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A REGULARIZATION METHOD FOR BACKWARD PROBLEMS OF SINGULARLY PERTURBED PARABOLIC AND FRACTIONAL DIFFUSION EQUATIONS

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Abstract In this paper, backward problems of singularly perturbed parabolic and fractional diffusion equations are studied from the additional temperature data at fixed time $t = T$. We analyze the ill-posedness of these two inverse problems, and apply the quasi-reversibility regularization method to solve these problems. Then we obtain the convergence rates of logarithmic and Hölder types for the backward problems. Finally, several one- and two-dimensional numerical examples are given to verify the effectiveness and feasibility of the proposed method.

Keywords Singularly perturbed equations, backward problems, quasi-reversibility method, convergence rates.

MSC(2010) 35R25, 65M30.

1. Introduction

Singularly perturbed problems play important roles in many application fields, for example, Navier-Stokes equations with high Reynolds numbers, heat transport problems with large Peclet numbers, drift-diffusion equations in modeling semiconductor devices, and magneto-hydrodynamic pipelines at Hartmann numbers; for references focusing on the numerical aspects see [2, 21, 38] and the references therein.

As early as the 1950s, Haber et al. studied the boundary value problem of singularly perturbed differential equations [12, 13]. In [34], the authors studied the initial boundary value problem for singularly perturbed Boussinesq-type equations and in [33], the authors studied a stable standard difference scheme for a singularly perturbed convection-diffusion equation under computer perturbations. In [27], Lukyanenko et al. studied the analytic-numerical approach to solving singularly perturbed parabolic equations with the use of dynamic adapted meshes; for more

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information on singularly perturbed differential equations, see [3, 15, 17, 22, 29, 32, 36, 37].

In [1], Bomba et al. studied the inverse singular perturbation problem of convection-diffusion type over a quadrilateral curve field and in [25], Lukyanenko et al. studied the inverse coefficient solution of nonlinear singularly perturbed reaction-diffusion-advection equations with final time data. In [24], the authors studied the coefficient inverse problem for a nonlinear singularly perturbed two-dimensional reaction-diffusion equation with the location of moving front data and in [5], Chaikovskii et al. analyzed the convergence of forward and inverse problems of the singular-perturbed time-varying reaction-advection diffusion equation. In [4], the authors studied the asymptotic expansion regularization for inverse source problems in two-dimensional singularly perturbed nonlinear parabolic PDEs; for more information on inverse problems of singularly perturbed differential equations, see [7, 8, 23, 26].

To overcome the ill-posedness of the inverse problems, we need to use various regularization methods. In 1969, Lattès and Lions [18] proposed the method of quasi-reversibility to stabilize ill-posed problems. The main principle of this method is to replace the original second-order ill-posed problem by a family of well-posed fourth-order problems depending on a regularization parameter. Later, Eldén [11] used this method to consider the inverse heat conduction problem.

This method has been applied to solve various inverse diffusion problems, for example, Dorroh and Ru [9] considered the Cauchy problem for the heat equation (also we can call it sideways heat equation), Duc et al. [10] considered the inverse source problem of time-space fractional parabolic equations, Wen et al. [42] considered the backward problem of time-fractional wave equations, and Wang et al. [40] considered the inverse space-dependent source for the time-fractional diffusion equation. From the references by Yang et al. [47, 48], we can see that the proposed inverse source problem is mildly ill-posed and the degree of the ill-posedness is equivalent to the second-order numerical differentiation. In [6], the author studied the iterated quasi-reversibility method to regularize ill-posed elliptic and parabolic problems. For more results on numerical methods and regularization methods for diffusion equations, one can see [19, 35, 41, 43–45].

In this paper, we consider the singularly perturbed time-fractional diffusion equation with Dirichlet boundary condition as follows

$$\begin{cases} \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} - D_t^\alpha u(x,t) = -\dfrac{\partial u(x,t)}{\partial x}, & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = a(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where u is an unknown function, $\Omega \subset \mathbb{R}^d$ $(d = 1,\ 2)$, $0 < \varepsilon \ll 1$, and $D_t^\alpha u(x,t)$ is the Caputo left-sided fractional derivative of order $\alpha \in (0,1]$ defined by

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{u_\tau(x,\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1, \\ \dfrac{\partial u(x,t)}{\partial t}, & \alpha = 1, \end{cases} \tag{1.2}$$

in which $\Gamma(\cdot)$ denotes the Gamma function. In particular, when $\alpha = 1$, above equation is the singularly perturbed parabolic equation.

Denote $f(x) = u(x,T)$ as the final time data. Since the measurement is noise-contaminated inevitably, we denote the noisy measurement of f as f^δ which satisfies

$$||f - f^\delta|| \leq \delta, \tag{1.3}$$

where $||\cdot||$ is the $L^2(0,1)$ norm throughout this paper.

In this paper, we consider the following two inverse problems:

(IP1): When $\alpha = 1$, the backward problem for the singularly perturbed parabolic equation.

(IP2): When $\alpha \in (0,1)$, the backward problem for the singularly perturbed fractional diffusion equation.

This paper is organized as follows. Section 2 presents some preliminary results. Section 3 discusses the ill-posedness of our proposed two inverse problems. The quasi-reversibility method is given in Section 4 and error estimates are obtained by *a priori* choice principle of the regularization parameters. Several numerical experiments are presented in Section 5 and Section 6 gives a short conclusion.

2. Preliminaries

In order to facilitate the following proofs on theoretical derivation, we give the following definition and lemmas.

Definition 2.1. [16,30] The Mittag-Leffer function is defined by

$$E_{\alpha,1}(z) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(s\alpha+1)},\ z \in \mathbb{C},$$

with an arbitrary constant $\alpha > 0$.

Lemma 2.1. *For $n \in \boldsymbol{N}^+$ and ε, $\mu > 0$, we have*

$$\lambda_n^\mu = \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2} \leq \frac{2\varepsilon}{\mu} + \frac{1}{\mu}.$$

Proof. (1) Suppose $\varepsilon^2 - \mu - \mu^2 n^2\pi^2 \geq 0$, then we have

$$\frac{2\varepsilon\mu n^2\pi^2 + \varepsilon}{2\mu + 2\mu^2 n^2\pi^2} \leq \frac{2\varepsilon\mu n^2\pi^2}{2\mu^2 n^2\pi^2} + \frac{\varepsilon}{2\mu} \leq \frac{\varepsilon}{\mu} + \frac{\varepsilon}{2\mu} = \frac{3\varepsilon}{2\mu},$$

and

$$\frac{\sqrt{\varepsilon^2 - \mu - \mu^2 n^2\pi^2}}{2\mu + 2\mu^2 n^2\pi^2} \leq \frac{\sqrt{\varepsilon^2}}{2\mu} \leq \frac{\varepsilon}{2\mu},$$

so combining above two inequalities gives

$$\frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2} \leq \frac{3\varepsilon}{2\mu} + \frac{\varepsilon}{2\mu} = \frac{2\varepsilon}{\mu}.$$

(2) Suppose $\varepsilon^2 - \mu - \mu^2 n^2\pi^2 < 0$, then we can obtain that

$$\begin{aligned}\lambda_n^\mu &= \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2}\\ &\le \frac{\sqrt{(2\varepsilon\mu n^2\pi^2 + \varepsilon)^2}}{2\mu + 2\mu^2 n^2\pi^2} + \frac{\sqrt{\mu + \mu^2 n^2\pi^2 - \varepsilon^2}}{2\mu + 2\mu^2 n^2\pi^2}\\ &\le \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon}{2\mu + 2\mu^2 n^2\pi^2} + \frac{1}{2\sqrt{\mu + \mu^2 n^2\pi^2}}\\ &\le \frac{\varepsilon}{\mu} + \frac{\varepsilon}{2\mu} + \frac{1}{2\sqrt{\mu}}\\ &\le \frac{3\varepsilon}{2\mu} + \frac{1}{\mu}.\end{aligned}$$

Thus

$$\frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2} \le \frac{2\varepsilon}{\mu} + \frac{1}{\mu}.$$

The proof is completed. □

Lemma 2.2. *For $n \in \boldsymbol{N}^+$, $x \in [0,1]$, and $0 < \varepsilon \ll 1$, we have*

$$|1 - e^{\frac{\mu\lambda_n^\mu x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}}| \le C_1, \quad |e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}| \le C_2,$$

where $C_1,\ C_2$ are constants.

Proof. First of all, let $f(x) = 1 - e^{\frac{\mu\lambda_n^\mu x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}}$ and $g(x) = e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}$. We know from Lemma 2.1 that λ_n^μ is uniformly convergent, and μ, ε are constants, so whether $\frac{\mu\lambda_n^\mu}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}$ and $2\varepsilon - 2\mu\lambda_n^\mu$ are greater than 0 or not, $f(x)$ and $g(x)$ are monotonic functions of x. Therefore, there exist constants C_1 and C_2 such that

$$|1 - e^{\frac{\mu\lambda_n^\mu x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}}| \le C_1,$$

and

$$|e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}| \le C_2.$$

The proof is completed. □

Lemma 2.3. [31] *For $\forall r \ge 0$,*

$$1 - e^{-r} \le r.$$

Lemma 2.4. *For $n \in \boldsymbol{N}^+$ and $\varepsilon,\ \mu > 0$, if $\mu \le 2\varepsilon^2$, then we have*

$$\lambda_n - \lambda_n^\mu \le \frac{2}{\varepsilon}\frac{\mu n^4\pi^4}{1 + \mu n^2\pi^2},$$

here $\lambda_n = \frac{4\varepsilon^2 n^2\pi^2 + 1}{4\varepsilon}$, $\lambda_n^\mu = \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2}$.

Proof. Note

$$\begin{aligned}\lambda_n - \lambda_n^\mu &= \frac{4\varepsilon^2 n^2\pi^2 + 1}{4\varepsilon} - \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2} \\ &= \frac{4\varepsilon^2\mu^2 n^4\pi^4 + 4\varepsilon^2\mu n^2\pi^2 + \mu + \mu^2 n^2\pi^2}{4\varepsilon\mu(1+\mu n^2\pi^2)} \\ &\quad - \frac{4\varepsilon^2\mu n^2\pi^2 + 2\varepsilon^2 + 2\varepsilon\sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{4\varepsilon\mu(1+\mu n^2\pi^2)} \\ &= \frac{4\varepsilon^2\mu^2 n^4\pi^4 + \mu^2 n^2\pi^2}{4\varepsilon\mu(1+\mu n^2\pi^2)} \\ &\quad - \frac{2\varepsilon^2 + 2\varepsilon\sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|} - \mu}{4\varepsilon\mu(1+\mu n^2\pi^2)}.\end{aligned}$$

Note that if $\mu \le 2\varepsilon^2$, then $2\varepsilon^2 + 2\varepsilon\sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|} - \mu > 0$. Thus

$$\begin{aligned}\lambda_n - \lambda_n^\mu &\le \frac{4\varepsilon^2\mu^2 n^4\pi^4 + \mu^2 n^2\pi^2}{4\varepsilon\mu(1+\mu n^2\pi^2)} \\ &\le \frac{4\mu^2 n^4\pi^4(\varepsilon^2 + \frac{1}{4n^2\pi^2})}{4\varepsilon\mu(1+\mu n^2\pi^2)} \\ &\le \frac{2}{\varepsilon}\frac{\mu n^4\pi^4}{1+\mu n^2\pi^2}.\end{aligned}$$

The proof is completed. □

Lemma 2.5. [20] *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \le E_{\alpha,1}(-\eta) < 1$. Moreover, $E_{\alpha,1}(-\eta)$ is completely monotonic, that is*

$$(-1)^n \frac{d^n}{d\eta^n} E_{\alpha,1}(-\eta) \ge 0, \quad \eta \ge 0.$$

Lemma 2.6. [46] *For any λ_k satisfying $\lambda_k > \lambda_1 > 0$, there exist positive constants $\underline{C}$, $\overline{C} > 0$ depending on α, T, λ_1 such that*

$$\frac{\underline{C}}{\lambda_k} \le E_{\alpha,1}(-\lambda_k T^\alpha) \le \frac{\overline{C}}{\lambda_k}.$$

3. Ill-posedness of the inverse problem

In this section, we will analyze the ill-posedness of **(IP1)** and **(IP2)**.

First of all, we will study **(IP1):** The backward problem for the singularly perturbed parabolic equation when $\alpha = 1$.

If $\alpha = 1$, Equation (1.1) can be changed into the following equation

$$\begin{cases} \varepsilon\dfrac{\partial^2 u(x,t)}{\partial x^2} - \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{\partial u(x,t)}{\partial x}, & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = a(x), & x \in \Omega. \end{cases} \tag{3.1}$$

Applying the method of separation of variables, we seek a solution of (3.1) of the form

$$u(x,t) = \sum_{n=1}^{\infty} \rho_n(t) y_n(x). \tag{3.2}$$

Substituting (3.1) into (3.2), we require that $y_n(x)$ satisfies the equation

$$\varepsilon y_n''(x) + y_n'(x) = -\lambda y_n(x), x \in (\Omega) \tag{3.3}$$

and the boundary conditions

$$y_n(0) = y_n(1) = 0,$$

where λ is an unknown constant. The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{4\varepsilon^2 n^2 \pi^2 + 1}{4\varepsilon}, \quad y_n(x) = \sin(n\pi x), \quad n \geq 1. \tag{3.4}$$

The solution of the direct problem (3.1) is given by formula

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{x}{2\varepsilon} - \lambda_n t} \sin(n\pi x), \tag{3.5}$$

here $\{a_n\}_{n=1}^{\infty}$ are the Fourier coefficients of the function $a(x)$. Then putting $t = T$ in (3.5) we obtain that

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n T} e^{-\frac{x}{2\varepsilon}} \sin(n\pi x),$$

therefore

$$a(x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n T} e^{\frac{x}{2\varepsilon}} \sin(n\pi x), \tag{3.6}$$

here $\{f_n\}_{n=1}^{\infty}$ are the Fourier coefficients of the function $f(x)$ as follows

$$f_n = \frac{2}{\pi} \int_0^1 f(\xi) \sin(n\pi\xi) d\xi.$$

From the right hand side of (3.6), we see that the coefficient function $e^{\lambda_n T}$ approaches infinity when $n \to \infty$. Therefore, for (3.6), the exact data function $f(x)$ must decay rapidly when $n \to \infty$, otherwise a small error in the measured data will explode and completely destroy the solution. In [14], a detailed proof of the ill-posedness of this inverse problem was given.

(IP2): Let $\alpha \in (0,1)$, we study the backward problem for the singularly perturbed fractional diffusion equation as follows.

Firstly, we give the following singularly perturbed fractional diffusion equation

$$\begin{cases} \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} - D_t^{\alpha} u = -\dfrac{\partial u(x,t)}{\partial x}, & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = a(x), & x \in \Omega. \end{cases} \tag{3.7}$$

Similarly by the method of separative of variables, the solution of the direct problem (3.7) is given by

$$u(x,t)=\sum_{n=1}^{\infty}a_n e^{-\frac{x}{2\varepsilon}}E_{\alpha,1}(-\lambda_n t^{\alpha})\sin(n\pi x), \tag{3.8}$$

where, $\{a_n\}_{n=1}^{\infty}$ are the Fourier coefficients of the function $a(x)$ and $\lambda_n=\frac{4\varepsilon^2n^2\pi^2+1}{4\varepsilon}$.

Then putting $t=T$ in (3.8), we obtain that

$$f(x)=\sum_{n=1}^{\infty}a_n E_{\alpha,1}(-\lambda_n T^{\alpha})e^{-\frac{x}{2\varepsilon}}\sin(n\pi x),$$

therefore

$$a(x)=\sum_{n=1}^{\infty}\frac{f_n}{E_{\alpha,1}(-\lambda_n T^{\alpha})}e^{\frac{x}{2\varepsilon}}\sin(n\pi x). \tag{3.9}$$

Because $\lambda_n>0$ and λ_n is monotonically increasing, it follows from Lemma 2.5 that $E_{\alpha,1}(-\lambda_n T^{\alpha})\to 0$ as $n\to\infty$. Hence, above backward problem is also an ill-posed problem.

4. Quasi-reversibility regularization method and error estimate

In this section, we propose the quasi-reversibility method to solve **(IP1)** and **(IP2)**.

4.1. Quasi-reversibility regularization method and error estimate for (IP1)

Add to the singularly perturbed parabolic equation the proposed quasi-reversibility regularization item ν_{xxt}:

$$\begin{cases}\varepsilon\dfrac{\partial^2\nu(x,t)}{\partial x^2}-\dfrac{\partial\nu(x,t)}{\partial t}=-\dfrac{\partial\nu(x,t)}{\partial x}-\mu\nu_{xxt}(x,t), & x\in\Omega,\quad t\in(0,T),\\ \nu(x,t)=0, & x\in\partial\Omega,\quad t\in(0,T),\\ \nu(x,0)=a(x), & x\in\Omega,\\ \nu(x,T)=f(x), & x\in\Omega,\end{cases} \tag{4.1}$$

where $\mu>0$ is a regularization parameter. Applying the method of separation of variables, we seek a solution of the form

$$\nu(x,t)=\sum_{n=1}^{\infty}\rho_n(t)y_n(x). \tag{4.2}$$

Substituting (4.1) into (4.2), we require that $y_n(x)$ satisfies the equation

$$(\varepsilon-\mu\lambda^{\mu})y_n''(x)+y_n'(x)=-\lambda^{\mu}y_n(x),\quad x\in(0,1) \tag{4.3}$$

and the boundary conditions

$$y_n(0) = y_n(1) = 0. \tag{4.4}$$

Using the method of solving ordinary differential equation (4.3), we get the following eigenvalue equation

$$(\varepsilon - \mu\lambda^\mu)m^2 + m + \lambda^\mu = 0,$$

consequently,

$$m = \frac{-1 \pm \sqrt{1 - 4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2)}}{2(\varepsilon - \mu\lambda^\mu)}.$$

Then the general solution for $y_n(x)$ is

$$y_n(x) = e^{\frac{-1}{2(\varepsilon-\mu\lambda^\mu)}}(C_1 e^{\frac{\sqrt{1-4(\varepsilon\lambda^\mu-\mu(\lambda^\mu)^2)}}{2(\varepsilon-\mu\lambda^\mu)}x} + C_2 e^{\frac{-\sqrt{1-4(\varepsilon\lambda^\mu-\mu(\lambda^\mu)^2)}}{2(\varepsilon-\mu\lambda^\mu)}x}),$$

here C_1 and C_2 are constants.

Case 1. If $1 - 4(\varepsilon\lambda_n^\mu - \mu(\lambda_n^\mu)^2) > 0$, the boundary conditions $y_n(0) = 0$ and $y_n(1) = 0$ imply that $C_1 = 0$ and $C_2 = 0$. Hence, we can get the solution $y_n(x) = 0$. This trivial solution is not satisfied.

Case 2. If $1 - 4(\varepsilon\lambda_n^\mu - \mu(\lambda_n^\mu)^2) = 0$, the unique solution satisfying the boundary conditions is $y_n(x) = 0$, which is the same as Case 1.

Case 3. If $1 - 4(\varepsilon\lambda_n^\mu - \mu(\lambda_n^\mu)^2) < 0$, the general solution using Euler's formula is

$$\begin{aligned} y_n(x) = &e^{\frac{-1}{2(\varepsilon-\mu\lambda^\mu)}}(C_1 \cos(\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)}x) \\ &+ C_2 \sin(\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)}x)). \end{aligned}$$

For above solution, boundary condition $y_n(0) = 0$ requires that $C_1 = 0$, so

$$y_n(x) = C_2 e^{\frac{-1}{2(\varepsilon-\mu\lambda^\mu)}} \sin(\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)}x).$$

On the other hand, boundary condition $y_n(1) = 0$ requires that

$$C_2 e^{\frac{-1}{2(\varepsilon-\mu\lambda^\mu)}} \sin(\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)}) = 0,$$

if $C_2 = 0$ then $y_n(x) = 0$; if $C_2 \neq 0$ then $\sin(\frac{\sqrt{4(\varepsilon\lambda^\mu-\mu(\lambda^\mu)^2)-1}}{2(\varepsilon-\mu\lambda^\mu)}) = 0$, so

$$\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)} = n\pi.$$

Therefore, we can obtain the non-trivial solution for (4.3) satisfying the above boundary conditions (4.4) if and only if

$$1 - 4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) < 0, \tag{4.5}$$

and

$$\frac{\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1}}{2(\varepsilon - \mu\lambda^\mu)} = n\pi. \tag{4.6}$$

Solving the equation (4.6) for λ^μ, we have

$$\sqrt{4(\varepsilon\lambda^\mu - \mu(\lambda^\mu)^2) - 1} = 2n\pi(\varepsilon - \mu\lambda^\mu).$$

By simple calculation, we can get

$$(4\mu + 4\mu^2 n^2\pi^2)(\lambda_n^\mu)^2 - (4\varepsilon + 8\mu\varepsilon n^2\pi^2)\lambda_n^\mu + (1 + 4\varepsilon^2 n^2\pi^2) = 0. \tag{4.7}$$

The eigenvalues and eigenfunctions are respectively

$$\lambda_n^\mu = \frac{2\varepsilon\mu n^2\pi^2 + \varepsilon + \sqrt{|\varepsilon^2 - \mu - \mu^2 n^2\pi^2|}}{2\mu + 2\mu^2 n^2\pi^2}, \quad y_n(x) = \sin(n\pi x), \quad n \geq 1. \tag{4.8}$$

The solution of the direct problem (4.1) is given by the formula:

$$\nu(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu} - \lambda_n^\mu t} \sin(n\pi x). \tag{4.9}$$

Then substituting the exact data $f(x)$ and the error data $f^\delta(x)$ at time $t = T$ respectively into (3.5), we can obtain

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^\mu T} e^{-\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x),$$

and

$$f^\delta(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^\mu T} e^{-\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x).$$

Therefore

$$a_\mu(x) = \sum_{n=1}^{\infty} f_n e^{\lambda_n^\mu T} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x),$$

and

$$a_\mu^\delta(x) = \sum_{n=1}^{\infty} f_n^\delta e^{\lambda_n^\mu T} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x), \tag{4.10}$$

here $\{f_n^\delta\}_{n=1}^\infty$ are the Fourier coefficients of the function $f^\delta(x)$.

By the same notation as [49], we also use $||\cdot||_p$ to denote the norm of the Sobolev space $H^p(\Omega)$ defined by

$$||f(\cdot)||_p^2 = \sum_{n=1}^{\infty} (1 + n^2\pi^2)^p f_n^2, \tag{4.11}$$

where, $p > 0$ is a constant and $\{f_n\}_{n=1}^\infty$ are the Fourier coefficients of the function $f(x)$. Moreover, we have the following *a priori* bound about $u(x,0)$

$$||u(\cdot,0)||_p \leq E. \tag{4.12}$$

Then, we have the following error estimate between $a(x)$ and $a_\mu^\delta(x)$.

Theorem 4.1. *Suppose that the noisy data satisfy the condition (1.3), δ is a constant and given by* (1.3), $\mu \le 2\varepsilon^2$, *with the regularization parameter μ chosen as*

$$\mu = \frac{(2\varepsilon+1)T}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})}, \tag{4.13}$$

and the a priori *assumption (4.12) holds. Then for $p > 0$, we have*

$$||a(\cdot) - a^{\delta}_{\mu}(\cdot)|| \le C' \max\left\{\left(\frac{(2\varepsilon+1)T}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})}\right)^{\frac{p}{4}}, \frac{(2\varepsilon+1)T}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})}\right\}E + C_2\frac{E}{(\ln\frac{E}{\delta})^{2p}}.$$

Proof. From (3.6) and (4.10), we have

$$\begin{aligned}
||a(\cdot) - a^{\delta}_{\mu}(\cdot)|| &\le ||a(\cdot) - a_{\mu}(\cdot)||^2 + ||a_{\mu}(\cdot) - a^{\delta}_{\mu}(\cdot)|| \\
&= ||\sum_{n=1}^{\infty} f_n e^{\lambda_n T} e^{\frac{x}{2\varepsilon}} \sin(n\pi x) - \sum_{n=1}^{\infty} f_n e^{\lambda_n^{\mu} T} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}} \sin(n\pi x)|| \\
&\quad + ||\sum_{n=1}^{\infty} f_n e^{\lambda_n^{\mu} T} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}} \sin(n\pi x) - \sum_{n=1}^{\infty} f_n^{\delta} e^{\lambda_n^{\mu} T} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}} \sin(n\pi x)|| \\
&= \underbrace{||\sum_{n=1}^{\infty}(f_n e^{\frac{x}{2\varepsilon}} - f_n e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}})(e^{\lambda_n T} - e^{\lambda_n^{\mu} T}) \sin(n\pi x)||}_{I_1} \\
&\quad + \underbrace{||\sum_{n=1}^{\infty}(f_n e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}} - f_n^{\delta} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}}) e^{\lambda_n^{\mu} T} \sin(n\pi x)||}_{I_2}.
\end{aligned}$$

Next, we will estimate I_1 and I_2. Note

$$\begin{aligned}
I_1 &= ||\sum_{n=1}^{\infty}(f_n e^{\frac{x}{2\varepsilon}} - f_n e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}}})(e^{\lambda_n T} - e^{\lambda_n^{\mu} T}) \sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty} f_n e^{\frac{x}{2\varepsilon}}(1 - e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^{\mu}} - \frac{x}{2\varepsilon}})(e^{\lambda_n T} - e^{\lambda_n^{\mu} T}) \sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty} f_n e^{\frac{x}{2\varepsilon}}(1 - e^{\frac{\varepsilon x}{2\varepsilon^2 - 2\mu\varepsilon\lambda_n^{\mu}} - \frac{\varepsilon x - \mu\lambda_n^{\mu} x}{2\varepsilon^2 - 2\mu\varepsilon\lambda_n^{\mu}}})(e^{\lambda_n T} - e^{\lambda_n^{\mu} T}) \sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty} f_n e^{\frac{x}{2\varepsilon}}(1 - e^{\frac{\mu\lambda_n^{\mu} x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^{\mu}}})(e^{\lambda_n T} - e^{\lambda_n^{\mu} T}) \sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty} f_n e^{\frac{x}{2\varepsilon}} e^{\lambda_n T}(1 - e^{\frac{\mu\lambda_n^{\mu} x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^{\mu}}})(1 + n^2\pi^2)^{\frac{p}{2}}(1 + n^2\pi^2)^{-\frac{p}{2}} \\
&\quad \times (1 - e^{-(\lambda_n T - \lambda_n^{\mu} T)}) \sin(n\pi x)|| \\
&\le \sup_n |1 - e^{\frac{\mu\lambda_n^{\mu} x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^{\mu}}}||(1 - e^{-(\lambda_n - \lambda_n^{\mu})T})(1 + n^2\pi^2)^{-\frac{p}{2}}|E.
\end{aligned}$$

By calculation, if $\mu \le 2\varepsilon^2$, we have

$$\lambda_n - \lambda_n^\mu \ge 0.$$

According to Lemma 2.3 and Lemma 2.4, we obtain

$$(1 - e^{-(\lambda_n - \lambda_n^\mu)T}) \le (\lambda_n - \lambda_n^\mu)T \le \frac{2T}{\varepsilon}\frac{n^4\pi^4}{1+\mu n^2\pi^2}. \tag{4.14}$$

Let $A(n\pi) = (1 - e^{-(\lambda_n - \lambda_n^\mu)T})(1+n^2\pi^2)^{-\frac{p}{2}}$. In order to estimate $A(n\pi)$, we will consider two cases.

Case 1. When $n\pi \ge \mu^{-\frac{1}{4}}$, we have

$$A(n\pi) \le (1+n^2\pi^2)^{-\frac{p}{2}} \le (n\pi)^{-p} \le \mu^{\frac{p}{4}}.$$

Case 2. When $1 < n\pi < \mu^{-\frac{1}{4}}$, we estimate $A(n\pi)$ by (4.14) as

$$A(n\pi) \le \frac{2T}{\varepsilon}\frac{\mu n^4\pi^4}{1+\mu n^2\pi^2}(1+n^2\pi^2)^{-\frac{p}{2}} \le \frac{2T}{\varepsilon}\mu n^4\pi^4(1+n^2\pi^2)^{-\frac{p}{2}} \le \frac{2T}{\varepsilon}\mu(n\pi)^{4-p}.$$

If $0 < p < 4$, note that $n\pi < \mu^{-\frac{1}{4}}$, and the above inequality becomes into

$$A(n\pi) \le \frac{2T}{\varepsilon}\mu(n\pi)^{4-p} \le \frac{2T}{\varepsilon}\mu^{\frac{p}{4}},$$

else if $p \ge 4$, note that $n\pi > 1$, and we get

$$A(n\pi) \le \frac{2T}{\varepsilon}\mu.$$

Now combining the above, we get

$$A(n\pi) \le \max\{\frac{2T}{\varepsilon}\mu^{\frac{p}{4}}, \frac{2T}{\varepsilon}\mu\}.$$

Therefore, we obtain

$$I_1 \le C_1\frac{2T}{\varepsilon}\max\{\mu^{\frac{p}{4}}, \mu\}E \le C'\max\{\mu^{\frac{p}{4}}, \mu\}E. \tag{4.15}$$

According to Lemma 2.1, Lemma 2.2 and (1.3), we conclude that

$$\begin{aligned}
I_2 &= ||\sum_{n=1}^{\infty} f_n e^{\lambda_n T} e^{\overline{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}}\sin(n\pi x) - \sum_{n=1}^{\infty} f_n^\delta e^{\lambda_n^\mu T} e^{\overline{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}}\sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty}(f_n - f_n^\delta)e^{\lambda_n^\mu T} e^{\overline{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}}\sin(n\pi x)|| \\
&\le \sup_n |e^{\overline{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}}||e^{\lambda_n^\mu T}|\delta \\
&\le C_2 e^{\frac{2\varepsilon+1}{\mu}T}\delta.
\end{aligned}$$

Thus, we have

$$||a(\cdot) - a_\mu^\delta(\cdot)|| \le I_1 + I_2 \le C'\max\{\mu^{\frac{p}{4}}, \mu\}E + C_2 e^{\frac{2\varepsilon+1}{\mu}T}\delta.$$

We plug (4.13) into the above equation to get:

$$||a(\cdot) - a^{\delta}_{\mu}(\cdot)|| \leq C' \max\left\{ \left(\frac{(2\varepsilon+1)T}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})} \right)^{\frac{p}{4}}, \frac{(2\varepsilon+1)T}{\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-2p})} \right\} E + C_2 \frac{E}{(\ln\frac{E}{\delta})^{2p}}.$$

Now, the proof of the theorem is completed. $\square$

Remark 4.1. Since the regularization parameter $\mu \to 0$ as the measured error $\delta \to 0$, we find

$$\lim_{\delta\to 0} ||a(\cdot) - a^{\delta}_{\mu}(\cdot)|| \to 0,$$

if $p > 0,\ \mu \leq 2\varepsilon^2$.

4.2. Quasi-reversibility regularization method and error estimate for (IP2)

The quasi-reversibility regularization term $D^{\alpha}_t \nu_{xx}$ is added to the singular perturbation fractional diffusion equation:

$$\begin{cases} \varepsilon\dfrac{\partial^2\nu}{\partial x^2} - D^{\alpha}_t\nu = -\dfrac{\partial\nu}{\partial x} - \mu D^{\alpha}_t\nu_{xx}, & x\in\Omega, \quad t\in(0,T), \\ \nu(x,t) = 0, & x\in\partial\Omega, \quad t\in(0,T), \\ \nu(x,0) = a(x), & x\in\Omega, \\ \nu(x,T) = f^{\delta}(x), & x\in\Omega, \end{cases} \tag{4.16}$$

where $\mu > 0$ is a regularization parameter.

Similar to Problem (4.1) we can get the same λ^{μ}_n, so the solution of Problem (4.16) is given by the following

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{x}{2\varepsilon-2\mu\lambda^{\mu}_n}} E_{\alpha,1}(-\lambda^{\mu}_n t^{\alpha}) \sin(n\pi x). \tag{4.17}$$

Then substituting the exact data $f(x)$ and the error data $f^{\delta}(x)$ at time $t = T$ respectively into (4.17), we can obtain

$$f(x) = \sum_{n=1}^{\infty} a_n E_{\alpha,1}(-\lambda^{\mu}_n T^{\alpha}) e^{-\frac{x}{2\varepsilon-2\mu\lambda^{\mu}_n}} \sin(n\pi x),$$

and

$$f^{\delta}(x) = \sum_{n=1}^{\infty} a_n E_{\alpha,1}(-\lambda^{\mu}_n T^{\alpha}) e^{-\frac{x}{2\varepsilon-2\mu\lambda^{\mu}_n}} \sin(n\pi x).$$

Therefore

$$a_{\mu}(x) = \sum_{n=1}^{\infty} \frac{f_n}{E_{\alpha,1}(-\lambda^{\mu}_n T^{\alpha})} e^{\frac{x}{2\varepsilon-2\mu\lambda^{\mu}_n}} \sin(n\pi x),$$

and

$$a^{\delta}_{\mu}(x) = \sum_{n=1}^{\infty} \frac{f^{\delta}_n}{E_{\alpha,1}(-\lambda^{\mu}_n T^{\alpha})} e^{\frac{x}{2\varepsilon-2\mu\lambda^{\mu}_n}} \sin(n\pi x). \tag{4.18}$$

Theorem 4.2. *Assume that the noisy data satisfy the condition (1.3), δ is a constant and given by* (1.3), $\mu \leq 2\varepsilon^2$, *with the regularization parameter μ chosen as*

$$\mu = (\frac{\delta}{E})^{\frac{2}{p+2}}, \tag{4.19}$$

and the a priori *assumption (4.12) holds. Then for $p > 0$, we have*

$$||a(\cdot) - a_\mu^\delta(\cdot)|| \leq C_4 \max\{\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}, \delta^{\frac{2}{p+2}} E^{\frac{p}{p+2}}\} + \frac{(2\varepsilon+1)C_2}{\underline{C}} \delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}}.$$

Proof. From (3.9) and (4.18), we have

$$\begin{aligned}
&||a(\cdot) - a_\mu(\cdot)|| \\
&\leq ||a(\cdot) - a_\mu(\cdot)||^2 + ||a_\mu(\cdot) - a_\mu^\delta(\cdot)|| \\
&= ||\sum_{n=1}^{\infty} \frac{f_n}{E_{\alpha,1}(-\lambda_n T^\alpha)} e^{\frac{x}{2\varepsilon}} \sin(n\pi x) - \sum_{n=1}^{\infty} \frac{f_n}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x)||^2 \\
&\quad + ||\sum_{n=1}^{\infty} \frac{f_n}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x) - \sum_{n=1}^{\infty} \frac{f_n^\delta}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x)|| \\
&= \underbrace{||\sum_{n=1}^{\infty} (\frac{f_n}{E_{\alpha,1}(-\lambda_n T^\alpha)} e^{\frac{x}{2\varepsilon}} - \frac{f_n}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}) \sin(n\pi x)||}_{I_3} \\
&\quad + \underbrace{||\sum_{n=1}^{\infty} \frac{f_n - f_n^\delta}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}} \sin(n\pi x)||}_{I_4}.
\end{aligned}$$

Next, we will estimate I_3 and I_4:

$$\begin{aligned}
I_3 &= ||\sum_{n=1}^{\infty} (\frac{f_n}{E_{\alpha,1}(-\lambda_n T^\alpha)} e^{\frac{x}{2\varepsilon}} - \frac{f_n}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} e^{\frac{x}{2\varepsilon - 2\mu\lambda_n^\mu}}) \sin(n\pi x)|| \\
&= ||\sum_{n=1}^{\infty} \frac{f_n e^{\frac{x}{2\varepsilon}}}{E_{\alpha,1}(-\lambda_n T^\alpha)} (1 - e^{\frac{\mu\lambda_n^\mu x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}})(1+n^2\pi^2)^{\frac{p}{2}}(1+n^2\pi^2)^{-\frac{p}{2}} \\
&\quad \times (1 - \frac{E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)}) \sin(n\pi x)|| \\
&\leq \sup_n |1 - e^{\frac{\mu\lambda_n^\mu x}{2\varepsilon^2 - 2\varepsilon\mu\lambda_n^\mu}} ||(1 - \frac{E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)})(1+n^2\pi^2)^{-\frac{p}{2}}| E.
\end{aligned}$$

Given that $\lambda_n > 0$, $\lambda_n^\mu > 0$, if $\mu \leq 2\varepsilon^2$, we have $\lambda_n - \lambda_n^\mu > 0$. We can see from Lemma 2.5 that the function $E_{\alpha,1}\alpha(-\eta)$ of η is monotonically decreasing, then

$$E_{\alpha,1}(-\lambda_n T^\alpha) < E_{\alpha,1}(-\lambda_n^\mu T^\alpha),$$

so,

$$\frac{E_{\alpha,1}(-\lambda_n T^\alpha)}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} < 1.$$

By Lemmas 2.4, 2.6 and (3.4), we have

$$\begin{aligned}1-\frac{E_{\alpha,1}(-\lambda_nT^\alpha)}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)} &\le 1-\frac{\frac{\underline{C}}{\lambda_n}}{\frac{\overline{C}}{\lambda_n^\mu}}\\ &=1-C\frac{\lambda_n^\mu}{\lambda_n}\\ &\le \widetilde{C}(1-\frac{\lambda_n^\mu}{\lambda_n})\\ &\le \widetilde{C}(\frac{\lambda_n-\lambda_n^\mu}{\lambda_n^\mu})\\ &\le \widetilde{C}(\frac{\frac{2}{\varepsilon}\frac{\mu n^4\pi^4}{1+\mu n^2\pi^2}}{\varepsilon n^2\pi^2+\frac{1}{4\varepsilon}})\\ &\le C_3\frac{\mu n^2\pi^2}{1+\mu n^2\pi^2}.\end{aligned}$$

Let $B(n\pi)=(1-\frac{E_{\alpha,1}(-\lambda_nT^\alpha)}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)})(1+n^2\pi^2)^{-\frac{p}{2}}$, to estimate $B(n\pi)$, we will consider two cases.

Case 1. For $n\pi\ge\mu^{-\frac{1}{2}}$, we can easily estimate $B(n\pi)$ as

$$B(n\pi)\le(1+n^2\pi^2)^{-\frac{p}{2}}\le(n\pi)^{-p}\le\mu^{\frac{p}{2}}.$$

Case 2. For $1<n\pi<\mu^{-\frac{1}{2}}$, we estimate $B(n\pi)$ by Lemma 2.3 as

$$B(n\pi)\le C_3\frac{\mu n^2\pi^2}{1+\mu n^2\pi^2}(1+n^2\pi^2)^{-\frac{p}{2}}\le C_3\mu n^2\pi^2(1+n^2\pi^2)^{-\frac{p}{2}}\le C_3\mu(n\pi)^{2-p}.$$

If $0<p<2$, note that $n\pi<\mu^{-\frac{1}{2}}$, above inequality becomes into

$$B(n\pi)\le C_3\mu(n\pi)^{2-p}\le C_3\mu^{\frac{p}{2}},$$

else if $p\ge 2$, note that $n\pi>1$, we get

$$B(n\pi)\le C_3\mu.$$

Now combining above, we get

$$B(n\pi)\le C_4\max\{\mu^{\frac{p}{2}},\mu\}.$$

Therefore, we can obtain

$$I_3\le C_4\max\{\mu^{\frac{p}{2}},\mu\}E. \tag{4.20}$$

According to Lemmas 2.1, 2.2, 2.6 and (1.3), we can get

$$\begin{aligned}I_4&=||\sum_{n=1}^{\infty}\frac{f_n-f_n^\delta}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)}e^{\frac{x}{2\varepsilon-2\mu\lambda_n^\mu}}\sin(n\pi x)||\\ &\le\sup_n|e^{\frac{x}{2\varepsilon-2\mu\lambda_n^\mu}}||\frac{1}{E_{\alpha,1}(-\lambda_n^\mu T^\alpha)}|\delta\\ &\le C_2\frac{\lambda_n^\mu}{\underline{C}}\delta\\ &\le\frac{(2\varepsilon+1)C_2}{\underline{C}}\frac{\delta}{\mu}.\end{aligned}$$

Thus, we have

$$||a(\cdot) - a^\delta_\mu(\cdot)|| \le I_1 + I_2 \le C_4 \max\{\mu^{\frac{p}{2}}, \mu\}E + \frac{(2\varepsilon+1)C_2}{\underline{C}}\frac{\delta}{\mu}.$$

Substituting (4.19) into the above equation to, we can achieve

$$\begin{aligned}||a(\cdot) - a^\delta_\mu(\cdot)|| \le & C_4 \max\{\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}, \delta^{\frac{2}{p+2}}E^{\frac{p}{p+2}}\} \\ & + \frac{(2\varepsilon+1)C_2}{\underline{C}}\delta^{\frac{p}{p+2}}E^{\frac{2}{p+2}}.\end{aligned}$$

The proof is completed. □

Remark 4.2. When the measured error $\delta \to 0$, for $p > 0$, $\mu \le 2\varepsilon^2$. We can obtain

$$\lim_{\delta\to 0}||a(\cdot) - a^\delta_\mu(\cdot)|| \to 0, \quad p > 0.$$

5. Numerical experiments

To verify the above theoretical results, we will give some numerical experiments in this section. The numerical examples are calculated as follows:

First, the accurate data function $f(x)$ is obtained by solving the direct problem with the exact initial value $a(x)$.

Then, we add a perturbation of the normal distribution to each resulting function $f(x)$, giving the vector $f^\delta(x)$,

$$f^\delta = f + \delta f \cdot randn(size(f)), \tag{5.1}$$

where "$randn(\cdot)$" is a normally distributed random variable with zero mean and unit standard deviation.

Finally, f^δ is applied to (4.10) and (4.18) to obtain the regularization solution. In order to illustrate the accuracy of the numerical solution, we compute the relative error in $L^2(\Omega)$ between the numerical and exact solutions by

$$E(a) = \frac{||a^\delta_\mu(\cdot) - a(\cdot)||_{L^2(0,1)}}{||a(\cdot)||_{L^2(0,1)}}, \tag{5.2}$$

where $a^\delta_\mu(x)$ is the regularization solution, and $a(x)$ is the exact solution.

5.1. Numerical experiments for (IP1)

Example 5.1. Consider the continuous initial value

$$a(x) = \sin(3\pi x)e^x, \quad x \in [0,1]. \tag{5.3}$$

In this example, let $\varepsilon = 0.001$. Numerical results at different noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are shown in Figure 1.

It can be seen from the figure that the numerical results are in agreement with the exact solution to a high noise level $\delta = 0.1$, The results show that the accuracy and stability of the proposed method is satisfactory for inverse initial value problem. In Table 1, we show the values of $E(a)$ under different ε and δ, and it can be observed that the numerical results are stable with the parameters ε.

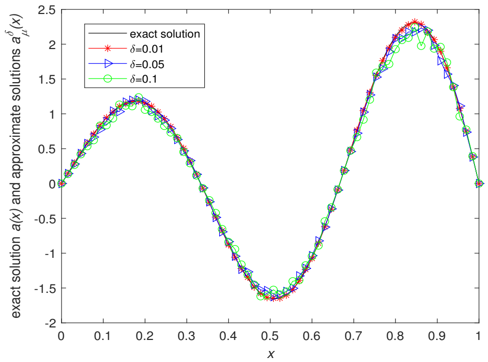


Figure 1. Exact solution and its approximation for Example 5.1 when $\varepsilon = 0.001$.

Table 1. The relative errors $E(a)$ with different singular value ε and noise level δ for Example 5.1.

ε	δ=0.01	δ=0.05	δ=0.1
0.001	0.0173	0.0403	0.0727
0.005	0.0151	0.0311	0.0581
0.008	0.0106	0.0302	0.0511

Example 5.2. Consider non-smooth initial value

$$a(x) = \begin{cases} -4x, & 0 \le x \le \frac{1}{4}, \\ 8x - 3, & \frac{1}{4} < x \le \frac{1}{2}, \\ -8x + 5, & \frac{1}{2} < x \le \frac{3}{4}, \\ 4x - 4, & \frac{3}{4} < x \le 1. \end{cases} \tag{5.4}$$

In this example, let $\varepsilon = 0.003$. The numerical results with various noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are illustrated in Figure 2. As can be seen from the figure, our numerical method is also very effective for reconstructing the initial values of the nonsmooth functions with cusps.

Example 5.3. Consider piecewise initial value

$$a(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{4}, \\ 1, & \frac{1}{4} < x \le \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le \frac{3}{4}, \\ 0, & \frac{3}{4} < x \le 1. \end{cases} \tag{5.5}$$

In this example, let $\varepsilon = 0.005$. Numerical results at different noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are displayed in Figure 3. It can be seen from the figure that the

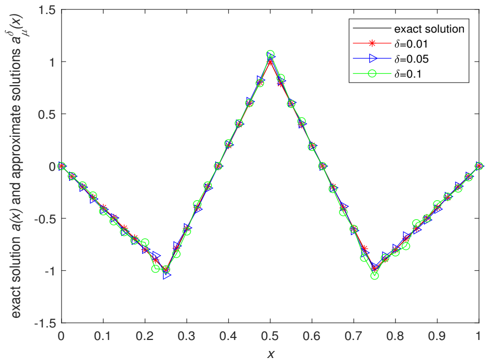


Figure 2. Exact solution and its approximation for Example 5.2 when $\varepsilon = 0.003$.

error between the approximate solution and the exact solution is very small and show that the approximate solution has good stability.

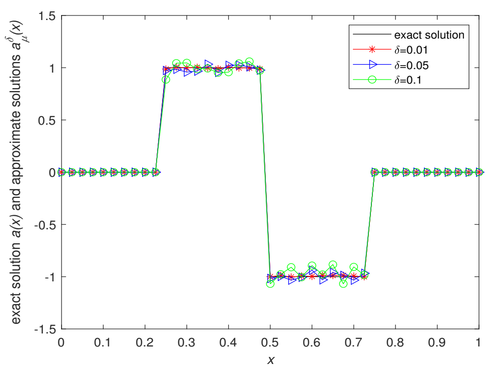


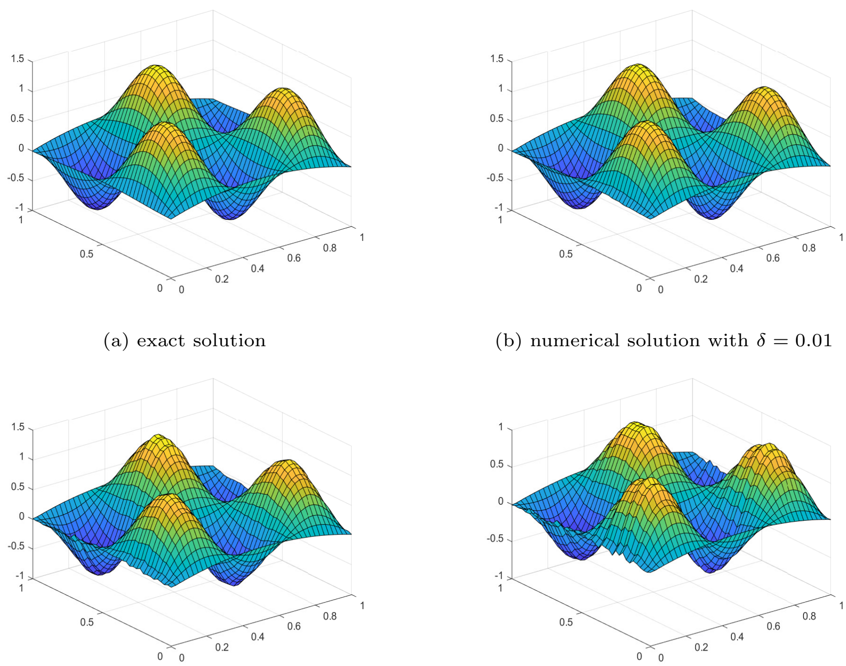
Figure 3. Exact solution and its approximation for Example 5.3 when $\varepsilon = 0.005$.

Example 5.4. Consider a two-dimensional example

$$a(x,y) = \sin(3\pi x)\sin(2\pi y) + x(1-x) + y^2(1-y)^2. \tag{5.6}$$

In this example, let $\varepsilon = 0.001$, $\mu = 6.737 \times 10^{-6}, 6.74 \times 10^{-6}, 6.75 \times 10^{-6}$. The numerical results with various noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are illustrated in Figure 4, and the corresponding numerical errors are shown in the Figure 5. From these figures, we can see that the proposed method is also very effective for two-dimensional examples.

From the figures above, we can see that the numerical results obtained by our regularization method are in good agreement with the exact values, even when relatively large errors are added to the measured data. It shows that our regularization method is very effective for retrieving the initial values of singularly perturbed parabolic equations.



(a) exact solution

(b) numerical solution with $\delta = 0.01$

(c) numerical solution with $\delta = 0.05$

(d) numerical solution with $\delta = 0.1$

Figure 4. Numerical results for source term in Example 5.4.

5.2. Numerical experiments for (IP2)

Example 5.5. Consider the continuous initial value

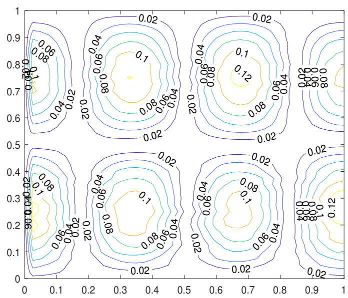
$$a(x) = \sin(5\pi x)e^x + x^{(1+\alpha)}(1-x)^{(2+\alpha)}, \quad x \in [0,1]. \tag{5.7}$$

In this example, let $\varepsilon = 0.001$, $\alpha = 0.8$. Numerical results at different noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are shown in Figure 6. As can be seen from the figure, the error between the regularized and exact solutions is very small, indicating that our numerical method is very efficient for reconstructing continuous initial values.

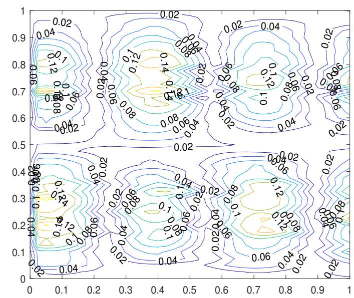
Example 5.6. Consider the non-smooth initial value

$$a(x) = \begin{cases} 0, & 0 \le x \le \dfrac{1}{4}, \\ 4x-1, & \dfrac{1}{4} < x \le \dfrac{1}{2}, \\ -4x+3, & \dfrac{1}{2} < x \le \dfrac{3}{4}, \\ 0, & \dfrac{3}{4} < x \le 1. \end{cases} \tag{5.8}$$

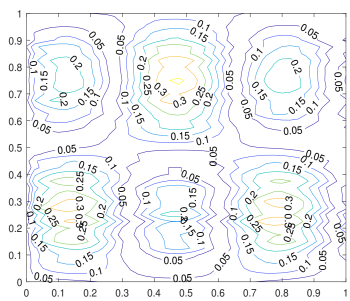
In this example, let $\varepsilon = 0.004$, $\alpha = 0.8$. Numerical results with different noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are shown in Figure 7. For fractional diffusion equations, our numerical method is also feasible to reconstruct the initial values containing cusps.



(a) absolute error to the source term with $\delta = 0.01$



(b) absolute error to the source term with $\delta = 0.05$



(c) absolute error to the source term with $\delta = 0.1$

Figure 5. The comparison of the numerical effects between the exact source term and its computed approximations for Example 5.4.

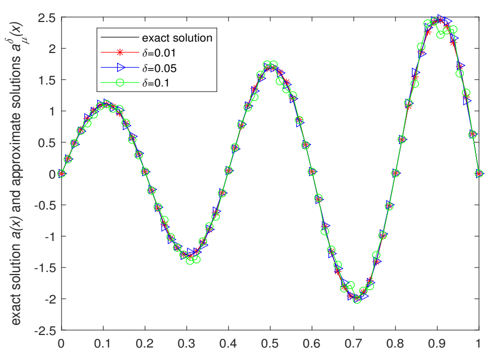


Figure 6. Exact solution and its approximation for Example 5.5 when $\varepsilon = 0.001, \alpha = 0.8$.

Table 2 reports the relative errors of Example 5.6 for different α and ε. From the table, we notice that the relative error $E(a)$ is relatively stable.

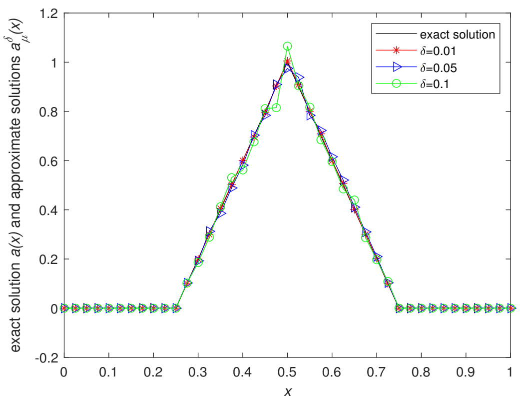


Figure 7. Exact solution and its approximation for Example 5.6 when $\varepsilon = 0.004, \alpha = 0.8$.

Table 2. The relative errors $E(a)$ with different fractional order α and noise level ε when $\delta = 0.01$ for Example 5.6.

α	ε=0.001	ε=0.005	ε=0.008
0.2	0.0072	0.0064	0.0052
0.5	0.0065	0.0056	0.0046
0.8	0.0058	0.0054	0.0043

Example 5.7. Consider piecewise continuous initial value

$$a(x) = \begin{cases} 0, & 0 \leq x \leq \dfrac{1}{4}, \\ \sin(0.25\pi)e^{\frac{1}{2}} + 0.5^{(3+2\alpha)}, & \dfrac{1}{4} < x \leq \dfrac{1}{2}, \\ \sin(5\pi x)e^{x} + x^{(1+\alpha)}(1-x)^{(2+\alpha)}, & \dfrac{1}{2} < x \leq 1. \end{cases} \tag{5.9}$$

In this example, let $\varepsilon = 0.008$, $\alpha = 0.8$. Numerical results with different noise levels $\delta = 0.01,\ 0.05,\ 0.1$ are displayed in Figure 8.

Example 5.8. Consider a two-dimensional example

$$a(x, y) = \cos(4\pi x)\cos(2\pi y) + (\frac{2x^3}{3} - x^2)e^{(2y^3-3y^2)}. \tag{5.10}$$

In this example, let $\varepsilon = 0.001$ and $\alpha = 0.8$, and we choose that $\mu = 6.7\times 10^{-6}$, 8×10^{-6}, 9×10^{-6} for different noise levels $\delta = 0.01,\ 0.05,\ 0.1$, respectively. The numerical results are illustrated in Figure 9, and the corresponding numerical errors are shown in the Figure 10. From these figures, we can see that the method used in this paper is also very effective for the two-dimensional fractional examples.

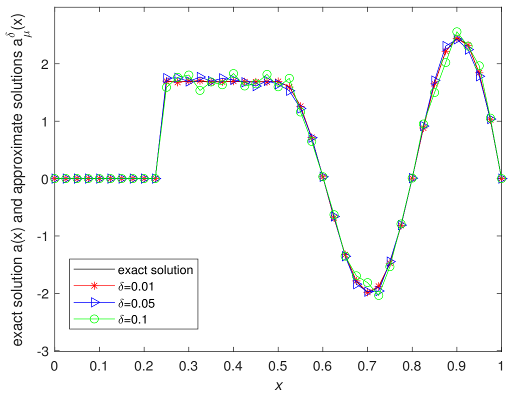
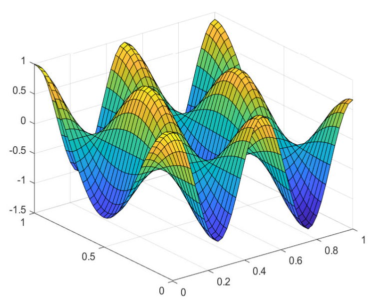
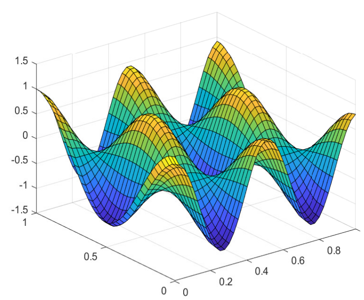


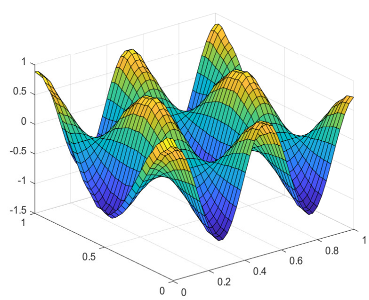
Figure 8. Exact solution and its approximation for Example 5.7 when $\varepsilon = 0.008, \alpha = 0.8$.



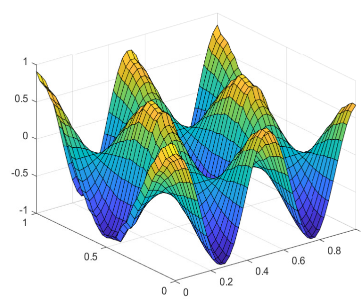
(a) exact solution



(b) numerical solution with $\delta = 0.01$



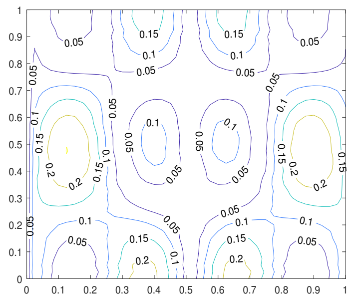
(c) numerical solution with $\delta = 0.05$



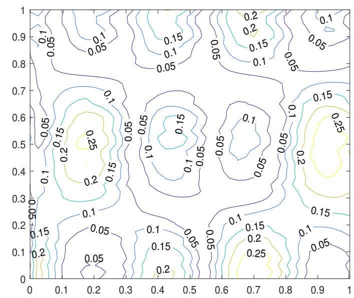
(d) numerical solution with $\delta = 0.1$

Figure 9. Numerical results for source term in Example 5.8 when $\alpha = 0.8$.

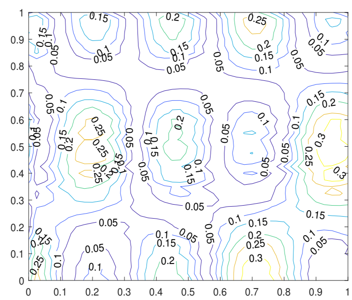
Combining Figures 1-10 with Tables 1-2, we can see that the numerical method presented in this paper is very effective for the inversion of the initial values of singularly perturbed parabolic and fractional diffusion equations, and has good convergence for both one- and two-dimensional numerical examples. Moreover, the numerical solutions are in good agreement with the exact solutions, even for the noise level up to $\delta = 0.1$.



(a) absolute error to the source term with $\delta = 0.01$



(b) absolute error to the source term with $\delta = 0.05$



(c) absolute error to the source term with $\delta = 0.1$

Figure 10. The comparison of the numerical effects between the exact source term and its computed approximations for Example 5.8 when $\alpha = 0.8$.

6. Conclusion

In this paper, we consider the problem of recovering the initial values of the singularly perturbed parabolic and fractional diffusion equations. Based on the quasi-reversibility method, we present regularized solutions and error estimates for both problems, respectively. Numerical examples of one- and two-dimensional show that the regularization method is effective and accurate under different noise levels.

In the future, we will study the singularly perturbed fractional diffusion wave equation $\alpha \in (1, 2)$, and try to analyze its existence, uniqueness and convergence estimates.

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