrt{c\gamma T} + 2c\gamma \right]}{2c\gamma T + \sqrt{c\gamma T}(T + c\gamma)}, \\ = \tilde{x}, & \text{if } \beta = \beta^*, \\ < \tilde{x}, & \text{if } \beta^* < \beta. \end{cases}$$

(ii) If $c^{**} \leq c < c^*$, $k < k^*$, $\beta < \frac{\alpha - \delta}{\gamma + 1} := \beta_1$, the relationship between β_1 and β^* needs to be discussed:

(1) if $\beta_1 > \beta^*$, we have:

$$x_1^* \begin{cases} > \tilde{x}, & \text{if } 0 < \beta < \beta^*, \\ = \tilde{x}, & \text{if } \beta = \beta^*, \\ < \tilde{x}, & \text{if } \beta^* < \beta < \beta_1. \end{cases}$$

(2) if $\beta_1 \leq \beta^*$, we have $x_1^* > \tilde{x}$.

The proof of Theorem 2.6 is completed. $\square$

3. Global attractivity of system (1.7)

3.1. Global attractivity of boundary equilibrium

We will analyze the global attractivity of E_2 using Lemma 3-5 and Lemma 6 from the reference cited as [4].

Theorem 3.1. *Suppose that $k > k^{**}$, $\alpha > \delta$, and $0 < \gamma < 2$ hold, $E_2(0, \gamma)$ is globally attractive, i.e.,*

$$\lim_{n \rightarrow \infty} x(n) = 0, \lim_{n \rightarrow \infty} y(n) = \gamma.$$

Proof. Lemma 4 in [4] states that if the condition $0 < \gamma < 2$ is satisfied, then

$$\lim_{n \rightarrow \infty} y(n) = \gamma.$$

Therefore, for sufficiently small ε , there exists a N_1 , such that for all $n > N_1$, for all

$$M_0 := \gamma - \varepsilon < y(n) < \gamma + \varepsilon := M_2. \quad (3.1)$$

Since

$$x(n+1) \leq x(n) \exp(\alpha - \delta - \beta x(n)),$$

then combined with Lemma 6 in [4], we get

$$\limsup_{n \rightarrow +\infty} x(n) \leq \frac{\exp(\alpha - \delta - 1)}{\beta}.$$

Therefore, for sufficiently small ε , there exists a N_2 , such that for all $n > N_2$, for all

$$x(n) \leq \frac{\exp(\alpha - \delta - 1)}{\beta} + \varepsilon := M_1. \quad (3.2)$$

Next, we prove that $\lim_{n \rightarrow \infty} x(n) = 0$. Based on system (1.7), for any $i \in \mathbb{N}$, it holds true that

$$\begin{aligned} \ln \frac{x(i+1)}{x(i)} &= \frac{\alpha}{1+ky(i)} - \delta - \beta x(i) - \frac{cy(i)}{1+x(i)+y(i)} \\ &\leq \frac{\alpha}{1+kM_0} - \delta - \beta x(i) - \frac{cy(i)}{1+M_1+M_2}, \end{aligned} \quad (3.3)$$

$$\ln \frac{y(i+1)}{y(i)} = \gamma - y(i). \quad (3.4)$$

Since $k > \frac{\alpha-\delta}{\delta\gamma} > \frac{\alpha-(\frac{c\gamma}{1+M_2+M_1}+\delta)}{M_0(\delta+\frac{c\gamma}{1+M_2+M_1})}$, i.e.,

$$\frac{\frac{\alpha}{1+kM_0} - \delta}{\gamma} < \frac{c}{1+M_1+M_2},$$

the following inequality holds:

$$\frac{\frac{\alpha}{1+kM_0} - \delta}{\gamma} < \frac{q}{p} < \frac{c}{1+M_1+M_2},$$

where p and q are positive numbers, then,

$$\frac{cp}{1 + M_1 + M_2} - q > 0, \quad (3.5)$$

and there exist $\delta > 0$ such that

$$p\left(\frac{\alpha}{1 + kM_0} - \delta\right) - q\gamma < -\delta < 0. \quad (3.6)$$

From (3.3)-(3.6), we have

$$\begin{aligned} & p \ln \frac{x(i+1)}{x(i)} - q \ln \frac{y(i+1)}{y(i)} \\ & \leq \left(p\left(\frac{\alpha}{1 + kM_0} - \delta\right) - q\gamma\right) - \left(p\beta x(i) + \left(\frac{cp}{1 + M_2 + M_2} - q\right)y(i)\right) \\ & \leq p\left(\frac{\alpha}{1 + kM_0} - \delta\right) - q\gamma \\ & < -\delta \\ & < 0. \end{aligned}$$

Summing up the above formulas, we obtain:

$$\begin{aligned} & \sum_{i=1}^n \left(p \ln \frac{x(i+1)}{x(i)} - q \ln \frac{y(i+1)}{y(i)}\right) \\ & = p \ln \frac{x(n)}{x(0)} - q \ln \frac{y(n)}{y(0)} \\ & < -\delta n, \end{aligned}$$

then

$$x(n) < x(0) \left(\frac{y(n)}{y(0)}\right)^{\frac{q}{p}} \exp\left(-\frac{\delta}{p}n\right). \quad (3.7)$$

According to inequality (3.1), we obtain:

$$x(n) < x(0) \left(\frac{M_2}{y(0)}\right)^{\frac{q}{p}} \exp\left(-\frac{\delta}{p}n\right), \forall n \geq N_2.$$

Thus,

$$\lim_{n \rightarrow \infty} x(n) = 0, \quad (3.8)$$

so Theorem 3.1 has been demonstrated. $\square$

3.2. Global attractivity of $E_1^*(x_1^*, \gamma)$

Theorem 3.2. Suppose that

$$\begin{aligned} & \tilde{c} < c < c^{**}, k < k^{**}, \beta > \beta^*, 0 < \gamma < 1, \\ & \delta < \alpha < 1 + \delta, \frac{\alpha}{1 + G_1^y k} - \delta - cG_1^y > 0 \end{aligned} \quad (3.9)$$

hold, where $G_1^y = \gamma + \varepsilon$, $E_1^*(x_1^*, \gamma)$ is globally attractive, i.e.,

$$\lim_{n \rightarrow \infty} x(n) = x_1^*, \lim_{n \rightarrow \infty} y(n) = \gamma. \quad (3.10)$$

Proof. Let

$$\begin{aligned} Q_1 &= \limsup_{n \rightarrow +\infty} x(n), & P_1 &= \liminf_{n \rightarrow +\infty} x(n), \\ Q_2 &= \limsup_{n \rightarrow +\infty} y(n), & P_2 &= \liminf_{n \rightarrow +\infty} y(n). \end{aligned}$$

Thus, to prove (3.10), we should prove the following equation:

$$Q_1 = P_1 = x_1^*, \quad Q_2 = P_2 = \gamma. \quad (3.11)$$

If there exist sequences $\{G_n^x\}, \{G_n^y\}, \{F_n^x\}, \{F_n^y\}$ satisfying:

$$F_n^x \leq P_1 \leq Q_1 \leq G_n^x, \quad F_n^y \leq P_2 \leq Q_2 \leq G_n^y, \quad n > 0,$$

and

$$\lim_{n \rightarrow +\infty} G_n^x = \lim_{n \rightarrow +\infty} F_n^x = x_1^*, \quad \lim_{n \rightarrow +\infty} G_n^y = \lim_{n \rightarrow +\infty} F_n^y = \gamma,$$

then Equation (3.11) holds up due to the convergence of the limit. Furthermore, we generate these four sequences using an iterative approach:

Iteration 1(i). We aim is to find $G_1^x, G_1^y > 0$, such that $Q_1 \leq G_1^x$ and $Q_2 \leq G_1^y$.

Scaling the first equation of system (1.7), it holds:

$$x(n+1) \leq x(n) \exp(\alpha - \delta - \beta x(n)), \quad n = 0, 1, 2, \dots \quad (3.12)$$

Combined with the following equation:

$$u(n+1) = u(n) \exp(\alpha - \delta - \beta u(n)), \quad (3.13)$$

where $u(n)$ is an arbitrary solution of the system, and $u(2) = x(2)$ always holds. Since $\delta < \alpha < 1 + \delta$, according to Lemma 4 in [4] that formula (3.13) satisfies $\lim_{n \rightarrow \infty} u(n) = \frac{\alpha - \delta}{\beta}$ and for all $n \geq 2$, $u(n) \leq \frac{1}{\beta}$ holds true. From the definition of Lemma 3 in [4], at this time, $f(u) = u \exp(\alpha - \delta - \beta u)$ does not decrease monotonically on $u \in (0, \frac{1}{\beta}]$, combined with Lemma 5 in [4], we can conclude that if $n \geq 2$, then $x(n) \leq u(n)$ is established. Therefore, we have

$$Q_1 = \limsup_{n \rightarrow +\infty} x(n) \leq \lim_{n \rightarrow +\infty} u(n) = \frac{\alpha - \delta}{\beta}.$$

Thus, for sufficiently small $\varepsilon > 0$, we have an integer $N_1 > 2$ which means if $n \geq N_1$, $x(n) \leq \frac{\alpha - \delta}{\beta} + \varepsilon := G_1^x$.

Taken from the second equation of system (1.7), we have:

$$y(n+1) \leq y(n) \exp(\gamma - y(n)), \quad n \geq N_1. \quad (3.14)$$

Combined with the following equation:

$$u(n+1) = u(n) \exp(\gamma - u(n)),$$

it is known that $0 < \gamma \leq 1$, so from Lemma 4 in [4], $n \geq N_1$, $u(n) \leq 1$ is established. According to the definition of Lemma 3 in [4], $f(u) = u \exp(\gamma - u)$ does not decrease

monotonically on $u \in (0, 1]$, combined with Lemma 5 in [4], we can conclude that if $n \geq N_1$, the $y(n) \leq u(n)$ holds. Consequently, we obtain

$$Q_2 = \limsup_{n \rightarrow +\infty} y(n) \leq \lim_{n \rightarrow +\infty} u(n) = \gamma.$$

Therefore, for ε above, if $n \geq N_2$, and N_2 is an integer, satisfying $N_2 > N_1$, then $y(n) \leq \gamma + \varepsilon := G_1^y$.

Iteration 1(ii). We aim is to find $F_1^x, F_1^y > 0$, such that $P_1 \geq F_1^x$ and $P_2 \geq F_1^y$.

Scaling the first formula of system (1.7), it holds:

$$x(n+1) \geq x(n) \exp \left(\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y - \beta x(n) \right), \quad n \geq N_2. \quad (3.15)$$

Combined with the following equation:

$$u(n+1) = u(n) \exp \left(\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y - \beta u(n) \right),$$

since $\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y > 0$, we can obtain from Lemma 4 in [4] that when $n \geq N_2$, $u(n) \leq \frac{1}{\beta}$ holds true. According to the definition of Lemma 3 in [4], $f(u) = u \exp(\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y - \beta u)$ does not decrease monotonically on $u \in (0, \frac{1}{\beta}]$, combined with Lemma 5 in [4], we can see that for any $n \geq N_2$, there exists $x(n) \geq u(n)$. Therefore, we obtain

$$P_1 = \liminf_{n \rightarrow +\infty} x(n) \geq \lim_{n \rightarrow +\infty} u(n) = \frac{\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y}{\beta}.$$

Hence, for sufficiently small $\varepsilon > 0$, if $n \geq N_3$, and N_3 is an integer, satisfying $N_3 > N_2$, then $x(n) \geq \frac{\frac{\alpha}{1+kG_1^y} - \delta - cG_1^y}{\beta} - \varepsilon := F_1^x$.

Scaling the second formula of system (1.7), it holds:

$$y(n+1) \geq y(n) \exp \left(\gamma - y(n) \right), \quad n \geq N_3, \quad (3.16)$$

combine with the relevant proofs above, it can be seen that:

$$P_2 = \liminf_{n \rightarrow +\infty} y(n) \geq \lim_{n \rightarrow +\infty} u(n) = \gamma.$$

Therefore, for ε above, if $n \geq N_4$, and N_4 is an integer, satisfying $N_4 > N_3$, then $y(n) \geq \gamma - \varepsilon := F_1^y$.

Iteration 2(i). We aim to find that $G_2^x \leq G_1^x, G_2^y \leq G_1^y$, such that $Q_1 \leq G_2^x, Q_2 \leq G_2^y$.

Scaling the first formula of system (1.7), it holds:

$$x(n+1) \leq x(n) \exp \left(\frac{\alpha}{1+kF_1^y} - \delta - \frac{cF_1^y}{1+G_1^y+G_1^x} - \beta x(n) \right), \quad n \geq N_4, \quad (3.17)$$

since $G_1^y > F_1^y$, we obtain $0 < \frac{\alpha}{1+kF_1^y} - \delta - \frac{cF_1^y}{1+G_1^y+G_1^x} \leq 1$, combined with Lemma 5 in [4], it is established:

$$Q_1 = \limsup_{n \rightarrow +\infty} x(n) \leq \frac{\frac{\alpha}{1+kF_1^y} - \delta - \frac{cF_1^y}{1+G_1^y+G_1^x}}{\beta}.$$

Therefore, for sufficiently small $\varepsilon > 0$, if $n \geq N_5$, and N_5 is an integer, satisfying $N_5 > N_4$, then $x(n) \leq \frac{\frac{\alpha}{1+kF_1^y} - \delta - \frac{cF_1^y}{1+G_1^y+G_1^x}}{\beta} + \frac{\varepsilon}{2} := G_2^x \leq G_1^x$.

Given by the second formula of system (1.7), it holds:

$$y(n+1) \leq y(n) \exp(\gamma - y(n)), \quad n \geq N_5, \quad (3.18)$$

we can obtain the following:

$$Q_2 = \limsup_{n \rightarrow +\infty} y(n) \leq \gamma.$$

Therefore, if $n \geq N_6$, and N_6 is an integer, satisfying $N_6 > N_5$, then $y(n) \leq \gamma + \frac{\varepsilon}{2} := G_2^y \leq G_1^y$.

Iteration 2(ii). We aim to find that $F_2^x \geq F_1^x$, $F_2^y \geq F_1^y$, such that $P_1 \geq F_2^x$, $P_2 \geq F_2^y$.

Scaling formula of system (1.7), it holds:

$$x(n+1) \geq x(n) \exp\left(\frac{\alpha}{1+kG_2^y} - \delta - \frac{cG_2^y}{1+F_1^y+F_1^x} - \beta x(n)\right), \quad n \geq N_6, \quad (3.19)$$

since $G_1^y \geq G_2^y$, we obtain $0 < \frac{\alpha}{1+kG_2^y} - \delta - \frac{cG_2^y}{1+F_1^y+F_1^x} \leq 1$, combine with the relevant proofs above, we can obtain

$$P_1 = \liminf_{n \rightarrow +\infty} x(n) \geq \frac{\frac{\alpha}{1+kG_2^y} - \delta - \frac{cG_2^y}{1+F_1^y+F_1^x}}{\beta}.$$

Hence, for sufficiently small $\varepsilon > 0$, if $n \geq N_7$, and N_7 is an integer, satisfying

$$N_7 > N_6, \text{ then } x(n) \geq \frac{\frac{\alpha}{1+kG_2^y} - \delta - \frac{cG_2^y}{1+F_1^y+F_1^x}}{\beta} - \frac{\varepsilon}{2} := F_2^x \geq F_1^x.$$

Scaling the second formula of system (1.7), it holds:

$$y(n+1) \geq y(n) \exp(\gamma - y(n)), \quad n \geq N_7, \quad (3.20)$$

we can obtain the following:

$$P_2 = \liminf_{n \rightarrow +\infty} y(n) \geq \gamma.$$

Therefore, for sufficiently small $\varepsilon > 0$, if $n \geq N_8$, and N_8 is an integer, satisfying $N_8 > N_7$, then $y(n) \geq \gamma - \frac{\varepsilon}{2} := F_2^y \geq F_1^y$.

Iteration 3. Continuing with the above steps, we end up with four sequences $\{G_n^x\}, \{G_n^y\}, \{F_n^x\}, \{F_n^y\}$ such that

$$\begin{aligned} G_n^x &= \frac{\alpha}{1+kF_{n-1}^y} - \delta - \frac{cF_{n-1}^y}{1+G_{n-1}^y+G_{n-1}^x} + \frac{\varepsilon}{n}, \\ G_n^y &= \gamma + \frac{\varepsilon}{n}, \\ F_n^x &= \frac{\alpha}{1+kG_n^y} - \delta - \frac{cG_n^y}{1+F_{n-1}^y+F_{n-1}^x} - \frac{\varepsilon}{n}, \\ F_n^y &= \gamma - \frac{\varepsilon}{n}. \end{aligned} \quad (3.21)$$

These four sequences satisfy the following inequalities:

$$F_n^x \leq P_1 \leq Q_1 \leq G_n^x, \quad F_n^y \leq P_2 \leq Q_2 \leq G_n^y, \quad n > 0,$$

Next, we prove by means of inductive method that $\{G_n^x\}$ and $\{G_n^y\}$ are monotonic and decreasing sequences, $\{F_n^x\}$ and $\{F_n^y\}$ are monotonic and increasing sequences.

Clearly, when $n = 1$, it holds:

$$G_1^x \geq G_2^x, \quad F_1^x \leq F_2^x, \quad G_1^y \geq G_2^y, \quad F_1^y \leq F_2^y.$$

According to mathematical induction, we let $F_{i-1}^x \leq F_i^x$, $G_{i-1}^x \geq G_i^x$, $G_{i-1}^y \geq G_i^y$, and $F_{i-1}^y \leq F_i^y$ hold. Then

$$\begin{aligned} G_i^x &= \frac{\alpha}{1 + kF_{i-1}^y} - \delta - \frac{cF_{i-1}^y}{1 + G_{i-1}^y + G_{i-1}^x} + \frac{\varepsilon}{i} \\ &\geq \frac{\alpha}{1 + kF_i^y} - \delta - \frac{cF_i^y}{1 + G_i^y + G_i^x} + \frac{\varepsilon}{i+1} \\ &= G_{i+1}^x; \\ G_i^y &= \gamma + \frac{\varepsilon}{i} \geq \gamma + \frac{\varepsilon}{i+1} = G_{i+1}^y; \\ F_i^x &= \frac{\alpha}{1 + kG_i^y} - \delta - \frac{cG_i^y}{1 + F_{i-1}^y + F_{i-1}^x} - \frac{\varepsilon}{i} \\ &\leq \frac{\alpha}{1 + kG_{i+1}^y} - \delta - \frac{cG_{i+1}^y}{1 + F_i^y + F_i^x} - \frac{\varepsilon}{i+1} \\ &= F_{i+1}^x; \\ F_i^y &= \gamma - \frac{\varepsilon}{i} \leq \gamma - \frac{\varepsilon}{i+1} = F_{i+1}^y. \end{aligned}$$

Therefore $\{G_n^x\}$ and $\{G_n^y\}$ are monotonically decreasing, $\{F_n^x\}$ and $\{F_n^y\}$ are monotonically increasing. Thus, all four sequences have limits. Let

$$\lim_{n \rightarrow +\infty} G_n^x = x_1, \quad \lim_{n \rightarrow +\infty} G_n^y = y_1, \quad \lim_{n \rightarrow +\infty} F_n^x = x_2, \quad \lim_{n \rightarrow +\infty} F_n^y = y_2.$$

Taking $n \rightarrow +\infty$ in (3.21), we obtain

$$\begin{aligned} x_1 &= \frac{\frac{\alpha}{1+k y_2} - \delta - \frac{c y_2}{1+y_1+x_1}}{\beta}, \\ y_1 &= \gamma, \\ x_2 &= \frac{\frac{\alpha}{1+k y_1} - \delta - \frac{c y_1}{1+y_2+x_2}}{\beta}, \\ y_2 &= \gamma. \end{aligned} \tag{3.22}$$

Equation (3.22) demonstrates that both (x_1, γ) and (x_2, γ) are solutions to system (1.7). Considering Theorem 3.2, system (1.7) has a single positive solution (x_1^*, γ) , so the equation has a positive solution $x_1 = x_2 = x_1^*$, $y_1 = y_2 = \gamma$. Therefore, we obtain $Q_1 = P_1 = x_1^*$ and $Q_2 = P_2 = \gamma$. Hence, the equilibrium $E_1^*(x_1^*, \gamma)$ is global attractive. $\square$

4. Bifurcation analysis

There are three types of bifurcations that will be discussed in this part, including transcritical bifurcations, pitchfork bifurcations, flip bifurcations and fold bifurcations of system at E_2 , E_2^* and E_3^* .

4.1. Transcritical and pitchfork bifurcations at $E_2(0, \gamma)$

Transcritical bifurcation is when two equilibria exchange stability as a parameter changes. Pitchfork bifurcation is when a stable equilibrium splits into two new equilibria as a parameter changes (the bifurcation may be stable or unstable). As the harvesting coefficient k increases, the internal equilibrium point E_2^* collides with the boundary equilibrium point E_2 , resulting in a change in the number of equilibrium points in the system, with E_2 transitioning from an unstable to a stable state. When $k = k^*$, i.e., $\beta = \frac{c\gamma}{1+2\gamma+\gamma^2}$, two stable equilibrium points E_1^* and E_2^* collide to form a heavy root E_3^* , which is connected to the boundary equilibrium point $E_2(0, \gamma)$ overlap, and as k further increases, E_3^* disappears. Furthermore, we can obtain that if $k = \tilde{k}$, $\alpha > \delta + \frac{c\gamma}{1+\gamma}$ and $\gamma \neq 2$ hold, the eigenvalues of $J(E_2)$ are $\lambda_1 = 1$ and $\lambda_2 \neq 1, -1$. In this case, transcritical and pitchfork bifurcations may happen when parameter values change in the area of

$$\Omega := \left\{ (\alpha, \beta, c, \gamma, \delta, k) \in \mathbb{R}^6 : \begin{array}{l} k = \tilde{k}, \alpha > \delta + \frac{c\gamma}{1+\gamma}, \\ \gamma, \beta, \delta, c > 0 \end{array} \right\}.$$

As $(\alpha, \beta, c, \gamma, \delta, k) \in \Omega$ and ξ stands for a vary small perturbation value such that $|\xi| \ll 1$, we obtain the following map,

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \exp \left(\frac{\alpha}{1 + (\tilde{k} + \xi)y} - \delta - \beta x - \frac{cy}{1 + x + y} \right) \\ y \exp(\gamma - y) \end{pmatrix}. \quad (4.1)$$

Expanding the system (4.1) using Taylor series expansion at the point $(x, y, \xi) = (0, 0, 0)$, we obtain:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 - \gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u, v, \xi) \\ g_1(u, v, \xi) \end{pmatrix}, \quad (4.2)$$

where

$$\begin{aligned} f_1(u, v, \xi) &= a_{200}u^2 + a_{110}uv + a_{101}u\xi + a_{300}u^3 + a_{210}u^2v + a_{201}u^2\xi \\ &\quad + a_{120}uv^2 + a_{111}uv\xi + a_{102}u\xi^2 + O\left((|u| + |v| + |\xi|)^4\right), \\ g_1(u, v, \xi) &= \frac{\gamma - 2}{2}v^2 - \frac{\gamma - 3}{6}v^3 + O\left((|u| + |v| + |\xi|)^4\right), \end{aligned}$$

with

$$a_{201} = -\frac{\gamma(\beta\gamma^2 + 2\beta\gamma - c\gamma + \beta + \gamma + 1)}{(1 + \gamma)^3},$$

$$\begin{aligned}
a_{200} &= -\frac{\beta\gamma^2 + 2\beta\gamma - c\gamma + \beta}{(1+\gamma)^2}, a_{101} = -\frac{\gamma}{1+\gamma}, \\
a_{110} &= -\frac{c\gamma^2\tilde{k}^2 + \alpha\gamma^2\tilde{k} + 2\alpha\gamma\tilde{k} + 2c\gamma\tilde{k} + \alpha\tilde{k} + c}{(\gamma\tilde{k} + 1)^2(1+\gamma)^2}, \\
a_{300} &= \frac{1}{2(1+\gamma)^4} \left(-\beta^2\gamma^4 - 2\beta(c-2\beta)\gamma^3 + \beta^2 \right. \\
&\quad \left. + (6\beta^2 - 4\beta c + c^2 - 2c)\gamma^2 + (4\beta^2 - 2\beta c - 2c)\gamma \right), \\
a_{210} &= \frac{1}{(1+\gamma)^4(\gamma\tilde{k} + 1)^2} \left((\tilde{k}(\beta-1)c + \beta\alpha)\tilde{k}\gamma^2 \right. \\
&\quad \left. - (c^2\tilde{k} + (-2\beta\tilde{k} + \alpha - 2\beta + 2)c - 4\beta\alpha)\tilde{k}\gamma^3 + (-2c^2\tilde{k} \right. \\
&\quad \left. + ((\beta+1)\tilde{k}^2 + (4\beta-2\alpha)\tilde{k} + \beta-1)c + 6\alpha\beta\tilde{k})\gamma^2 + (-c^2 \right. \\
&\quad \left. + ((2\beta-\alpha+2)\tilde{k} + 2\beta)c + 4\alpha\beta\tilde{k})\gamma + (\beta+1)c + \alpha\beta\tilde{k} \right), \\
a_{201} &= \frac{1}{(1+\gamma)^4(\gamma\tilde{k} + 1)^4} \left(\gamma^4 c(c+2\gamma+2)\tilde{k}^4 + 4\left(\frac{\alpha\gamma^4}{2} + \left(\frac{c}{2} + 2\right)\alpha \right. \right. \\
&\quad \left. \left. + 2c\right)\gamma^3 + \frac{(c+4)\alpha\gamma}{2} + \frac{\alpha}{2}\right)\gamma\tilde{k}^3 + ((\alpha^2 + 2\alpha)\gamma^4 + (4\alpha^2 + (4c \\
&\quad + 8)\alpha + 12c)\gamma^3 + (6\alpha^2 + (8c+12)\alpha + 6c^2 + 12c)\gamma^2 + 4\alpha \\
&\quad (c+\alpha+2)\gamma + \alpha^2 + 2\alpha)\tilde{k}^2 + 4c\left(\left(\frac{\alpha}{2} + 2\right)\gamma^2 + (c+\alpha+2)\gamma \right. \\
&\quad \left. + \frac{\alpha}{2}\right)\tilde{k} + c(c+2\gamma+2) \right), \\
a_{210} &= \frac{1}{(1+\gamma)^3(\gamma\tilde{k} + 1)^2} \left(c\gamma^3\tilde{k}^2 + \alpha\gamma^3\tilde{k} - \gamma^3\tilde{k}^2 + 2\alpha\gamma^2\tilde{k} + 2c\gamma^2\tilde{k} \right. \\
&\quad \left. - \gamma^2\tilde{k}^2 + \alpha\gamma\tilde{k} - 2\gamma^2\tilde{k} + c\gamma - 2\gamma\tilde{k} - \gamma - 1 \right), \\
a_{102} &= -\frac{\gamma^2}{2(1+\gamma)^2}.
\end{aligned}$$

For map (4.2), suppose a centre manifold $W_1^c(0,0,0)$ be approximated by

$$W_1^c(0,0,0) = \left\{ (u, v, \xi) \in \mathbb{R}^3 : v = k_{11}u^2 + k_{21}u\xi + k_{31}\xi^2 + O(|u| + |\xi|)^3 \right\}.$$

It is evident that $k_{11} = k_{21} = k_{31} = 0$. Hence, the map associated with the central manifold is

$$G_1 : u \rightarrow u + a_{200}u^2 + a_{101}u\xi + a_{300}u^3 + a_{201}u^2\xi + a_{102}u\xi^2 + O(|u| + |\xi|)^4.$$

A simple analysis provides

$$\begin{aligned}
G_1(0,0) &= 0, & \frac{\partial G_1}{\partial u}(0,0) &= 1, \\
\frac{\partial G_1}{\partial \xi}(0,0) &= 0, & \frac{\partial^3 G_1}{\partial u^3}(0,0) &= 6a_{300} \neq 0, \\
\frac{\partial^2 G_1}{\partial u \partial \xi}(0,0) &= a_{101} < 0, & \frac{\partial^2 G_1}{\partial u^2}(0,0) &= 2a_{200}.
\end{aligned}$$

Note that

$$\Psi_1 := -\frac{\frac{\partial^2 G_1}{\partial u^2}(0,0)}{\frac{\partial^2 G_1}{\partial u \partial \xi}(0,0)} = -\frac{2a_{200}}{a_{101}}, \quad \Psi_2 := -\frac{\frac{\partial^3 G_1}{\partial u^3}(0,0)}{\frac{\partial^2 G_1}{\partial u \partial \xi}(0,0)} = -\frac{6a_{300}}{a_{101}}.$$

Thus, we have the following outcome:

Theorem 4.1. *System (1.7) can undergo a transcritical bifurcation at $E_2(0, \gamma)$ if $(\alpha, \beta, c, \gamma, \delta, k) \in \Omega$ and $\beta \neq \frac{c\gamma}{1+2\gamma+\gamma^2}$.*

Remark 4.1. In biology, transcritical bifurcation can be used to detect crucial tipping points in ecosystems. These points mark the shift of a system from one state to another, a shift that may result in the extinction of a population. Through the identification of tipping points, we can predict the behavior of ecosystems under different conditions and consequently enhance the efficacy of ecological management and conservation measures.

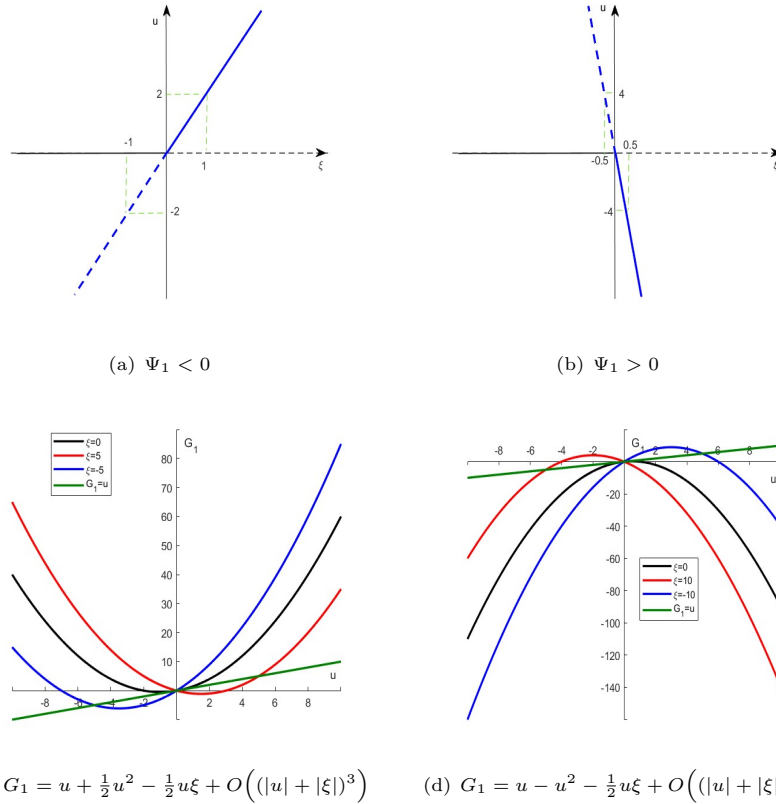


Figure 2. The transcritical bifurcation diagram of system (1.7) at E_2 with (a) $\alpha = 4$, $\delta = 1$, $k = 1$, $\gamma = 1$, $c = 2$ and $\beta = 0.25$, (b) $\alpha = 4$, $\delta = 1$, $k = 1$, $\gamma = 1$, $c = 2$ and $\beta = 1.5$.

In the u - ξ plane, $E_2(0, \gamma)$ transitions from unstable (resp., stable) to semi-stable and finally stable (resp., unstable) if $\Psi_1 > 0$ (resp., $\Psi_1 < 0$) holds. We illustrate

the transcritical bifurcation (see Figure 2(c) and 2(d)) with parameters from Figure 2(a) and 2(b).

Theorem 4.2. Suppose that $(\alpha, \beta, c, \gamma, \delta, k) \in \Omega$ and $\beta = \frac{c\gamma}{1+2\gamma+\gamma^2}$. Then system (1.7) can undergo a pitchfork bifurcation at $E_2(0, \gamma)$.

Remark 4.2. From a biological perspective, pitchfork bifurcation causes the system to transition from multiple equilibrium states to a single equilibrium state. Although this transition diminishes the volatility of the system, it also reduces its adaptability and diversity, making it more vulnerable to environmental changes.

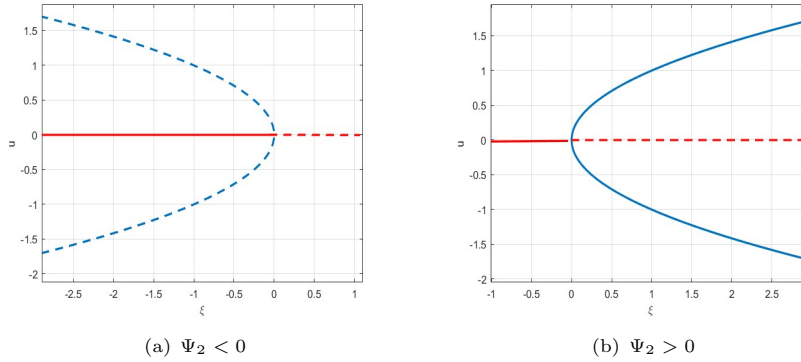


Figure 3. The pitchfork bifurcation diagram of system (1.7) at E_2 with (a) $\alpha = 4$, $\delta = 1$, $k = 1$, $\gamma = 1$, $c = 2$ and $\beta = 1$, (b) $\alpha = 4$, $\delta = 1$, $k = 1$, $\gamma = 1$, $c = 2$ and $\beta = 2$.

Refer to Figure 3 for a pitchfork bifurcation diagram on the $u - \xi$ plane. In addition, $\Psi_2 < 0$ indicates a fixed point curve on the left of $\xi = 0$ (refer to Figure 3(a)), whereas $\Psi_2 > 0$ indicates a fixed point curve on the right (refer to Figure 3(b)).

4.2. Fold bifurcations at $E_3^*(x_3^*, \gamma)$

Fold bifurcation is the process where two equilibrium points (one stable and one unstable) meet and annihilate as the parameters change. The fixed points $E_1^*(x_1^*, \gamma)$ and $E_2^*(x_2^*, \gamma)$ bifurcate from $E_3^*(x_3^*, \gamma)$ when $0 < c < c^*$, merge at $E_3^*(x_3^*, \gamma)$ when $c = c^*$ and when $c > c^*$ vanishes. In addition, according to Theorem 2.4, if $c = c^*$ and $k < \frac{\alpha - \beta - \delta - \beta\gamma}{\beta\gamma^2 + \beta\gamma + \delta\gamma}$, $\alpha > \beta + \delta + \beta\gamma$, and $\gamma \neq 2$, the eigenvalues of the Jacobian matrix of $J(E_3^*)$ are $\lambda_1 = 1$ and $\lambda_2 \neq 1, -1$. Thus, when all parameters satisfy the set Ω_{FB1} , system (1.7) would undergo a fold bifurcation at $E_3^*(x_3^*, \gamma)$:

$$\Omega_{FB1} := \left\{ (\alpha, \beta, c, \gamma, \delta, k) \in \mathbb{R}^6 : \begin{array}{l} c = c^*, \gamma \neq 2, k < \frac{\alpha - \beta - \delta - \beta\gamma}{\beta\gamma^2 + \beta\gamma + \delta\gamma}, \\ \alpha > \beta + \delta + \beta\gamma, k \geq 0, \\ \beta > 0, \gamma > 0, \delta > 0, c > 0 \end{array} \right\}.$$

Again, we analyze fold bifurcation using c as the bifurcation parameter. Assume $c = c^* + \zeta$, and ζ represents a small perturbation parameter, we employ the change

of variables $u_1 = x - x_3^*$ and $v_1 = y - \gamma$ to shift $E_3^*(x_3^*, \gamma)$ to the origin and to convert system (1.7) into

$$\begin{pmatrix} u_1 \\ \zeta \\ v_1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & a_1 & b_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ \zeta \\ v_1 \end{pmatrix} + \begin{pmatrix} f_2(u_1, \zeta, v_1) \\ 0 \\ g_2(u_1, \zeta, v_1) \end{pmatrix}, \quad (4.3)$$

where

$$\begin{aligned} f_2(u_1, \zeta, v_1) &= b_{200}u_1^2 + b_{110}u_1\zeta + b_{101}u_1v_1 + b_{020}\zeta^2 + b_{011}\zeta v_1 \\ &\quad + b_{002}v_1^2 + O(|u_1| + |\zeta| + |v_1|)^3, \\ g_2(u_1, \zeta, v_1) &= \frac{\gamma - 2}{2}v_1^2 + O(|u_1| + |\zeta| + |v_1|)^3, \end{aligned}$$

with

$$\begin{aligned} a_1 &= -\frac{\gamma x_3^*}{1 + \gamma + x_3^*}, \quad b_{020} = \frac{\gamma^2 + x_3^*}{2(1 + \gamma + x_3^*)^2}, \\ b_1 &= \frac{x_3^*}{(1 + x_3^* + \gamma)^3(\gamma k + 1)^2} \left(c^* \gamma^2 k^2 (x_3^* + 1) + (\alpha \gamma^2 + 2(x_3^* + 1)) \right. \\ &\quad \left. \times (c^* + \alpha) \gamma + \alpha (x_3^* + 1)^2 k + c^* (x_3^* + 1) \right), \\ b_{200} &= \frac{1}{(1 + x_3^* + \gamma)^4} \left(x_3^* (\gamma + x_3^* + 1)^2 - 2(x_3^{*2} + (2 + (c^* + 2)\gamma)x_3^* \right. \\ &\quad \left. + (\gamma + 1)^2)(\gamma + x_3^* + 1)^2 \beta + ((2 + (c^* + 2)\gamma)x_3^* + 2(\gamma + 1)^2)\gamma c^* \right), \\ b_{110} &= \frac{1}{(\gamma + x_3^* + 1)^3} \left(\gamma(-\beta x_3^{*3} - 2\beta(\gamma + 1)x_3^{*2} + (-\beta\gamma^2 + (c^* - 2\beta + 1)\gamma \right. \\ &\quad \left. - \beta + 1)x_3^* + (\gamma + 1)^2) \right), \\ b_{101} &= \frac{1}{(1 + x_3^* + \gamma)^4(\gamma k + 1)^2} \left(\alpha \beta k x_3^{*5} + (\beta c^* \gamma^2 k^2 + 2k\beta(c^* + 2\alpha)\gamma \right. \\ &\quad + (4\alpha\beta - \alpha)k + \beta c^*)x_3^{*4} + (2\beta c^* \gamma^3 k^2 + 3k\beta(c^* k + \frac{4}{3}c^* + 2\alpha)\gamma^2 \\ &\quad + (((6\beta - \alpha)c^* + 12\alpha\beta - 4\alpha)k + 2\beta c^*)\gamma + (6\alpha\beta - 4\alpha)k + 3\beta c^*)x_3^{*3} \\ &\quad + (\beta c^* \gamma^4 k^2 - k(c^*(c^* - 4\beta + 2)k - 2\beta(c^* + 2\alpha))\gamma^3 + (c^*(3\beta - 1)k^2 \\ &\quad + (-2c^{*2} + (8\beta - 2\alpha - 4)c^* + 12\alpha\beta - 6\alpha)k + \beta c^*)\gamma^2 \\ &\quad + (((6\beta - 2\alpha - 2)c^* + 12\alpha(\beta - 1))k - c^*(c^* - 4\beta + 2))\gamma + (4\alpha\beta - 6\alpha)k \\ &\quad + c^*(3\beta - 1))x_3^{*2} + ((c^*(\beta - 2)k + \alpha\beta)k\gamma^4 - k(c^*(c^* - 2\beta + 4)k \\ &\quad + (-2\beta + \alpha + 4)c^* - 4\alpha(\beta - 1))\gamma^3 + (c^*(\beta - 2)k^2 + (-2c^{*2} \\ &\quad + (4\beta - 2\alpha - 8)c^* + 6\alpha(\beta - 2))k + c^*(\beta - 2))\gamma^2 + (((2\beta - \alpha - 4)c^* \\ &\quad + 4\alpha(\beta - 3))k - c^*(c^* - 2\beta + 4))\gamma + \alpha(\beta - 4)k + c^*(\beta - 2))x_3^* \\ &\quad \left. - (k(c^* k + \alpha)\gamma^2 + 2k(c^* + \alpha)\gamma + \alpha k + c^*)(\gamma + 1)^2 \right), \\ b_{011} &= \frac{x_3^*}{(1 + x_3^* + \gamma)^3(\gamma k + 1)^2} \left(((x_3^* + 1)(c^* - 1)k + \alpha)k\gamma^3 + 2(x_3^* + 1) \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\left(-\frac{x_3^*}{2} - \frac{1}{2} \right) k + c^* + \alpha - 1 \right) k \gamma^2 + (x_3^* + 1) \left((\alpha - 2)(x_3^* + 1)k + c^* \right. \\
& \left. - 1 \right) \gamma - (x_3^* + 1)^2 \Big), \\
b_{002} = & \frac{x_3^*}{2(1+x_3^*+\gamma)^4(\gamma k+1)^4} \Big(\gamma^4(x_3^*+1)(2\gamma+(x_3^*+1)(c^*+2))c^*k^4 \\
& + 4\left(\frac{\alpha\gamma^4}{2} + \frac{1}{2}((x_3^*+1)((c^*+4)\alpha+4c^*)\gamma^3) + (x_3^*+1)^2((c^*+3)\alpha \right. \\
& + c^{*2}+2c)\gamma^2 + \frac{\alpha}{2}((x_3^*+1)^3(c^*+4)\gamma) + \frac{\alpha}{2}((x_3^*+1)^4)\gamma k^3 \\
& + ((\alpha^2+2\alpha)\gamma^4+4(x_3^*+1)(\alpha^2+(c^*+2)\alpha+3c^*)\gamma^3+6(x_3^*+1)^2 \\
& \times (\alpha^2+(\frac{4c^*}{3}+2)\alpha+c^{*2}+2c^*)\gamma^2+4\alpha(x_3^*+1)^3(c^*+\alpha+2)\gamma \\
& + \alpha(x_3^*+1)^4(\alpha+2))k^2+4((2+\frac{\alpha}{2})\gamma^2+(x_3^*+1)(c^*+\alpha+2)\gamma \\
& \left. + \frac{\alpha(x_3^*+1)^2}{2}(x_3^*+1)c^*k+(x_3^*+1)(2\gamma+(x_3^*+1)(c^*+2))c^* \right).
\end{aligned}$$

Then with the transformation,

$$\begin{pmatrix} u_1 \\ \zeta \\ v_1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & -\frac{1}{a_1} & 0 \\ 0 & 0 & -\frac{\gamma}{b_1} \end{pmatrix} \begin{pmatrix} u_2 \\ \tilde{\zeta} \\ v_2 \end{pmatrix},$$

system (4.3) becomes

$$\begin{pmatrix} u_2 \\ \tilde{\zeta} \\ v_2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \begin{pmatrix} u_2 \\ \tilde{\zeta} \\ v_2 \end{pmatrix} + \begin{pmatrix} f_3(u_1, \zeta, v_1) \\ 0 \\ g_3(u_1, \zeta, v_1) \end{pmatrix}, \quad (4.4)$$

where

$$\begin{aligned}
f_3(u_1, \zeta, v_1) = & -b_{200}u_1^2 - b_{110}u_1\zeta - c_{101}u_1v_1 - c_{020}\zeta^2 - c_{011}\zeta v_1 \\
& - \left(b_{002} + \frac{\beta(\gamma-2)}{2\gamma} \right) v_1^2 + O\left((|u_1| + |\zeta| + |v_1|)^3\right), \\
g_3(u_1, \zeta, v_1) = & \frac{\beta(2-\gamma)}{2\gamma} v_1^2 + O\left((|u_1| + |\zeta| + |v_1|)^3\right).
\end{aligned}$$

Therefore, in a similar manner to the previous approach, we can figure out the centre manifold $W_2^c(0, 0, 0)$ of the map (4.4) at the origin and in a limited neighborhood of $\tilde{\zeta} = 0$,

$$W_2^c(0, 0, 0) = \left\{ (u_2, \tilde{\zeta}, v_2) \in \mathbb{R}^3 : v_2 = t_{11}u_2 + t_{21}u_2\tilde{\zeta} + t_{31}\tilde{\zeta}^2 + O\left((|u_2| + |\tilde{\zeta}|)^3\right) \right\}.$$

It is easy to see that $t_{11} = t_{21} = t_{31} = 0$. Thus, the mapping that is limited to the center manifold can be expressed as

$$G_2 : u_2 \rightarrow u_2 + \tilde{\zeta} + h_{11}u_2^2 + h_{21}u_2\tilde{\zeta} + h_{31}\tilde{\zeta}^2 + O\left((|u_2| + |\tilde{\zeta}|)^3\right),$$

where $h_{11} = -b_{200}$, $h_{21} = -\frac{b_{110}}{a_1}$, and $h_{31} = \frac{b_{020}}{a_1}$. Note that

$$\begin{aligned} G_2(0,0) &= 0, & \frac{\partial G_2}{\partial u_2}(0,0) &= 1, & \frac{\partial G_2}{\partial \tilde{\zeta}}(0,0) &= 1, \\ \frac{\partial^2 G_2}{\partial u_2^2}(0,0) &= 2h_{11} \neq 0, & \frac{\partial^2 G_2}{\partial u_2 \partial \tilde{\zeta}}(0,0) &= h_{21} \neq 0. \end{aligned}$$

This results in the subsequent outcome of fold bifurcation.

Theorem 4.3. *System (1.7) exhibits a fold bifurcation at $E_3^*(x_3^*, \gamma)$ when values of $(\alpha, \beta, c, \gamma, \theta)$ fall inside the set Ω_{FB1} (see Figure 4).*

Remark 4.3. Biologically, at the fold bifurcation point, a stable equilibrium point E_1^* and an unstable equilibrium point E_2^* meet, merge and ultimately vanish. This implies that the system transitions from having two steady states (one stable and one unstable) to having only one steady state. The occurrence of a fold bifurcation implies that all values of the parameters must have a fixed point and will never be destroyed, i.e., the fixed point may change its stability at different parameter values.

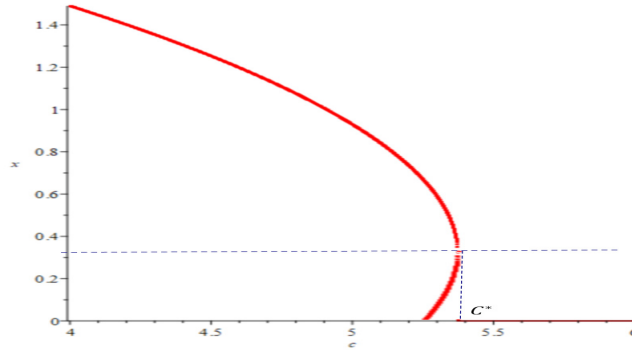


Figure 4. The fold bifurcation diagram of system (1.7) at $E_3^*(x_3^*, \gamma)$ around the critical value $c = c^* = 5.37$ with $\alpha = 4$, $\beta = 1$, $\delta = 1$, $\gamma = 1$ and $k = 0.1$.

4.3. Flip bifurcation at $E_1^*(x_1^*, \gamma)$

Flip bifurcation describes a system where a stable periodic solution changes to a solution with a doubled period under a specific parameter value. For example, a population changes from a stable state to a complex periodic or chaotic fluctuation under seasonal fluctuation. In this section, we discuss that system (1.7) undergoes flip bifurcation around its positive equilibrium $E_1^*(x_1^*, \gamma)$. For this, β is taken as bifurcation parameter. For a flip bifurcation to occur near $E_1^*(x_1^*, \gamma)$, its two eigenvalues of $J(E_1^*)$ must satisfy one is -1 and one is neither 1 nor -1. Therefore, the necessary conditions for the appearance of flip bifurcation near $E_1^*(x_1^*, \gamma)$ of system (1.7) can be defined using Ω_{FB2} as follows:

$$\Omega_{FB2} := \left\{ (\alpha, \beta, c, \gamma, \delta, k) \in \mathbb{R}^6 : \begin{aligned} &\beta = \beta^*, \tilde{c} < c < c^{**}, \\ &\gamma \neq 2, A_1 < 0, 0 < k < k^{**} \end{aligned} \right\}.$$

Let us analyze the subsequent perturbation of system (1.7),

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \exp\left(\frac{\alpha}{1+ky} - \delta - (\beta^* + \eta)x - \frac{cy}{1+x+y}\right) \\ y \exp(\gamma - y) \end{pmatrix}. \quad (4.5)$$

First, we apply the transformation $u_3 = x - x_1^*$, $v_3 = y - \gamma$ to translate the fixed point to the origin and to transform the map (4.5) into

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & j_{010} \\ 0 & 1 - \gamma \end{pmatrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + \begin{pmatrix} f_4(u_3, v_3, \eta) \\ g_4(u_3, v_3, \eta) \end{pmatrix}, \quad (4.6)$$

where

$$\begin{aligned} f_4(u_3, v_3, \eta) &= -x_1^{*2}\eta + j_{200}u_3^2 + j_{110}u_3v_3 + j_{101}u_3\eta + j_{020}v_3^2 \\ &\quad + j_{011}v_3\eta + \frac{x_1^{*2}}{2}\eta^2 + O\left((|u_3| + |v_3| + |\eta|)^3\right), \\ g_4(u_3, v_3, \eta) &= \frac{\gamma - 2}{2}v_3^2 + O\left((|u_3| + |v_3| + |\eta|)^3\right), \end{aligned}$$

with

$$\begin{aligned} j_{010} &= \frac{x_1^*}{(1+x_1^*+\gamma)^3(\gamma k+1)^2} \left(c\gamma^2 k^2 (x_1^*+1) + (\alpha\gamma^2 + 2(x_1^*+1)) \right. \\ &\quad \left. \times (c+\alpha)\gamma + \alpha(x_1^*+1)^2 k + c(x_1^*+1) \right), \\ j_{200} &= \frac{1}{(1+x_1^*+\gamma)^4} \left(x_1^*(\gamma+x_1^*+1)^2 - 2(x_1^{*2} + (2+(c+2)\gamma)x_1^* \right. \\ &\quad \left. + (\gamma+1)^2)(\gamma+x_1^*+1)^2\beta^* + ((2+(c+2)\gamma)x_1^* + 2(\gamma+1)^2)\gamma c \right), \\ j_{101} &= \frac{x_1^*}{(\gamma+x_1^*+1)^2} \left(\beta^*\gamma^2 x_1^* + 2\beta^*\gamma x_1^{*2} + \beta^*x_1^{*3} + 2\beta^*\gamma x_1^* 2\beta^* x_1^{*2} - c\gamma x_1^* \right. \\ &\quad \left. + \beta^*x_1^* - 2\gamma^2 - 4\gamma x_1^* - 2x_1^{*2} - 2\gamma - 4x_1^* - 2 \right), \\ j_{110} &= \frac{1}{(1+x_1^*+\gamma)^4(\gamma k+1)^2} \left(\alpha\beta^* k x_1^{*5} + (\beta^* c \gamma^2 k^2 + 2k\beta^*(c+2\alpha)\gamma \right. \\ &\quad + (4\alpha\beta^* - \alpha)k + \beta^* c) x_1^{*4} + (2\beta^* c \gamma^3 k^2 + 3k\beta^*(ck + \frac{4}{3}c + 2\alpha)\gamma^2 \\ &\quad + (((6\beta^* - \alpha)c + 12\alpha\beta^* - 4\alpha)k + 2\beta^* c)\gamma + (6\alpha\beta^* - 4\alpha)k + 3\beta^* c) x_1^{*3} \\ &\quad + (\beta^* c \gamma^4 k^2 - k(c(c-4\beta^*+2)k - 2\beta^*(c+2\alpha))\gamma^3 + (c(3\beta^*-1)k^2 \\ &\quad + (-2c^2 + (8\beta^* - 2\alpha - 4)c + 12\alpha\beta^* - 6\alpha)k + \beta^* c)\gamma^2 + (((6\beta^* - 2\alpha \\ &\quad - 2)c + 12\alpha(\beta^* - 1))k - c(c-4\beta^*+2))\gamma + (4\alpha\beta^* - 6\alpha)k + c(3\beta^* \\ &\quad - 1))x_1^{*2} + ((c(\beta^* - 2)k + \alpha\beta^*)k\gamma^4 - k(c(c-2\beta^*+4)k + (-2\beta^* + \alpha \\ &\quad + 4)c - 4\alpha(\beta^* - 1))\gamma^3 + (c(\beta^* - 2)k^2 + (-2c^2 + (4\beta^* - 2\alpha - 8)c \\ &\quad + 6\alpha(\beta^* - 2))k + c(\beta^* - 2))\gamma^2 + (((2\beta^* - \alpha - 4)c + 4\alpha(\beta^* - 3))k \\ &\quad - c(c-2\beta^*+4))\gamma + \alpha(\beta^* - 4)k + c(\beta^* - 2))x_1^* - (k(ck + \alpha)\gamma^2 \\ &\quad \left. + 2k(c+\alpha)\gamma + \alpha k + c)(\gamma+1)^2 \right), \end{aligned}$$

$$\begin{aligned}
j_{011} &= \frac{x_1^{*2}}{(1+x_1^*+\gamma)^2(\gamma k+1)^2} \left(c\gamma^2 k^2 x_1^* + c\gamma^2 k^2 + \alpha\gamma^2 k + 2\alpha\gamma k x_1^2 \right. \\
&\quad \left. + \alpha k x_1^{*2} + 2c\gamma k x_1^* + 2\alpha\gamma k + 2\alpha k x_1^* + 2c\gamma k + \alpha k + c x_1^* + c \right), \\
j_{020} &= \frac{x_1^*}{2(1+x_1^*+\gamma)^4(\gamma k+1)^4} \left(\gamma^4(x_1^*+1)(2\gamma+(x_1^*+1)(c+2))ck^4 \right. \\
&\quad + 4\left(\frac{\alpha\gamma^4}{2} + \frac{1}{2}((x_1^*+1)((c+4)\alpha+4c)\gamma^3) + (x_1^*+1)^2((c+3)\alpha \right. \\
&\quad \left. + c^2 + 2c)\gamma^2 + \frac{\alpha}{2}((x_1^*+1)^3(c+4)\gamma) + \frac{\alpha}{2}((x_1^*+1)^4))\gamma k^3 \right. \\
&\quad + ((\alpha^2 + 2\alpha)\gamma^4 + 4(x_1^*+1)(\alpha^2 + (c+2)\alpha + 3c)\gamma^3 + 6(x_1^*+1)^2 \\
&\quad \times (\alpha^2 + (\frac{4c}{3} + 2)\alpha + c^2 + 2c)\gamma^2 + 4\alpha(x_1^*+1)^3(c+\alpha+2)\gamma \\
&\quad + \alpha(x_1^*+1)^4(\alpha+2))k^2 + 4((2 + \frac{\alpha}{2})\gamma^2 + (x_1^*+1)(c+\alpha+2)\gamma \\
&\quad \left. + \frac{\alpha(x_1^*+1)^2}{2}(x_1^*+1)ck + (x_1^*+1)(2\gamma+(x_1^*+1)(c+2))c \right).
\end{aligned}$$

Subsequently, we employ the invertible transformation

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} j_{010} & j_{010} \\ 0 & 2-\gamma \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix}.$$

Equation (4.6) can be expressed as

$$\begin{pmatrix} u_4 \\ v_4 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1-\gamma \end{pmatrix} \begin{pmatrix} u_4 \\ v_4 \end{pmatrix} + \begin{pmatrix} f_5(u_3, v_3, \eta) \\ g_5(u_3, v_3, \eta) \end{pmatrix}, \quad (4.7)$$

where

$$\begin{aligned}
f_5(u_3, v_3, \eta) &= W_{11}\eta + W_{21}u_3^2 + W_{31}v_3^2 + W_{41}\eta^2 + W_{51}u_3v_3 \\
&\quad + W_{61}u_3\eta + W_{71}v_3\eta + O\left((|u_3| + |v_3| + |\eta|)^3\right), \\
g_5(u_3, v_3, \eta) &= -\frac{1}{2}v_3^2 + O\left((|u_4| + |v_4| + |\eta|)^3\right),
\end{aligned}$$

with

$$\begin{aligned}
W_{11} &= -\frac{x_1^{*2}}{j_{010}}, & W_{21} &= \frac{j_{200}}{j_{010}}, & W_{31} &= \frac{j_{020}}{j_{010}} + \frac{1}{2}, \\
W_{41} &= \frac{x_1^{*2}}{2j_{010}}, & W_{51} &= \frac{j_{110}}{j_{010}}, & W_{61} &= \frac{j_{101}}{j_{010}}, & W_{71} &= \frac{j_{011}}{j_{010}}.
\end{aligned}$$

Next, we can get the approximation of the center manifold $W_3^c(0, 0, 0)$ of (4.7) at the origin in a small neighborhood of $\eta = 0$,

$$W_3^c(0, 0, 0) = \left\{ (u_4, v_4) \in \mathbb{R}^3 : \begin{aligned} v_4 &= m_{11}u_4^2 + m_{12}u_4\eta + m_{13}\eta^2 \\ &\quad + O\left((|u_4| + |\eta|)^3\right) \end{aligned} \right\},$$

where $m_{11} = m_{12} = m_{13} = 0$. The restriction of (4.7) on the center manifold $W_3^c(0, 0, 0)$ is estimated as follows:

$$G_3 : u_4 \rightarrow -u_4 + W_{11}\eta + W_{21}j_{010}^2 u_4^2 + W_{41}\eta^2 + W_{61}j_{010} u_4 \eta + O\left((|u_4| + |\eta|)^4\right),$$

then the two discriminatory quantities $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are given by

$$\begin{aligned}\mathfrak{L}_1 &= \left(\frac{\partial G_3}{\partial u_4 \partial \eta} + \frac{1}{2} \frac{\partial G_3}{\partial \eta} \frac{\partial^2 G_3}{\partial u_4^2} \right) \Big|_{(u_4, \eta) = (0, 0)} = W_{61} j_{010}, \\ \mathfrak{L}_2 &= \left(\frac{1}{6} \frac{\partial^3 G_3}{\partial u_4^3} + \left(\frac{1}{2} \frac{\partial^2 G_3}{\partial u_4^2} \right)^2 \right) \Big|_{(u_4, \eta) = (0, 0)} = W_{21}^2 j_{010}^4.\end{aligned}$$

Therefore, we obtain the subsequent outcome for flip bifurcation.

Theorem 4.4. Suppose that $(\alpha, \beta, c, \gamma, \delta, k) \in \Omega_{FB2}$. If $\mathfrak{L}_1 \neq 0$ and $\mathfrak{L}_2 \neq 0$, then system (1.7) undergoes flip bifurcation at $E_2^*(x_2^*, \gamma)$ when the parameter β varies around β^* .

Remark 4.4. Flip bifurcation is a biological phenomenon that is distinguished by fluctuations in population size. When the parameters are altered, the discrete-time system transitions from a fixed point to a cycle of period-2, i.e., given particular conditions, the two species can exist together in the period-2 cycle. Hence, the occurrence of a flip bifurcation in system (1.7) signifies the presence of a stable cycle between the first and second species, providing a strategy for the long-term coexistence of the two species.

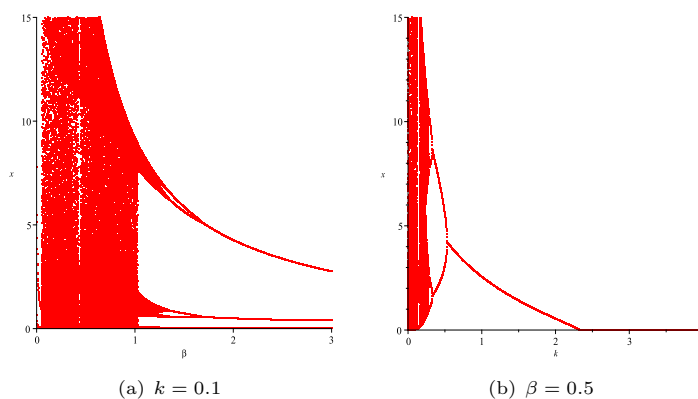


Figure 5. Flip bifurcation diagram of system (1.7) at $E_1^*(x_1^*, \gamma)$ with $\alpha = 5$, $c = 1$, $k = 0.1$, $\delta = 1$, and $\gamma = 1$.

Figure 5(a) shows the flip bifurcation diagram of system (1.7) at $E_1^*(x_1^*, \gamma)$ with $\alpha = 5$, $c = 1$, $k = 0.1$, $\delta = 1$, and $\gamma = 1$. Figure 5(b) is a bifurcation diagram with k as the bifurcation parameter when the first species is chaotic. From an ecological perspective, when the first species is in a chaotic state, the low-intensity fear effect is not enough to significantly change the population dynamics, and the first species may still maintain a chaotic state. Moderate fear effects reduce individual activity and foraging, indirectly reducing the population growth rate, reducing

excessive population fluctuations and stabilizing population dynamics. However, when the fear effect is too strong, the pressure of the second species on the first species will weaken its ability to acquire resources and reproduce, eventually leading to extinction. This result reveals the important influence of fear effects on population dynamics and highlights the importance of maintaining moderate stress in ecosystem management.

5. Chaos control

Chaos control refers to the non-periodic and irregular behavior of a system. In this state, the population size will show highly irregular fluctuations and is difficult to predict. We use the chaotic hybrid control method [32] to control the chaotic phenomenon caused by flip bifurcation. The relevant control system is expressed as follows:

$$\begin{cases} x_{n+1} = \rho x_n \exp\left(\frac{\alpha}{1+ky_n} - \delta - \beta x_n - \frac{cy_n}{1+x_n+y_n}\right) + (1-\rho)x_n, \\ y_{n+1} = \rho y_n \exp(\gamma - y_n) + (1-\rho)y_n, \end{cases} \quad (5.1)$$

where $\rho \in (0, 1)$ denotes the external control parameter. The Jacobian matrix for system (5.1) at $E_1^*(x_1^*, \gamma)$ is

$$J(E_1^*) = \begin{pmatrix} 1 + \rho \left(-\beta x_1^* + \frac{c\gamma x_1^*}{(1+x_1^*+\gamma)^2} \right) & -\frac{\alpha k}{(1+k\gamma)^2} - \frac{c\rho x_1^*(1+x_1^*)}{1+x_1^*+\gamma} \\ 0 & 1 - \rho\gamma \end{pmatrix}.$$

Its eigenvalues are $1 + \rho \left(-\beta x_1^* + \frac{c\gamma x_1^*}{(1+x_1^*+\gamma)^2} \right) < 1$ and $1 - \rho\gamma < 1$. From the proof of Theorem 2.6, we know that $-\beta x_1^* + \frac{c\gamma x_1^*}{(1+x_1^*+\gamma)^2} < 0$. Therefore, we have obtained the subsequent outcome.

Theorem 5.1. *In system (5.1), the positive equilibria $E_1^*(x_1^*, \gamma)$ is locally asymptotically stable when*

$$0 < \rho < \min \left\{ 1, \frac{2}{\gamma}, \frac{2}{\beta x_1^* - \frac{c\gamma x_1^*}{(1+x_1^*+\gamma)^2}} \right\}.$$

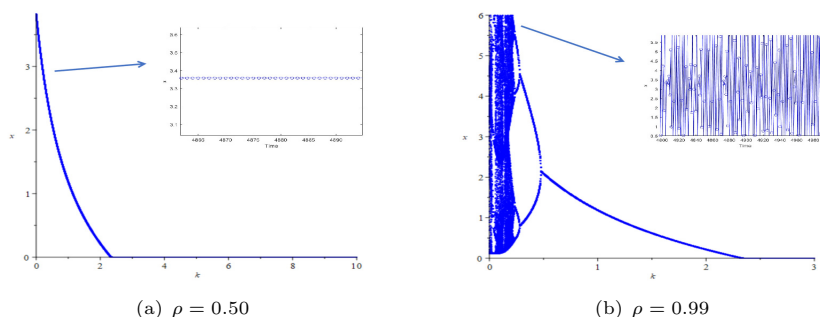
Since chaos exists in the bifurcation diagram with k as the bifurcation parameter, we now still use k as the branching parameter and try to verify whether the chaos control can get rid of the chaos phenomenon. we set the rest of parameter values as follows: $(\alpha, \delta, \beta, \gamma, c) = (5, 1, 0.5, 1, 1)$. Therefore, system (5.1) can be expressed as

$$\begin{cases} x_{n+1} = \rho x_n \exp\left(\frac{5}{1+ky_n} - 1 - 0.5x_n - \frac{y_n}{1+x_n+y_n}\right) + (1-\rho)x_n, \\ y_{n+1} = \rho y_n \exp(1 - y_n) + (1-\rho)y_n. \end{cases} \quad (5.2)$$

From Figure 5, it can be seen that when $0 < k < 0.5$, the second species is in a chaotic state. Therefore, taking $k = 0.10, 0.15, 0.20, 0.25, 0.30, 0.35$, and combined with Theorem 5.1, the value range of ρ can be obtained from Table 2 when $0.1 \leq k \leq 0.4$. Figure 6 (a) and (b) demonstrate that chaos control is effective.

Table 2. The corresponding range of ρ under different k values.

Values of bifurcation parameter k	The corresponding range of ρ
0.10	$0 < \rho < 0.594627540174$
0.15	$0 < \rho < 0.637084903349$
0.20	$0 < \rho < 0.677986786037$
0.25	$0 < \rho < 0.720980533525$
0.30	$0 < \rho < 0.756710127051$
0.35	$0 < \rho < 0.812215724496$

**Figure 6.** Diagrams of bifurcation for controlled system (5.2).

6. Numerical simulations

In this section, we illustrate and examine the theoretical results through numerical simulations.

Example 6.1. (the effect of fear effect on E_2) When the values $(\alpha, \delta, \beta, c, k, \gamma)$ are set to $(1, 0.5, 1, 11, 1, 1.5)$, it is evident that the condition mentioned above fulfills Theorem 3.1. The following three sets of beginning values are considered: $(0.1, 0.8)$, $(0.6, 0.2)$, and $(1.2, 0.7)$. As depicted in Figure 7(a), it is evident that E_2 exhibits global asymptotic stability, i.e., the result of Theorem 3.1 is correct. In addition, to investigate the influence of the fear effect on the stability of E_2 , we assigned the coefficient k values of 1 and 5. Based on Figure 7(b), an increase in the fear effect leads to an overall increase in the stability of E_2 . it also speeds up the extinction of the first species.

Example 6.2. (the effect of fear effect on E_1^*) We consider parameter values as below: $\alpha = 2, \delta = 1, \beta = 1, c = 2.5, k = 1, \gamma = 1$. The initial values (x_0, y_0) are set as follows: $(0.3, 0.8)$, $(0.6, 0.2)$, and $(1.2, 0.7)$. According to the requirements specified in Theorem 3.2, it can be inferred that when k falls within the range of $(0, 3.333)$, $E_1^*(x_1^*, \gamma)$ is globally stable. To verify the validity of Theorem 3.2, we select the value of k to 0.5, as depicted in Figure 8. At this particular value of k , it is clear that E_1^* remains globally stable. In addition, to examine the influence of

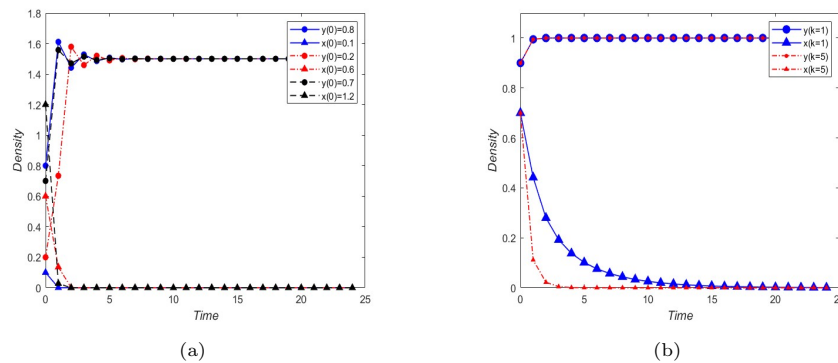


Figure 7. (a) Global stability of E_1 with $\alpha = 2.1$, $\delta = 0.5$, $c = 11$, $\beta = 1$ and $\gamma = 0.5$. (b) The fear effect on E_1 with $k = 1$ and $k = 5$.

different ranges of fear effects on E^* , we choose $k = 2$ and $k = 4$. The results show that when $k \in (0, k^{**})$, increasing k will result in decreasing the density of the first species but does not lead to the extinction of the first species, but when k greater than k^{**} will cause the first species, which could have stabilized coexistence with the second species, to tend to extinction.

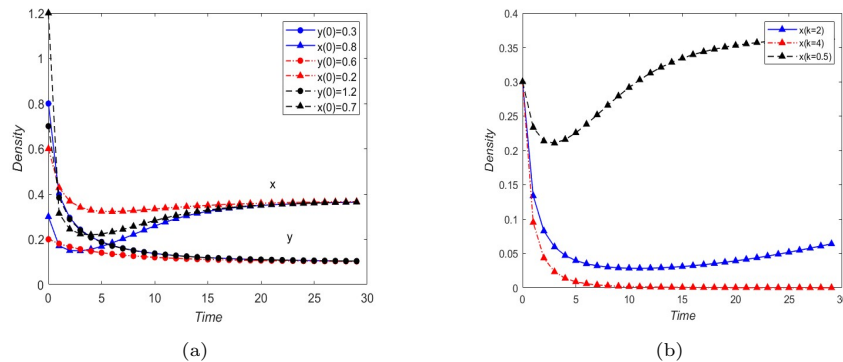


Figure 8. (a) Global stability of E_1^* with $\alpha = 2$, $\delta = 1.5$, $c = 0.5$, $\beta = 1$ and $\gamma = 1$. (b) The density map corresponds to x when $k = 0.5$, $k = 2$ and $k = 4$.

Example 6.3. (the fear effect on the first species) When $\gamma < 2$, i.e., the second species is stable, at this point, according to Figure 9(a), an appropriate k decreases the population density of the first species while transitioning it from a state of chaos to stability, and an excessively large k results in the extinction of the first species, which is consistent with the continuum model (1.5). When $\gamma > 2$, i.e., the second species is unstable, a phenomenon that is not present in the continuum system (1.5), Figure 9(b)-(d) illustrates the influence of fear effect on the first species in all three cases when the second species exhibits chaotic behavior. When the value of $\alpha - \delta - \frac{cM_2}{1+m_0} > 2$, the fear effect has a stabilizing influence on the population of the first species. On the other hand, when the value lies between $0 < \alpha - \delta - \frac{cM_2}{1+m_0} < 2$ and $\alpha - \delta > 2$, the density of the first species continues to decrease and eventually

leads to extinction as the fear effect increases.

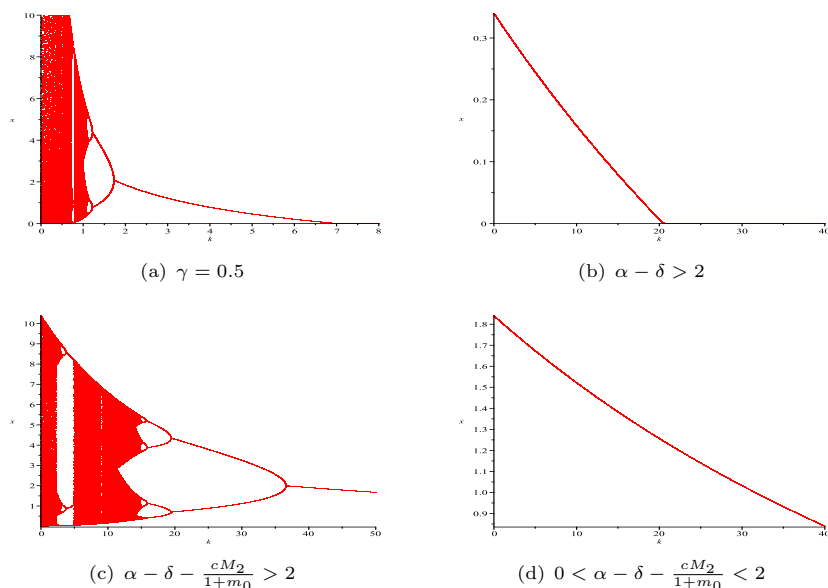


Figure 9. The bifurcation plot of x corresponding to the bifurcation parameter k .

Example 6.4. (the Beddington-DeAngelis functional response on the first species) To explore the impact of Beddington-DeAngelis functional response on system (1.7), the parameters in system (1.7) are set to be the same as those in system (1.4). From Figure 10(a) and 10(b), it can be seen that the density of the first species in system (1.7) is greater than that of the first species in system (1.4), i.e., the Beddington-DeAngelis functional response reduces the extinction rate of the first species. This research finding shows that system (1.7) may be more advantageous in terms of resource utilization, reproduction, survival rate, etc., and also reflects from the side that system (1.7) is more in line with the living environment of the species. In addition, Figure 10(c) shows that both k and c have an impact on the bifurcation of system (1.7).

7. Summary and discussion

A discrete amensalism system with Beddington-DeAngelis functional response and fear effect on the first species is proposed. Previously, Zhu and Chen [51] discussed the system (1.5), which is continuous-time and determined the local and global stabilities by analyzing the phase diagram of the system. In this work, we obtain the discrete-time system (1.7) by applying the piecewise constant parameter method in the model, discussed by Zhu and Chen. For system (1.7), we first discuss the local stability of all fixed points. Then, we analyze the bifurcation phenomenon of the system with the help of the existence, local stability, and global stability of equilibrium points. This analysis reveals a wide range of dynamical behaviors, including pitchfork bifurcation, transcritical bifurcation, fold bifurcation, flip bifurcation, and chaos. Finally, extensive numerical simulations are conducted to confirm these analytical results. The complex dynamical behaviors exhibited by a discrete system help to understand the dynamics of an amensalism system with Beddington-DeAngelis

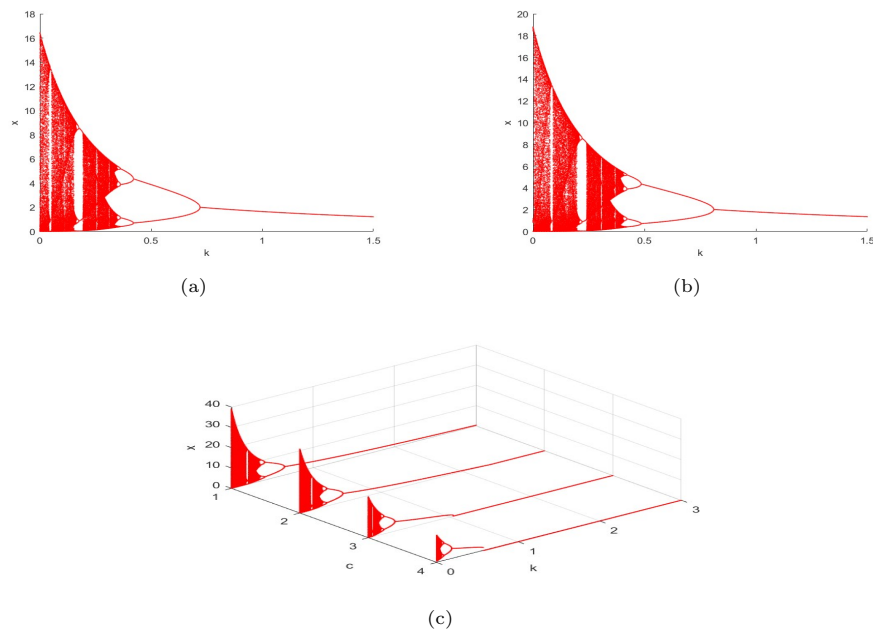


Figure 10. (a) represent bifurcation plot of system (1.4) with $e_1 = 4.3$, $e_2 = 0.3$, $a_2 = 1$, $c = 0.2$; (b) represent bifurcation plot of system (1.7) with $\alpha = 5$, $\beta = 0.5$, $c = 1$, $\delta = 1$, $\gamma = 1$; (c) represent bifurcation plot of system (1.7) in (k, c, x) -space.

functional response and fear effects on the first species. Additionally, numerical simulations have revealed several intriguing phenomena:

Compared with the system that does not contain the Beddington-DeAngelis functional response and the fear effect by Zhou [50], system (1.7) exhibits transcritical bifurcation, indicating that system (1.7) has stronger ecological adaptability and diversity maintenance capabilities, and can exhibit stronger self-regulation capabilities when facing environmental changes and ecological disturbances. Moreover, in comparison to system (1.4) that excludes the Beddington-DeAngelis functional response examined by Li [25], system (1.7) has more equilibrium points, which also leads to the occurrence of fold bifurcation at E_3^* . The occurrence of a fold bifurcation implies that the fixed point may change its stability at different parameter values. Therefore, the introduction of Beddington-DeAngelis functional response can change the number and stability of equilibrium points and produce more complex dynamic behaviors. Furthermore, although the fixed point structure of system (1.7) is identical to that of the continuous counterpart (1.5), the same fixed points are not always stable in the same region. According to Zhu and Chen [51], E_2^* is always a saddle and E_1^* is always a stable node. Here we find that the type of E_2^* depends on the value of γ and the type of E_1^* depends on the value of γ , β , c and k (see Theorem 2.6). Furthermore, system (1.7) exhibits several bifurcation phenomena that are absent in the continuous system (1.5). These include flip bifurcation and fold bifurcation, which are distinct to discrete systems. Therefore, compared with continuous systems, discrete systems exhibit richer dynamic behaviors and more effective numerical simulation results.

Theorems 3.1 and 3.2 provide detailed and rigorous demonstrations. It may

be inferred that when $0 < k < k^{**}$, the two species can coexist stably. When $k > k^{**}$, the second species exists and the first species becomes extinct, which shows that the implementation of suitable fear effect practices significantly contributes to the preservation of stable coexistence within system (1.7). If the fear effect exceeds a certain threshold, the first species will become extinct as a result. Hence, this suggests that ecological managers should evaluate the intensity of the fear effect in the ecosystem and keep it within an acceptable range by monitoring behavioral, physiological and population data when intervening. In agriculture or forestry management, fear effects can be used to control pests. For example, simulating the presence of natural enemies (such as making threatening sounds) can reduce pest damage to crops.

In the experiment on the Tibetan alpine meadow [44], the authors observed through data detection that the presence of grasshoppers would reduce the birth rate of caterpillars and delay their development. However, the existing data can only intuitively reflect the existence of the fear effect. It is unclear how to further explore the specific impact of the fear effect and what functional reactions are used between the two populations to characterize their impact. Therefore, more in-depth experimental research is still needed. In the future, based on the existing experiments and combined with the results of this paper, the relevant experimental parameters of amensalism can be adjusted to further observe and analyze the specific impact of the fear effect on the partial harm population. In general, the research results in the paper are particularly instructive for experiments with small populations, non-overlapping generations, and those that can be observed under laboratory conditions.

The chaos phenomenon is caused by flip bifurcation, which is unique to discrete systems. When chaos occurs in the second species, due to the persistence of the second species, the fear effect can either stabilize the first species or accelerate the extinction of the first species. In addition, we control the flip bifurcation generated at E_1^* through a chaos control strategy, so that E_1^* tends to be stable, i.e., the two species can coexist stably.

From an ecological perspective, despite the large size of predator populations, prey are able to survive under saturated fear pressure due to physiological effects. Studies have shown that fear can cause predators to coexist in stable or oscillating forms, so the sensitivity of prey to predators may contribute to the protection of individual animals and the reduction of predator population densities [13, 36]. However, no scholar has incorporated this saturated fear cost into the study of biased system dynamics, and we intend to further explore the subject in subsequent research.

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