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DYNAMICS OF AN SIRS EPIDEMIC MODEL WITH TIME DELAY AND FREE BOUNDARIES

Yindi Zhang[1] and Meng Zhao[1,†]

Abstract In this paper, an SIRS epidemic model with time delay and free boundaries is studied. At first, we prove the global existence and uniqueness of the solution. And then we obtain criteria for spreading and vanishing. Moreover, the long-time behavior of the solution is given by a spreading-vanishing dichotomy. Finally, the numerical simulations are provided to illustrate our results. Our results indicate that the time delay can slow down the spreading of epidemic.

Keywords An SIRS epidemic model, time delay, free boundary, spreading-vanishing dichotomy.

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1. Introduction

Recently, people have attached great importance to the spread of epidemics. The study of epidemic models plays a crucial role in controlling and preventing epidemics. During the past several decades, many epidemic models have been widely proposed. In 1927, Kermack and McKendrick [17] studied the classical SIR model. However, for certain infectious diseases (such as, plague and cholera), the recovered individuals may experience a loss of immunity and become susceptible individuals again. This process can be described by the SIRS models, which have been investigated by many researchers from various aspects in recent years. For example, Anderson and May [2] proposed an SIRS model to study the dynamics of the infection and presented some numerical analysis. In 1992, Mena-Lorcat and Hethcote [26] considered five SIRS epidemiological models for populations of varying size.

The above models can already describe the spreading process of diseases well. To describe the effects of disease immunity, we can introduce temporal delays into those models. In [36], Wen and Yang considered the following time-delayed SIRS model with a linear incidence rate:

$$\begin{cases} S'(t) = b - kS(t) - \beta S(t)I(t) + \gamma I(t-\tau)e^{-k\tau}, \\ I'(t) = \beta S(t)I(t) - \gamma I(t) - kI(t), \\ R'(t) = \gamma I(t) - \gamma I(t-\tau)e^{-k\tau} - kR(t), \end{cases} \tag{1.1}$$

where S, I and R represent the susceptible, infectious, and recovered individuals respectively; b is the constant birth rate, k is the natural death rate, β represents

†The corresponding author.
1College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730000, China
Email: 3251267342@qq.com(Y. Zhang), zhaom@nwnu.edu.cn(M. Zhao)

the contact rate, γ is the recovery rate, these parameters are positive constants. The term $\gamma I(t-\tau)e^{-k\tau}$ indicates that an individual has survived from natural death in a recovery pool before becoming susceptible again, where $\tau \geq 0$ is the length of immunity period. They found a positive constant $\widehat{R}_0$ such that the disease-free equilibrium is globally asymptotically stable for any time delay if $\widehat{R}_0 \leq 1$, and the endemic equilibrium is locally asymptotically stable for any time delay if $\widehat{R}_0 > 1$ and $\gamma < k$. Subsequently, Xu et al. [38] replaced βSI by a saturation incidence rate $\beta SI/(1+\alpha I)$ in model (1.1) and considered the modified model. The results in [38] showed that there exists a positive constant $\widetilde{R}_0$ such that the disease-free equilibrium is globally asymptotically stable for any all $\tau > 0$ when $\widetilde{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable under some conditions if $\widetilde{R}_0 > 1$. And numerical simulations indicated that the solutions are represented by small amplitude oscillations near the endemic equilibrium for $\widetilde{R}_0 > 1$ and a certain τ, and the amplitude of these oscillations is increasing in the immunity period τ. Furthermore, when τ increases to a certain value, the oscillatory dynamics return to the stable steady-state form. For other problems related to time delay, we can refer to [3, 7, 8, 13, 20, 25] and references therein.

Due to the fact that individuals can diffuse randomly, then spatial diffusion can not be ignored during modeling. Based on [19], Sounvoravong [27] et al. studied the following diffusive SIRS epidemic model with time delay and nonlinear incidence rate:

$$\begin{cases} S_t - d\Delta S = b - kS - \beta f(I)S + \gamma I(t-\tau,x)e^{-k\tau}, & t>0,\ x\in\Omega, \\ I_t - d\Delta I = \beta f(I)S - \gamma I - kI, & t>0,\ x\in\Omega, \\ R_t - d\Delta R = \gamma I - \gamma I(t-\tau,x)e^{-k\tau} - kR, & t>0,\ x\in\Omega, \\ \dfrac{\partial S}{\partial n} = \dfrac{\partial I}{\partial n} = \dfrac{\partial R}{\partial n}, & t>0,\ x\in\partial\Omega, \\ I(\theta,x) = I_0(\theta,x), & \theta\in[-\tau,0],\ x\in\overline{\Omega}, \\ S(0,x) = S_0(x), R(0,x) = R_0(x), & x\in\overline{\Omega}, \end{cases} \tag{1.2}$$

where $S(t,x)$, $I(t,x)$ and $R(t,x)$ stand for the population densities of the susceptible, infectious and recovered individuals at $t > 0$ and $x \in \Omega$ respectively, the constant $d > 0$ is the diffusion rate. While, a nonlinear incidence rate is denoted by the function $f(I)$, which satisfies the properties:

$$f(0) = 0,\ f'(I) > 0,\ f''(I) < 0,\ \lim_{I\to\infty} f(I) = c < \infty,\ \forall I \geq 0.$$

The results in [27] showed that there exists a positive constant $\overline{R}_0$ such that the disease-free equilibrium is globally asymptotically stable and the disease will die out if $\overline{R}_0 < 1$, and the endemic equilibrium is locally asymptotical stability if $\overline{R}_0 > 1$.

However, the results of all the above work indicate that diseases always spread regardless of the initial condition, which contradicts the observed phenomenon of the spreading of diseases. Moreover, above works can not describe the spreading front well, which is important in studying the spreading of diseases. To overcome this shortcoming, free boundary conditions are first introduced by Du and Lin [10] to study the spreading of invasive species. Subsequently, many researchers used this free boundary problems to analyze the related mathematical models. For example, Kim et al. [18] considered an SIR epidemic model with free boundary, they proved the global existence and uniqueness of the solution and provided sufficient conditions for the disease vanishing and spreading; Huang and Wang [15] studied the reaction-diffusion system for an SIR epidemic model with a free boundary. For other problems related to free boundary, we can refer to [1,4,9,11,14,22,23,30,35,37,40,41] and references therein.

During using mathematical models to describe some biological problems, the time from birth to maturation can be not ignored since it may significantly affect the dynamics of the systems. There are some works considering free boundary problems with time delay, such as, [5,6,24,28,29]. In [29], the authors investigated a Lotka-Volterra weak competition model with time delays and free boundary, they obtained the long-time behavior of the solution and the spreading-vanishing criteria, and they estimated the upper and lower bounds of asymptotic spreading speed using the corresponding semi-wave theory.

Inspired by the above works, we investigate the following SIRS epidemic model with time delay and free boundaries:

$$\begin{cases} S_t - dS_{xx} = b - kS - \beta SI + \gamma I(t-\tau, x)e^{-k\tau}, & t > 0,\ x \in \mathbb{R}, \\ I_t - dI_{xx} = \beta SI - \gamma I - kI, & t > 0,\ x \in (g(t), h(t)), \\ R_t - dR_{xx} = \gamma I - \gamma I(t-\tau, x)e^{-k\tau} - kR, & t > 0,\ x \in (g(t), h(t)), \\ I(t,x) = R(t,x) = 0, & t \geq 0,\ x \in \mathbb{R}\backslash(g(t), h(t)), \\ g'(t) = -\mu I_x(t, g(t)),\ g(0) = -h_0, & t > 0, \\ h'(t) = -\mu I_x(t, h(t)),\ h(0) = h_0, & t > 0, \\ S(0,x) = S_0(x), & x \in \mathbb{R}, \\ I(\theta, x) = I_0(\theta, x),\ g(\theta) = -h(\theta), & -\tau \leq \theta \leq 0, g(\theta) < x < h(\theta), \\ R(0,x) = R_0(x), & -h(0) < x < h(0), \end{cases} \tag{1.3}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundary to be determined, μ is the expanding capability of the free boundary. Assume that the initial functions $S_0(x)$, $I_0(\theta, x)$ and $R_0(x)$ satisfy

$$\begin{cases} S_0(x) \in C^{1,2}(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ S_0(x) > 0,\ x \in \mathbb{R}, \\ I_0(\theta, x) \in C^{1,2}([-\tau, 0] \times [-h(\theta), h(\theta)]),\ R_0(x) \in C^2(-h(0), h(0)), \\ I_0(\theta, \pm h(\theta)) = 0,\ \forall \theta \in [-\tau, 0],\ R_0(\pm h(0)) = 0, \\ I_0(\theta, x) > 0,\ \forall (\theta, x) \in [-\tau, 0] \times (-h(\theta), h(\theta)), \\ R_0(x) > 0,\ x \in (-h(0), h(0)), \\ I_0(\theta, x) = 0,\ \forall \theta \in [-\tau, 0],\ x \notin (-h(\theta), h(\theta)), \\ R_0(x) = 0,\ x \notin (-h(0), h(0)), \end{cases} \tag{1.4}$$

as well as the compatible condition

$$[-h(\theta), h(\theta)] \subset [-h_0, h_0],\ \theta \in [-\tau, 0]. \tag{1.5}$$

Denote

$$\mathcal{R}_0 = \frac{\beta b}{k(k+\gamma)}. \tag{1.6}$$

The main results of this paper are as follows.

Theorem 1.1 (Global existence and uniqueness)**.** *For any given $h_0 > 0$ and $S_0(x)$, $I_0(\theta, x)$ and $R_0(x)$ satisfying* (1.4) *and* (1.5)*, problem* (1.3) *admits a unique solution $(S(t,x), I(t,x), R(t,x), g(t), h(t))$ defined for all $t > 0$.*

Theorem 1.2 (Spreading-vanishing dichotomy). *Assume that the conditions in Theorem 1.1 hold. Let* (S, I, R, g, h) *be the unique solution of* (1.3)*. Then one of the followings must hold:*

(i) *Vanishing: If* $h_\infty - g_\infty < \infty$*, then*

$$\lim_{t\to\infty} S(t,x) = \frac{b}{k} \text{ in } C_{loc}(\mathbb{R}), \ \lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} = 0;$$

(ii) *Spreading: If* $h_\infty - g_\infty = \infty (\mathcal{R}_0 > 1)$*, and* $k > \gamma(1 - e^{-k\tau})$*, then*

$$\lim_{t\to\infty} (S(t,x), I(t,x), R(t,x)) = (S_*, I_*, R_*) \text{ locally uniformly in } \mathbb{R},$$

where (S_*, I_*, R_*) *is given* (4.1).

Theorem 1.3 (Spreading-vanishing criteria). *In Theorem 1.2, the dichotomy can be determined as follows:*

(i) *If* $\mathcal{R}_0 \le 1$ *and* $\|N_0\|_\infty \le \frac{b}{k}$*, then disease will vanish;*

(ii) *If* $\mathcal{R}_0 > 1$*, then there exists* $h^* > 0$ *independent of* $(S_0(x), I_0(\theta, x), R_0(x))$ *such that spreading happens when* $h_0 \ge h^*$*, and if* $h_0 < h^*$ *and* $\|N_0\|_\infty \le \frac{b}{k}$*, then there exists* $\mu^* \ge \mu_* > 0$ *depending on* $(S_0(x), I_0(\theta, x), R_0(x))$ *such that spreading happens when* $\mu > \mu^*$*, and vanishing happens when* $\mu \le \mu_*$ *and* $\mu = \mu^*$.

The rest of this paper is organised as follows. In Section 2, we prove the global existence and uniqueness of the solution of problem (1.3). The criteria for spreading and vanishing will be established in Section 3. Then, we will show the long-time behavior of solution (S, I, R, g, h) for problem (1.3) in Section 4. In Section 5, we give some numerical simulations for the spreading and vanishing of diseases. The last section is a brief discussion.

2. Existence and uniqueness

In this section, we first prove the local existence and uniqueness of the solution. Then we use some suitable estimates to show that the solution is defined for all $t > 0$.

Theorem 2.1. *For any given* $\alpha \in (0, 1)$ *and* $(S_0(x), I_0(\theta, x), R_0(x))$ *satisfying* (1.4) *and* (1.5)*, there exists* $T > 0$ *such that problem* (1.3) *admits a unique solution*

$$(S, I, R, g, h) \in \mathcal{C}_T \times [C^{1+\frac{\alpha}{2}, 2+\alpha}(\overline{\Gamma}_T)]^2 \times [C^{1+\frac{\alpha}{2}}([0, T])]^2,$$

moreover,

$$\begin{aligned}&\|S\|_{C^{1+\frac{\alpha}{2},2+\alpha}(\mathcal{C}_T)} + \|I\|_{C^{1+\frac{\alpha}{2},2+\alpha}(\overline{\Gamma}_T)} + \|R\|_{C^{1+\frac{\alpha}{2},2+\alpha}(\overline{\Gamma}_T)} \\ &+ \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C,\end{aligned}$$

where $\mathcal{C}_T = L^\infty([0, T] \times \mathbb{R}) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}([0, T] \times \mathbb{R})$, $\Gamma_T = [0, T] \times (g(t), h(t))$*. Here* C *and* T *depend on* $h(0)$, $g(\theta)$, $h(\theta)$, α, $\|S_0\|_{C^2(\mathbb{R})}$, $\|S_0\|_{L^\infty(\mathbb{R})}$, $\|I_0\|_{C^2([g(\theta),h(\theta)])}$ *and* $\|R_0\|_{C^2([-h(0),h(0)])}$.

Proof. The proof of the local existence and uniqueness of the solution is similar to that in [29, Theorem 2.1] and [16, Theorem 2.1]. For the sake of completeness, we give the details as follows.

Step 1. We first straighten the free boundaries. Let

$$y = \frac{2x - h(t) - g(t)}{h(t) - g(t)},$$

and

$$\begin{aligned}
S(t,x) &= S(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: u(t,y),\ t > 0, -\infty \le y \le +\infty,\\
I(t,x) &= I(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: v(t,y),\ t > 0, -1 \le y \le 1,\\
R(t,x) &= R(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: w(t,y),\ t > 0, -1 \le y \le 1,\\
I(\theta,x) &= I(\theta, \frac{(h(\theta)-g(\theta))y + h(\theta) + g(\theta)}{2}) =: v(\theta,y),\ -\tau \le \theta \le 0,\ -1 \le y \le 1,
\end{aligned}$$

we have

$$\begin{aligned}
S_t &= u_t + u_y y_t = u_t - \rho(t,y)u_y,\ S_{xx} = \varphi(t)u_{yy},\\
I_t &= v_t + v_y y_t = v_t - \rho(t,y)v_y,\ I_{xx} = \varphi(t)v_{yy},\\
R_t &= w_t + w_y y_t = w_t - \rho(t,y)w_y,\ R_{xx} = \varphi(t)w_{yy},
\end{aligned}$$

where $\rho(t,y) = \frac{h'(t)-g'(t)}{h(t)-g(t)}y + \frac{h'(t)+g'(t)}{h(t)-g(t)}$, $\varphi(t) = \frac{4}{(h(t)-g(t))^2}$.

Then problem (1.3) becomes the following problem:

$$\begin{cases}
\quad u_t - d\varphi(t)u_{yy} - \rho(t,y)u_y & \\
= b - ku - \beta uv + \gamma v(t-\tau,y)e^{-k\tau}, & 0 < t < T,\ y \in \mathbb{R},\\
v_t - d\varphi(t)v_{yy} - \rho(t,y)v_y = \beta uv - \gamma v - kv, & 0 < t < T,\ |y| < 1,\\
\quad w_t - d\varphi(t)w_{yy} - \rho(t,y)w_y & \\
= \gamma v - kw - \gamma v(t-\tau,y)e^{-k\tau}, & 0 < t < T,\ |y| < 1,\\
g'(t) = -\mu\psi(t)v_y(t,-1), g(0) = -h_0, & 0 < t < T,\\
h'(t) = -\mu\psi(t)v_y(t,1), h(0) = h_0, & 0 < t < T,\\
v(t,y) = w(t,y) = 0, & 0 < t < T,\ |y| \ge 1,\\
u(0,y) = u_0(y), & y \in \mathbb{R},\\
v(\theta,y) = v_0(\theta,y) > 0, & -\tau \le \theta \le 0,\ -1 < y < 1,\\
w(0,y) = w_0(y), & -1 < y < 1,
\end{cases} \tag{2.1}$$

where $\psi(t) = \frac{2}{h(t)-g(t)}$.

Let $g^* = -\mu I_0'(-h_0)$, $h^* = -\mu I_0'(h_0)$, $T_1 = \min\left\{\tau, \frac{h_0}{2(4+|h^*|+|g^*|)}\right\}$. For $0 < T \le T_1$, we define $\Omega_T = [0,T] \times [-1,1]$ and

$$\begin{aligned}
D_{1T} &= \{v \in C(\Omega_T) : v(t,\pm 1) = 0, v(\theta,y) = v_0, \|v - v_0\|_{C(\Omega_T)} \le 1\},\\
D_{2T} &= \{w \in C(\Omega_T) : w(t,\pm 1) = 0, w(0,y) = w_0, \|w - w_0\|_{C(\Omega_T)} \le 1\},\\
D_{3T} &= \{g \in C^1([0,T]) : g(0) = -h_0, g'(0) = g^*, \|g' - g^*\|_{C([0,T])} \le 1\},
\end{aligned}$$

$$D_{4T} = \{h \in C^1([0,T]) : h(0) = h_0, h'(0) = h^*, \|h' - h^*\|_{C([0,T])} \leq 1\}.$$

It is easy to see that $D_T := D_{1T} \times D_{2T} \times D_{3T} \times D_{4T}$ is a bounded and closed convex set of $(C(\Omega_T))^2 \times (C([0,T]))^2$.

In view of the choice of T, for any given $(v,w,g,h) \in D_T$, we can extend $(v,w,g,h) \in D_T$ to new functions, denoted by themselves, such that $(v,w,g,h) \in D^*_{T_1} := D^*_{1T_1} \times D^*_{2T_1} \times D^*_{3T_1} \times D^*_{4T_1}$, where

$$\begin{aligned}
D^*_{1T_1} &= \{v \in C(\Omega_{T_1}) : v(t,\pm 1) = 0, v(\theta,y) = v_0, \|v - v_0\|_{C(\Omega_{T_1})} \leq 2\},\\
D^*_{2T_1} &= \{w \in C(\Omega_{T_1}) : w(t,\pm 1) = 0, w(0,y) = w_0, \|w - w_0\|_{C(\Omega_{T_1})} \leq 2\},\\
D^*_{3T_1} &= \{g \in C^1([0,T_1]) : g(0) = -h_0, g'(0) = g^*, \|g' - g^*\|_{C([0,T_1])} \leq 2\},\\
D^*_{4T_1} &= \{h \in C^1([0,T_1]) : h(0) = h_0, h'(0) = h^*, \|h' - h^*\|_{C([0,T_1])} \leq 2\}.
\end{aligned}$$

Therefore, for $(g,h) \in D_{3T} \times D_{4T}$, we can extend it to $D^*_{3T_1} \times D^*_{4T_1}$ and

$$|g(t) + h_0| + |h(t) - h_0| \leq T_1(\|g'\|_{C([0,T_1])}) + \|g'\|_{C([0,T_1])}) \leq \frac{h_0}{2},$$

then

$$h_0 \leq h(t) - g(t) \leq 3h_0, \ \forall t \in [0,T_1].$$

For any given $(v,w,g,h) \in D_T$, we first extend it to $D^*_{T_1}$. Then we consider the following problem

$$\begin{cases} u_t - d\varphi(t)u_{yy} - \rho(t,y)u_y = b - ku - \beta uv + \gamma v(t-\tau,x)e^{-k\tau}, \ 0 < t \leq T_1, \ y \in \mathbb{R},\\ u(0,y) = u_0(y), \ y \in \mathbb{R}. \end{cases}$$

Applying the standard partial differential equation theory in [12], this problem has a unique solution $u \in L^\infty([0,T_1]\times\mathbb{R}) \cap C^{\frac{1+\alpha}{2},1+\alpha}([0,T_1]\times\mathbb{R})$. For above (u,v,w,g,h), we consider

$$\begin{cases} \tilde{v}_t - d\varphi(t)\tilde{v}_{yy} - \rho(t,y)\tilde{v}_y = \beta uv - \gamma v - kv, & 0 < t \leq T_1, \ |y| < 1,\\ \quad \tilde{w}_t - d\varphi(t)\tilde{w}_{yy} - \rho(t,y)\tilde{w}_y & \\ = \gamma v - kw - \gamma v(t-\tau,x)e^{-k\tau}, & 0 < t \leq T_1, \ |y| < 1,\\ \tilde{v}(t,y) = \tilde{w}(t,y) = 0, & 0 < t \leq T_1, \ |y| \geq 1,\\ \tilde{v}(\theta,y) = \tilde{v}_0(\theta,y) > 0, & -\tau \leq \theta \leq 0, \ -1 < y < 1,\\ \tilde{w}(0,y) = \tilde{w}_0(0,y), & -1 < y < 1. \end{cases} \tag{2.2}$$

By L^p theory and Sobolev embedding theorem, problem (2.2) admits a unique positive solution $(\tilde{v},\tilde{w}) \in [C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_{T_1})]^2$, which satisfies

$$\begin{aligned}
\|\tilde{v}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_{T_1})} &\leq C(T_1^{-1})\|\tilde{v}\|_{W^{1,2}_p(\overline{D}_{T_1})} \leq C_1(T_1, T_1^{-1}),\\
\|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_{T_1})} &\leq C(T_1^{-1})\|\tilde{w}\|_{W^{1,2}_p(\overline{D}_{T_1})} \leq C_1(T_1, T_1^{-1}),
\end{aligned}$$

where C_1 is a constant depending on T_1, T_1^{-1}, α, $h(\theta)$, p, β, b, γ, $\|u_0\|_{L^\infty(\mathbb{R})}$, $\|v_0\|_{C^{1,2}([-\tau,0]\times[-h(\theta),h(\theta)])}$ and $\|w_0\|_{C^2([-h(0),h(0)])}$. Obviously, when $0 < T < T_1$, we have $(\tilde{v},\tilde{w}) \in [C^{\frac{1+\alpha}{2},1+\alpha}(\overline{D}_T)]^2$ and

$$\|\tilde{v}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T)} \leq C_1, \tag{2.3}$$

$$\|\tilde{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T)} \le C_1, \tag{2.4}$$

where C_1 is independent of T.

For $0 < T < T_1$, defining

$$\tilde{g}(t) = -h_0 - \int_0^t \mu\tilde{v}_y(s,-1)ds,\ \tilde{h}(t) = h_0 - \int_0^t \mu\tilde{v}_y(s,1)ds,\ 0 \le t \le T.$$

We have

$$\tilde{g}'(t) = -\mu\tilde{v}_y(t,-1),\ \tilde{h}'(t) = -\mu\tilde{v}_y(t,1),\ -\tilde{g}(0) = \tilde{h}(0) = h_0,$$

and then $\tilde{g}'(t), \tilde{h}'(t) \in C^{\frac{\alpha}{2}}([0,T])$,

$$\|\tilde{g}'\|_{C^{\frac{\alpha}{2}}([0,T])}, \|\tilde{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} \le \mu C_1 := C_2, \tag{2.5}$$

where C_2 depends on μ, h_0, C_1.

Now, for any given $(v,w,g,h) \in D_T$, we can define the mapping $\mathcal{F} : D_T \to [C(\Omega_T)]^2 \times [C^1([0,T])]^2$ by

$$\mathcal{F}(v,w,g,h) = (\tilde{v},\tilde{w},\tilde{g},\tilde{h}).$$

It is clear that $(v,w,g,h) \in D_T$ is a fixed point of $\mathcal{F}$ if and only if it solves (2.1). At first, we prove that $\mathcal{F}$ is a self-mapping on D_T for $T > 0$ sufficiently small. By (2.3), (2.4) and (2.5), we know that $\mathcal{F}$ is continuous and compact, and

$$\begin{aligned}
&\|\tilde{g}' - g^*\|_{C([0,T])} \le \|\tilde{g}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C_2 T^{\frac{\alpha}{2}},\\
&\|\tilde{h}' - h^*\|_{C([0,T])} \le \|\tilde{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} T^{\frac{\alpha}{2}} \le C_2 T^{\frac{\alpha}{2}},\\
&\|\tilde{v} - v_0\|_{C(\Omega_T)} \le \|\tilde{v}\|_{C^{\frac{1+\alpha}{2},0}(\Omega_T)} T^{\frac{1+\alpha}{2}} \le C_1 T^{\frac{1+\alpha}{2}},\\
&\|\tilde{w} - w_0\|_{C(\Omega_T)} \le \|\tilde{w}\|_{C^{\frac{1+\alpha}{2},0}(\Omega_T)} T^{\frac{1+\alpha}{2}} \le C_1 T^{\frac{1+\alpha}{2}}.
\end{aligned}$$

If we take

$$0 < T \le \min\left\{\tau, T_1, C_2^{-\frac{2}{\alpha}}, C_1^{-\frac{2}{1+\alpha}}\right\},$$

then $\mathcal{F}$ maps D_T into itself. It follows from the Schauder fixed point theorem that $\mathcal{F}$ has at least one fixed point $(v,w,g,h) \in D_T$. Therefore, (2.1) has at least one solution (u,v,w,g,h) and

$$\begin{aligned}
&u \in L^\infty([0,T]\times\mathbb{R}) \cap C^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R}),\ v,w \in C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T),\\
&g,h \in C^{1+\frac{\alpha}{2}}([0,T]),\ g'(t) < 0,\ h'(t) > 0 \text{ in } [0,T].
\end{aligned}$$

Hence, problem (1.3) has a solution

$$(S,I,R,g,h) \in \mathcal{C}_T \times [C^{\frac{1+\alpha}{2},1+\alpha}(\overline{\Gamma}_T)]^2 \times [C^{1+\frac{\alpha}{2}}([0,T])]^2.$$

Step 2. Let $(S_i,I_i,R_i,g_i,h_i)(i=1,2)$ be the solution of (1.3), which is defined for $t \in [0,T]$ with $0 < T \ll 1$. Making the same transformation as in Step 1, we have

$$S_i(t,x) = S_i(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: u_i(t,y),\ (t,y) \in [0,T]\times[-\infty,\infty],$$

$$I_i(t,x) = I_i(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: v_i(t,y),\ (t,y) \in [0,T] \times [-1,1],$$
$$R_i(t,x) = R_i(t, \frac{(h(t)-g(t))y + h(t) + g(t)}{2}) =: w_i(t,y),\ (t,y) \in [0,T] \times [-1,1].$$

Letting

$$U = u_1 - u_2,\ V = v_1 - v_2,\ W = w_1 - w_2,\ G = g_1 - g_2,\ H = h_1 - h_2,$$

we have that $(U(t,y), V(t,y), W(t,y), G(t), H(t))$ satisfies

$$\begin{cases} \quad U_t - d\varphi_1(t)U_{yy} - \rho_1(t,y)U_y + [\beta v_1 + k]U \\ = (\rho_1(t,y) - \rho_2(t,y))u_{2y} + d(\varphi_1(t) - \varphi_2(t))u_{2yy} \\ \quad + \gamma V(t-\tau)e^{-k\tau} - \beta u_2 V, & 0 < t < T,\ y \in \mathbb{R}, \\ \quad V_t - d\varphi_1(t)V_{yy} - \rho_1(t,y)V_y - (\beta u_1 - \gamma - k)V \\ = (\rho_1(t,y) - \rho_2(t,y))v_{2y} \\ \quad + d(\varphi_1(t) - \varphi_2(t))v_{2yy} + \beta v_2 U, & 0 < t < T,\ |y| < 1, \\ \quad W_t - d\varphi_1(t)W_{yy} - \rho_1(t,y)W_y + kW \\ = (\rho_1(t,y) - \rho_2(t,y))w_{2y} + d(\varphi_1(t) - \varphi_2(t))w_{2yy} \\ \quad + \gamma V - \gamma V(t-\tau)e^{-k\tau}, & 0 < t < T,\ |y| < 1, \\ U(0,y) = 0, & y \in \mathbb{R}, \\ V(t,\pm 1) = 0,\ W(t,\pm 1) = 0, & 0 \le t \le T, \\ V(\theta,y) = 0, & -\tau \le \theta \le 0,\ |y| < 1, \\ W(0,y) = 0, & |y| < 1, \end{cases}$$

and

$$\begin{cases} G'(t) = -\mu\psi_1(t)V_y(t,-1) - \mu(\psi_1(t) - \psi_2(t))v_{2y}(t,-1), & 0 < t < T, \\ H'(t) = -\mu\psi_1(t)V_y(t,1) - \mu(\psi_1(t) - \psi_2(t))v_{2y}(t,1), & 0 < t < T, \\ G(0) = H(0) = 0, \end{cases}$$

where

$$\rho_i(t,y) = \frac{h_i'(t) - g_i'(t)}{h_i(t) - g_i(t)}y + \frac{h_i'(t) + g_i'(t)}{h_i(t) - g_i(t)},$$
$$\varphi_i(t) = \frac{4}{(h_i(t) - g_i(t))^2},\ \psi_i = \frac{2}{h_i(t) - g_i(t)},\ i = 1,2.$$

In terms of $T \le \tau$, we can derive $V(t-\tau,y) = 0$ for $(t,y) \in \Omega_T$. Applying the L^p theory, we can derive that

$$\|U\|_{L^\infty([0,T]\times\mathbb{R})} \le C_3(\|G,H\|_{C^1([0,T])} + \|V\|_{C(\Omega_T)}),$$
$$\|V\|_{W_p^{1,2}(\Omega_T)} \le C_4(\|G,H\|_{C^1([0,T])} + \|U\|_{L^\infty([0,T]\times\mathbb{R})}),$$
$$\|W\|_{W_p^{1,2}(\Omega_T)} \le C_5(\|G,H\|_{C^1([0,T])} + \|V\|_{C(\Omega_T)}),$$

where C_3, C_4, C_5 depend on C_1, C_2, d, b, γ, β, $h(\theta)$, $\|w_0\|_{C^2([-h(0),h(0)])}$, $\|u_0\|_{L^\infty(\mathbb{R})}$ and $\|v_0\|_{C^{1,2}([-\tau,0]\times[-h(\theta),h(\theta)])}$.

By similar arguments in the proof of [32, Theorem 1.1], we can obtain that

$$[V]_{C^{\frac{\alpha}{2},\alpha}(\Omega_T)}, [V_y]_{C^{\frac{\alpha}{2},\alpha}(\Omega_T)} \le C_6\|V\|_{W_p^{1,2}(\Omega_T)},$$

where C_6 is independent of T^{-1}. Therefore,

$$[V_y]_{C^{\frac{\alpha}{2},\alpha}(\Omega_T)} \le C_4C_6(\|G,H\|_{C^1([0,T])} + \|U\|_{L^\infty([0,T]\times\mathbb{R})}).$$

Combining with the definition of $G'(t)$ and $H'(t)$, we obtain

$$\begin{aligned}[G',H']_{C^{\frac{\alpha}{2}}([0,T])} &\le \mu[\psi_1 V_y(t,\pm 1)]_{C^{\frac{\alpha}{2}}([0,T])} + \mu[(\psi_1-\psi_2)v_{2y}(t,\pm 1)]_{C^{\frac{\alpha}{2}}([0,T])} \\ &\le C_7(\|G,H\|_{C^1([0,T])} + \|U\|_{L^\infty([0,T]\times\mathbb{R})}),\end{aligned}$$

where C_7 depends on C_4, C_6 and μ.

Due to $G(0) = G'(0) = 0$ and $H(0) = H'(0) = 0$, it follows that, for $T \ll 1$,

$$\begin{aligned}\|G,H\|_{C^1([0,T])} &\le (1+T)T^{\frac{\alpha}{2}}[G',H']_{C^{\frac{\alpha}{2}}([0,T])} \\ &\le (1+T)C_7T^{\frac{\alpha}{2}}(\|G,H\|_{C^1([0,T])} + \|U\|_{L^\infty([0,T]\times\mathbb{R})}) \\ &\le 2C_7T^{\frac{\alpha}{2}}\|U\|_{L^\infty([0,T]\times\mathbb{R})}.\end{aligned}$$

Meanwhile,

$$\begin{aligned}\|V\|_{C(\Omega_T)} &\le T^{\frac{\alpha}{2}}[V]_{C^{\frac{\alpha}{2},0}(\Omega_T)} \\ &\le C_4C_6T^{\frac{\alpha}{2}}(\|G,H\|_{C^1([0,T])} + \|U\|_{L^\infty([0,T]\times\mathbb{R})}) \\ &\le 2C_8T^{\frac{\alpha}{2}}\|U\|_{L^\infty([0,T]\times\mathbb{R})},\end{aligned}$$

where C_8 depends on C_4, C_6 and C_7. Therefore,

$$\|U\|_{L^\infty([0,T]\times\mathbb{R})} \le C_3(\|G,H\|_{C^1([0,T])} + \|V\|_{C(\Omega_T)}) \le C_9T^{\frac{\alpha}{2}}\|U\|_{L^\infty([0,T]\times\mathbb{R})},$$

where C_9 depends on C_7 and C_8. Therefore, for $0 < T \ll 1$, we have $u_1 = u_2$. Consequently, $v_1 = v_2$, $w_1 = w_2$, $g_1 = g_2$ and $h_1 = h_2$.

Step 3. By the Schauder estimates, we have additional regularity for the solution (u,v,w,g,h) of (2.1), namely, $u \in L^\infty([0,T]\times\mathbb{R}) \cap C_{loc}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\mathbb{R})$,$v,w \in C^{1+\frac{\alpha}{2},2+\alpha}(\Omega_T)$ and $g,h \in C^{1+\frac{\alpha}{2}}([0,T])$. Hence, (u,v,w,g,h) is a unique local classical solution of the problem (2.1), and then (S,I,R,g,h) is a unique local classical solution of the problem (1.3). This proof is completed. □

Lemma 2.1. *Let (S,I,R,g,h) be a solution to problem (1.3) defined for $t \in (0,T_0)$ with $T_0 \in (0,+\infty)$. Then*

$$\begin{aligned}&0 < S(t,x) \le M_1,\ 0 < t < T_0,\ x \in \mathbb{R}, \\ &0 < I(t,x), R(t,x) \le M_1,\ 0 < t < T_0,\ g(t) < x < h(t), \\ &-M_2 \le g'(t) < 0, 0 < h'(t) < M_2,\ 0 < t < T_0,\end{aligned}$$

where $M_i (i = 1,2)$ is independent of T_0 and will be determined later.

Proof. Applying the strong maximum principle, we obtain that $S(t,x) > 0$ in $(0,T_0)\times\mathbb{R}$ and $I(t,x), R(t,x) > 0$ in $(0,T_0)\times(g(t),h(t))$. Let $N(t,x) = S(t,x) + I(t,x) + R(t,x)$. By the direct calculations, we have that $N(t,x)$ satisfies

$$\begin{cases} N_t - dN_{xx} = b - kN, & 0 < t < T_0,\ x \in \mathbb{R}\setminus\{h(t),g(t)\}, \\ N_x(t,g(t)-0) \le N_x(t,g(t)+0), & 0 < t < T_0, \\ N_x(t,h(t)-0) \le N_x(t,h(t)+0), & 0 < t < T_0, \\ N(0,x) = N_0(x), & x \in \mathbb{R}, \end{cases}$$

where $N_0(x) = S_0(x) + I_0(x) + R_0(x)$. Let $\overline{N}(t,x)$ be the solution of

$$\begin{cases} \overline{N}_t - d\overline{N}_{xx} = b - k\overline{N}, & 0 < t < T_0,\ x \in \mathbb{R}, \\ \overline{N}(0,x) = N_0(x), & x \in \mathbb{R}. \end{cases}$$

It follows from the comparison principle that we have

$$N(t,x) \le \overline{N}(t,x) \le \max\left\{\frac{b}{k}, \|N_0\|_\infty\right\} =: M_1.$$

Therefore,

$$\begin{aligned} &0 < S(t,x) \le M_1,\ 0 < t < T_0,\ x \in \mathbb{R}, \\ &0 < I(t,x), R(t,x) \le M_1,\ 0 < t < T_0,\ g(t) < x < h(t). \end{aligned}$$

By the Hopf boundary lemma, we have $g'(t) < 0,\ h'(t) > 0,\ t \in (0,T_0)$. In the following, we will prove that $h'(t) \le M_2$ for all $t \in (0,T_0)$, where M_2 is independent of T_0. Inspired by the arguments in [29, Lemma 2.2], we define

$$\begin{aligned} &\Omega_M := \{(t,x) : -\tau \le t < T_0,\ h(t) - M^{-1} < x < h(t)\}, \\ &z(t,x) = M_1[2M(h(t) - x) - M^2(h(t) - x)^2],\ (t,x) \in \Omega_M, \end{aligned}$$

where M will be determined later. Direct calculations show that, for $(t,x) \in \Omega_M$,

$$\begin{aligned} &z_t - dz_{xx} - \beta SI + \gamma I + kI \\ = &\ 2M_1 M h'(t)(1 - M(h(t) - x)) + 2dM_1M^2 - \beta SI + (\gamma + k)I \\ \ge &\ 2dM_1M^2 - \beta M_1^2. \end{aligned}$$

If $M \ge \sqrt{\frac{\beta M_1}{2d}}$, then

$$z_t - dz_{xx} - \beta SI + \gamma I + kI \ge 0.$$

It is easy to check that

$$\begin{aligned} &z(t, h(t) - M^{-1}) = M_1 \ge I(t, h(t) - M^{-1}), \\ &z(t, h(t)) = 0 = I(t, h(t)),\ t \in [-\tau, T_0). \end{aligned}$$

In the following, we choose some suitable M independent of T_0 such that

$$I_0(\theta, x) \le z(\theta, x) \text{ for } (\theta, x) \in [-\tau, 0] \times [h(\theta) - M^{-1}, h(\theta)] \tag{2.6}$$

holds. In fact, for $(\theta, x) \in [-\tau, 0] \times [h(\theta) - M^{-1}, h(\theta)]$,

$$\begin{aligned} z(\theta, x) &= M_1[2M(h(\theta) - x) - M^2(h(\theta) - x)^2] \\ &\ge M_1 M(h(\theta) - x)[2 - M(h(\theta) - h(\theta) + M^{-1})] \\ &= M_1 M(h(\theta) - x). \end{aligned}$$

Since $I_0(\theta, h(\theta)) = 0$, we have

$$I_0(\theta, x) = -\int_x^{h(\theta)} (I_0)_y(\theta, y)dy \le -(h(\theta) - x) \min_{[-\tau,0]\times[0,h(\theta)]} (I_0)_x(\theta, x).$$

If

$$M \geq \max\left\{\max_{[-\tau,0]}\frac{1}{h(\theta)}, -\frac{1}{M_1}\min_{[-\tau,0]\times[0,h(\theta)]}(I_0)_x(\theta,x)\right\},$$

then we can have that (2.6) holds.

Let

$$M := \max\left\{\sqrt{\frac{\beta M_1}{2d}}, \max_{[-\tau,0]}\frac{1}{h(\theta)}, -\frac{1}{M_1}\min_{[-\tau,0]\times[0,h(\theta)]}(I_0)_x(\theta,x)\right\}.$$

By the maximum principle, we have that $I(t,x) \leq z(t,x)$ in Ω_M. It follows that

$$I_x(t,h(t)) \geq z_x(t,h(t)) = -2M_1M.$$

Therefore,

$$h'(t) = -\mu I_x(t,h(t)) \leq 2\mu M_1 M := M_2.$$

Similarly, we have $g'(t) \geq -M_2$. This proof is completed. □

Proof of Theorem 1.1. This proof can be done by the similar arguments as in [10, Theorem 2.3] and [28, Lemma 2.2]. We provide a detailed proof as follows.

Let $[0, T_{\max})$ be the maximal time interval in which the solution exists. By Theorem 2.1, $T_{\max} > 0$. Arguing indirectly we assume that $T_{\max} < \infty$. By Lemma 2.1, there exist M_1 and M_2 independent of $T_{\max}$ such that

$$\begin{aligned}
&0 < S(t,x) \leq M_1,\ (t,x) \in [0,T_{\max}) \times \mathbb{R},\\
&0 < I(t,x), R(t,x) \leq M_1,\ (t,x) \in [0,T_{\max}) \times [g(t),h(t)],\\
&-M_2 \leq g'(t) < 0, -h_0 - M_2 t \leq g(t) \leq -h_0,\ t \in [0,T_{\max}),\\
&0 < h'(t) < M_2, h_0 \leq h(t) \leq h_0 + M_2 t,\ t \in [0,T_{\max}).
\end{aligned}$$

We now fix $\delta_0 \in (0, T_{\max})$ and $M > T_{\max}$. By standard L^p estimates, the Sobolev embedding theorem, and the Hölder estimates for parabolic equations, we can find $M_3 > 0$ depending only on δ_0, M, M_1 and M_2 such that

$$\|S(t,\cdot)\|_{C^2[0,\infty)}, \|I(,\cdot)\|_{C^2[g(t),h(t)]}, \|R(t,\cdot)\|_{C^2[g(t),h(t)]} \leq M_3,\ t \in [\delta_0, T_{\max}).$$

Then it follows from the proof of Theorem 2.1 that there exists a $\epsilon > 0$ depending on $M_i (i = 1,2,3)$ such that the solution of problem (1.3) with the initial time be $T_{\max} - \frac{\epsilon}{2}$ can be extended uniquely to the time $T_{\max} - \frac{\epsilon}{2} + \epsilon = T_{\max} + \frac{\epsilon}{2}$. But this contradicts the assumption, thus $T_{\max} = \infty$. This proof is completed. □

3. Criteria for spreading and vanishing

This section will be divided into two cases: $\mathcal{R}_0 \leq 1$ and $\mathcal{R}_0 > 1$.

3.1. The case of $\mathcal{R}_0 \leq 1$

Lemma 3.1. *If $\mathcal{R}_0 \leq 1$ and $\|N_0\|_\infty \leq \frac{b}{k}$, then $h_\infty - g_\infty < \infty$.*

Proof. It follows from $\|N_0\|_\infty \leq \frac{b}{k}$ and the proof of Lemma 2.1 that

$$S(t,x) \leq \frac{b}{k} \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

In view of $\mathcal{R}_0 \leq 1$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{g(t)}^{h(t)} I(t,x)dx \\
=&\int_{g(t)}^{h(t)} I_t(t,x)dx + h'(t)I(t,h(t)) - g'(t)I(t,g(t)) \\
=&\int_{g(t)}^{h(t)} \left[dI_{xx} + \beta SI - \gamma I - kI\right] dx \\
=&\int_{g(t)}^{h(t)} \left[(\beta S - \gamma - k)I\right] dx + \int_{g(t)}^{h(t)} dI_{xx}dx \\
\leq&\int_{g(t)}^{h(t)} \left[(\beta\frac{b}{k} - \gamma - k)I\right] dx + \int_{g(t)}^{h(t)} dI_{xx}dx \\
\leq&\int_{g(t)}^{h(t)} (\gamma + k)(\mathcal{R}_0 - 1)Idx + \int_{g(t)}^{h(t)} dI_{xx}dx \\
\leq& \, d\int_{g(t)}^{h(t)} I_{xx}dx \\
=&-\frac{d}{\mu}[h'(t) - g'(t)] \\
\leq&0,
\end{aligned}$$

which implies

$$h(t) - g(t) \leq 2h_0 + \frac{\mu}{d}\int_{-h_0}^{h_0} I_0(0,x)dx < \infty \text{ for } t > 0.$$

Thus we finish the proof. □

Lemma 3.2. *If $h_\infty - g_\infty < \infty$, then $\lim_{t\to\infty} S(t,x) = b/k$ in $C_{loc}(\mathbb{R})$, and*

$$\lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} = 0. \tag{3.1}$$

Proof. By [21, Lemma 3.3], we have

$$\lim_{t\to\infty} \|I(t,x)\|_{C([g(t),h(t)])} = 0.$$

Thus, for any given $\varepsilon_1 > 0$, there exists $T_1 > 0$ such that $I(t,x) < \varepsilon_1$ for $t \geq T_1$ and $x \in [g(t),h(t)]$. For above ε_1, we can use the comparison principle to obtain that

$$R(t,x) \leq \frac{\gamma\varepsilon_1}{k} \text{ for } t \in [T_1,\infty) \text{ and } x \in [g(t),h(t)].$$

By the arbitrariness of ε_1, we have

$$\lim_{t\to\infty} \|R(t,x)\|_{C([g(t),h(t)])} = 0.$$

Therefore, (3.1) is proved.

Let $N(t,x) = S(t,x) + I(t,x) + R(t,x)$, then N satisfies

$$\begin{cases} N_t - dN_{xx} = b - kN, & t > 0,\ x \in (g(t), h(t)), \\ N(t, g(t)) > 0,\ N(t, h(t)) > 0, & t > 0, \\ N(0,x) = S_0(x) + I_0(x) + R_0(x), & x \in [-h(0), h(0)], \end{cases}$$

it follows from [39, Lemma 2.6(iii)] that

$$\lim_{t\to\infty} N(t,x) = \frac{b}{k} =: N^* \text{ locally uniformly in } \mathbb{R}.$$

Clearly,

$$\lim_{t\to\infty} S(t,x) \le \frac{b}{k} \text{ locally uniformly in } \mathbb{R}. \tag{3.2}$$

Noting that $I(t,x) = 0$ for $t > 0$ and $x \in \mathbb{R}\backslash(g(t), h(t))$, it follows that $I(t,x) \le \varepsilon_1$ for $t \ge T_1$ and $x \in \mathbb{R}$. For $T_2 \in [T_1 + \tau, \infty)$, we have S satisfies

$$\begin{cases} S_t - dS_{xx} \ge b - kS - \beta\varepsilon_1 S, & t > T_2,\ x \in \mathbb{R}, \\ S(T_2, x) > 0, & x \in \mathbb{R}. \end{cases}$$

Let $\underline{S}$ be the unique solution of

$$\begin{cases} \underline{S}_t - d\underline{S}_{xx} = b - k\underline{S} - \beta\varepsilon_1 \underline{S}, & t > T_2,\ x \in \mathbb{R}, \\ \underline{S}(T_2, x) = S(T_2, x), & x \in \mathbb{R}. \end{cases}$$

It is obvious that $\lim_{t\to\infty} \underline{S}(t,x) = \frac{b}{k+\beta\varepsilon_1}$ in $\mathbb{R}$. By a comparison argument and the arbitrariness of ε_1, we have $\liminf_{t\to\infty} S(t,x) \ge b/k$ in $C_{loc}(\mathbb{R})$. Combining with (3.2), we have $\lim_{t\to\infty} S(t,x) = b/k$ in $C_{loc}(\mathbb{R})$. The proof is completed. □

3.2. The case of $\mathcal{R}_0 > 1$

We first give the following comparison principle, which will be used later.

Lemma 3.3 (Comparison principle). *Suppose that* $T \in (0,\infty)$, $\overline{g}, \overline{h} \in C^1([0,T])$ *and* $\overline{g} < \overline{h}$ *in* $[0,T]$, $\overline{S} \in C([0,T]\times[0,\infty)) \cap C^{1,2}((0,T]\times(0,\infty))$, $\overline{I}, \overline{R} \in C(\overline{\Gamma}^*_T) \cap C^{1,2}(\Gamma^*_T)$ *with* $\Gamma^*_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \le T, \overline{g} < x < \overline{h}\}$, *and*

$$\begin{cases} \overline{S}_t - d\overline{S}_{xx} \ge b - k\overline{S} - \beta\overline{S}\overline{I} + \gamma\overline{I}(t-\tau,x)e^{-k\tau}, & t > 0,\ x \in \mathbb{R}, \\ \overline{I}_t - d\overline{I}_{xx} \ge \beta\overline{S}\overline{I} - (\gamma + k)\overline{I}, & t > 0,\ x \in (\overline{g}(t), \overline{h}(t)), \\ \overline{R}_t - d\overline{R}_{xx} \ge \gamma\overline{I} - \gamma\overline{I}(t-\tau,x)e^{-k\tau} - k\overline{R}, & t > 0,\ x \in (\overline{g}(t), \overline{h}(t)), \\ \overline{I}(t,x) = \overline{R}(t,x) = 0, & t \ge 0,\ x \in \mathbb{R}\backslash(\overline{g}(t), \overline{h}(t)), \\ \overline{g}'(t) \le -\mu\overline{I}_x(t, \overline{g}(t)),\ \overline{g}(0) = -\overline{h}(0), & t > 0, \\ \overline{h}'(t) \ge -\mu\overline{I}_x(t, \overline{h}(t)),\ \overline{h}(0) = \overline{h}(0) & t > 0, \\ \overline{S}(0,x) \ge S_0(x), & x \in \mathbb{R}, \\ \overline{I}(\theta,x) \ge I_0(\theta,x), & -\tau \le \theta \le 0,\ \overline{g}(\theta) < x < \overline{h}(\theta), \\ \overline{R}(0,x) \ge R_0(x), & -\overline{h}(0) < x < \overline{h}(0). \end{cases}$$

If $[g(\theta), h(\theta)] \subseteq [\overline{g}(\theta), \overline{h}(\theta)]$ *in* $[-\tau, 0]$, *then we have*

$$S(t,x) \le \overline{S}(t,x),\ \overline{g}(t) \le g(t),\ h(t) \le \overline{h}(t),\ t \in (0,T], x \in (0,\infty),$$

and

$$I(t,x) \le \overline{I}(t,x),\ R(t,x) \le \overline{R}(t,x),\ (t,x) \in (0,T] \times (g(t),h(t)).$$

Proof. The proof is the same as that of [30, Lemma 4.1] and [34, Lemma 3.1]. We omit the details. □

Consider the following eigenvalue problem:

$$\begin{cases} -d\phi'' = a\phi + \lambda\phi, & x \in (-L,L), \\ \phi(-L) = \phi(L) = 0, \end{cases} \tag{3.3}$$

where d and a are positive constants. We denote the principal eigenpair by $(\lambda, \phi(x))$. Then

$$\lambda = d\frac{\pi^2}{4L^2} - a \text{ and } \phi(x) = \cos(\frac{\pi}{2L}x).$$

Let $(\lambda_1, \phi(x))$ be the principal eigenpair of (3.3) with $a = \beta b/k - \gamma - k$. By $\mathcal{R}_0 > 1$, we have $a = \beta b/k - \gamma - k > 0$. Then there exists some $h^* > 0$ such that $\lambda_1(L) > 0$ for $L < h^*$, $\lambda_1(L) = 0$ for $L = h^*$ and $\lambda_1(L) < 0$ for $L > h^*$, where

$$h^* = \frac{\pi}{2}\sqrt{\frac{d}{\beta b/k - \gamma - k}}.$$

Lemma 3.4. *If $h_\infty - g_\infty < \infty$, then $h_\infty - g_\infty \le 2h^*$.*

Proof. Assume on the contrary that $2h^* < h_\infty - g_\infty < \infty$. By Lemma 3.2 and $\mathcal{R}_0 > 1$, for any $\varepsilon > 0$ satisfying $\beta(b/k - \varepsilon) - \gamma - k > 0$, there exists $T \gg 1$ such that $S(t,x) \ge b/k - \varepsilon$ and

$$h(T) - g(T) > \pi\sqrt{\frac{d}{\beta(b/k-\varepsilon) - \gamma - k}} \text{ for } (t,x) \in (T,\infty) \times [g(T),h(T)].$$

Therefore, I satisfies

$$\begin{cases} I_t - dI_{xx} \ge (\beta(b/k-\varepsilon) - \gamma - k)I, & t > T,\ x \in (g(T),h(T)), \\ I(t,g(T)) > 0,\ I(t,h(T)) > 0, & t > T, \\ I(0,x) \ge 0, & x \in (g(T),h(T)). \end{cases} \tag{3.4}$$

Let $(\lambda_1, \phi(x))$ be the eigenpair of (3.3) with $L = \frac{h(T)-g(T)}{2}$ and $a = \beta b/k - \gamma - k - \beta\varepsilon$, then $\lambda_1(L) < 0$. We define

$$\underline{I}(t,x) = m\phi\left(x - \frac{g(T)+h(T)}{2}\right),\ t \ge 0,\ x \in [g(T),h(T)],$$

where m will be determined later.

Direct calculations yields that

$$\begin{aligned} & \underline{I}_t - d\underline{I}_{xx} - (\beta\frac{b}{k} - \gamma - k - \beta\varepsilon)\underline{I} \\ = & -dm\phi'' - m(\beta\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi \\ = & \ m\left[(\beta\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi + \lambda_1\phi\right] - m(k\frac{b}{k} - \gamma - k - \beta\varepsilon)\phi \end{aligned}$$

$$= m\lambda_1\phi$$
$$< 0.$$

It is easy to check that

$$\underline{I}(t, g(T)) = \underline{I}(t, h(T)) = 0.$$

If we choose some sufficiently small m such that

$$I(0, x) \geq \underline{I}(0, x) \text{ for } x \in [g(T), h(T)],$$

then we can apply the comparison principle to get that

$$I(t, x) \geq \underline{I}(t, x) \text{ for } t \geq T \text{ and } x \in [g(T), h(T)].$$

Hence,

$$\lim_{t\to\infty} \|I(t, x)\|_{C([g(t),h(t)])} > 0,$$

which is a contradiction to (3.1). We complete the proof. □

Corollary 3.1. *If $h_0 \geq h^*$, then spreading always happens.*

Lemma 3.5. *If $h_0 < h^*$, then there exists $\mu^0 > 0$ such that spreading occurs if $\mu > \mu^0$.*

Proof. This lemma can be proved by similar arguments in [33, Lemma 3.2]. We give the details below. Consider the following auxiliary free boundary problem

$$\begin{cases} V_t - dV_{xx} = -(\gamma + k)V, & t > 0,\ r(t) < x < s(t), \\ V(t, x) = 0, & t \geq 0,\ x = r(t) \text{ or } s(t), \\ r'(t) = -\mu V_x(t, r(t)), r(0) = -h_0, & t > 0, \\ s'(t) = -\mu V_x(t, s(t)), s(0) = h_0, & t > 0, \\ V(0, x) = I_0(x), & x \in [-h(0), h(0)]. \end{cases} \tag{3.5}$$

The proof of the existence and uniqueness of problem (3.5) is similar to that of problem (1.3), it is easy to show that (3.5) admits a unique global solution (V, r, s), and $s'(t) > 0$, $r'(t) < 0$ for $t > 0$. To clarify the dependence of the solutions on the parameter μ, we write (I^μ, g^μ, h^μ) and (V^μ, r^μ, s^μ) in place of (I, g, h) and (V, r, s). By using Lemma 3.3, we have

$$\begin{aligned} &I^\mu(t, x) \geq V^\mu(t, x), \\ &h^\mu \geq s^\mu(t), g^\mu \leq r^\mu(t) \text{ for } t \geq 0 \text{ and } x \in [r^\mu(t), s^\mu(t)]. \end{aligned} \tag{3.6}$$

In the following, we will prove that for all large μ,

$$s^\mu(2) - r^\mu(2) \geq 2h^*. \tag{3.7}$$

We first choose the smooth functions $\underline{s}(t)$ and $\underline{r}(t)$ such that $\underline{s}(0) = -\underline{r}(0) = h(0)$, $\underline{s}'(t) < 0$, $\underline{r}'(t) > 0$ for $t > 0$ and $\underline{s}(2) - \underline{r}(2) = 2h^*$. We next consider the following initial-boundary value problem

$$\begin{cases} \underline{V}_t - d\underline{V}_{xx} = -(\gamma + k)\underline{V}, & t > 0,\ \underline{r}(t) < x < \underline{s}(t), \\ \underline{V}(t, \underline{r}(t)) = 0,\ \underline{V}(t, \underline{s}(t)) = 0, & t \geq 0, \\ \underline{V}(0, x) = \underline{V}_0(x), & x \in [-h(0), h(0)], \end{cases} \tag{3.8}$$

where

$$0 < \underline{V}_0(x) \leq I_0(x),\ \underline{V}_0(-h(0)) = \underline{V}_0(h(0)) = 0 \text{ for all } x \in [-h(0), h(0)]. \quad (3.9)$$

It follows from the standard theory for parabolic equations that problem (3.8) has a unique positive solution $\underline{V}(t,x)$. By the Hopf lemma, we have $\underline{V}_x(t, \underline{s}(t)) < 0$ and $\underline{V}_x(t, \underline{r}(t)) > 0$ for all $t \in [0,2]$. By the choice of $\underline{s}(t)$, $\underline{r}(t)$ and $\underline{V}_0(x)$, there exists $\mu^0 > 0$ such that, for all $\mu > \mu^0$,

$$\underline{s}'(t) \leq -\mu \underline{V}_x(t, \underline{s}(t)),\ \underline{r}'(t) \geq -\mu \underline{V}_x(t, \underline{r}(t)) \text{ for all } t \in [0,2]. \quad (3.10)$$

In addition, for system (3.5), we can establish the comparison principle analogous with lower solution to Lemma 3.3 by the same argument. Noting that $\underline{s}(0) = h_0 < s^\mu(0)$, $\underline{r}(0) = -h_0 > r^\mu(0)$, it follows from (3.5), (3.8), (3.9) and (3.10) that

$$V^\mu(t,x) \geq \underline{V}(t,x), s^\mu(t) \geq \underline{s}(t), r^\mu(t) \leq \underline{r}(t) \text{ for } t \in [0,2] \text{ and } x \in [\underline{r}(t), \underline{s}(t)],$$

which implies that $s^\mu(2) - r^\mu(2) \geq \underline{s}(2) - \underline{r}(2) = 2h^*$. Thus, (3.7) holds. It follows that

$$h_\infty - g_\infty = \lim_{t \to \infty} [s^\mu(t) - r^\mu(t)] > s^\mu(2) - r^\mu(2) \geq 2h^*.$$

Hence, we obtain the desired result by Corollary 3.1. We complete the proof. □

Lemma 3.6. *If $\|N_0\|_\infty \leq b/k$ and $h_0 < h^*$, there exists $\mu_0 > 0$ such that vanishing happens when $\mu < \mu_0$.*

Proof. By $\|N_0\|_\infty \leq b/k$ and the proof of Lemma 2.1, it is easy to see that $0 \leq S(t,x) \leq b/k$ for $t > 0$ and $x \in \mathbb{R}$. Then I satisfies

$$\begin{cases} I_t - dI_{xx} \leq (\beta b/k - \gamma - k)I, & t > 0,\ x \in (g(t), h(t)), \\ g'(t) = -\mu I_x(t, g(t)),\ g(0) = -h_0, & t > 0, \\ h'(t) = -\mu I_x(t, h(t)),\ h(0) = h_0, & t > 0, \\ I(0,x) = I_0(x), & x \in (-h(0), h(0)). \end{cases} \quad (3.11)$$

In the following, we construct an upper solution of (3.11). Let $(\lambda_1, \phi(x))$ be the eigenpair of (3.3) with $L = h_0$, $a = \beta b/k - \gamma - k$, then $\lambda_1(L) > 0$. Inspired by the arguments in [31, Lemma 3.4], we define

$$\sigma(t) = h_0(1 + 2m - me^{-mt}),\ t > 0,$$
$$\overline{I}(t,x) = Ke^{-mt}\phi\left(\frac{h_0 x}{\sigma(t)}\right),\ t \geq 0,\ x \in [-\sigma(t), \sigma(t)],$$

where positive constants K and m will be determined later.

Direct calculations show that, for $(t,x) \in (0,\infty) \times [-\sigma(t), \sigma(t)]$,

$$\begin{aligned} &\overline{I}_t - d\overline{I}_{xx} - (\beta\frac{b}{k} - \gamma - k)\overline{I} \\ = &Ke^{-mt}\left[-m\phi - \frac{h_0 x\sigma'}{\sigma^2(t)}\phi' - d\phi''\frac{h_0^2}{\sigma^2(t)} - (\beta\frac{b}{k} - \gamma - k)\phi\right] \\ = &Ke^{-mt}\left[-m\phi - m\phi''\frac{h_0^2}{\sigma^2(t)} - (\beta\frac{b}{k} - \gamma - k)\phi\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \end{aligned}$$

$$= Ke^{-mt}\left\{-m\phi + \frac{h_0^2}{\sigma^2(t)}\left[(\beta\frac{b}{k}-\gamma-k)\phi + \lambda_1\phi\right] - (\beta\frac{b}{k}-\gamma-k)\phi\right\} - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi'$$
$$= Ke^{-mt}\left[-m\phi + (\beta\frac{b}{k}-\gamma-k)\phi\left(\frac{h_0^2}{\sigma^2(t)}-1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\phi\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi'$$
$$= Ke^{-mt}\phi\left[-m + \left(\beta\frac{b}{k}-\gamma-k\right)\left(\frac{h_0^2}{\sigma^2(t)}-1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right] - Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi'.$$

By $Ke^{-mt}\frac{h_0 x\sigma'}{\sigma^2(t)}\phi' \leq 0$, we have

$$\begin{aligned}&\overline{I}_t - d\overline{I}_{xx} - (\beta\frac{b}{k}-\gamma-k)\overline{I}\\ &\geq Ke^{-mt}\phi\left[-m + \left(\beta\frac{b}{k}-\gamma-k\right)\left(\frac{h_0^2}{\sigma^2(t)}-1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)}\right]\\ &=:\Delta_1.\end{aligned}$$

If we choose small enough m such that

$$-m + \left(\beta\frac{b}{k}-\gamma-k\right)\left(\frac{h_0^2}{\sigma^2(t)}-1\right) + \lambda_1\frac{h_0^2}{\sigma^2(t)} > 0,$$

then $\Delta_1 \geq 0$. Now we choose K sufficiently large such that

$$\overline{I}(0,x) = K\phi\left(\frac{h_0 x}{\sigma(0)}\right) \geq I_0(0,x) \text{ for } x \in [-h(0), h(0)].$$

Obviously, $\overline{I}(t,-\sigma(t)) = \overline{I}(t,\sigma(t)) = 0$. A simple calculation shows, for $t > 0$,

$$-\mu\overline{I}_x(t,\sigma(t)) = -\frac{\mu h_0}{\sigma(t)}Ke^{-mt}\phi'(h_0) \leq -\mu Ke^{-mt}\phi'(h_0) \leq h_0 m^2 e^{-mt} = \sigma'(t)$$

provided that $\mu \leq -\frac{h_0 m^2}{K\phi'(h_0)} := \mu_0$. Similarly, we can obtain $-\sigma'(t) \leq -\mu\overline{I}_x(t,-\sigma(t))$ for $t > 0$. By Lemma 3.3, we have

$$g(t) \geq -\sigma(t),\ h(t) \leq \sigma(t),\ I(t,x) \leq \overline{I}(t,x) \text{ for } t > 0 \text{ and } x \in [g(t), h(t)].$$

It follows that $h_\infty - g_\infty \leq 2\lim_{t\to\infty}\sigma(t) = 2h_0(1+m) < \infty$. We have finished the proof. □

From Lemma 3.5 and Lemma 3.6, we can obtain the following criteria for spreading and vanishing.

Lemma 3.7. *If $\|N_0\|_\infty \leq \frac{b}{k}$ and $h_0 < h^*$, then there exists $\mu^* \geq \mu_* > 0$ such that spreading happens if $\mu > \mu^*$, and vanishing happens if $\mu \leq \mu_*$ and $\mu = \mu^*$.*

Proof. This proof is similar to that of [34, Theorem 5.2] and [10, Theorem 3.9]. We omit the details. □

Proof of Theorem 1.3. This theorem can be obtained by Lemmas 3.1 and 3.7. □

4. Long-time behaviors

Lemma 4.1. *If $h_\infty - g_\infty = \infty$, then $h_\infty = \infty$ and $g_\infty = -\infty$.*

Proof. This lemma is proved by the similar arguments in [22, Lemma 3.10]. We give the outline of the proof below.

Assume on the contrary that $g_\infty = -\infty$ and $h_\infty < \infty$. At first, by using [21, Lemma 3.3], we have

$$\lim_{t\to\infty} \|I(t,\cdot)\|_{C([-L,h(t)])} = 0 \text{ for any given } L > 0.$$

Then, by applying the argument in the Step 3 of the proof in [22, Lemma 3.10], we can have that for any given constant $L > 0$ and small $\varepsilon > 0$, there exist $T_1 > 0$ and $l_1 < l_2 < 0$ satisfying $l_2 - l_1 = L$ such that

$$S(t,x) \geq \frac{b}{k} - \varepsilon \text{ for all } t \geq T_1 \text{ and } x \in [l_1, l_2].$$

We choose l_1 and l_2 satisfying $l_2 - l_1 \geq \pi\sqrt{\frac{d}{\beta(b/k-\varepsilon)-\gamma-k}}$, for small $\varepsilon > 0$ and large $T > T_1$, we have

$$\begin{cases} I_t - dI_{xx} \geq (\beta(b/k-\varepsilon) - \gamma - k)I, & t > T,\ x \in [l_1, l_2], \\ I(t,x) > 0, & t > T,\ x = l_1 \text{ or } l_2, \\ I(0,x) \geq 0, & x \in [l_1, l_2]. \end{cases}$$

By $l_2 - l_1 \geq \pi\sqrt{\frac{d}{\beta(b/k-\varepsilon)-\gamma-k}}$, we can argue as in the proof of Lemma 3.4 to obtain that

$$\liminf_{t\to\infty} I(t,x) > 0 \text{ for } x \in [l_1, l_2],$$

which is a contradiction.

Therefore, if $h_\infty - g_\infty = \infty$, then $h_\infty = \infty$. Similarly, we can prove $g_\infty = -\infty$. This proof is completed. □

Lemma 4.2. *Let (S, I, R, g, h) be the unique solution of problem (1.3). If spreading happens and $k > \gamma(1 - e^{-k\tau})$, then*

$$\lim_{t\to\infty} (S(t,x), I(t,x), R(t,x)) = (S_*, I_*, R_*) \text{ locally uniformly in } \mathbb{R},$$

where

$$(S_*, I_*, R_*) = \left(\frac{\gamma+k}{\beta}, \frac{\beta b - k(\gamma+k)}{\beta[k+\gamma(1-e^{-k\tau})]}, \frac{\gamma(1-e^{-k\tau})}{k} I_*\right). \tag{4.1}$$

Proof. **Step 1.** Let $N(t,x) = S(t,x) + I(t,x) + R(t,x)$, then N satisfies

$$\begin{cases} N_t - dN_{xx} = b - kN, & t > 0,\ x \in (g(t), h(t)), \\ N(t, g(t)) > 0,\ N(t, h(t)) > 0, & t > 0, \\ N(0,x) = S_0(x) + I_0(x) + R_0(x), & x \in [-h(0), h(0)], \end{cases}$$

it follows from [39, Lemma 2.6(iii)] that

$$\lim_{t\to\infty} N(t,x) = \frac{b}{k} =: N_* \text{ locally uniformly in } \mathbb{R}.$$

Clearly,

$$\lim_{t\to\infty} S(t,x) \leq \frac{b}{k} =: \overline{S}_1 \text{ locally uniformly in } \mathbb{R}.$$

For any small $\varepsilon > 0$ and large l, it follows from $h_\infty = -g_\infty = \infty$ that there exists T_1 such that

$$g(t) < -l,\ h(t) > l,\ N(t,x) \le N_* + \varepsilon \text{ for } t > T_1 \text{ and } x \in [-l, l].$$

By the second equation of (1.3), we have I satisfies

$$\begin{cases} I_t - dI_{xx} \le [\beta(N_* + \varepsilon) - \gamma - k - \beta I]I, & t > T_1,\ x \in [-l, l], \\ I(t,x) \ge 0, & t > T_1,\ x \le -l \text{ or } x \ge l, \\ I(0,x) > 0, & x \in [-l, l]. \end{cases}$$

It follows from [34, Proposition 8.1] that

$$\limsup_{t\to\infty} I(t,x) \le \frac{\beta(N_* + \varepsilon) - \gamma - k}{\beta} \text{ for } x \in [-l, l].$$

By the arbitrariness of ε and l, we have

$$\limsup_{t\to\infty} I(t,x) \le \frac{\beta N_* - \gamma - k}{\beta} = \frac{\gamma + k}{\beta}(\mathcal{R}_0 - 1) =: \overline{I}_1 \text{ locally uniformly in } \mathbb{R}. \quad (4.2)$$

By the third equation of (1.3), it follows from [39, Lemma 2.6(ii)] that

$$\limsup_{t\to\infty} R(t,x) \le \frac{\gamma(1 - e^{-k\tau})}{k}\overline{I}_1 =: \overline{R}_1 \text{ locally uniformly in } \mathbb{R}. \quad (4.3)$$

Thanks to $\mathcal{R}_0 > 1$, we have $\overline{I}_1$ and $\overline{R}_1$ are positive.

By (4.2), for any small $\varepsilon_1 > 0$ and large l, there exists $T_2 > T_1$ such that

$$I(t,x) \le \overline{I}_1 + \varepsilon_1 \text{ for } t > T_2 \text{ and } x \in [-l, l]. \quad (4.4)$$

Therefore, by the first equation of (1.3), we have S satisfies

$$\begin{cases} S_t - dS_{xx} \ge b - kS - \beta(\overline{I}_1 + \varepsilon_1)S, & t > T_2,\ x \in [-l, l], \\ S(T_2, x) > 0, & x \in [-l, l], \end{cases}$$

then

$$\liminf_{t\to\infty} S(t,x) \ge \frac{b}{k + \beta(\overline{I}_1 + \varepsilon_1)} \text{ for } x \in [-l, l].$$

By the arbitrariness of ε_1 and l, we have

$$\liminf_{t\to\infty} S(t,x) \ge \frac{b}{k + \beta\overline{I}_1} =: \underline{S}_1 \text{ locally uniformly in } \mathbb{R}.$$

Step 2. For any small $\varepsilon_2 > 0$ and large l, it follows from $h_\infty = -g_\infty = \infty$ that there exists T_3 such that

$$g(t) < -l,\ h(t) > l,\ N(t,x) \ge N_* - \varepsilon_2 \text{ for } t > T_3 \text{ and } x \in [-l, l].$$

By (4.3), for above ε_2 and large l, there exists $T_4 > T_3$ such that

$$R(t,x) \le \overline{R}_1 + \varepsilon_2 \text{ for } t > T_4 \text{ and } x \in [-l, l].$$

By the second equation of (1.3), I satisfies

$$\begin{cases} I_t - dI_{xx} \geq [\beta(N_* - \varepsilon_2) - \beta(\overline{R}_1 + \varepsilon_2) - \gamma - k - \beta I]I, & t > T_4,\ x \in [-l, l], \\ I(t,x) \geq 0, & t > T_4,\ x \leq -l \text{ or } x \geq l, \\ I(0,x) > 0, & x \in [-l, l]. \end{cases}$$

It follows from [34, Proposition 8.1] that

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta(N_* - \varepsilon_2) - \beta(\overline{R}_1 + \varepsilon_2) - \gamma - k}{\beta} \text{ for } x \in [-l, l].$$

By the arbitrariness of ε_2 and l, we have

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta N_* - \beta\overline{R}_1 - \gamma - k}{\beta} = \overline{I}_1 - \overline{R}_1 =: \underline{I}_1 \text{ locally uniformly in } \mathbb{R}. \quad (4.5)$$

By the third equation of (1.3), it follows from [39, Lemma 2.6(i)] that

$$\liminf_{t\to\infty} R(t,x) \geq \frac{\gamma(1-e^{-k\tau})}{k}\underline{I}_1 =: \underline{R}_1 \text{ locally uniformly in } \mathbb{R}. \quad (4.6)$$

Since $k > \gamma(1 - e^{-k\tau})$ and $\mathcal{R}_0 > 1$, we have $\underline{I}_1$ and $\underline{R}_1$ are positive.

By (4.5) and (4.4), for any small $\varepsilon_3 > 0$ and large l, there exists T_5 such that

$$\underline{I}_1 - \varepsilon_3 \leq I(t,x) \leq \overline{I}_1 + \varepsilon_3 \text{ for } t > T_5 \text{ and } x \in [-l, l]. \quad (4.7)$$

Thus by (4.7), S satisfies, for $T_6 > T_5 + \tau$,

$$\begin{cases} S_t - dS_{xx} \leq b - kS - \beta(\underline{I}_1 - \varepsilon_3)S + \gamma(\overline{I}_1 + \varepsilon_3)e^{-k\tau}, & t > T_6,\ x \in [-l, l], \\ S(T_6, x) > 0, \quad x \in [-l, l], \end{cases}$$

then

$$\limsup_{t\to\infty} S(t,x) \leq \frac{b + \gamma(\overline{I}_1 + \varepsilon_3)e^{-k\tau}}{k + \beta(\underline{I}_1 - \varepsilon_3)} \text{ for } x \in [-l, l].$$

By the arbitrariness of ε_3 and l, we have

$$\limsup_{t\to\infty} S(t,x) \leq \frac{b + \gamma\overline{I}_1 e^{-k\tau}}{k + \beta\underline{I}_1} =: \overline{S}_2 \text{ locally uniformly in } \mathbb{R}.$$

Step 3. By the similar arguments in Step 1, we can obtain

$$\limsup_{t\to\infty} I(t,x) \leq \frac{\beta N_* - \beta\underline{R}_1 - \gamma - k}{\beta} = \overline{I}_1 - \underline{R}_1 =: \overline{I}_2 \text{ locally uniformly in } \mathbb{R}.$$

By the third equation of (1.3), it follows from [39, Lemma 2.6(ii)] that

$$\limsup_{t\to\infty} R(t,x) \leq \frac{\gamma(1-e^{-k\tau})}{k}\overline{I}_2 =: \overline{R}_2 \text{ locally uniformly in } \mathbb{R}.$$

Due to $k > \gamma(1 - e^{-k\tau})$ and $\mathcal{R}_0 > 1$, we have $\overline{I}_2$ and $\overline{R}_2$ are positive.

By the first equation of (1.3) and the similar arguments in Step 1, we can obtain

$$\liminf_{t\to\infty} S(t,x) \ge \frac{b+\gamma \underline{I}_1 e^{-k\tau}}{k+\beta \overline{I}_2} =: \underline{S}_2 \text{ locally uniformly in } \mathbb{R}.$$

Step 4. By the similar arguments in Step 2, we can obtain

$$\liminf_{t\to\infty} I(t,x) \ge \frac{\beta N_* - \beta \overline{R}_2 - \gamma - k}{\beta} = \overline{I}_1 - \overline{R}_2 =: \underline{I}_2 \text{ locally uniformly in } \mathbb{R}.$$

By the third equation of (1.3), it follows from [39, Lemma 2.6(ii)] that

$$\liminf_{t\to\infty} R(t,x) \ge \frac{\gamma(1-e^{-k\tau})}{k}\underline{I}_2 =: \underline{R}_2 \text{ locally uniformly in } \mathbb{R}.$$

Since $k > \gamma(1-e^{-k\tau})$ and $\mathcal{R}_0 > 1$, we have $\underline{I}_2$ and $\underline{R}_2$ are positive.

By the first equation of (1.3) and the similar arguments in Step 2, we can obtain

$$\limsup_{t\to\infty} S(t,x) \le \frac{b+\gamma \overline{I}_2 e^{-k\tau}}{k+\beta \underline{I}_2} =: \overline{S}_3 \text{ locally uniformly in } \mathbb{R}.$$

Step 5. Repeating above arguments, we can obtain six sequences $\{\overline{S}_n\}$, $\{\underline{S}_n\}$, $\{\overline{I}_n\}$, $\{\underline{I}_n\}$, $\{\overline{R}_n\}$, and $\{\underline{R}_n\}$ satisfying

$$\underline{S}_n \le \liminf_{t\to\infty} S(t,x) \le \limsup_{t\to\infty} S(t,x) \le \overline{S}_n \text{ locally uniformly in } \mathbb{R},$$
$$\underline{I}_n \le \liminf_{t\to\infty} I(t,x) \le \limsup_{t\to\infty} I(t,x) \le \overline{I}_n \text{ locally uniformly in } \mathbb{R},$$
$$\underline{R}_n \le \liminf_{t\to\infty} R(t,x) \le \limsup_{t\to\infty} R(t,x) \le \overline{R}_n \text{ locally uniformly in } \mathbb{R},$$

where

$$\overline{S}_n = \frac{b+\gamma \overline{I}_{n-1} e^{-k\tau}}{k+\beta \underline{I}_{n-1}},\ \overline{I}_n = \frac{\beta N_* - \gamma - k - \beta \underline{R}_{n-1}}{\beta},\ \overline{R}_n = \frac{\gamma(1-e^{-k\tau})}{k}\overline{I}_n,$$
$$\underline{I}_n = \frac{\beta N_* - \gamma - k - \beta \overline{R}_n}{\beta},\ \underline{R}_n = \frac{\gamma(1-e^{-k\tau})}{k}\underline{I}_n,\ \underline{S}_n = \frac{b+\gamma \underline{I}_{n-1} e^{-k\tau}}{k+\beta \overline{I}_n}.$$

Moreover,

$$\underline{S}_1 < \underline{S}_2 < \cdots < \underline{S}_n < \cdots < \overline{S}_n < \cdots < \overline{S}_2 < \overline{S}_1,$$
$$\underline{I}_1 < \underline{I}_2 < \cdots < \underline{I}_n < \cdots < \overline{I}_n < \cdots < \overline{I}_2 < \overline{I}_1,$$
$$\underline{R}_1 < \underline{R}_2 < \cdots < \underline{R}_n < \cdots < \overline{R}_n < \cdots < \overline{R}_2 < \overline{R}_1,$$

then we have

$$\overline{I}_\infty = \frac{\beta N_* - \gamma - k - \beta \underline{R}_\infty}{\beta},\ \overline{R}_\infty = \frac{\gamma(1-e^{-k\tau})}{k}\overline{I}_\infty,\ \overline{S}_\infty = \frac{b+\gamma \overline{I}_\infty e^{-k\tau}}{k+\beta \underline{I}_\infty},$$
$$\underline{I}_\infty = \frac{\beta N_* - \gamma - k - \beta \overline{R}_\infty}{\beta},\ \underline{R}_\infty = \frac{\gamma(1-e^{-k\tau})}{k}\underline{I}_\infty,\ \underline{S}_\infty = \frac{b+\gamma \underline{I}_\infty e^{-k\tau}}{k+\beta \overline{I}_\infty}.$$

By direct computations, we have $(\underline{S}_\infty, \underline{I}_\infty, \underline{R}_\infty) = (\overline{S}_\infty, \overline{I}_\infty, \overline{R}_\infty) = (S_*, I_*, R_*)$. Therefore, this lemma has been proved. □

Proof of Theorem 1.2. This theorem can be obtained by Lemmas 3.2 and 4.2. □

5. Numerical simulation

To support theoretical results in previous sections, we will use MATLAB to make some numerical simulations in this section.

Now we assume that the coefficient and initial functions in (1.3) are as follows:

$$d = 1,\ h(\theta) = h_0 e^{\theta},\ I_0(x) = \cos(\frac{\pi}{2h(\theta)}x),\ R_0(x) = \cos(\frac{\pi}{2h(0)}x).$$

Example 5.1 (The case of $\mathcal{R}_0 \le 1$)**.** Let $b = 1$, $\beta = 0.5$, $k = 0.6$, $\gamma = 0.5$, $h_0 = 1.5$, then we have that $\mathcal{R}_0 = 0.7576 < 1$ by (1.6).

The simulation results are showed in Figure 1 and 2, we can observe that the disease will die out and $h_\infty - g_\infty < \infty$ if $\mathcal{R}_0 \le 1$.

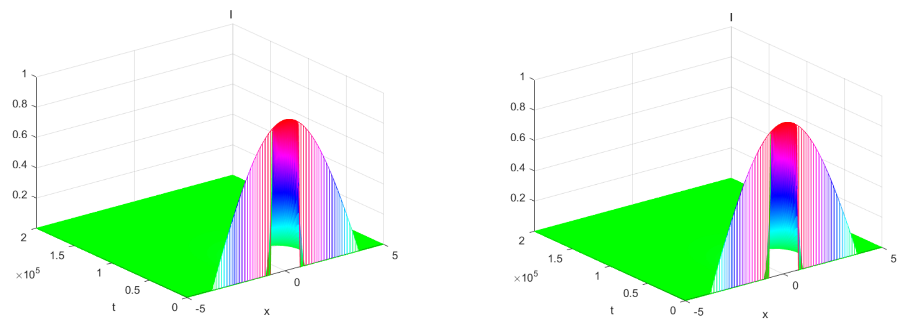

Figure 1. The profiles of $\tau = 0.5$ and $\tau = 0.8$.

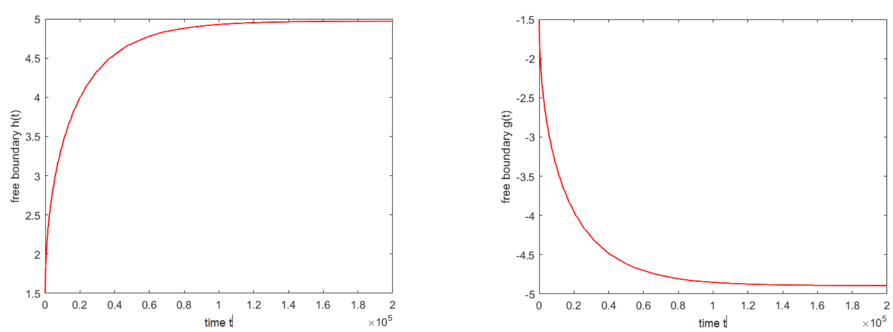

Figure 2. Vanishing of the free boundary h(t) and g(t).

Example 5.2 (The case of $h_0 \ge h_*$ and $\mathcal{R}_0 > 1$)**.** Let $b = 1$, $\beta = 1$, $k = 0.5$, $\gamma = 0.6$, $h_0 = 1.8$. By direct calculations, we can obtain that $h_* = 1.6558$, and $\mathcal{R}_0 = 1.8182$ by (1.6). Then $h_0 > h_*$ and $\mathcal{R}_0 > 1$. Moreover,

$$(S_*, I_*, R_*) = \begin{cases} (1.1, 0.7112, 0.1888), & \tau = 0.5, \\ (1.1, 0.6449, 0.2551), & \tau = 0.8. \end{cases}$$

The simulation results are showed in Figure 3, it is easy to see that the solution $I(t,x)$ keeps positive and tends to an equilibrium I_* if $\mathcal{R}_0 > 1$ and $h_0 \ge h_*$. Moreover, we can observe that the equilibrium I_* is decreasing with the increasing of immunity τ.

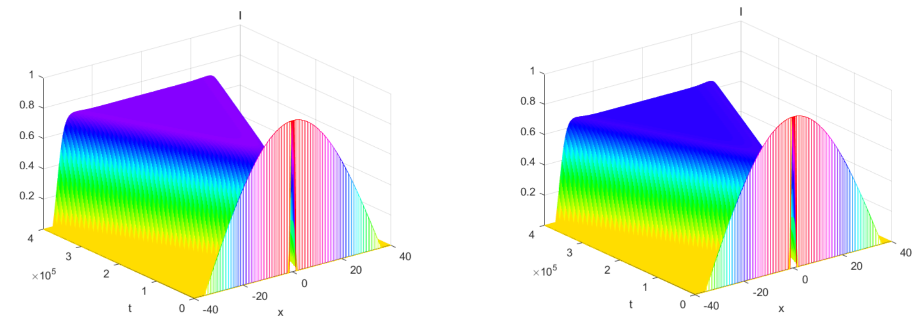

Figure 3. The profiles of $\tau = 0.5$ and $\tau = 0.8$.

Example 5.3 (The case of $h_0 < h_*$ and $\mathcal{R}_0 > 1$)**.** Let $b = 1$, $\beta = 1$, $k = 0.5$, $\gamma = 0.6$, $h_0 = 1.5$. By direct calculations, we can obtain that $h_* = 1.5279$, and $\mathcal{R}_0 = 1.8182$ by (1.6). Then $h_0 < h_*$ and $\mathcal{R}_0 > 1$. Moreover,

$$(S_*, I_*, R_*) = \begin{cases} (1.1, 0.7112, 0.1888), & \tau = 0.5, \\ (1.1, 0.6449, 0.2551), & \tau = 0.8. \end{cases}$$

The simulation results are showed in Figure 4 and 5. From Figure 4, we can find the solution $I(t,x)$ keeps positive and tends to an equilibrium I_* for some large $\mu = 10$ if $\mathcal{R}_0 > 1$ and $h_0 < h_*$. And we can observe that the equilibrium I_* is decreasing with the increasing of immunity τ. From Figure 5, it is easy to see that the disease will die out for some small $\mu = 1$ if $\mathcal{R}_0 > 1$ and $h_0 < h_*$. This means that, when $\mathcal{R}_0 > 1$ and $h_0 < h_*$, whether the disease spreads or not depends on the expanding capability μ of the spreading front.

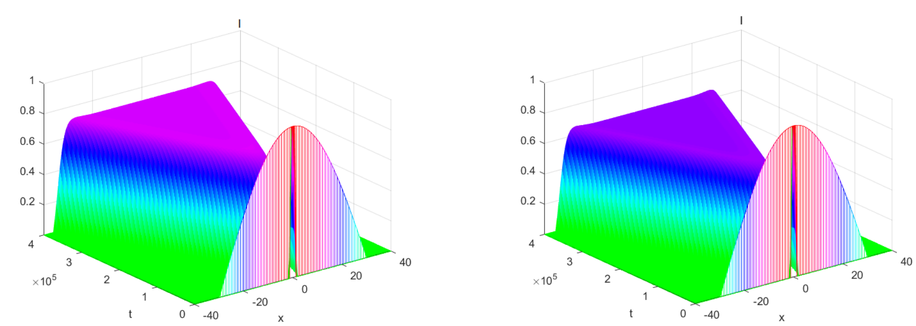

Figure 4. The profiles of $\tau = 0.5$ and $\tau = 0.8$ with $\mu = 10$.

Example 5.4 (The effect of τ on the spreading speed of $h(t)$ and $g(t)$)**.** Let $h_0 = 1.8$, $h(\theta) = h_0 e^{\theta}$, $I_0(x) = \cos(\frac{\pi}{2h(\theta)} x)$.

The simulation results are showed in Figure 6. From Figure 6, we can observe that the spreading speeds of the spreading fronts $h(t)$ and $g(t)$ are decreasing with the increasing of delay τ, which implies that the time delay can slow down the spreading of epidemic.

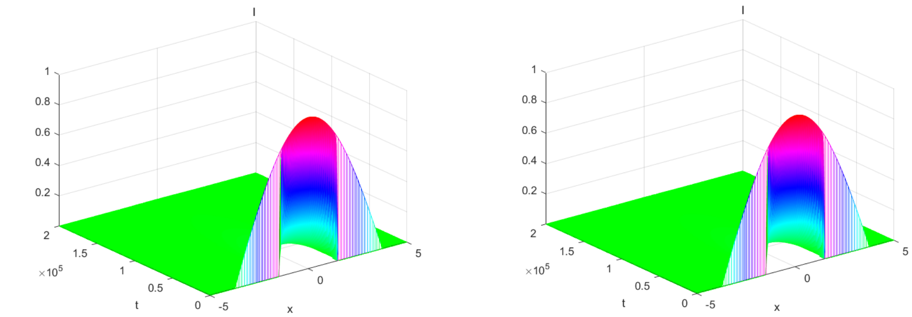

Figure 5. The profiles of $\tau = 0.5$ and $\tau = 0.8$ with $\mu = 1$.

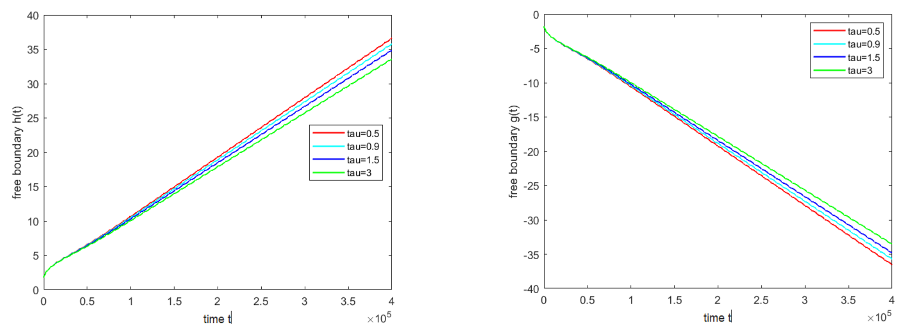

Figure 6. Spreading of the free boundary $h(t)$ and $g(t)$.

6. Discussion

This paper considers an SIRS epidemic model with time delay and free boundaries. We first prove the global existence and uniqueness of the solution. Then we show the long-time behavior of the solution can be determined by the following spreading-vanishing dichotomy:

(i) Vanishing: If $h_\infty - g_\infty < \infty$, then

$$\lim_{t\to\infty} S(t,x) = \frac{b}{k} \text{ in } C_{loc}(\mathbb{R}), \ \lim_{t\to\infty} \|I(t,x) + R(t,x)\|_{C([g(t),h(t)])} = 0;$$

(ii) Spreading: If $h_\infty - g_\infty = \infty$ and $k > \gamma(1 - e^{-k\tau})$, then

$$\lim_{t\to\infty} (S(t,x), I(t,x), R(t,x)) = (S_*, I_*, R_*) \text{ locally uniformly in } \mathbb{R}.$$

Furthermore, we obtain the following criteria for spreading and vanishing hold:

(i) If $\mathcal{R}_0 \le 1$ and $\|N_0\|_\infty \le \frac{b}{k}$, then disease will vanish;

(ii) If $\mathcal{R}_0 > 1$, then there exists $h^* > 0$ such that spreading happens when $h_0 \ge h^*$, and if $h_0 < h^*$ and $\|N_0\|_\infty \le \frac{b}{k}$, then there exists $\mu^* \ge \mu_* > 0$ such that spreading happens when $\mu > \mu^*$, and vanishing happens when $\mu \le \mu_*$ and $\mu = \mu^*$.

Finally, some numerical simulations are provided to illustrate our results.

Since it is difficult to establish the corresponding semi-wave theory, we do not give the precise estimation of the spreading speed of the spreading front if spreading happens. We will study it in the future. However, the numerical simulation in Example 5.4 shows that the spreading speeds of the spreading fronts $h(t)$ and $g(t)$ are decreasing with the increasing of delay τ, which implies that the time delay can slow down the spreading of epidemic. Furthermore, our results in figure 3 and 4 indicate that time delay can affect the value of equilibrium point. Moreover, the results in figure 6 show that the region $(g(t), h(t))$ of infectious individuals is decreasing with the increasing of time delay.

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