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BIFURCATION ANALYSIS IN A MODIFIED LESLIE-GOWER WITH NONLOCAL COMPETITION AND BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

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Abstract In this paper, a diffusive predator-prey system with nonlocal competition and Beddington-DeAngelis functional response is considered. After analyzing the influence of the selected parameters on the existence, multiplicity and stability of the nonhomogeneous steady-state solution, it is obtained that there is an unstable positive nonconstant steady-state in the neighborhood of the positive constant steady-state. Compared with the system without nonlocal competition, the system with nonlocal competition can generate Hopf-Hopf bifurcation under certain conditions. Through the qualitative analysis, the normal form at the Hopf-Hopf bifurcation singularity is calculated to analyze the different dynamic properties exhibited by the system in different parameter regions. In order to illustrate the feasibility of the obtained results and the dependence of the dynamic behavior on the nonlocal competition, numerical simulations are carried out. Through the numerical simulations, it is further shown that under certain conditions, the nonlocal competition will lead to the generation of stable spatially inhomogeneous periodic solutions and stable spatially inhomogeneous quasi-periodic solutions.

Keywords Predator-prey, nonlocal competition, steady-state bifurcation, Hopf-Hopf bifurcation.

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1. Introduction

In the real world, the predator-prey relationship is seen as a common population interaction in nature. For many scholars, the study of predator-prey relationship is also an important bifurcation of biological mathematics [18, 34, 37, 43]. Many scholars have studied the dynamic properties of some systems from different perspectives [16,25,35,36], among them, the dynamic properties of the Leslie-Gower system and the various modifications of it has received extensive attention [22, 24, 27]. In

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this paper, we are considering the modified Leslie-Gower system

$$\begin{cases} \dfrac{du}{dt} = r_1 u \left(1 - \dfrac{u}{K}\right) - \varphi(u)v, \\ \dfrac{dv}{dt} = r_2 v \left(1 - \dfrac{\beta v}{u+b}\right), \\ u(0) > 0, \quad v(0) > 0, \end{cases} \tag{1.1}$$

where u denotes prey population densities; v denotes predator population densities; $\varphi(u)$ is functional response. The carrying capacity of the predator is proportional to the prey and other food. Many scholars have studied the system (1.1) with Holling type functional response [1,28,39]. In [33], the authors study a spatial predator-prey system with Leslie-Gower and Holling II functional response. In [7], the authors considered a Leslie-Gower system with Holling II functional response and time delay. In these studies, since the Holling type functional response is dependent on prey, it is impossible to simulate the interference between predators. Therefore, some biologists believe that in many cases, when predators have to compete or cooperate to obtain food, the functional response in the predator-prey system often depends on the predator. In the study of many scholars, it is shown that the functional response in the predator-prey system depends on the predator quite frequently [4, 5, 9, 20]. After a lot of experiments further observations, many scholars have shown that predators do interfere with each other's activities, so that prey changes its behavior, thereby increasing the threat to predators and generating competitive effects. In [32], the author studied the Beddington-DeAngelis, Crowley-Martin, Hassell-Varley functional response functions, which are predator-dependent. It is verified that Beddington-DeAngelis and Hassell-Varley systems are more suitable for predicting the asymptotic feeding rates at high prey abundance independent of predator abundance and Crowley-Mart systen is more suitable for predicting the asymptote dependent on predator abundance. Since this paper considers the asymptotic feeding rate at high prey abundance independent of predator abundance, we choose Beddington-DeAngelis functional response. In [42], a modified Leslie-Gower with diffusion and Beddington-DeAngelis functional response is studied by Yang, is in the following form

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \Delta u + r_1 u \left(1 - \dfrac{u}{K}\right) - \dfrac{\alpha u v}{p + u + h v}, & x \in (0, l\pi), t > 0, \\ \dfrac{\partial v}{\partial t} = D_2 \Delta v + r_2 v \left(1 - \dfrac{\beta v}{u+b}\right), & x \in (0, l\pi), t > 0, \\ u_x(t,0) = v_x(t,0) = 0, u_x(t,l\pi) = v_x(t,l\pi) = 0, & t > 0, \\ u(0,x) = u_0(x) \geq 0, v(0,x) = v_0(x) \geq 0, & x \in [0, l\pi]. \end{cases} \tag{1.2}$$

Here, u denotes prey population densities; v denotes predator population densities.

In the prey-predator system, for the interaction between different types of populations, there will be complex patterns. In some spatio-temporal models of prey-predator systems, there may be some important mechanisms such as diffusion, advection and gliding. In order to facilitate the study, we assume that the prey and the predator are in an isolated patch. Additionally in this paper only consider the diffusion of the spatial domain and ignore the impact of immigration. Generally speaking, for the competition between biological populations, when they compete for common resources, their competitive effect may depend on the population density of such resources near them. Therefore, considering that the interaction among species is nonlocal, it is reasonable to introduce nonlocal competition in the population system. When the reaction-diffusion system with nonlocal competition term is extended in space, complex dynamic behaviors including spatially inhomogeneous quasi-periodic solutions will be generated. The nonlocal competition term describes the mobility of a species in its spatial position. By introducing an integral term with an appropriate kernel function into the model, the nonlocal competition term is incorporated into the predator-prey system studied. In [3, 10], by modifying the parameters $\frac{u}{K}$ as $\frac{1}{K}\int_\Omega G(x,y)u(y,t)dy$, the authors introduced the nonlocal competition effect in prey, where the average kernel function $G(x,y) = \frac{1}{|\Omega|}$ with $\Omega = (0, l\pi)$. In [30], Y. H. Peng and G. Y. Zhang consider a predator-prey system with herding behavior and nonlocal prey competition. The influence of nonlocal competition term on system dynamics in bounded domain is studied. Then, they obtain the conditions of Hopf bifurcation and Turing bifurcation in the system with nonlocal competition. Finally, it is concluded that nonlocal competition can destroy the stability of predator-prey system. At present, many scholars have considered the predator-prey models with nonlocal competition [6, 26, 38, 46]. Their results show that nonlocal competition will involve more complex spatiotemporal dynamical properties and nonlocal competition plays an important role in the generation of Hopf-Hopf bifurcation.

Therefore, in order to better describe the predator's feeding situation, the interaction between species are nonlocal, we consider a modified Leslie-Gower with diffusion and Beddington-DeAngelis functional response in system (1.2) including a nonlocal competition as the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1\Delta u + r_1 u\left(1 - \dfrac{1}{K}\displaystyle\int_0^{l\pi}\dfrac{1}{l\pi}u(y,t)\mathrm{d}y\right) - \dfrac{\alpha uv}{p+u+hv}, & x \in (0, l\pi), t > 0, \\ \dfrac{\partial v}{\partial t} = D_2\Delta v + r_2 v\left(1 - \dfrac{\beta v}{u+b}\right), & x \in (0, l\pi), t > 0, \\ u_x(t,0) = v_x(t,0) = 0, u_x(t,l\pi) = v_x(t,l\pi) = 0, & t > 0, \\ u(0,x) = u_0(x) \geq 0, v(0,x) = v_0(x) \geq 0, & x \in [0, l\pi]. \end{cases} \tag{1.3}$$

Taking wolves and rabbits as an example, we assume that rabbits are the main food source of wolves, wolves are the main predators of rabbits and the forest capacity will limit their growth. In order to facilitate the study, we assume that the wolves and rabbits are in an isolated patch. Further in this paper, we only consider the diffusion of the spatial domain and ignore the impact of immigration. In Table 1, we give the meaning of parameters. All parameters involved with the model are positive. The carrying capacity of the predator is proportional to the prey and other food.

By transforming the parameters as follows

$$u = \widetilde{u},\ \beta v = \widetilde{v},\ r_1 t = \widetilde{t},\ \frac{h}{\beta} = s,\ \frac{\alpha}{r_1\beta} = a,\ \frac{r_2}{r_1} = c,\ \frac{D_1}{r_1} = d_1,\ \frac{D_2}{r_1} = d_2,$$

Table 1. The meaning of the parameters of the system (1.3).

Parameters	The meanings of parameters	Unit
u	The rabbits population densities	$ind/km^2 \times year^{-1}$
v	The wolves population densities	$ind/km^2 \times year^{-1}$
D_1	Rabbits corresponding diffusive rates	$year^{-1}$
D_2	Wolves corresponding diffusive rates	$year^{-1}$
K	The carrying capacity of the rabbits	$ind/km^2 \times year^{-1}$
r_1	The growth rate of rabbits	$ind/km^2 \times year^{-1}$
r_2	The growth rate of wolves	$ind/km^2 \times year^{-1}$
β	The number of rabbits required to support one wolves at equilibrium when v equals $\frac{u+b}{\beta}$	$ind \times year^{-1}$
α	The wolves consume the maximum amount of rabbits per unit time	$ind \times year^{-1}$
p	Half-saturation constant	
h	Measure the degree to which rabbits interferes with each other	
b	Measures the extent to which environment provides protection to rabbits	

the system (1.3) (drop the "$\sim$") can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u\left(1 - \dfrac{1}{K}\displaystyle\int_0^{l\pi} \dfrac{1}{l\pi} u(y,t)\mathrm{d}y\right) - \dfrac{auv}{p+u+sv}, & x \in (0, l\pi), t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + cv\left(1 - \dfrac{v}{u+b}\right), & x \in (0, l\pi), t > 0, \\ u_x(t,0) = v_x(t,0) = 0, u_x(t,l\pi) = v_x(t,l\pi) = 0, & t > 0, \\ u(0,x) = u_0(x) \geq 0, v(0,x) = v_0(x) \geq 0, & x \in [0, l\pi]. \end{cases} \tag{1.4}$$

For more convenient calculation, many scholars consider the normal form of the Hopf-Hopf bifurcation of the system in the case of resonance, weak resonance or non-resonance [21,29,31,40,45]. In [12], the author mainly studies the infinite-level normal form of supernormal singularities of vector fields with non-resonance double Hopf bifurcation. In [8], the author establishes the normal form of double-Hopf bifurcation in the case of non-resonance or weak resonance, where the ratio of the two amplitudes $\omega_1 : \omega_2$ is not $1, 2, 3, 4$. In addition, according to the spatial pattern, the derivation process of the normal form is divided into three cases. In this paper, in order to facilitate the calculation, we consider the case of non-resonance.

The structure of this paper is as follows. In the second part, we analyze the steady-state bifurcation. In the third part, we study the stability of the positive equilibrium. In the fourth part, we study the existence of Hopf-Hopf bifurcation and the normal form of Hopf-Hopf bifurcation. In the fifth part, we carry out numerical simulation and then analyze the properties of the equilibrium points in the seven bifurcation regions to verify the correctness of the theoretical results. In the sixth part, a brief conclusion of this paper is given.

2. Steady-state bifurcation analysis

In this section, we mainly study the Steady-State bifurcation. First, we study the constant steady states of (1.4) which satisfy

$$\begin{cases} d_1\Delta u + u(1 - \dfrac{1}{K}\dfrac{M*u}{l\pi}) - \dfrac{auv}{p+u+sv} = 0, & x \in (0, l\pi), \\ d_2\Delta v + cv(1 - \dfrac{v}{u+b}) = 0, & x \in (0, l\pi), \\ u_x(0) = v_x(0) = 0, u_x(l\pi) = v_x(l\pi) = 0, & x \in [0, l\pi], \end{cases} \tag{2.1}$$

where

$$M * u = \int_0^{l\pi} u(y,t)\mathrm{d}y.$$

Obviously, system (1.4) has three boundary constant steady states: $E_0 = (0, 0)$, $E_1 = (K, 0)$, $E_2 = (0, b)$. By computing, we can obtain that

$$(1 + s)u^2 + (-K + aK + p + bs - Ks)u + abK - pK - bsK = 0,$$

then we can get the following assumption

Assumption 2.1. *One of the following conditions holds:*

(i) $p > \max\{K - aK - bs + Ks, ab - bs\}$;

(ii) $p = K - aK - bs + Ks$;

(iii) $\dfrac{-2K - 2aK - 2bs - 2Ks + 4\sqrt{aK(K+b)(1+s)}}{2} < p < K - aK - bs + Ks.$

In this paper, we mainly discuss the coexistence of wolves and rabbits density, i.e. there exists positive equilibrium point $E^* = (u^*, v^*)$. Then we discuss the stability of E^*. Define $H^2(\Omega) = \{u(x) : \frac{\partial^k u}{\partial x^k} \in L^2(\Omega), k = 0, 1, 2\}$, $\mathbb{X} = \{u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0, x \in \partial\Omega\}$, $\mathbb{Y} = L^2(\Omega)$. Let

$$\mathcal{F}(h_1, h_2) = \begin{bmatrix} d_1\Delta + \left(1 - \dfrac{1}{l\pi K}\displaystyle\int_0^{l\pi} h_1 \mathrm{d}y\right) - h_1\dfrac{1}{l\pi K}M - \dfrac{ah_2(p+sh_2)}{(p+h_1+sh_2)^2} & -\dfrac{ah_1(p+h_1)}{(p+h_1+sh_2)^2} \\ \dfrac{ch_2{}^2}{(b+h_1)^2} & d_2\Delta + c + \dfrac{2ch_2}{h_1+b} \end{bmatrix},$$

where $(h_1, h_2) \in \mathbb{X}^2$, then we can get

$$\mathcal{F}(u^*, v^*) = \begin{bmatrix} d_1\Delta - u^*\dfrac{1}{l\pi K}M + \dfrac{au^*(u^*+b)}{(p+u^*+su^*+bs)^2} & -\dfrac{au^*(p+u^*)}{(p+u^*+su^*+bs)^2} \\ c & d_2\Delta - c \end{bmatrix}.$$

Define the normalized eigenfunction α_{mk} corresponding to λ_m, where $m \geq 0$, $1 \leq k \leq n_m$. Then a complete orthonormal basis of $L^2(\overline{\Omega})$ is formed by the set $\{\alpha_m : m \geq 0,\ 1 \leq k \leq n_m\}$. When $n_0 = 1$, we can get $\alpha_{01}(x) \equiv \frac{1}{\sqrt{|\Omega|}} > 0$, where $x \in \Omega$. Define $\mathcal{A}_m = (\alpha_{m1},\ \alpha_{m2}, \cdots, \alpha_{mn_m})^{\mathrm{T}}$, $-\Delta\mathcal{A}_m = F_m\mathcal{A}_m$, where $F_m = (f_{mks})$ is an $n_m \times n_m$ matrix and λ_m is the only eigenvalue of F_m. Define $u = \sum_{m=0}^{\infty} u_m\mathcal{A}_m$, where $u_m = (u_{m1}, u_{m2}, \cdots, u_{mn_m})$ and $u_{mk} = (u^1_{mk}, u^2_{mk})^{\mathrm{T}} \in \mathbb{C}^2$ for $k \in \{1, 2, \cdots, n_m\}$. For $\mathcal{F}(u^*, v^*)u = hu$, we can get

$$\begin{cases} \left(1 - \dfrac{u^*}{K}\right) u^1_{mk}\alpha_{mk} - u^* \dfrac{u^1_{mk}}{Kl\pi} M * \alpha_{mk} - \dfrac{a(u^*+b)(p+su^*+bs)}{(p+u^*+su^*+bs)^2} u^1_{mk}\alpha_{mk} \\ -hu^1_{mk}\alpha_{mk} - d_1 \displaystyle\sum_{s=1}^{n_m} u^1_{ms} f_{msk}\alpha_{mk} - \dfrac{au^*(p+u^*)}{(p+u^*+su^*+bs)^2} u^2_{mk}\alpha_{mk} = 0, \\ cu^1_{mk}\alpha_{mk} - cu^2_{mk}\alpha_{mk} - hu^2_{mk}\alpha_{mk} - d_2 \displaystyle\sum_{s=1}^{n_m} u^2_{ms} f_{msk}\alpha_{mk} = 0, \end{cases}$$

where $m \in \mathbb{N}_0$ and $k \in \{1, 2, \cdots, n_m\}$,

$$\begin{cases} \left(1 - \dfrac{u^*}{K}\right) u^1_{mk}\alpha_{mk} - \dfrac{a(u^*+b)(p+su^*+bs)}{(p+u^*+su^*+bs)^2} u^1_{mk}\alpha_{mk} - hu^1_{mk}\alpha_{mk} \\ -d_1 \displaystyle\sum_{s=1}^{n_m} u^1_{ms} f_{msk}\alpha_{mk} - \dfrac{au^*(p+u^*)}{(p+u^*+su^*+bs)^2} u^2_{mk}\alpha_{mk} = 0, \\ cu^1_{mk}\alpha_{mk} - cu^2_{mk}\alpha_{mk} - hu^2_{mk}\alpha_{mk} - d_2 \displaystyle\sum_{s=1}^{n_m} u^2_{ms} f_{msk}\alpha_{mk} = 0, \end{cases}$$

where $m \in \mathbb{N}$ and $k \in \{1, 2, \cdots, n_m\}$.

Then we can get $B_m(h)\mathbf{u}_m = 0$, where $\mathbf{u}_m = (u^1_{m1}, u^1_{m2}, \cdots, u^1_{mnm}, u^2_{m1}, u^2_{m2}, \cdots, u^2_{mnm}) \in \mathbb{C}^{2nm}$ and

$$B_m(h) = \begin{bmatrix} [l_1 - h]Id_{nm\times nm} - d_1F_m^T & -\dfrac{l_2}{c} Id_{nm\times nm} \\ cId_{nm\times nm} & (-c-h)Id_{nm\times nm} - d_2F_m^T \end{bmatrix},$$

with $l_1 = 1 - \frac{u^*}{K} - \frac{a(u^*+b)(p+su^*+bs)}{(p+u^*+su^*+bs)^2}$, $l_2 = \frac{cau^*(p+u^*)}{(p+u^*+su^*+bs)^2} > 0$, because of $1 - \frac{u^*}{K} - \frac{a(u^*+b)}{p+u^*+su^*+bs} = 0$, then we can get $l_1 = \frac{a(u^*+b)u^*}{(p+u^*+su^*+bs)^2} > 0$. Define $\Gamma : \mathbb{C} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\Gamma(h, \lambda, l_1, l_2) \triangleq h^2 + (d_1\lambda + d_2\lambda + c - l_1)h + (d_1\lambda - l_1)(d_2\lambda + c) + l_2 = 0$.

Similarly, for $m = 0$ we can also obtain

$$B_0(h) = \begin{bmatrix} l_1 - h - \dfrac{u^*}{K} & -\dfrac{l_2}{c} \\ c & -c-h \end{bmatrix},$$

$\Gamma_0(h, l_1, l_2) \triangleq h^2 + (-l_1 + \frac{u^*}{K} + c)h + c\frac{u^*}{K} - l_1c + l_2 = 0$.

Therefore, the eigenvalues of $\mathcal{F}(u^*, v^*)$ are all the roots of $\Gamma(h, \lambda_m, l_1, l_2) = 0$ for $m \in \mathbb{N}$ and $\Gamma_0(h, l_1, l_2) = 0$.

Remark 2.1. The system of (1.4) without nonlocal competition is

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u - \dfrac{u^2}{K} - \dfrac{auv}{p+u+sv}, & x \in (0, l\pi), t > 0, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + cv(1 - \dfrac{v}{u+b}), & x \in (0, l\pi), t > 0, \\ u_x(t,0) = v_x(t,0) = 0, u_x(t, l\pi) = v_x(t, l\pi) = 0, & t > 0, \\ u(0,x) = u_0(x) \geq 0, v(0,x) = v_0(x) \geq 0, & x \in [0, l\pi]. \end{cases} \tag{2.2}$$

Similarly, for a positive constant steady state solution $E^{*\prime}$ of the system (2.2), we can get the eigenvalues of $\mathcal{F}'(u^*, v^*)$ are all the roots of $\Gamma'(h, \lambda_m, l_1, l_2) = 0$, $m \in \mathbb{N}_0$, where

$$\Gamma'(h, \lambda, l_1, l_2) = h^2 + (d_1\lambda + d_2\lambda - l_1 + c + \frac{u^*}{K})h + (d_1\lambda - l_1 + \frac{u^*}{K})(d_2\lambda + c) + l_2.$$

Then, we study the effects of parameters (l_1, l_2) on the existence and stability of nonhomogeneous steady-state solutions. Define $\lambda_* > 0$ is a simple eigenvalue of the linear operator $-\Delta$ with the homogeneous Neumann boundary conditions and α_* is the corresponding eigenfunction satisfying $\|\alpha_*\|_{L^2(\Omega)} = 1$. Firstly, we give the following assumption

Assumption 2.2. $(d_1\lambda_* - {l_1}^*)(d_2\lambda_* + c) + {l_2}^* = 0,\ {l_1}^* \neq (d_1 + d_2)\lambda_* + c,\ {l_2}^* + c\frac{u^*}{K} - {l_1}^* c \neq 0,\ \mathit{for\ any}\ m \in \mathbb{N}\backslash\{*\}, (d_1\lambda_m - {l_1}^*)(d_2\lambda_m + c) + {l_2}^* \neq 0.$

According to Assumption 2.2, we can get $d_1\lambda_* - {l_1}^* \neq 0$.

Remark 2.2. For the system (2.2), we give the following assumption.

Assumption 2.3. $(d_1\lambda_* - {l_1}^* + \frac{u^*}{K})(d_2\lambda_* + c) + {l_2}^* = 0,\ {l_1}^* \neq (d_1 + d_2)\lambda_* + c + \frac{u^*}{K},\ \mathit{for\ any}\ m \in \mathbb{N}\backslash\{*\},\ (d_1\lambda_m - {l_1}^* + \frac{u^*}{K})(d_2\lambda_m + c) + {l_2}^* \neq 0.$

Define

$$\mathcal{B}^*(h, l_1, l_2) = \begin{bmatrix} -d_1\lambda_* + l_1 - h & -\dfrac{l_2}{c} \\ c & -h - c - d_2\lambda_* \end{bmatrix},$$

then we can get $\mathcal{B}^*(0, {l_1}^*, {l_2}^*)\bar{b} = (0,0)^{\mathrm{T}}$, $\mathcal{B}^{*\mathrm{T}}(0, {l_1}^*, {l_2}^*)\bar{a} = (0,0)^{\mathrm{T}}$, where

$$\bar{b} = \begin{bmatrix} 1 \\ \dfrac{c}{{l_2}^*}({l_1}^* - d_1\lambda_*) \end{bmatrix}, \qquad \bar{a} = \begin{bmatrix} \dfrac{{l_2}^*}{d_1\lambda_* - {l_1}^*} \\ \dfrac{{l_2}^*}{c} \end{bmatrix}.$$

Then we have $\bar{a}^{\mathrm{T}} \cdot \bar{b} = {l_1}^* - c - (d_1 + d_2)\lambda_*$, $\bar{a}^{\mathrm{T}} \cdot \mathcal{B}^*(0, l_1, l_2)\bar{b} = l_2 - (c + d_2\lambda_*)(l_1 - d_1\lambda_*) \triangleq \mathcal{G}(l_1, l_2)$.

Although l_1 and l_2 are variables, changing some parameters in the system (1.4) can still fix E^*. Next, in the neighborhood of positive constant steady state E^*, we will investigate the existence of positive nonconstant steady state in the system (1.4). i.e. solve a nonlinear functional equation $F(\mathbf{u}, l_1, l_2) = 0$, where $\mathbf{u}$ is near E^*

in $\mathbb{X}^2$ and (l_1, l_2) is near $({l_1}^*, {l_2}^*)$ in $\mathbb{R}^2$, where

$$F(\mathbf{u}, l_1, l_2) = \begin{bmatrix} d_1\Delta u + u\left(1 - \dfrac{1}{Kl\pi}M * u\right) - \dfrac{auv}{p+u+sv} \\ d_2\Delta v + cv\left(1 - \dfrac{v}{u+b}\right) \end{bmatrix}.$$

Define

$$\mathcal{S}_{l_1,l_2}\bar{\mathbf{u}} = \mathcal{F}(u^*, v^*)\bar{\mathbf{u}} = \begin{bmatrix} d_1\Delta u_1 - u^*\dfrac{1}{Kl\pi}M * u_1 + l_1 u_1 - \dfrac{l_2}{c}u_2 \\ cu_1 + d_2\Delta u_2 - cu_2 \end{bmatrix},$$

$$\mathcal{T}_{l_1,l_2}(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = \begin{bmatrix} -\dfrac{1}{Kl\pi}(u_1 M * v_1 + v_1 M * u_1) + \dfrac{2av^*(p+sv^*)}{(p+u^*+sv^*)^3}u_1v_1 + \dfrac{2asu^*(p+u^*)}{(p+u^*+sv^*)^3}u_2v_2 \\ -\dfrac{ap^2 + apu^* + av^*ps + 2asu^*v^*}{(p+u^*+sv^*)^3}(u_1v_2 + u_2v_1) \\ -\dfrac{2cv^{*2}}{(u^*+b)^3}u_1v_1 - \dfrac{2c}{u^*+b}u_2v_2 + \dfrac{2cv^*}{(u^*+b)^2}(u_1v_2 + u_2v_1) \end{bmatrix},$$

$$\mathcal{B}_{l_1,l_2}(\bar{\mathbf{u}}, \bar{\mathbf{v}}, \bar{\mathbf{w}}) = \begin{bmatrix} \dfrac{2ap^2 + 4au^*v^*s + 2au^*p - 3as^2v^{*2}}{(p+u^*+sv^*)^4}(w_2u_1v_1 + w_1u_1v_2 + w_1u_2v_1) \\ +\dfrac{-2asu^{*2} + 4as^2u^*v^* + 2asp^2 + 2as^2pv^*}{(p+u^*+sv^*)^4}(w_1u_2v_2 + w_2u_1v_2 + w_2u_2v_1) \\ -\dfrac{6as^2u^*(p+u^*)}{(p+u^*+sv^*)^4}w_2u_2v_2 - \dfrac{6av^*(p+sv^*)}{(p+u^*+sv^*)^4}w_1u_1v_1 \\ \dfrac{6cv^{*2}}{(u^*+b)^4}w_1u_1v_1 - \dfrac{4cv^*}{(u^*+b)^3}(w_2u_1v_1 + w_1u_1v_2 + w_1u_2v_1) \\ +\dfrac{2c}{(u^*+b)^2}(w_1u_2v_2 + w_2u_1v_2 + w_2u_2v_1) \end{bmatrix},$$

for $\bar{\mathbf{u}} = (u_1, u_2)^{\mathrm{T}}$, $\bar{\mathbf{v}} = (v_1, v_2)^{\mathrm{T}}$, $\bar{\mathbf{w}} = (w_1, w_2)^{\mathrm{T}}$.

We can get $\bar{a}^{\mathrm{T}} \cdot \mathcal{S}_{l_1,l_2}(\bar{b}\varphi_*) = \mathcal{G}(l_1, l_2)\varphi_*$. In [11], define the adjoint operator $\mathcal{S}^*_{l_1,l_2}$ of $\mathcal{S}_{{l_1}^*,{l_2}^*}$ is $\mathcal{F}^{\mathrm{T}}(u^*, v^*)$, then we can get Ker $\mathcal{S}_{{l_1}^*,{l_2}^*} = \mathrm{span}\{\bar{b}\varphi_*\}$, Ker $\mathcal{S}^*_{{l_1}^*,{l_2}^*} = \mathrm{span}\{\bar{a}\varphi_*\}$. By assumption Assumption 2.2, we have $\mathbb{X}^2 = \mathrm{Ker}\ \mathcal{S}_{{l_1}^*,{l_2}^*} \oplus \mathbb{X}_0$, $\mathbb{Y}^2 = \mathrm{Ker}\ \mathcal{S}^*_{{l_1}^*,{l_2}^*} \oplus \mathbb{Y}_0$, where $\mathbb{X}_0 = \{\mathbf{w} \in \mathbb{X}^2 : \langle \varphi_*, \bar{b}^{\mathrm{T}} \cdot \mathbf{w}\rangle = 0\}$, $\mathbb{Y}_0 = \{\mathbf{w} \in \mathbb{Y}^2 : \langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot \mathbf{w}\rangle = 0\}$. Define $P : \mathbb{Y}^2 \to \mathbb{Y}_0$, $I - P : \mathbb{Y}^2 \to \mathrm{Ker}\mathcal{S}^*_{{l_1}^*,{l_2}^*}$. Then we can get the bifurcation equation corresponding to the equation $F(\mathbf{u}, l_1, l_2) = 0$ by performing the Lyapunov-Schmidt reduction is

$$\begin{cases} PF(E^* + z\bar{b}\varphi_* + \varsigma, l_1, l_2) = 0, \\ (I-P)F(E^* + z\bar{b}\varphi_* + \varsigma, l_1, l_2) = 0, \end{cases} \tag{2.3}$$

where $z \in \mathbb{R}$, $\varsigma \in \mathbb{X}_0$, for any $\mathbf{u} \in \mathbb{X}^2$, $P\mathbf{u} = \mathbf{u} - \frac{\langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot \mathbf{u}\rangle}{\bar{a}^{\mathrm{T}} \cdot \bar{b}}\bar{b}\varphi_*$. Define a real number field $\mathbb{R}$, an open neighborhood $\mathcal{N}$ of 0 in $\mathbb{R}$, an open neighborhood Π of $({l_1}^*, {l_2}^*)$ in

$\mathbb{R}^2$. Then define a continuously differentiable map $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)^{\mathrm{T}} : \mathcal{N} \times \Pi \to \mathbb{X}_0$, then we can get $\mathcal{W}(0, l_1, l_2) = 0$, for $z \in \mathbb{R}$. Let $\varsigma = \mathcal{W}(z, l_1, l_2)$, we can get

$$PF(E^* + z\bar{b}\varphi_* + \mathcal{W}(z, l_1, l_2), l_1, l_2) = 0,\ \langle \varphi_*, \bar{b}^{\mathrm{T}} \cdot \mathcal{W}(z, l_1, l_2)\rangle = 0 \tag{2.4}$$

and

$$\mathcal{J}(z, l_1, l_2) \triangleq (I - P)F(E^* + z\bar{b}\varphi_* + \mathcal{W}(z, l_1, l_2), l_1, l_2) = 0, \tag{2.5}$$

where $(z, l_1, l_2) \in \mathcal{N} \times \Pi$. We can know the equation (2.5) is the bifurcation map of $F(\mathbf{u}, l_1, l_2) = 0$ and satisfies $\mathcal{J}(0, l_1, l_2) = 0$, $\mathcal{J}_z(0, l_1, l_2) = 0$. Define a reduced mapping g from $\mathbb{R}^3$ to $\mathbb{R}$, where $g(z, l_1, l_2) = \langle \varphi_*, \mathcal{J}(z, l_1, l_2)\rangle = \langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot F(E^* + z\bar{b}\varphi_* + \mathcal{W}(z, l_1, l_2), l_1, l_2)\rangle$. Then we have

$$g(0, l_1, l_2) = 0,\ g(z, l_1, l_2) = \mathcal{G}_{l_1,l_2} z + \frac{1}{2}\sigma_1 z^2 + \frac{1}{6}\sigma_2 z^3 + \circ(|l_1 - {l_1}^*|, |l_2 - {l_2}^*|, |z|^3),$$

where $\sigma_1 = \langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot \mathcal{T}_{{l_1}^*,{l_2}^*}(\bar{b}\varphi_*, \bar{b}\varphi_*)\rangle$, $\sigma_2 = 3\langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot \mathcal{T}_{{l_1}^*,{l_2}^*}(\bar{b}\varphi_*, \mathcal{W}_{zz}(0, {l_1}^*, {l_2}^*))\rangle + \langle \varphi_*, \bar{a}^{\mathrm{T}} \cdot \mathcal{F}_{{l_1}^*,{l_2}^*}(\bar{b}\varphi_*, \bar{b}\varphi_*, \bar{b}\varphi_*)\rangle$. Then we give the following two cases.

Case I. $\sigma_1 \neq 0$, from the implicit function theorem, we can get there exists a positive constant δ_1 and a continuously differentiable function $j_1 : ({l_1}^* - \delta_1, {l_1}^* + \delta_1) \times ({l_2}^* - \delta_1, {l_2}^* + \delta_1) \to \mathbb{R}$, such that $g(j_1(l_1, l_2), l_1, l_2) = 0$ for $(l_1, l_2) \in ({l_1}^* - \delta_1, {l_1}^* + \delta_1) \times ({l_2}^* - \delta_1, {l_2}^* + \delta_1)$. Then, we have $j_1(l_1, l_2) = -\frac{2\mathcal{G}_{l_1,l_2}}{\sigma_1} + o(|l_1 - {l_1}^*|, |l_2 - {l_2}^*|)$. Then, we have the following theorem.

Theorem 2.1. *Assume the assumptions Assumption 2.1 and Assumption 2.2 hold, if* $\sigma_1 \neq 0$, *then the system* (2.1) *admits a nonconstant positive steady state* $\mathbf{E}_{l_1,l_2} \in \mathbb{X}^2$, *where*

$$\mathbf{E}_{l_1,l_2} = E^* + j_1(l_1, l_2)\bar{b}\varphi_* + \mathcal{W}(j_1(l_1, l_2), l_1, l_2), \tag{2.6}$$

for $(l_1, l_2) \in ({l_1}^* - \delta_1, {l_1}^* + \delta_1) \times ({l_2}^* - \delta_1, {l_2}^* + \delta_1)$, *satisfying* $\mathcal{G}(l_1, l_2) \neq 0$ *and* $\mathbf{E}_{l_1,l_2} \to E^*$ *as* $(l_1, l_2) \to ({l_1}^*, {l_2}^*)$.

Case II. $\sigma_1 = 0$, $\sigma_2 \neq 0$, we can compute $W_{zz}(0, {l_1}^*, {l_2}^*)$ to obtain σ_2. Define $I^{\delta',1}_{l_1,l_2} = \{(l_1, l_2) \in ({l_1}^* - \delta', {l_1}^* + \delta') \times ({l_2}^* - \delta', {l_2}^* + \delta') : \mathcal{G}(l_1, l_2) > 0\}$, $I^{\delta',2}_{l_1,l_2} = \{(l_1, l_2) \in ({l_1}^* - \delta', {l_1}^* + \delta') \times ({l_2}^* - \delta', {l_2}^* + \delta') : \mathcal{G}(l_1, l_2) < 0\}$, for any $\delta' > 0$. Then, we have the following theorem.

Theorem 2.2. *Assume the assumptions Assumption 2.1 and Assumption 2.2 hold, if* $\sigma_1 = 0$ *and* $\sigma_2 < 0(\sigma_2 > 0)$, *we can get there exist a constant* $\delta'^* > 0$ *and from* $I^{\delta'^*,1}_{l_1,l_2}(I^{\delta'^*,2}_{l_1,l_2})$ *to* $\mathbb{R}$, *there exist two continuously differentiable mappings* $j_1^{\pm}$, *then we can get the system* (2.1) *admits two nonconstant positive steady states* $\mathbf{E}^{\pm}_{l_1,l_2} \in \mathbb{X}^2$, *where*

$$\mathbf{E}^{\pm}_{l_1,l_2} = E^* + j_1^{\pm}(l_1, l_2)\bar{b}\varphi_* + \mathcal{W}(j_1^{\pm}(l_1, l_2), l_1, l_2) \tag{2.7}$$

and $\mathbf{E}^{\pm}_{l_1,l_2} \to E^*$ *as* $(l_1, l_2) \to ({l_1}^*, {l_2}^*)$.

From equation (2.6) and equation (2.7), we can get

$$\int_0^{l\pi} (\mathbf{E}_{l_1,l_2} - E^*)\varphi_*(x)dx = j_1(l_1, l_2)\bar{b},$$

$$\lim_{(l_1,l_2)\to({l_1}^*,{l_2}^*)} \frac{\mathbf{E}_{l_1,l_2} - E^*}{j_1(l_1, l_2)} = \varphi_*\bar{b},$$

where $z = j_1(l_1, l_2)$ is the root of $g(z, l_1, l_2)$ and satisfies $j_1({l_1}^*, {l_2}^*) = 0$.

In [11], from *Lemma* 3 we can get there exists a $\mathcal{E}' \subseteq \mathcal{E}$, then we can get for each $(l_1, l_2) \in \mathcal{E}'$, the sign of the real part of an eigenvalue h_{l_1,l_2} of $\mathcal{F}(\mathbf{E}_{l_1,l_2})$ can be determined by the sign of Re h^*, where $h^* \in \mathbb{C}$ is such that $\Gamma_m(h^*, \lambda_m, {l_1}^*, {l_2}^*) = 0$ for some $m \in \mathbb{N}_0$. Then, we have the following theorem.

Theorem 2.3. *One of the following conditions holds:*
(i) ${l_1}^* > \min\{c + \frac{u^*}{K}, \frac{u^*}{K} + \frac{l_2}{c}\}$*;*
(ii) ${l_1}^* > (d_1 + d_2)\lambda_m + c$ *for some* $m \in \mathbb{N}$*;*
(iii) $(d_1\lambda_m - {l_1}^*)(c + d_2\lambda_m) + {l_2}^* < 0$ *for some* $m \in \mathbb{N}$*, for* $m = 0$*,* $\Gamma_m(h^*, \lambda_m, {l_1}^*, {l_2}^*) = 0$ *has at least one root with a positive real part, then we can get the nonconstant steady states* $\mathbf{E}_{l_1,l_2}$ *given in Theorem 2.1 and Theorem 2.2 are unstable.*

3. Stability analysis

In this paper, we assume that the region $\Omega = (0, l\pi)$ and the kernel function $G(x, y) = \frac{1}{l\pi}$. Denotes $\mathbb{N}$ as positive integer set and $\mathbb{N}_0$ as nonnegative integer set. Obviously, we can get the positive equilibrium exists under the assumption Assumption 2.1. We mainly consider the dynamic properties near the positive equilibrium point.

By translation, let $u = u - u^*, v = v - v^*$, the linearized equation of the system (1.4) at (u^*, v^*) is given by

$$\begin{cases} u_t = d_1\Delta u - u^*\dfrac{1}{K}\displaystyle\int_0^{l\pi} u(t, y)\mathrm{d}y + (1 - \dfrac{u^*}{K})\dfrac{u^*}{p + u^* + sv^*}\ u \\ \qquad -\dfrac{u^*(1 - \frac{u^*}{K})(u^* + p)}{v^*(p + u^* + sv^*)}\ v, \\ v_t = d_2\Delta v + cu - cv, \\ u_x(0, t) = v_x(0, t) = 0, \quad u_x(l\pi, t) = v_x(l\pi, t) = 0, \end{cases} \tag{3.1}$$

for $x \in (0, l\pi), t > 0$. Under zero Neumann boundary conditions, we can get the eigenvalues of $-\Delta$ is $\frac{k^2}{l^2}$, $k \in \mathbb{N}_0$. By simple calculation, we can get the characteristic equation of the system (3.1) is

$$\mathcal{P}_k(\lambda) := \lambda^2 - T_k(c)\lambda + D_k(c) = 0, \tag{3.2}$$

where

$$\begin{aligned} T_0(c) &= \frac{u^*(1 - \frac{u^*}{K})}{p + u^* + sv^*} - \frac{u^*}{K} - c, \\ D_0(c) &= \frac{cu^*(1 - \frac{u^*}{K})(u^* + p)}{v^*(p + u^* + sv^*)} - \frac{cu^*(1 - \frac{u^*}{K})}{p + u^* + sv^*} + \frac{cu^*}{K}, \\ T_k(c) &= -(d_1 + d_2)\frac{k^2}{l^2} + \frac{u^*(1 - \frac{u^*}{K})}{p + u^* + sv^*} - c, \\ D_k(c) &= \frac{cu^*(1 - \frac{u^*}{K})(u^* + p)}{v^*(p + u^* + sv^*)} - \frac{cu^*(1 - \frac{u^*}{K})}{p + u^* + sv^*} \\ &\quad + (cd_1 - \frac{u^*(1 - \frac{u^*}{K})d_2}{p + u^* + sv^*})\frac{k^2}{l^2} + \frac{d_1 d_2 k^4}{l^4}, \quad k \in \mathbb{N}. \end{aligned}$$

Therefore, the roots of the above equation are

$$\lambda_{1,2} = \frac{T_k(c) \pm \sqrt{{T_k}^2(c) - 4D_k(c)}}{2}, \quad k \in \mathbb{N}_0.$$

If the characteristic roots satisfy $\mathrm{Re}(\lambda_{1,2}) < 0$, then (u^*, v^*) is locally asymptotically stable. Define $r = \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*}$. Rewrite $D_k(c)$ as a quadratic function $f(m)(m = \frac{k^2}{l^2})$, where $f(m) = \frac{cr(u^*+p)}{v^*} - cr + (cd_1 - rd_2)m + d_1 d_2 m^2$. Then we can get $f(0) = \frac{cr(u^*+p)}{v^*} - cr > 0$. Define $\Delta_f = (cd_1 + rd_2)^2 - \frac{4d_1d_2cr(u^*+p)}{v^*}$, $f_{\min} = \frac{-\Delta_f}{4d_1d_2}$, $m_1 = \frac{-(cd_1-d_2r)-\sqrt{(cd_1+rd_2)^2-\frac{4d_1d_2cr(u^*+p)}{v^*}}}{2d_1d_2}$ and $m_2 = \frac{-(cd_1-d_2r)+\sqrt{(cd_1+rd_2)^2-\frac{4d_1d_2cr(u^*+p)}{v^*}}}{2d_1d_2}$. Then we can get the following theorem

Theorem 3.1. *For the system* (1.4), *assume Assumption 2.1 and* $\frac{r(u^*+p)}{v^*} - r + \frac{u^*}{K} > 0$ *hold, we can get the following results*

(*i*) *When* $c \geq \max\{\frac{d_2 r}{d_1}, r\}$, *the equilibrium* $E^*(u^*, v^*)$ *is locally asymptotically stable.*

(*ii*) *When* $r < c < \frac{d_2 r}{d_1}$ *and* $\Delta_f < 0$, *the equilibrium* $E^*(u^*, v^*)$ *is locally asymptotically stable.*

(*iii*) *When* $r < c < \frac{d_2 r}{d_1}$, $\Delta_f > 0$ *and there exists no* m *such that* $f(m) < 0$, *the equilibrium* $E^*(u^*, v^*)$ *is locally asymptotically stable.*

Proof. Under the conditions (*i*), (*ii*) and (*iii*), we can obtain that when Assumption 2.1 and $\frac{r(u^*+p)}{v^*} - r + \frac{u^*}{K} > 0$ hold, $T_k < 0$ and $D_k > 0$ for all $k \in \mathbb{N}_0$, which means that the equilibrium $E^*(u^*, v^*)$ is locally asymptotically stable. □

Remark 3.1. Using the same method, we can obtain that the characteristic equation of the system (3.1) without nonlocal competition is

$$\mathcal{O}_k(\lambda) := \lambda^2 - M_k(c)\lambda + B_k(c) = 0, \tag{3.3}$$

where

$$\begin{cases} M_k(c) = 1 - \dfrac{2u^*}{K} - \dfrac{(p+sv^*)(1-\frac{u^*}{K})}{p+u^*+sv^*} - c - (d_1+d_2)\dfrac{k^2}{l^2}, \\ B_k(c) = \dfrac{cu^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - c(1 - \dfrac{2u^*}{K} - \dfrac{(p+sv^*)(1-\frac{u^*}{K})}{p+u^*+sv^*}) + \dfrac{cd_1k^2}{l^2} \\ \qquad + \dfrac{d_1d_2k^4}{l^4} - (1 - \dfrac{2u^*}{K} - \dfrac{(p+sv^*)(1-\frac{u^*}{K})}{p+u^*+sv^*})\dfrac{d_2k^2}{l^2}, \quad k \in \mathbb{N}_0. \end{cases}$$

Then, the two roots of the above equation are

$$\lambda_{1,2} = \frac{M_k(c) \pm \sqrt{{M_k}^2(c) - 4B_k(c)}}{2}, \quad k \in \mathbb{N}_0.$$

To generate Hopf-Hopf bifurcation, there are k_1 and k_2 to make $M_k = 0$, which is monotonically contradiction with M_k about k, so it is impossible to generate Hopf-Hopf bifurcation. Therefore, Hopf-Hopf bifurcation may occur when the nonlocal competition is added to the system.

4. Hopf-Hopf bifurcation analysis

4.1. Existence of Hopf-Hopf bifurcation

In the following article, we will compute the canonical form of the Hopf-Hopf bifurcation for $k_1 = 0, k_2 \neq 0$, i.e. $(0, k_2)$-mode Hopf-Hopf bifurcation. We study the Hopf-Hopf bifurcation, but T_k is monotonic with respect to k, so there can only be $T_0 = 0$ and $T_k = 0,\ k \in \mathbb{N}$. Define $r = \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} > 0$, by simple calculation, we can get

$$\begin{aligned} c &= c_0 = r - \frac{u^*}{K}, \\ d_2 &= {d_2}^* = \frac{u^*}{K} \cdot \frac{l^2}{k^2} - d_1. \end{aligned} \tag{4.1}$$

If we take $(d_2,\ c)$ as a parameter and let $\lambda_{1,2}(d_2,\ c) = \alpha(d_2,\ c) \pm i\omega(d_2,\ c)$ be the pair of roots of the equation (3.2) near $(d_2,\ c) = ({d_2}^*,\ c_0)$ satisfying $\alpha(d_2,\ c) = 0$ and $\omega(d_2,\ c) = \omega_n$, for $n = 1, 2$. In the following, we give the lemma to verify the transversality condition

Lemma 4.1. *Assume Assumption 2.1 holds, then we can get*
$Re[\frac{\partial\lambda}{\partial c}|_{(d_2,\ c)=({d_2}^*,\ c_0)}] < 0,\ Re[\frac{\partial\lambda}{\partial d_2}|_{(d_2,\ c)=({d_2}^*,\ c_0)}] < 0.$

Proof. By equation (3.2), we have

$$\begin{aligned} (\frac{\partial\lambda}{\partial c})^{-1} &= \frac{2\lambda - T_0(c)}{-\lambda - (\frac{u^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{u^*}{K})}, \\ (\frac{\partial\lambda}{\partial d_2})^{-1} &= \frac{2\lambda - T_k(c)}{-\frac{k^2}{l^2}\lambda - (-\frac{u^*(1-\frac{u^*}{K})k^2}{(p+u^*+sv^*)l^2} + \frac{d_1k^4}{l^4})}. \end{aligned}$$

Then

$$\begin{aligned} &[\mathrm{Re}(\frac{\partial\lambda}{\partial c})^{-1}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ =& \mathrm{Re}[\frac{2\lambda - T_0(c)}{-\lambda - (\frac{u^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{u^*}{K})}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ =& \mathrm{Re}[\frac{2\lambda(\lambda - (\frac{u^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{u^*}{K}))}{-\lambda^2 + (\frac{u^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{u^*}{K})^2}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ =& [\frac{-2{\omega_n}^2}{{\omega_n}^2 + (\frac{u^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{u^*}{K})^2}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ <& 0, \\ &[\mathrm{Re}(\frac{\partial\lambda}{\partial d_2})^{-1}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ =& \mathrm{Re}[\frac{2\lambda - T_k(c)}{-\frac{k^2}{l^2}\lambda - (-\frac{u^*(1-\frac{u^*}{K})k^2}{(p+u^*+sv^*)l^2} + \frac{d_1k^4}{l^4})}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \\ =& \mathrm{Re}[\frac{2\lambda(\frac{k^2}{l^2}\lambda - (-\frac{u^*(1-\frac{u^*}{K})k^2}{(p+u^*+sv^*)l^2} + \frac{d_1k^4}{l^4}))}{-\frac{k^4}{l^4}\lambda^2 + (-\frac{u^*(1-\frac{u^*}{K})k^2}{(p+u^*+sv^*)l^2} + \frac{d_1k^4}{l^4})^2}]_{(d_2,\ c)=({d_2}^*,\ c_0)} \end{aligned}$$

$$= [\frac{-\frac{2{\omega_n}^2 k^2}{l^2}}{\frac{k^4}{l^4}{\omega_n}^2 + (-\frac{u^*(1-\frac{u^*}{K})k^2}{(p+u^*+sv^*)l^2} + \frac{d_1 k^4}{l^4})^2}]_{(d_2,\ c)=({d_2}^*,\ c_0)}$$
$$< 0.$$

□

In [13], we can know that when $(d_2, c) = ({d_2}^*, c_0)$ at the positive equilibrium E^* the system (1.4) undergoes $(k_1,\ k_2)$-mode Hopf-Hopf bifurcation, where $k_1 = 0$, $k_2 \neq 0$.

Define

$$\Lambda_2 := \{k \in \mathbb{N} |\ \max\{0, (r - \sqrt{\frac{rc(u^*+p)}{v^*}})\frac{l^2}{d_1}\} < k^2 < \min\{\frac{u^* l^2}{K d_1}, (r + \sqrt{\frac{rc(u^*+p)}{v^*}})\frac{l^2}{d_1}\}\}.$$

After that, we have the following theorem

Theorem 4.1. *For the system* (1.4), *assume Assumption 2.1 and* $\frac{r(u^*+p)}{v^*} - r + \frac{u^*}{K} > 0$ *hold, we can get the following results*

(*i*) *If* $\Lambda_2 = \emptyset$, *the system* (1.4) *does not undergo Hopf-Hopf bifurcation.*

(*ii*) *If* $\Lambda_2 \neq \emptyset$, *the system* (1.4) *undergoes Hopf-Hopf bifurcation at the point* $(d_2,\ c) = ({d_2}^*,\ c_0)$.

Proof. By equation (3.2), when $(d_2,\ c) = ({d_2}^*,\ c_0)$, we can get $T_0(c) = 0$ and $T_k(c) = 0$. Obviously, when $\frac{r(u^*+p)}{v^*} - r + \frac{u^*}{K} > 0$, we can get $D_0(c) > 0$. When $\Lambda_2 \neq \emptyset$, we assume that there is a unique $k^* \in \Lambda_2$ makes $D_{k^*}(c) > 0$. Then we prove $D_{k^*}(c) > 0$.

Let

$$f(\frac{{k^*}^2}{l^2}) = \frac{cu^*(1-\frac{u^*}{K})(u^*+p)}{v^*(p+u^*+sv^*)} - \frac{cu^*(1-\frac{u^*}{K})}{p+u^*+sv^*} + \frac{cd_1{k^*}^2}{l^2} - \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*}\frac{d_2{k^*}^2}{l^2} + \frac{d_1 d_2 {k^*}^4}{l^4},$$

because of $(d_2,\ c) = ({d_2}^*,\ c_0)$ and $r = \frac{u^*(1-\frac{u^*}{K})}{p+u^*+sv^*}$, we can get $f(\frac{{k^*}^2}{l^2}) = \frac{r(r-\frac{u^*}{K})(u^*+p)}{v^*} - r^2 + 2r\frac{d_1{k^*}^2}{l^2} - \frac{{d_1}^2{k^*}^4}{l^4}$.

After that we can know $f(\frac{r}{d_1}) = \frac{r(r-\frac{u^*}{K})(u^*+p)}{v^*} > 0$, then we can get there have a $k^* \in \Lambda_2$ makes $D_{k^*}(c) > 0$. i.e. When $\Lambda_2 \neq \emptyset$, the system (1.4) undergoes Hopf-Hopf bifurcation at the point $(d_2,\ c) = ({d_2}^*,\ c_0)$. When $\Lambda_2 = \emptyset$, the system (1.4) does not undergo Hopf-Hopf bifurcation. □

4.2. Property of Hopf-Hopf bifurcation

Define the Banach space of continuous maps $\mathcal{C} = C([-r, 0]; \mathbb{X}^m)$ with $m \in N$ with the sup norm. We consider the nonlocal term as an independent variable and separate it. Then, in the phase space $\mathcal{C}$, we consider the abstract PFDE with nonlocal effect

$$\dot{u}(t) = L(\xi)\Delta u(t) + M(\xi)u(t) + \hat{M}(\xi)\hat{u}(t) + B(u(t), \hat{u}(t), \xi), \tag{4.2}$$

where $u(t) \in \mathcal{C}$, $\hat{u}(x,t) := \int_\Omega G(x,\eta)u(\eta,t)d\eta$ represents the nonlocal effect, where $G(x,y)$ is the kernel function, $\xi = (\xi_1, \xi_2)$ is the varying parameter belonging to a neighborhood of $(0,0) \in \mathbb{R}^2$, $L(\xi) = diag\,(d_1(\xi), d_2(\xi), \ldots, d_m(\xi))$, where $d_i(0) > 0$, $1 \le i \le m$, $M, \hat{M} : V \to M(\mathcal{C}, X^m)$ is C^1 smooth, $B : \mathcal{C} \times \mathcal{C} \times V \to X^m$ is C^k smooth for $k > 3$, where $B(0,0,0) = 0$, $DB(0,0,0) = 0$. Then we can get the linearized equation of equation (4.2) at the zero equilibrium is

$$\dot{u}(t) = L_0 \Delta u(t) + M(0)u(t) + \hat{M}(0)\hat{u}(t). \tag{4.3}$$

Define the Banach space

$$\mathcal{BC} = \left\{ \phi : [-r, 0] \to X^m \,\middle|\, \phi \text{ is continuous on } [-r, 0)\,, \lim_{\theta \to 0^-} \phi(\theta) \text{ exists} \right\}$$

and the operation

$$\langle v, \gamma_k \rangle = \left(\langle v_1, \gamma_k \rangle, \langle v_2, \gamma_k \rangle \ldots, \langle v_m, \gamma_k \rangle \right)^{\mathrm{T}}, k \in \mathbb{N}_B,$$

where $v = (v_1, v_2, \ldots v_m)^{\mathrm{T}} \in \mathcal{C}$,

$$\mathbb{N}_B = \begin{cases} \mathbb{N}_0 \text{ for homogeneous Neumann boundary conditions,} \\ \mathbb{N} \text{ for homogeneous Dirichlet boundary conditions.} \end{cases}$$

Then we consider equation (4.2) with Neumann boundary conditions on the spatial domain $\Omega = (0, l\pi)$ for some $l > 0$. Define

$$\tilde{M}_k(\psi)\gamma_k = M(0)(\psi\gamma_k) + \hat{M}(0)(\psi\hat{\gamma}_k),$$

where $\tilde{M}_k : C \to \mathbb{C}^m$, $\hat{\gamma}_k = \int_\Omega G(x,\eta)\gamma_k(\eta)\mathrm{d}\eta$, $\psi \in \mathbb{C}$, $k \in \mathbb{N}_B$.

Define a $m \times m$ matrix-valued function of bounded variation $\rho_k \in BV([-r,0], \mathbb{C}^{m\times m})$, such that $-\mu_k L_0 \phi(0) + \tilde{M}_k \phi = \int_{-r}^0 \mathrm{d}\rho_k(\theta)\phi(\theta)$, for $\phi \in C$. Then we can get the linear equation (4.3) is equivalent to a sequence of functional differential equations on $\mathbb{C}^m$,

$$\dot{g}(t) = -\mu_k L_0 g(t) + \tilde{M}_k g_t, \tag{4.4}$$

where $g_t(\cdot) = \langle u_t(\cdot), \gamma_k \rangle \in C$, with the characteristic equation is $\det \Delta_{k_i}(\lambda) = 0$, *where* $\Delta_k(\lambda) = \lambda I + \mu_k L_0 - M_k\left(e^\lambda I\right) - \hat{M}_k\left(e^\lambda I\right)$, $k \in \mathbb{N}_B$.

Define the adjoint bilinear form on $C^* \times C$ is

$$(\phi, \psi)_k = \phi(0)\psi(0) - \int_{-r}^0 \int_0^\theta \phi(\eta - \theta)\mathrm{d}\rho_k(\theta)\psi(\eta)\mathrm{d}\eta, \tag{4.5}$$

for $\phi \in C^*$, $\psi \in C$, where $C^* \triangleq C([0,r]; \mathbb{C}^{m*})$.

Define the characteristic equation

$$\det \Delta(\lambda) = 0, \text{ where } \Delta(\lambda) = \lambda I - L_0\Delta - M(0)\left(e^\lambda I\right) - \hat{M}(0)\left(e^\lambda I\right) \tag{4.6}$$

and a sequence of characteristic equations

$$\det \Delta_{k_i}(\lambda) = 0, \tag{4.7}$$

where $\Delta_{k_i}(\lambda)=\lambda I+\mu_{k_i}L_0-M_{k_i}\left(e^{\lambda}I\right)-\hat{M}_{k_i}\left(e^{\lambda}I\right),\ k_i\in\mathbb{N}_B,\ i=1,2.$

Let $\Lambda=\Lambda_1\cup\Lambda_2$, where $\Lambda_1=\{\pm i\omega_1\}$, $\Lambda_2=\{\pm i\omega_2\}$. Using Λ_i to decompose the phase space C, we can get $C=P_i\oplus Q_i$, where P_i denotes the eigenspace generalized by the eigenfunction corresponding to Λ_i, $Q_i=\{\psi\in C:(\phi,\psi)=0\}$ for all $\phi\in P_i{}^*,\ i=1,2$, where $P_i{}^*$ representing the generalized eigenspace corresponding to the formal adjoint differential equation of equation (4.4) with $k=k_i,\ i=1,2$. In $P_i,\ P_i{}^*$, i=1,2, we choose the basis

$$\Psi_{k_i}=\left(\psi_i,\bar{\psi}_i\right),\qquad \Phi_{k_i}=\begin{pmatrix}\phi_i\\ \bar{\phi}_i\end{pmatrix},\tag{4.8}$$

satisfy $(\Phi_{k_i},\ \Psi_{k_i})_{k_i}=I$, where I is the identity matrix,

$$\dot{\Psi}_{k_i}=\Psi_{k_i}B_i\ and\ -\dot{\Phi}_{k_i}=B_i\Phi_{k_i},\tag{4.9}$$

where $B_i=\mathrm{diag}(i\omega_i,-i\omega_i),\ i=1,2$. Simplify the normal form expression, we can get

$$\Psi_i=\begin{pmatrix}\psi_i\gamma_{k_i}\\ \psi_i\hat{\gamma}_{k_i}\end{pmatrix},\ i=1,2.\tag{4.10}$$

From [19,23], we can know that

$$\begin{aligned}&\psi_i(\theta)=\psi_i(0)e^{\mathrm{i}\omega_i\theta},\ \theta\in[-r,0],\\ &\phi_i(s)=\phi_i(0)e^{-\mathrm{i}\omega_i s},\ s\in[0,r],\ i=1,2.\end{aligned}\tag{4.11}$$

The phase space $\mathcal{C}$ can be decomposed by Λ as below

$$\mathcal{C}=\mathcal{P}\oplus\mathcal{Q},\ \mathcal{P}=\mathrm{Im}\ \pi,\ \mathcal{Q}=\mathrm{Ker}\ \pi,$$

where $\dim\mathcal{P}=4$, $\pi:\mathcal{C}\to\mathcal{P}$ is the projection defined by

$$\pi\psi=\sum_{i=1,2}\Psi_{k_i}(\Phi_{k_i},\langle\psi(\cdot),\gamma_{k_i}\rangle)_{k_i}\gamma_{k_i}.\tag{4.12}$$

In [23], the projection operator in equation (4.12) is extended to the phase space $\mathcal{BC}$, expressed by π. Decomposition of phase space $\mathcal{BC}$, we can get

$$\mathcal{BC}=\mathcal{P}\oplus Ker\pi.$$

In the space $\mathcal{BC}$, rewrite equation (4.2) as an abstract ODE

$$\frac{\mathrm{d}v}{\mathrm{d}t}=Av+X_0F\left(v,\hat{v},\xi\right),\tag{4.13}$$

where

$$F(v,\hat{v},\xi)=(M(\xi)-M(0))v+(\hat{M}(\xi)-\hat{M}(0))\hat{v}+(L(\xi)-L(0))\Delta v(0)+B(v,\hat{v},\xi),\tag{4.14}$$

for $v,\ \hat{v}\in\mathcal{C},\ \xi\in V.$

In [14], define $\mathcal{C}_0^1 \triangleq \{\psi \in \mathcal{C} : \dot{\psi} \in \mathcal{C}, \psi(0) \in dom(\Delta)\}$, $A : \mathcal{C}_0^1 \subset \mathcal{BC} \to \mathcal{BC}$, $A\psi = \dot{\psi} + X_0[M(0)\psi + \hat{M}(0)\hat{\psi} + L_0\Delta\psi(0) - \dot{\psi}(0)]$, where $\hat{\psi} = \int_\Omega G(x,\eta)\psi(\eta,t)\mathrm{d}\eta$. Define $\mathcal{Q}^1 \triangleq \mathcal{C}_0^1 \cap \mathcal{Q}$. Let $g = (g_1, \bar{g}_1, g_2, \bar{g}_2)^{\mathrm{T}} \in \mathbb{C}^4$, when π commutes with A in $\mathcal{C}_0^1$, we can get the abstract ODE in $\mathcal{BC}$ is equivalent to

$$
\begin{aligned}
\dot{g}_1 &= i\omega_1 g_1 + \phi_1(0)\langle F((\Psi_{k_1}\gamma_{k_1}, \Psi_{k_2}\gamma_{k_2})g + y, (\Psi_{k_1}\hat{\gamma}_{k_1}, \Psi_{k_2}\hat{\gamma}_{k_2})g + \hat{y}, \xi), \gamma_{k_1}\rangle,\\
\dot{\bar{g}}_1 &= -i\omega_1 \bar{g}_1 + \bar{\phi}_1(0)\langle F((\Psi_{k_1}\gamma_{k_1}, \Psi_{k_2}\gamma_{k_2})g + y, (\Psi_{k_1}\hat{\gamma}_{k_1}, \Psi_{k_2}\hat{\gamma}_{k_2})g + \hat{y}, \xi), \gamma_{k_1}\rangle,\\
\dot{g}_2 &= i\omega_2 g_2 + \phi_2(0)\langle F((\Psi_{k_1}\gamma_{k_1}, \Psi_{k_2}\gamma_{k_2})g + y, (\Psi_{k_1}\hat{\gamma}_{k_1}, \Psi_{k_2}\hat{\gamma}_{k_2})g + \hat{y}, \xi), \gamma_{k_2}\rangle,\\
\dot{\bar{g}}_2 &= -i\omega_2 \bar{g}_2 + \bar{\phi}_2(0)\langle F((\Psi_{k_1}\gamma_{k_1}, \Psi_{k_2}\gamma_{k_2})g + y, (\Psi_{k_1}\hat{\gamma}_{k_1}, \Psi_{k_2}\hat{\gamma}_{k_2})\bar{g} + \hat{y}, \xi), \gamma_{k_2}\rangle,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= A_1 y + (I-\pi)X_0 F((\Psi_{k_1}\gamma_{k_1}, \Psi_{k_2}\gamma_{k_2})g + y, (\Psi_{k_1}\hat{\gamma}_{k_1}, \Psi_{k_2}\hat{\gamma}_{k_2})g + \hat{y}, \xi),
\end{aligned}
\tag{4.15}
$$

where $\hat{\gamma}_{k_i} = \int_\Omega G(x,\eta)\gamma_{k_i}(\eta,t)d\eta$, $\hat{y} = \int_\Omega G(x,\eta)y(\eta,t)d\eta$ for $y \in \mathcal{Q}^1$, A_1 is the restriction of A on $\mathcal{Q}^1$.

Define $\tilde{M}(\xi)(\psi) = M(\xi)\psi + \hat{M}(\hat{\psi})$, where $\psi \in \mathcal{C}$, $\tilde{M} \in M(\mathcal{C}, \mathbb{X}^m)$. Then taylor expansion for $\tilde{M}(\xi)$ and $L(\xi)$ *at* $\xi = 0$, we can get

$$
\begin{aligned}
&\tilde{M}(\xi)\psi = \tilde{M}(0)\psi + \frac{1}{2}\tilde{M}_1(\xi)\psi + \cdots, \text{ for } \psi \in \mathcal{C},\\
&L(\xi) = L(0) + \tfrac{1}{2}L_1(\xi) + \cdots,
\end{aligned}
\tag{4.16}
$$

where $\tilde{M}(0)\psi = M(0)\psi + \hat{M}(0)(\hat{\psi})$, $\tilde{M}_1\psi = M_1\psi + \hat{M}_1(\hat{\psi})$ and $M_1 : V \to M(\mathcal{C}, \mathbb{R}^m)$, $L_1 : V \to \mathbb{R}^{m\times m}$ are linear operators, where V is a neighborhood of $(0,0)$. Define B in equation (4.14) can be rewritten as

$$
B(v, \hat{v}, 0) = \frac{1}{2!}Q(V,V) + \frac{1}{3!}C(V,V,V) + O(||V||^4),
\tag{4.17}
$$

where $V = \begin{pmatrix} v \\ \hat{v} \end{pmatrix}$, $\hat{v} = \int_\Omega G(x,\eta)v(\eta,t)d\eta$, $v \in \mathcal{C}$, $Q(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ are symmetric multilinear forms. In order to facilitate the calculation, write $Q(V,V)$ as Q_{VV} and $C(V,V,V)$ as C_{VVV}.

Based on the above content, ignoring the influence of the higher-order terms (≥ 2) of the smaller parameters ξ_1, ξ_2 and the influence of ξ_1, ξ_2 on the third-order terms of the normal form. In the following theorem, we give the formula of the third-order normal form of equation (4.2).

Theorem 4.2. *Assume Assumption 2.1 holds, we can get the normal forms of equation* (4.2) *restricted on the center manifold up to the third order term are*

$$
\dot{g} = Bg + \frac{1}{2}q_2^1(g,0,0,\xi) + \frac{1}{3!}q_3^1(g,0,0,0) + h.o.t.,
\tag{4.18}
$$

is equivalent to

$$
\begin{aligned}
\dot{g}_1 &= \mathrm{i}\omega_1 g_1 + n_1(\xi)g_1 + n_{2100}g_1^2\bar{g}_1 + n_{1011}g_1 g_2\bar{g}_2 + h.o.t.,\\
\dot{\bar{g}}_1 &= -\mathrm{i}\omega_1\bar{g}_1 + \overline{n_1(\xi)}\bar{g}_1 + \overline{n_{2100}}g_1\bar{g}_1^2 + \overline{n_{1011}}\bar{g}_1 g_2\bar{g}_2 + h.o.t.,\\
\dot{g}_2 &= \mathrm{i}\omega_2 g_2 + m_2(\xi)g_2 + m_{0021}g_2^2\bar{g}_2 + m_{1110}g_1\bar{g}_1 g_2 + h.o.t.,\\
\dot{\bar{g}}_2 &= -\mathrm{i}\omega_2\bar{g}_2 + \overline{m_2(\xi)}\bar{g}_2 + \overline{m_{0021}}g_2\bar{g}_2^2 + \overline{m_{1110}}g_1\bar{g}_1\bar{g}_2 + h.o.t..
\end{aligned}
\tag{4.19}
$$

In the appendix, we give the final formula for calculating the coefficients $n_1(\xi)$, $m_2(\xi)$, n_{2100}, n_{1011}, m_{0021}, m_{1110}.

Let

$$g_1 = \beta_1 \cos(\theta) + \mathrm{i}\beta_1 \sin(\theta),\ g_2 = \beta_2 \cos(\theta) + \mathrm{i}\beta_2 \sin(\theta)$$

and transforming $\sqrt{|Re(n_{2100})|\beta_1}\ \mathrm{sign}(Re(n_{2100})) \to \beta_1$, $\sqrt{|Re(m_{0021})|\beta_2} \to \beta_2$, we can get the normal form equation (4.19) can be written as

$$\begin{aligned}\dot{\beta}_1 &= \beta_1(\epsilon_1(\xi) + \beta_1^2 + b\beta_2^2),\\ \dot{\beta}_2 &= \beta_2(\epsilon_2(\xi) + c\beta_1^2 + d\beta_2^2),\end{aligned} \tag{4.20}$$

where

$$\epsilon_1(\xi) = Re(n_1(\xi))\ \mathrm{sign}\ (Re(n_{2100})),\ \epsilon_2(\xi) = Re(m_2(\xi)),$$
$$b = \frac{Re(n_{1011})}{|Re(m_{0021})|}\ \mathrm{sign}\ (Re(n_{2100})),\ c = \frac{Re(m_{1110})}{|Re(n_{2100})|},\ d = \pm 1.$$

Table 2. The twelve unfoldings [15] of the system (4.20).

	Ia	Ib	II	III	IVa	IVb	V	VIa	VIb	VIIa	VIIb	VIII
d	+1	+1	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1
c	+	+	-	+	-	-	+	-	-	+	+	-
b	+	+	+	-	-	-	+	+	+	-	-	-
d-cb	+	-	+	+	+	-	-	+	-	+	-	-

Table 3. The correspondence between Original system and Planar system.

Planar system	Original system
E_0	Positive constant steady state
E_1	Spatially homogeneous periodic solution
E_2	Spatially nonhomogeneous periodic solution
E_3	Spatially nonhomogeneous quasi-periodic solution

In [15], in Table 2 give the system (4.20) has twelve cases according to different signs of d, c, b and $d - bc$. In Table 3, after analyzing the phase portrait and bifurcation diagram of the system (4.20) corresponding to each case, we can get the dynamic behavior of equation (4.2) near the Hopf-Hopf bifurcation singularity.

5. Numerical simulations

Because the difference can reduce the irregular fluctuations between the data and make the fluctuations curve more stable, we use the forward difference method in MATLAB for numerical simulation. We choose c and d_2 as the variable parameters.

In order to study the influence of nonlocal competition on the system (1.4), we select some of the same parameters as in [41]. After assuming other parameters, we fix the following parameters of the system (1.4), then we can obtain the following paremeters in our system

$$K = 50, \quad a = 1, \quad s = 0.1, \quad p = 5, \quad b = 0.01, \quad k = 1, \quad l = 5, \quad d_1 = 7.$$

Because $p > \max\{K - ab - bs + Ks, ab - bs\}$, we can know Assumption 2.1(i) holds. Then we can get the the positive constant steady state $E^* \approx (15.0615386, 15.0715386)$, $K - ab - bs + Ks = 4.9990000$, $ab - bs = 0.0090000$ and $\frac{d_2 r}{d_1} = 0.0369987$. There exist critical values $c_0 \approx 0.1867237$, ${d_2}^* \approx 0.5307693$, $\omega_1 \approx 0.2939612$, $\omega_2 \approx 0.2793454$. When $d_2 = {d_2}^*$, $c = c_0$, all eigenvalues of $\mathcal{P}_k(\lambda)$ have negative real parts other than two pairs of purely imaginary roots $\pm i\omega_1$, $\pm i\omega_2$, by Theorem 4.1, $\Lambda_2 \neq \emptyset$, then we can get near (u^*, v^*) the system (1.4) undergoes $(0, k_2)$-mode Hopf-Hopf bifurcation.

As shown in Figure 1, it can be known that in the system without nonlocal competition there are no two intersecting bifurcation curves when $d_2 > 0$ and $c > 0$, which will not produce Hopf-Hopf bifurcation. When $d_2 > 0$, $c > 0$ and $k = 1$ the system with nonlocal competition has two intersecting bifurcation curves, which will produce Hopf-Hopf bifurcation, verify the Remark 3.1.

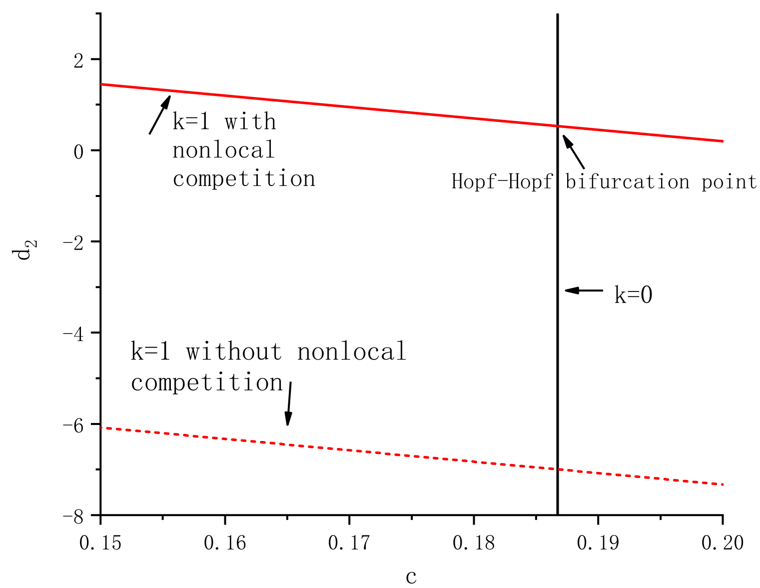


Figure 1. Hopf bifurcation curves of models with and without nonlocal competition.

Further the equation (7.5) become

$$\psi_1(0) = \begin{pmatrix} 1 \\ 0.2874839 - 0.4525891i \end{pmatrix}, \ \psi_2(0) = \begin{pmatrix} 1 \\ 0.3201712 - 0.4300863i \end{pmatrix},$$
$$\phi_1(0) = \begin{pmatrix} 0.5000000 - 0.3175992 \\ 1.1047548i \end{pmatrix}^{\mathrm{T}}, \ \phi_2(0) = \begin{pmatrix} 0.5000000 - 0.3722174i \\ 1.1625573i \end{pmatrix}^{\mathrm{T}}.$$

By equation (7.3) and equation (7.4), we can get the coefficients of normal form up to the third-order

$$
\begin{aligned}
&n_1(\xi) = (-0.5000000 + 0.7871556i)\xi_2,\\
&m_2(\xi) = -\,(0.0200000 + 0.0148887i)\,\xi_1 - (0.5000000 - 0.7903399i)\,\xi_2,\\
&n_{2100} = -0.0007925 - 0.0001348i,\ n_{1011} = -0.0017482 + 0.0004029i,\\
&m_{0021} = -0.0014502 + 0.0007949i,\ m_{1110} = -0.0008523 - 0.0000779i.
\end{aligned}
$$

By the system (4.20), the corresponding planar system is

$$
\begin{aligned}
\dot{\beta}_1 &= \beta_1({\beta_1}^2 + 1.2054893{\beta_2}^2 + 0.5000000\xi_2),\\
\dot{\beta}_2 &= -\beta_2\left(1.0754108{\beta_1}^2 + {\beta_2}^2 + 0.0200000\xi_1 + 0.5000000\xi_2\right).
\end{aligned} \tag{5.1}
$$

The equilibria of the system (5.1) are

$$
\begin{aligned}
&E_0 = (0,\ 0),\ E_1 = (\sqrt{-0.5000000\xi_2},\ 0),\ \text{for } \xi_2 < 0,\\
&E_2 = (0,\ \sqrt{-0.0200000\xi_1 - 0.5000000\xi_2}),\ \text{for } \xi_2 < -0.04000000\xi_1,\\
&E_3 = (\sqrt{-0.0813431\xi_1 - 0.3466463\xi_2},\ \sqrt{0.0674772\xi_1 - 0.1272129\xi_2}),\\
&\text{for } \ -0.0813431\xi_1 - 0.3466463\xi_2 > 0,\ 0.0674772\xi_1 - 0.1272129\xi_2 > 0.
\end{aligned}
$$

Because $b = 1.2054893,\ c = -1.0754108,\ d = -1,\ d - bc > 0$, then the *Case VIa* of the unfoldings in [15] occurs. In [2], by computing, we can get the critical bifurcation lines in (d_2, c)-plane

$$
\begin{aligned}
&\mathcal{L}_0^+ : c = c_0,\ \text{for } d_2 > {d_2}^*,\\
&\mathcal{L}_0^- : c = c_0,\ \text{for } 0 < d_2 < {d_2}^*,\\
&\mathcal{L}_1^+ : c = c_0 - 0.04000000(d_2 - {d_2}^*),\ \text{for } d_2 > {d_2}^*,\\
&\mathcal{L}_1^- : c = c_0 - 0.04000000(d_2 - {d_2}^*),\ \text{for } \ 0 < d_2 < {d_2}^*,\\
&\mathcal{H}_1 : c = c_0 - 0.2346574(d_2 - {d_2}^*),\ \text{for } \ d_2 > {d_2}^*,\\
&\mathcal{H}_2 : c = c_0 + 0.5304279(d_2 - {d_2}^*),\ \text{for } \ 0 < d_2 < {d_2}^*,\\
&\mathcal{R}\ : c = c_0 - 0.6782028(d_2 - {d_2}^*),\ \text{for } \ d_2 > {d_2}^*(\text{Hopf bifurcation curve}).
\end{aligned} \tag{5.2}
$$

As in Figure 2-Left, the (d_2, c)-plane is divided into seven regions around $({d_2}^*, c_0)$ and in Figure 2-Right, we give the corresponding phase portraits in seven regions. The equilibria of the planar system (5.1) correspond to the positive constant steady state, periodic and quasi-periodic solutions of the system (1.4), we can see in Table 3.

In the following Table 4, when (d_2, c) close to $({d_2}^*, c_0) \approx (0.5307693, 0.1867237)$ at $E^* = (15.0615386, 15.0715386)$, with $\omega_1 \approx 0.2939612$, $\omega_2 \approx 0.2793454$, we shall analyze the dynamical behaviors of the system (1.4) with nonlocal competition and the system without nonlocal competition, when the parameters $(d_2,\ c)$ fall in these seven regions.

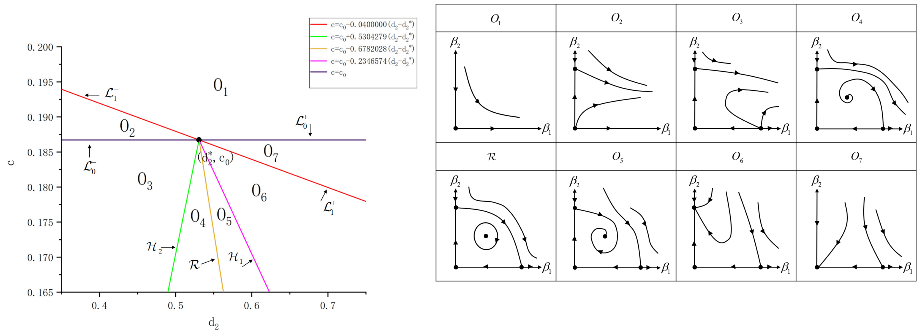


Figure 2. Left: the bifurcation regions of the system (5.1) near $({d_2}^*, c_0)$ in(d_2, c)-plane; Right: the corresponding phase portraits in O_1-O_7 and the corresponding phase portraits on Hopf bifurcation curve $\mathcal{R}$.

By numerical simulation, we verified that there has an unstable positive constant steady state E_0 in O_1. There have an unstable positive constant steady state E_0 and an unstable spatially nonhomogeneous periodic solution E_2 in O_2. There have an unstable positive constant steady state E_0, an unstable spatially homogeneous periodic solution E_1 and an unstable spatially nonhomogeneous periodic solution E_2 in O_3. There have an unstable positive constant steady state E_0, an unstable spatially homogeneous periodic solution E_1, an unstable spatially nonhomogeneous periodic solution E_2 and an unstable spatially nonhomogeneous quasi-periodic solution E_3 in O_4. There have an unstable positive constant steady state E_0, an unstable spatially homogeneous periodic solution E_1, an unstable spatially nonhomogeneous periodic solution E_2 and a stable spatially nonhomogeneous quasi-periodic solution E_3 in O_5. Then we can know that the spatially nonhomogeneous quasi-periodic solution with multiple time-frequencies, which are peak alternating with a single period, this shown that the wolves and rabbits will first concentrate at one side of the habitat and then shift to the other side. There have an unstable positive constant steady state E_0 and an unstable spatially homogeneous periodic solution E_1, a stable spatially nonhomogeneous periodic solution E_2 in O_6. This shows that the density of wolves and rabbits is unevenly distributed in space and changes in a certain period. There have an unstable spatially homogeneous periodic solution E_1, a stable positive constant steady state E_0 in O_7. This shows that the density of wolves and rabbits is evenly distributed in space and gradually tends to a positive equilibrium point.

6. Conclusion

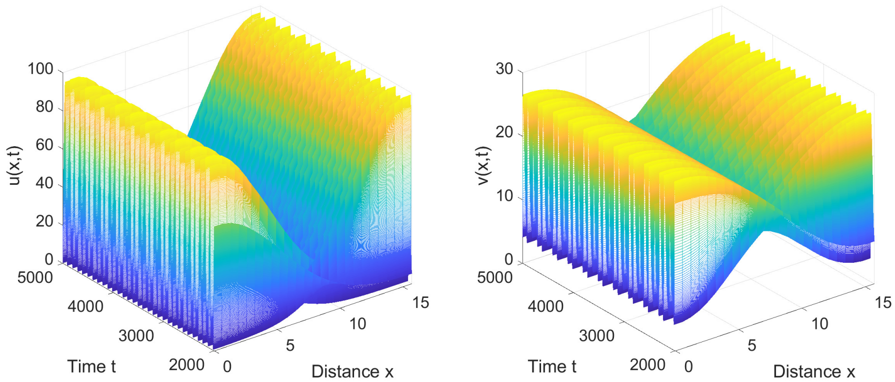
In this paper, we consider a diffusive predator-prey system with nonlocal competition. By selecting appropriate parameters (l_1, l_2), we study the effects of parameters (l_1, l_2) on the existence, multiplicity and stability of nonhomogeneous steady states. Then we study the existence and stability of positive nonconstant steady states in the neighborhood of the positive constant steady state E^* in (1.4). By solving the nonlinear functional equation $F(\mathbf{u}, l_1, l_2) = 0$, we obtain that there have unstable positive nonconstant steady states in the neighborhood of the positive constant steady state E^*. The linear coefficients c and d_2 are selected as the parameters.

Table 4. The comparison between the system with nonlocal competition and the system without nonlocal competition in O_1, O_2, O_3, O_4, O_5, O_6, O_7.

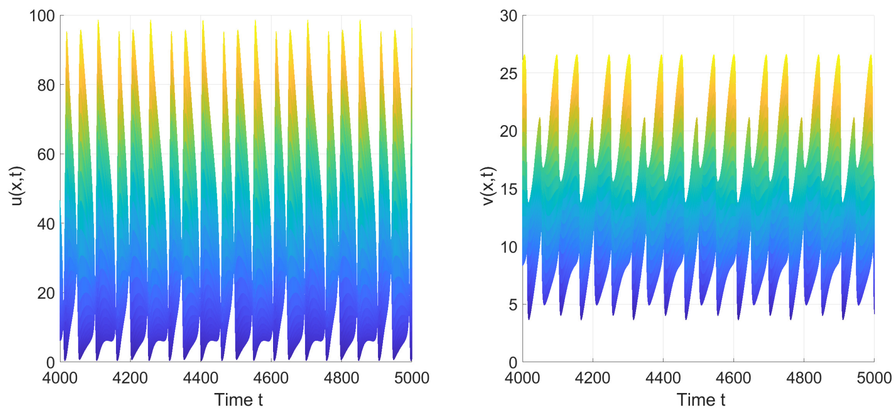
<table>
<tr><th></th><th>with nonlocal competition</th><th>without nonlocal competition</th></tr>
<tr><td>O_1</td><td>Unstable positive constant steady state E_0.</td><td rowspan="4"></td></tr>
<tr><td>O_2</td><td>Unstable positive constant steady state E_0; Unstable spatially nonhomogeneous periodic solution E_2.</td></tr>
<tr><td>O_3</td><td>Unstable positive constant steady state E_0; Unstable spatially homogeneous periodic solution E_1; Unstable spatially nonhomogeneous periodic solution E_2.</td></tr>
<tr><td>O_4</td><td>Unstable positive constant steady state E_0; Unstable spatially homogeneous periodic solution E_1; Unstable spatially nonhomogeneous periodic solution E_2; Unstable spatially nonhomogeneous quasi-periodic solution E_3.</td></tr>
<tr><td>O_5</td><td>Unstable positive constant steady state E_0; Unstable spatially homogeneous periodic solution E_1; Unstable spatially nonhomogeneous periodic solution E_2; Stable spatially nonhomogeneous quasi-periodic solution E_3. (see Figure 3)</td><td>Stable spatially homogeneous periodic solution. (see Figure 4)</td></tr>
<tr><td>O_6</td><td>Unstable positive constant steady state E_0; Unstable spatially homogeneous periodic solution E_1; Stable spatially nonhomogeneous periodic solution E_2. (see Figure 5)</td><td rowspan="2">Stable positive constant steady state E_0. (see Figure 7)</td></tr>
<tr><td>O_7</td><td>Unstable spatially homogeneous periodic solution E_1; Stable positive constant steady state E_0. (see Figure 6)</td></tr>
</table>

By analyzing the two pairs of pure imaginary roots of the characteristic equation and the third-order normal form of the diffusion predator-prey system with nonlocal competition, the normal form at the Hopf-Hopf bifurcation singularity is calculated. Marching polar coordinate transformation for the normal form. According to the critical bifurcation value, the plane region is divided into seven regions. The local stability of the equilibrium point of each part and the existence of Hopf-Hopf bifurcation are studied. Finally, the numerical simulation is carried out by MATLAB to verify the correctness of the theoretical analysis.

Our research shows that after adding nonlocal competition to a modified Leslie-Gower with diffusion and Beddington-DeAngelis functional response system can produce two intersecting Hopf bifurcation curves, i.e. produce Hopf-Hopf bifurcation. In the system without nonlocal competition, there will be no two intersecting Hopf bifurcation curves. And after adding the nonlocal competition to the model,



(a) Numerical simulations of u and v.(Left:(u, x, t)-plane, Right:(v, x, t)-plane.)



(b) The projection of Figure 3(a).(Left:(u, t)-plane, Right:(v, t)-plane.)

Figure 3. Numerical simulations of (1.4) for parameters $(d_2, c) = (0.7707693,\ 0.00267237) \in O_5$, with initial values $u(x, 0) = u^* + 0.0045000cos(\frac{3}{5}x)$, $v(x, 0) = v^* + 0.0005000cos(\frac{3}{5}x)$. In the Figure 3(a) and Figure 3(b), we can see that the spatially nonhomogeneous quasi-periodic solution is locally asymptotically stable in O_5.

when the parameters c and d_2 change, we can obtain the dynamic characteristics of the spatial distribution of the predator and prey in different regions. For example, when the parameters c and d_2 are in the O_5, the spatially nonhomogeneous quasi-periodic solution with multiple time-frequencies, which are peak alternating with a single period, this shown that the wolves and rabbits will first concentrate at one side of the habitat and then shift to the other side. When the parameters c and d_2 are in the O_6, the density of wolves and rabbits is unevenly distributed in space and oscillates within a certain period. When the parameters c and d_2 are in the O_7, the density of wolves and rabbits is evenly distributed in space and gradually tends to a positive equilibrium point. And we can know that the limit cycle appearing through the Hopf bifurcation curve $\mathcal{R}$ must be generated after the parameter enters the region O_5. This indicates that nonlocal competition can induce some new dynamic phenomena in model (1.4), produce locally asymptotically stable spatially nonhomogeneous periodic solution, locally asymptotically stable spatially nonhomo-

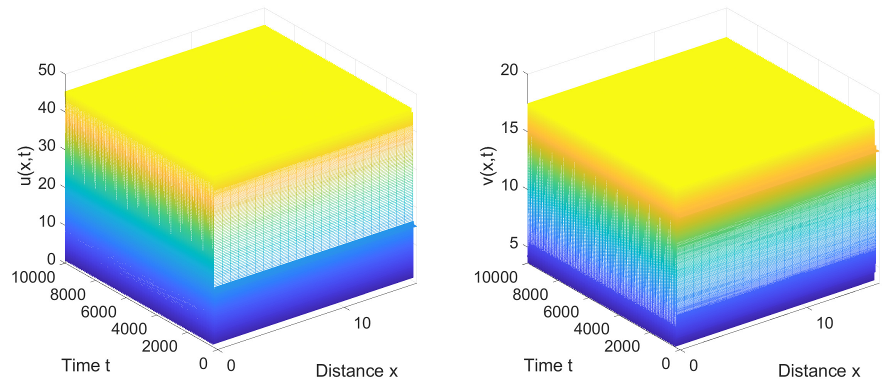


Figure 4. Numerical simulations of the system without nonlocal competition for parameters $(d_2, c) = (0.7707693, 0.00267237) \in O_5$, with initial values $u(x, 0) = u^* + 0.0045000cos(\frac{3}{5}x)$, $v(x, 0) = v^* + 0.0005000cos(\frac{3}{5}x)$. (Left:$(u, x, t)$-plane, Right: (v, x, t)-plane). In the image, we can see that the spatially homogeneous periodic solution is locally asymptotically stable in O_5.

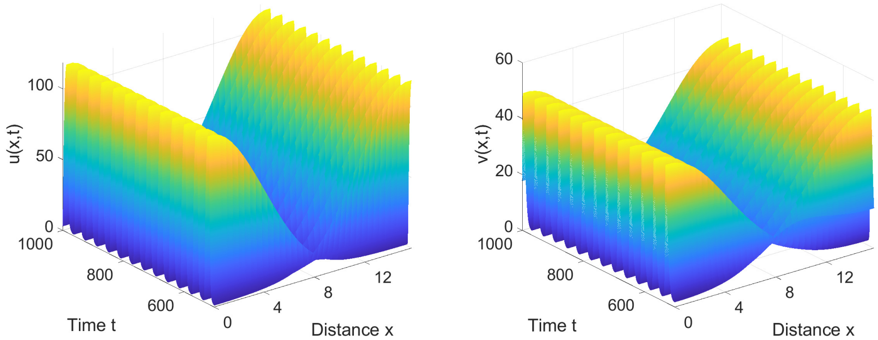
geneous quasi-periodic solutions and the limit cycles in the Hopf bifurcation curve $\mathcal{R}$.

Nonlocal competition between organisms is affected by the mobility of species populations in their spatial locations. With the permission of this nonlocal interaction, prey populations obtain limited food resources at their locations and nearby locations. In order to better study the influence of nonlocal competition on the dynamics with spatial heterogeneity, in this paper, we consider adding nonlocal competition to the modified Leslie-Gower with diffusion and Beddington-Diangelis functional response system. Through our research, it is shown that the addition of nonlocal competition to the model will induce some new dynamic phenomena, resulting in locally asymptotically stable spatial non-homogeneous periodic solutions, locally asymptotically stable spatial non-homogeneous quasi-periodic solutions, limit cycles in the Hopf bifurcation curve $\mathcal{R}$, which will help us to better study the relationship between biological populations. However, in this paper, we introduce the nonlocal competition effect including the average kernel function $G(x, y) = \frac{1}{|\Omega|}$ with $\Omega = (0, l\pi)$ in the prey. Although it will induce some new dynamic phenomena, in the real world, the competition between populations may be non-average. Therefore, it will be better to consider the non-average kernel function, which will also become the content of our further research. In addition, spatial pattern formation is also an important research direction in reaction-diffusion systems [17, 44], which we will consider in future studies.

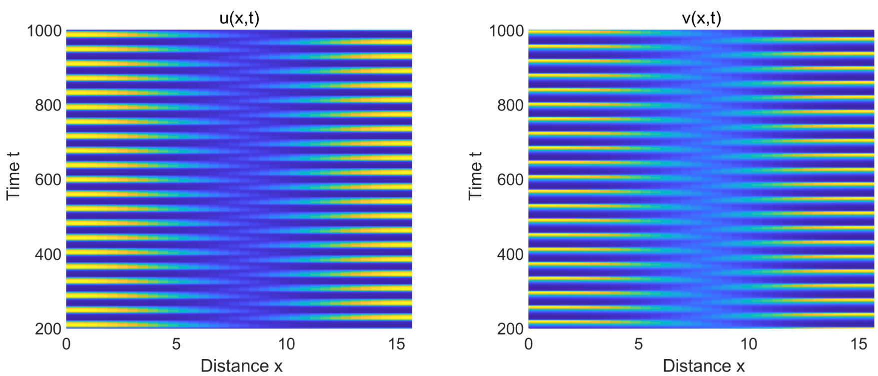
7. Appendix

7.1. Detailed calculations the coefficients in equation (4.19)

$$
\begin{aligned}
m_1(\xi) &= \frac{1}{2}\phi_1(0)(\langle \tilde{M}_1(\xi)(\psi_1\gamma_{k_1}), \gamma_{k_1}\rangle - \mu_{k_1} L_1(\xi)\psi_1(0)), \\
n_2(\xi) &= \frac{1}{2}\phi_2(0)(\langle \tilde{M}_1(\xi)(\psi_2\gamma_{k_2}), \gamma_{k_2}\rangle - \mu_{k_2} L_1(\xi)\psi_2(0)),
\end{aligned} \tag{7.1}
$$



(a) Numerical simulations of u and v.(Left:(u, x, t)-plane, Right:(v, x, t)-plane.)



(b) The projection of Figure 5(a) .(Left:(x, t)-plane, Right:(x, t)-plane.)

Figure 5. Numerical simulations of (1.4) for parameters $(d_2, c) = (0.5807693,\ 0.1857237) \in O_6$, with initial values $u(x, 0) = u^* + 0.0004000cos(\frac{2}{5}x)$, $v(x, 0) = v^* + 0.0002000cos(\frac{2}{5}x)$. In the Figure 5(a) and Figure 5(b), we can see that the spatially nonhomogeneous periodic solution is locally asymptotically stable in O_6.

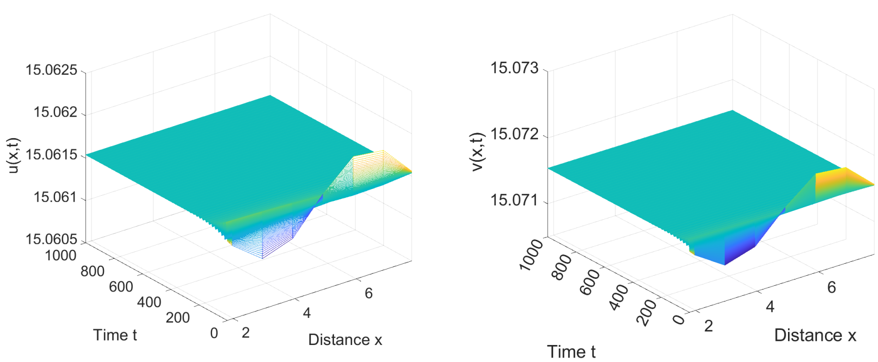
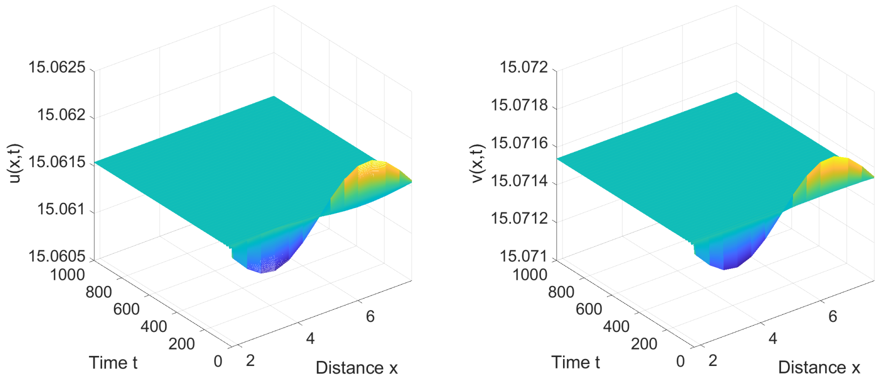
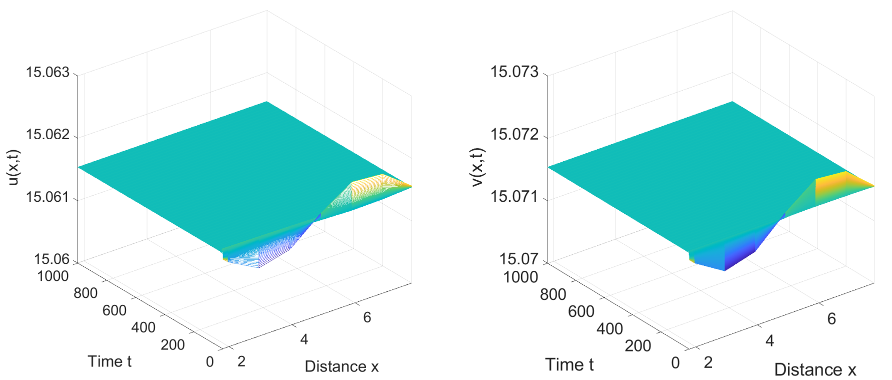


Figure 6. Numerical simulations of (1.4) for parameters $(d_2, c) = (0.5450693,\ 0.1862237) \in O_7$, with initial values $u(x, 0) = u^* + 0.0004500cos(\frac{1}{5}x)$, $v(x, 0) = v^* + 0.0005000cos(\frac{1}{5}x)$. (Left:$(u, x, t)$-plane, Right:$(v, x, t)$-plane.). In the image, we can see that the constant value solution is locally asymptotically stable in O_7.



(a) Numerical simulations of the system without nonlocal competition for parameters $(d_2, c) = (0.5807693, 0.1857237) \in O_6$, with initial values $u(x,0) = u^* + 0.0004000cos(\frac{2}{5}x)$, $v(x,0) = v^* + 0.0002000cos(\frac{2}{5}x)$.(Left:$(u,x,t)$-plane, Right:$(v,x,t)$-plane.)



(b) Numerical simulations of the system without nonlocal competition for parameters $(d_2, c) = (0.5450693, 0.1862237) \in O_7$, with initial values $u(x,0) = u^* + 0.0004500cos(\frac{1}{5}x)$, $v(x,0) = v^* + 0.0005000cos(\frac{1}{5}x)$.(Left:$(u,x,t)$-plane, Right:$(v,x,t)$-plane.)

Figure 7. Numerical simulations of the system without nonlocal competition for parameters in O_6, O_7. In the image, we can see that the constant value solution is locally asymptotically stable in O_6, O_7.

$$
\begin{aligned}
n_{2100} = {} & \frac{1}{2}\phi_1(0)[\langle C_{\Psi_1\Psi_1\Psi_1}, \gamma_{k_1}\rangle + \frac{2}{i\omega_1}(-\langle Q_{\Psi_1\Psi_1}, \gamma_{k_1}\rangle\phi_1(0) + \langle Q_{\Psi_1\Psi_1}, \gamma_{k_1}\rangle\bar{\phi}_1(0)) \\
& \times\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_1}\rangle + \frac{1}{i\omega_1}(\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_1}\rangle\phi_1(0) + \frac{1}{3}\langle Q_{\bar{\Psi}_1\bar{\Psi}_1}, \gamma_{k_1}\rangle\bar{\phi}_1(0)) \\
& \times\langle Q_{\Psi_1\Psi_1}, \gamma_{k_1}\rangle + \frac{2}{i\omega_2}(-\langle Q_{\Psi_1\Psi_2}, \gamma_{k_1}\rangle\phi_2(0) + \langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_2}\rangle \\
& + (\frac{1}{i(2\omega_1 - \omega_2)}\langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_1}\rangle\phi_2(0) \\
& + \frac{1}{i(2\omega_1 + \omega_2)}\langle Q_{\bar{\Psi}_1\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_1\Psi_1}, \gamma_{k_2}\rangle \\
& + 2\langle Q_{\Psi_1 R_{1100}}, \gamma_{k_1}\rangle + 2\langle Q_{\bar{\Psi}_1 R_{2000}}, \gamma_{k_1}\rangle],
\end{aligned}
$$

$$
\begin{aligned}
n_{1011} = & \frac{1}{2}\phi_1(0)[2\langle C_{\Psi_1\Psi_2\bar{\Psi}_2}, \gamma_{k_1}\rangle + \frac{2}{i\omega_1}(-\langle Q_{\Psi_1\Psi_1}, \gamma_{k_1}\rangle\phi_1(0) + \langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_1}\rangle\bar{\phi}_1(0)) \\
& \times\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_1}\rangle + (-\frac{2}{i\omega_2}\langle Q_{\Psi_1\Psi_2}, \gamma_{k_1}\rangle\phi_1(0) + \frac{2}{i(2\omega_1 - \omega_2)}\langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_1}\rangle\bar{\phi}_1(0)) \\
& \times\langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_1}\rangle + (\frac{2}{i\omega_2}\langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_1}\rangle\phi_1(0) + \frac{2}{i(2\omega_1 + \omega_2)}\langle Q_{\bar{\Psi}_1\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_1(0)) \\
& \times\langle Q_{\Psi_1\Psi_2}, \gamma_{k_1}\rangle + (\frac{2}{i\omega_2}(-\langle Q_{\Psi_1\Psi_2}, \gamma_{k_1}\rangle\phi_2(0) + \langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle \\
& + (\frac{2}{i(\omega_1 - 2\omega_2)}\langle Q_{\Psi_2\Psi_2}, \gamma_{k_1}\rangle\phi_2(0) + \frac{2}{i\omega_1}\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_2}\rangle \\
& + (\frac{2}{i\omega_1}\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_1}\rangle\phi_2(0) + \frac{2}{i(\omega_1 + 2\omega_2)}\langle Q_{\bar{\Psi}_2\bar{\Psi}_2}, \gamma_{k_1}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_1\Psi_2}, \gamma_{k_2}\rangle \\
& + 2\langle Q_{\Psi_1 R_{0011}}, \gamma_{k_1}\rangle + 2\langle Q_{\Psi_2 R_{1001}}, \gamma_{k_1}\rangle + 2\langle Q_{\bar{\Psi}_2 R_{1010}}, \gamma_{k_1}\rangle],
\end{aligned}
$$

$$
\begin{aligned}
m_{0021} = & \frac{1}{2}\phi_2(0)[\langle C_{\Psi_2\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle + \frac{2}{i\omega_2}(-\langle Q_{\Psi_2\Psi_2}, \gamma_{k_2}\rangle\phi_2(0) + \langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_2(0)) \\
& \times\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle + \frac{1}{i\omega_2}(\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle\phi_2(0) + \frac{1}{3}\langle Q_{\bar{\Psi}_2\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_2(0))\langle Q_{\Psi_2\Psi_2}, \gamma_{k_2}\rangle \\
& + \frac{2}{i\omega_1}(-\langle Q_{\Psi_1\Psi_2}, \gamma_{k_2}\rangle\phi_1(0) + \langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_2}\rangle\bar{\phi}_1(0))\langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_1}\rangle \\
& + (\frac{1}{i(2\omega_2 - \omega_1)}\langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_2}\rangle\phi_1(0) \\
& + \frac{1}{i(2\omega_2 + \omega_1)}\langle Q_{\bar{\Psi}_1\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_1(0))\langle Q_{\Psi_2\Psi_2}, \gamma_{k_1}\rangle \\
& + 2\langle Q_{\Psi_2 R_{0011}}, \gamma_{k_2}\rangle + 2\langle Q_{\bar{\Psi}_2 R_{0020}}, \gamma_{k_2}\rangle],
\end{aligned}
$$

$$
\begin{aligned}
m_{1110} = & \frac{1}{2}\phi_2(0)[2\langle C_{\Psi_1\bar{\Psi}_1\Psi_2}, \gamma_{k_2}\rangle + \frac{2}{i\omega_2}(-\langle Q_{\Psi_2\Psi_2}, \gamma_{k_2}\rangle\phi_2(0) + \langle Q_{\Psi_2\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_2(0)) \\
& \times\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_2}\rangle + (-\frac{2}{i\omega_1}\langle Q_{\Psi_1\Psi_2}, \gamma_{k_2}\rangle\phi_2(0) + \frac{2}{i(2\omega_2 - \omega_1)}\langle Q_{\Psi_1\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_2(0)) \\
& \times\langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_2}\rangle + (\frac{2}{i\omega_1}\langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_2}\rangle\phi_2(0) + \frac{2}{i(2\omega_2 + \omega_1)}\langle Q_{\bar{\Psi}_1\bar{\Psi}_2}, \gamma_{k_2}\rangle\bar{\phi}_2(0)) \\
& \times\langle Q_{\Psi_1\Psi_2}, \gamma_{k_2}\rangle + (\frac{2}{i\omega_1}(-\langle Q_{\Psi_1\Psi_2}, \gamma_{k_2}\rangle\phi_1(0) \\
& + \langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_2}\rangle\bar{\phi}_1(0))\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_1}\rangle \\
& + (\frac{2}{i(\omega_2 - 2\omega_1)}\langle Q_{\Psi_1\Psi_1}, \gamma_{k_2}\rangle\phi_1(0) + \frac{2}{i\omega_2}\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_2}\rangle\bar{\phi}_1(0))\langle Q_{\bar{\Psi}_1\Psi_2}, \gamma_{k_1}\rangle \\
& + (\frac{2}{i\omega_2}\langle Q_{\Psi_1\bar{\Psi}_1}, \gamma_{k_2}\rangle\phi_1(0) + \frac{2}{i(\omega_2 + 2\omega_1)}\langle Q_{\bar{\Psi}_1\bar{\Psi}_1}, \gamma_{k_2}\rangle\bar{\phi}_1(0))\langle Q_{\Psi_1\Psi_2}, \gamma_{k_1}\rangle \\
& + 2\langle Q_{\Psi_2 R_{1100}}, \gamma_{k_2}\rangle + 2\langle Q_{\Psi_1 R_{0110}}, \gamma_{k_2}\rangle + 2\langle Q_{\bar{\Psi}_1 R_{1010}}, \gamma_{k_2}\rangle],
\end{aligned}
$$

where

$$
R_{z_1z_2z_3z_4} = \begin{pmatrix} r_{z_1z_2z_3z_4} \\ \hat{r}_{z_1z_2z_3z_4} \end{pmatrix}, \tag{7.2}
$$

with $\hat{r}_{z_1z_2z_3z_4} = \int_\Omega G(x,\eta) r_{z_1z_2z_3z_4}(\eta)d\eta$ and $r_{z_1z_2z_3z_4}$, $z_1+z_2+z_3+z_4 = 2$, $z_1, z_2, z_3, z_4 \in \mathbb{N}_0$.

Define $\tilde{M}_1^k(\psi)\gamma_k = \tilde{M}_1^k(\psi\gamma_k)$, $\varphi_i := \langle \Psi_i, \gamma_{k_i}\rangle$, $\gamma_0(x) = 1$, $\gamma_k(x) = \sqrt{2}\cos\frac{k}{l}x$, $k \in \mathbb{N}$, $\langle \gamma_n(x), \gamma_m(x)\rangle = \frac{1}{l\pi}\int_0^{l\pi}\gamma_n(x)\gamma_m(x)\mathrm{d}x = \delta_{nm} := \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases}$, $r_z^k := \langle r_z, \gamma_k\rangle$, $z \in \mathbb{N}_0^4$, $|z| = 2$, $k \in \mathbb{N}_0$, $R_z^k := \langle R_z, \gamma_k\rangle$, $z \in \mathbb{N}_0^4$, $|z| = 2$, $k \in \mathbb{N}_0$.

For $k_1 = 0, k_2 \neq 0$, we have that

$$\begin{aligned}
&\langle \gamma_{k_1}^2, \gamma_{k_1}\rangle = \langle \gamma_{k_2}^2, \gamma_{k_1}\rangle = \langle \gamma_{k_1}\gamma_{k_2}, \gamma_{k_2}\rangle = 1,\\
&\langle \gamma_{k_1}^2, \gamma_{k_2}\rangle = \langle \gamma_{k_2}^2, \gamma_{k_2}\rangle = \langle \gamma_{k_1}\gamma_{k_2}, \gamma_{k_1}\rangle = 0,\\
&\langle \gamma_{k_1}^3, \gamma_{k_1}\rangle = 1, \quad \langle \gamma_{k_1}\gamma_{k_2}^2, \gamma_{k_1}\rangle = \langle \gamma_{k_1}^2\gamma_{k_2}, \gamma_{k_2}\rangle = 1,\\
&\langle Q_{\Psi_1 R_{1100}}\gamma_{k_1}, \gamma_{k_1}\rangle = Q_{\varphi_1}R_{1100}^0, \quad \langle Q_{\Psi_1 R_{0011}}\gamma_{k_1}, \gamma_{k_1}\rangle = Q_{\varphi_1}R_{0011}^0,\\
&\langle Q_{\bar{\Psi}_1 R_{2000}}\gamma_{k_1}, \gamma_{k_1}\rangle = Q_{\bar{\varphi}_1}R_{2000}^0, \quad \langle Q_{\Psi_2 R_{1100}}\gamma_{k_2}, \gamma_{k_2}\rangle = Q_{\varphi_2}R_{1100}^0,\\
&\langle Q_{\Psi_1 R_{0110}}\gamma_{k_1}, \gamma_{k_2}\rangle = Q_{\varphi_1}R_{0110}^{k_2}, \quad \langle Q_{\bar{\Psi}_1 R_{1010}}\gamma_{k_1}, \gamma_{k_2}\rangle = Q_{\bar{\varphi}_1}R_{1010}^{k_2},\\
&\langle Q_{\Psi_2 R_{1001}}\gamma_{k_2}, \gamma_{k_1}\rangle = Q_{\varphi_2}R_{1001}^{k_2}, \quad \langle Q_{\bar{\Psi}_2 R_{1010}}\gamma_{k_2}, \gamma_{k_1}\rangle = Q_{\bar{\varphi}_2}R_{1010}^{k_2},\\
&\langle Q_{\Psi_2 R_{0011}}\gamma_{k_2}, \gamma_{k_2}\rangle = Q_{\varphi_2}(R_{0011}^0 + \frac{1}{\sqrt{2}}R_{0011}^{2k_2}),\\
&\langle Q_{\bar{\Psi}_2 R_{0020}}\gamma_{k_2}, \gamma_{k_2}\rangle = Q_{\bar{\varphi}_2}(R_{0020}^0 + \frac{1}{\sqrt{2}}R_{0020}^{2k_2}).
\end{aligned}$$

Then the functions r_{2000}^0, r_{1100}^0, r_{0020}^0, $r_{0020}^{2k_2}$, r_{0011}^0, $r_{0011}^{2k_2}$, $r_{1010}^{k_2}$, $r_{1001}^{k_2}$, $r_{0110}^{k_2}$ are

$$\begin{aligned}
r_{2000}^0(\theta) =& \frac{1}{2}[2i\omega_1 I - \int_{-r}^0 e^{2i\omega_1\theta}\mathrm{d}\rho_0(\theta)]^{-1}Q_{\varphi_1\varphi_1}e^{2i\omega_1\theta} - \frac{1}{2i\omega_1}[\psi_1(\theta)\phi_1(0) + \frac{1}{3}\bar{\psi}_1(\theta)\\
&\times \bar{\phi}_1(0)]Q_{\varphi_1\varphi_1},\\
r_{1100}^0(\theta) =& [\int_{-r}^0 \mathrm{d}\rho_0(\theta)]^{-1}[-I + \psi_1(0)\phi_1(0) + \bar{\psi}_1(0)\bar{\phi}_1(0)]Q_{\varphi_1\bar{\varphi}_1},\\
r_{0020}^0(\theta) =& \frac{1}{2}[2i\omega_2 I - \int_{-r}^0 e^{2i\omega_2\theta}\mathrm{d}\rho_0(\theta)]^{-1}Q_{\varphi_2\varphi_2}e^{2i\omega_2\theta} + \frac{1}{2}[\frac{1}{i(\omega_1 - 2\omega_2)}\psi_1(\theta)\phi_1(0)\\
&- \frac{1}{i(\omega_1 + 2\omega_2)}\bar{\psi}_1(\theta)\bar{\phi}_1(0)]Q_{\varphi_2\varphi_2},\\
r_{0020}^{2k_2}(\theta) =& \frac{1}{2\sqrt{2}}[2i\omega_2 I - \int_{-r}^0 e^{2i\omega_2\theta}\mathrm{d}\rho_{2k_2}(\theta)]^{-1}Q_{\varphi_2\varphi_2}e^{2i\omega_2\theta},\\
r_{0011}^0(\theta) =& [\int_{-r}^0 \mathrm{d}\rho_0(\theta)]^{-1}[-I + \psi_1(0)\phi_1(0) + \bar{\psi}_1(0)\bar{\phi}_1(0)]Q_{\varphi_2\bar{\varphi}_2},\\
r_{1010}^{k_2}(\theta) =& [i(\omega_1 + \omega_2)I - \int_{-r}^0 e^{i(\omega_1+\omega_2)\theta}\mathrm{d}\rho_{k_2}(\theta)]^{-1}Q_{\varphi_1\varphi_2}e^{i(\omega_1+\omega_2)\theta}\\
&- [\frac{1}{i\omega_1}\psi_2(\theta)\phi_2(0) + \frac{1}{i(\omega_1 + 2\omega_2)}\bar{\psi}_2(\theta)\bar{\phi}_2(0)]Q_{\varphi_1\varphi_2},\\
r_{1001}^{k_2}(\theta) =& [i(\omega_1 - \omega_2)I - \int_{-r}^0 e^{i(\omega_1-\omega_2)\theta}\mathrm{d}\rho_{k_2}(\theta)]^{-1}Q_{\varphi_1\bar{\varphi}_2}e^{i(\omega_1-\omega_2)\theta} - [\frac{1}{i(\omega_1 - 2\omega_2)}
\end{aligned}$$

$$\times\, \psi_2(\theta)\phi_2(0) + \frac{1}{i\omega_1}\bar{\psi}_2(\theta)\bar{\phi}_2(0)]Q_{\varphi_1\bar{\varphi}_2},$$

$$r_{0011}^{2k_2}(\theta) = -\frac{1}{\sqrt{2}}[\int_{-r}^{0}\mathrm{d}\rho_{2k_2}(\theta)]^{-1}Q_{\varphi_2\overline{\varphi_2}}, \quad r_{0110}^{k_2}(\theta) = \overline{r_{1001}^{k_2}(\theta)}. \tag{7.3}$$

Then we can get the coefficients $n_{2100},\ n_{1011},\ m_{0021},\ m_{1110}$ of equation (4.19) are

$$\begin{aligned}
n_{2100} =\ & \frac{1}{2}\phi_1(0)C_{\varphi_1\varphi_1\bar{\varphi}_1} - \frac{\phi_1(0)}{2i\omega_1}[2(Q_{\varphi_1\varphi_1}\phi_1(0) - Q_{\varphi_1\bar{\varphi}_1}\bar{\phi}_1(0))Q_{\varphi_1\bar{\varphi}_1} \\
& -(Q_{\varphi_1\bar{\varphi}_1}\phi_1(0) + \frac{1}{3}Q_{\bar{\varphi}_1\bar{\varphi}_1}\bar{\phi}_1(0))Q_{\varphi_1\varphi_1}] + \phi_1(0)(Q_{\varphi_1}R_{1100}^{0} + Q_{\bar{\varphi}_1}R_{2000}^{0}), \\
n_{1011} =\ & \phi_1(0)C_{\varphi_1\varphi_2\bar{\varphi}_2} - \phi_1(0)[\frac{1}{i\omega_1}(Q_{\varphi_1\varphi_1}\phi_1(0) - Q_{\varphi_1\bar{\varphi}_1}\bar{\phi}_1(0))Q_{\varphi_2\bar{\varphi}_2} - (\frac{1}{i\omega_1}Q_{\varphi_2\bar{\varphi}_2} \\
& \times\phi_2(0) + \frac{1}{i(\omega_1+2\omega_2)}Q_{\bar{\varphi}_2\bar{\varphi}_2}\bar{\phi}_2(0))Q_{\varphi_1\varphi_2} - (\frac{1}{i(\omega_1-2\omega_2)}Q_{\varphi_2\varphi_2}\phi_2(0) \\
& +\frac{1}{i\omega_1}Q_{\varphi_2\bar{\varphi}_2}\bar{\phi}_2(0))Q_{\varphi_1\bar{\varphi}_2}] + \phi_1(0)(Q_{\varphi_1}R_{0011}^{0} + Q_{\varphi_2}R_{1001}^{k_2} + R_{\bar{\varphi}_2}R_{1010}^{k_2}), \\
m_{0021} =\ & \frac{3}{4}\phi_2(0)C_{\varphi_2\varphi_2\bar{\varphi}_2} - \frac{1}{2}\phi_2(0)[-(\frac{1}{i(2\omega_2-\omega_1)}Q_{\varphi_1\bar{\varphi}_2}\phi_1(0) \\
& +\frac{1}{i(2\omega_2+\omega_1)}Q_{\bar{\varphi}_1\bar{\varphi}_2}\bar{\phi}_1(0))Q_{\varphi_2\varphi_2} \\
& +\frac{2}{i\omega_1}(Q_{\varphi_1\varphi_2}\phi_1(0) - Q_{\bar{\varphi}_1\varphi_2}\bar{\phi}_1(0))Q_{\varphi_2\bar{\varphi}_2}] + \phi_2(0)[Q_{\varphi_2}(R_{0011}^{0} + \frac{1}{\sqrt{2}} \\
& \times R_{0011}^{2k_2}) + Q_{\bar{\varphi}_2}(R_{0020}^{0} + \frac{1}{\sqrt{2}}R_{0020}^{2k_2})], \\
m_{1110} =\ & \phi_2(0)C_{\varphi_1\bar{\varphi}_1\varphi_2} - \phi_2(0)[\frac{1}{i\omega_1}(Q_{\varphi_1\varphi_2}\phi_1(0) - Q_{\bar{\varphi}_1\varphi_2}\bar{\phi}_1(0))Q_{\varphi_1\bar{\varphi}_1} + (\frac{1}{i\omega_1}Q_{\varphi_1\varphi_2} \\
& \times\phi_2(0) - \frac{1}{i(2\omega_2-\omega_1)}Q_{\varphi_1\bar{\varphi}_2}\bar{\phi}_2(0))Q_{\bar{\varphi}_1\varphi_2} - (\frac{1}{i\omega_1}Q_{\bar{\varphi}_1\varphi_2}\phi_2(0) \\
& +\frac{1}{i(2\omega_2+\omega_1)}Q_{\bar{\varphi}_1\bar{\varphi}_2}\bar{\phi}_2(0))Q_{\varphi_1\varphi_2}] \\
& +\phi_2(0)(Q_{\varphi_1}R_{0110}^{k_2} + Q_{\bar{\varphi}_1}R_{1010}^{k_2} + Q_{\varphi_2}R_{1100}^{0}).
\end{aligned} \tag{7.4}$$

7.2. Detailed calculations the variables in n_{2100}, n_{1011}, m_{0021}, m_{1110}

Let

$$d_2 = {d_2}^* + \xi_1, \qquad c = c_0 + \xi_2,$$

$$V(t) = (u(t), v(t))^{\mathrm{T}}, \qquad \hat{V}(t) = \frac{1}{l\pi}\int_0^{l\pi} V(y,t)\,\mathrm{d}y,$$

as equation (4.2), transform the linearized the system (1.4) at (u^*, v^*)

$$\dot{V}(t) = L(\xi)\Delta V(t) + M(\xi)V(t) + \hat{M}(\xi)\hat{V}(t) + B\left(V(t), \hat{V}(t), \xi\right),$$

where

$$L_0(\xi) = \begin{pmatrix} d_1 & 0 \\ 0 & {d_2}^* + \xi_1 \end{pmatrix},$$

$$M(\xi) = \begin{pmatrix} r & \dfrac{-(u^*+p)r}{v^*} \\ c_0 + \xi_2 & -c_0 - \xi_2 \end{pmatrix}, \qquad \hat{M}(\xi) = \begin{pmatrix} -\dfrac{u^*}{K} & 0 \\ 0 & 0 \end{pmatrix},$$

$$B(u, \hat{u}, \xi) = \begin{pmatrix} u_1 + u^* - \dfrac{1}{K}(u_1 + u^*)(\hat{u}_1 + u_0) - \dfrac{a(u_1+u^*)(u_2+v^*)}{p + u_1 + u^* + s(u_2 + v^*)} - ru_1 + \\ \dfrac{r(u^*+p)}{v^*} u_2 + \dfrac{u^*}{K}\hat{u}_1 \\ (c_0 + \xi_2)(u_2 + v^*)(1 - \dfrac{u_2 + v^*}{u_1 + u^* + b}) - (c_0 + \xi_2)(u_1 - u_2) \end{pmatrix},$$

and $u = (u_1, u_2)^T$, $\hat{u} = (\hat{u}_1, \hat{u}_2)^{\mathrm{T}} \triangleq \frac{1}{l\pi}\int_0^{l\pi} u(\eta, t)\,\mathrm{d}\eta$, $\xi = (\xi_1, \xi_2)$. Then

$$L(0) = \begin{pmatrix} d_1 & 0 \\ 0 & {d_2}^* \end{pmatrix}, \qquad L_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & \xi_1 \end{pmatrix},$$

$$M(0) = \begin{pmatrix} r & \dfrac{-r(u^*+p)}{v^*} \\ c_0 & -c_0 \end{pmatrix}, \qquad M_1(\xi) = \begin{pmatrix} 0 & 0 \\ \xi_2 & -\xi_2 \end{pmatrix},$$

$$\hat{M}(0) = \begin{pmatrix} -\dfrac{u_0}{K} & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{M}_1(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$Q(V, V) = \begin{pmatrix} -\dfrac{2av^*(p+sv^*)}{(p+u^*+sv^*)^3}{u_1}^2 + \dfrac{2sau^*(u^*+p)}{(p+u^*+sv^*)^3}{u_2}^2 - \dfrac{2ap^2 + 2apu^* + 2apsv^* + 4asu^*v^*}{(p+u^*+sv^*)^3}u_1u_2 \\ -\dfrac{2}{K}u_1\hat{u} \\ -\dfrac{2c}{u^*+b}{u_2}^2 - \dfrac{2c}{u^*+b}{u_1}^2 + \dfrac{4c}{u^*+b}u_1u_2 \end{pmatrix},$$

$$C(V, V, V) = \begin{pmatrix} \dfrac{-6av^*(p+sv^*)}{(p+u^*+sv^*)^4}{u_1}^3 - \dfrac{6au^*s^2(p+u^*)}{(p+u^*+sv^*)^4}{u_2}^3 + \dfrac{6ap^2 + 12asu^*v^* + 6apu^* - 6as^2{v^*}^2}{(p+u^*+sv^*)^4}{u_1}^2u_2 \\ + \dfrac{-6as{u^*}^2 + 12as^2u^*v^* + 6asp^2 + 6as^2pv^*}{(p+u^*+sv^*)^4}u_1{u_2}^2 \\ \dfrac{6c}{(u^*+b)^2}{u_1}^3 - \dfrac{12c}{(u^*+b)^2}{u_1}^2u_2 + \dfrac{6c}{(u^*+b)^2}u_1{u_2}^2 \end{pmatrix},$$

where

$$V = \begin{pmatrix} u \\ \hat{u} \end{pmatrix}.$$

For $(0,\ k_2)$-mode Hopf-Hopf bifurcation, by equation (4.9) and equation (4.11), we can get the eigenfunctions ψ_i, $\bar{\psi}_i$, ϕ_i, $\bar{\phi}_i$, which are satisfiable equation $\phi_i\psi_i = 1$,

$\phi_i\psi_j = 0$, for $i, j = 1, 2$, $i \neq j$, where

$$\psi_1 = \begin{pmatrix} 1 \\ s_1 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} 1 \\ s_2 \end{pmatrix}, \quad \phi_1 = \begin{pmatrix} \frac{1}{S_1} \\ \frac{s_3}{S_1} \end{pmatrix}^{\mathrm{T}}, \quad \phi_2 = \begin{pmatrix} \frac{1}{S_2} \\ \frac{s_4}{S_2} \end{pmatrix}^{\mathrm{T}}, \tag{7.5}$$

with

$$\begin{aligned} s_1 &= \frac{c_0}{i\omega_1 + c_0}, & s_3 &= -\frac{r(u^* + p)}{v^*(i\omega_1 + c_0)}, \\ s_2 &= \frac{c_0}{c_0 + \frac{{d_2}^* {k_2}^2}{l^2} + i\omega_2}, & s_4 &= -\frac{r(u^* + p)}{v^*(i\omega_2 + c_0 + \frac{{d_2}^* {k_2}^2}{l^2})}, \\ S_1 &= 1 - \frac{c_0 r(u^* + p)}{v^*(i\omega_1 + c_0)^2}, & S_2 &= 1 - \frac{c_0 r(u^* + p)}{v^*(i\omega_2 + c_0 + \frac{{d_2}^* {k_2}^2}{l^2})^2}. \end{aligned}$$

For $k_1 = 0,\ k_2 = 1$, we can know that the system (1.4) satisfies Theorem 4.2, then the normal form of the system (1.4) can be derived (7.3) and (7.4).

References

[1] M. A. Aziz-Alaoui and M. D. Okiye, *Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes*, Applied Mathematics Letters, 2003, 16(7), 1069–1075.

[2] Q. An and W. Jiang, *Turing-Hopf bifurcation and Spatio-Temporal patterns of a ratio-dependent Holling-Tanner model with diffusion*, International Journal of Bifurcation and Chaos, 2018, 28(9), 1850108.

[3] N. F. Britton, *Aggregation and the competitive exclusion principle*, Journal of Theoretical Biology, 1989, 136(1), 57–66.

[4] R. S. Cantrell and C. Cosner, *On the dynamics of predator-prey models with the Beddington-DeAngelis functional response*, Journal of Mathematical Analysis and Applications, 2001, 257(1), 206–222.

[5] J. Cao, L. Ma and P. Hao, *Bifurcation analysis in a modified Leslie-Gower predator-prey model with Beddington-Deangelis functional response*, Journal of Applied Analysis and Computation, 2023, 13(5), 3026–3053.

[6] S. Chen and J. Yu, *Stability and bifurcation on predator-prey systems with nonlocal prey competition*, Discrete and Continuous Dynamical Systems, 2018, 38(1), 43–62.

[7] X. P. Chen, C. D. Huang, J. D. Cao, X. Y. Shi and A. Luo, *Hopf bifurcation in the delayed fractional Leslie-Gower model with Holling-type II functional response*, Journal of Applied Analysis and Computation, 2023, 13(5), 2555–2571.

[8] Y. F. Du, B. Niu, Y. X. Guo and J. J. Wei, *Double Hopf bifurcation in delayed reaction-diffusion systems*, Journal of Dynamics and Differential Equations, 2020, 32(1), 313–358.

[9] M. Fan and Y. Kuang, *Dynamics of a nonautonomous predator-prey system with the Beddington-DeAngelis functional response*, Journal of Mathematical Analysis and Applications, 2004, 295(1), 15–39.

[10] J. Furter and M. Grinfeld, *Local vs. non-local interactions in population dynamics*, Journal of Mathematical Biology, 1989, 27(1), 65–80.

[11] J. P. Gao and S. J. Guo, *Patterns in a modified Leslie-Gower model with Beddington-DeAngelis functional response and nonlocal prey competition*, International Journal of Bifurcation and Chaos, 2020, 30(5), 2050074.

[12] M. Gazor, B. Hamzi and A. Shoghi, *The infinite level normal forms for non-resonant double Hopf singularities*, System and Control Letters, 2023, 176, 105529.

[13] D. X. Geng, W. H. Jiang, Y. Lou and H. B. Wang, *Spatiotemporal patterns in a diffusive predator-prey system with nonlocal intraspecific prey competition*, Studies in Applied Mathematics, 2021, 148(1), 396–432.

[14] D. Geng and H. Wang, *Normal form formulations of double-Hopf bifurcation for partial functional differential equations with nonlocal effect*, Journal of Differential Equations, 2021, 741–785.

[15] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[16] G. Guo and J. Wang, *Pattern formation and qualitative analysis for a vegetation-water model with diffusion*, Nonlinear Analysis: Real World Applications, 2024, 76, 104008.

[17] G. Guo, J. You and K. A. Abbakar, *Pattern dynamics in a water-vegetation model with cross-diffusion and nonlocal delay*, Mathematical Methods in the Applied Sciences, 2025, 48, 3190–3213.

[18] S. Guo, *Global Hopf bifurcation of state-dependent delay differential equations*, International Journal of Bifurcation and Chaos, 2023, 33(06), 2350074.

[19] J. Hale and S. V. Lunel, *An Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, Springer-Verlag, New York, 1993.

[20] T. W. Hwang, *Global analysis of the predator-prey system with Beddington-DeAngelis functional response*, Journal of Mathematical Analysis and Applications, 2003, 281(1), 395–401.

[21] G. R. Itovich and J. L. Moiola, *Non-resonant double Hopf bifurcations: The complex case*, Journal of Sound and Vibration, 2009, 322(1–2), 358–380.

[22] X. Jia, K. Huang and C. Li, *Bifurcation analysis of a modified Leslie-Gower predator-prey system*, International Journal of Bifurcation and Chaos, 2023, 33(02), 2350024.

[23] W. Jiang, Q. An and J. Shi, *Formulation of the normal form of Turing-Hopf bifurcation in partial functional differential equations*, Journal of Differential Equations, 2020, 268(10), 6067–6102.

[24] T. T. Liu, L. J. Chen, F. D. Chen and Z. Li, *Dynamics of a Leslie-Gower model with weak Allee effect on prey and fear effect on predator*, International Journal of Bifurcation and Chaos, 2023, 33(01), 2350008.

[25] Y. Ma and R. Yang, *Hopf-Hopf bifurcation in a predator-prey model with nonlocal competition and refuge in prey*, Discrete and Continuous Dynamical Systems-Series B, 2023, 29(6), 2582–2609.

[26] S. M. Merchant and W. Nagata, *Instabilities and spatiotemporal patterns behind predator invasions with nonlocal prey competition*, Theoretical Population Biology, 2011, 80(4), 289–297.

[27] Y. Mi, C. Song and Z. Wang, *Global boundedness and dynamics of a diffusive predator-prey model with modified Leslie-Gower functional response and density-dependent motion*, Communications in Nonlinear Science and Numerical Simulation, 2023, 119, 107115.

[28] A. F. Nindjin, M. A. Aziz-Alaoui and M. Cadivel, *Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay*, Nonlinear Analysis Real World Applications, 2006, 7(5), 1104–1118.

[29] L. Pei, Q. Wang and H. Shi, *Bifurcation dynamics of the modified physiological model ofartificial pancreas with insulin secretion delay*, Nonlinear Dynamics, 2011, 63(3), 417–427.

[30] Y. H. Peng and G. Y. Zhang, *Dynamics analysis of a predator-prey model with herd behavior and nonlocal prey competition*, Mathematics and Computers in Simulation, 2020, 170, 366–378.

[31] Z. L. Shen and C. R. Zhang, *Double Hopf bifurcation of coupled dissipative Stuart-Landau oscillators with delay*, Applied Mathematics and Computation, 2014, 227, 553–566.

[32] G. T. Skalski and J. F. Gilliam, *Functional responses with predator interference: Viable alternatives to the Holling type II model*, Ecology, 2001, 82(11), 3083–3092.

[33] F. Souna, S. Djilali, S. Alyobi, A. Zeb, N. Gul, S. Alsaeed and K. S. Nisar, *Spatiotemporal dynamics of a diffusive predator-prey system incorporating social behavior*, AIMS Mathematics, 2023, 8(7), 15723–15748.

[34] F. Wang and R. Yang, *Spatial pattern formation driven by the cross-diffusion in a predator-prey model with Holling type functional response*, Chaos, Solitons and Fractals, 2023, 174, 113890.

[35] F. Wang and R. Yang, *Dynamics of a delayed reaction-diffusion predator-prey model with nonlocal competition and double Allee effect in prey*, International Journal of Biomathematics, 2025, 18(02), 2350097.

[36] F. Wang, R. Yang and X. Zhang, *Turing patterns in a predator-prey model with double Allee effect*, Mathematics and Computers in Simulation, 2024, 220, 170–191.

[37] J. Wang, J. Shi and J. Wei, *Dynamics and pattern formation in a diffusive predator-prey system with strong Allee effect in prey*, Journal of Differential Equations, 2011, 251(4–5), 1276–1304.

[38] L. Y. Wang, W. L. Wang and R. Z. Yang, *Stability switch and Hopf bifurcations for a diffusive plankton system with nonlocal competition and toxic effect*, AIMS Mathematics, 2023, 8(4), 9716–9739.

[39] T. P. Weng, *Stability analysis of diffusive predator-prey model with modified Leslie-Gower and Holling-type III schemes*, Applied Mathematics and Computation, 2011, 218(7), 3733–3745.

[40] J. Xu and L. J. Pei, *The nonresonant double Hopf bifurcation in delayed neural network*, International Journal of Computer Mathematics, 2008, 85(6), 925–935.

[41] R. Yang and C. Zhang, *Dynamics in a diffusive predator prey system with a constant prey refuge and delay*, Nonlinear Analysis Real World Applications, 2016, 31, 1–22.

[42] W. Yang, *Global asymptotical stability and persistent property for a diffusive predator-prey system with modified Leslie-Gower functional response*, Nonlinear Analysis-Real World Applications, 2013, 14(3), 1323–1330.

[43] F. Yi, J. Wei and J. Shi, *Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system*, Journal of Differential Equations, 2009, 246(5), 1944–1977.

[44] J. You and G. Guo, *Pattern formation for a reversible biochemical reaction model with cross-diffusion and Michalis saturation*, Journal of Mathematical Chemistry, 2025. DOI: 10.1007/s10910-025-01705-0.

[45] Y. Zhang and J. Xu, *Classification and computation of non-resonant double Hopf bifurcations and solutions in delayed van der Pol-Duffing system*, International Journal of Nonlinear Sciences and Numerical Simulation, 2005, 6(1), 63–68.

[46] F. Zhu and R. Yang, *Bifurcation in a modified Leslie-Gower model with nonlocal competition and fear effect*, Discrete and Continuous Dynamical Systems - B, 2025. DOI: 10.3934/dcdsb.2024195.