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EXACT NULL CONTROLLABILITY OF A FRACTIONAL NONLOCAL DELAY EVOLUTION SYSTEM*

Shouguo Zhu[1,†] and Gang Li[2]

Abstract We delve into the exact null controllability problem of a fractional nonlocal weighted delay abstract system. For this strategy, we launch the resolvent trick and the approximation solvability method to construct control-state approximation sequence pairs twice to explore the problem without involving the compactness of semigroups and nonlocal items and the Lipschitz restriction on nonlinear terms and nonlocal parts or the noncompactness measure condition. Our work extends and generalizes previous results about exact null controllability problems of all evolution systems. Moreover, a significant diffusion model is displayed to show the applicability and validity of our mentioned outcomes. Finally, the conclusion of this paper is offered.

Keywords φ-Riemann-Liouville fractional derivatives, exact null controllability, nonlocal conditions, approximation solvability method.

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1. Introduction

Since fractional calculus can supply plenteous toolkits to model real-world physical problems with memory characteristics, many researchers have been involved in the focus of fractional systems and abundant inspiring findings have been garnered and reported, as shown in [4, 6, 13, 20, 22, 36]. Considering that fractional derivatives involving a function ψ are more elastic and convenient in modeling practical application than derivatives alone (see [11] and [15]) and Riemann-Liouville type derivatives systems are more accurate and suitable in describing some real-world materials with hereditary features than Caputo type (see [12] and [28]), we here restrict our attention and exploration to ψ-Riemann-Liouville type fractional evolution systems.

In 1991, Byszewski in [7] initially proposed and handled the nonlocal problems subjected to abstract models. Later on, many scholars devoted themselves to the investigation of nonlocal abstract equations. In most of the existing results about nonlocal evolution systems, compactness hypothesis was usually required. In order

†The corresponding author.
1 Wuxi Institute of Technology, Wuxi 214121, China
2 School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China
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Email: sgzhu2015@163.com (S. Zhu), gli@yzu.edu.cn(G. Li)

to drop the compactness, the noncompactness measure approach was utilized in many fruitful results (see [19,37]). Recently, without involving the noncompactness measure condition, the approximation solvability trick was launched to deal with abstract systems incorporating nonlocal terms, see [24,25,33,38,39,41].

On the other hand, the controllability problems of evolution systems were investigated by many authors, see [16,22,24,30–32]. The notion of controllability can be separated into exact controllability, exact null controllability and approximate controllability. Exact null controllability problems governed by abstract systems are of tremendous importance in their practical applications and aim at seeking a control-state pair (u, x) to drive the systems to zero state. [5] and [9] employed the compactness of the semigroup or resolvent and the boundedness of inverse operator $(L_0^b)^{-1}$ on $L^2(J,U)/\ker(L_0^b)$ to tackle the problems. Under the compactness restriction on the semigroup, [10] discovered a more accurate criteria, the boundedness of operator $(L_0)^{-1}N_0^b$, to explore them. Subsequently, this criteria was expeditiously adopted in plenty of results, as demonstrated in [1, 14, 17, 34]. Unfortunately, as for the exact null controllability problems without involving compactness assumption on semigroups or the noncompactness measure condition, they have not been treated. They become the targets of our work.

Let H and U be two Hilbert spaces. We thus, in this paper, show interest in exploring the exact null controllability problem of the following fractional nonlocal weighted delay abstract system:

$$\begin{cases} D^{\alpha,\psi}x(s) = Ax(s) + F(s,\widetilde{x}_s) + Bu(s), \; s \in (0,b], \\ \widetilde{x}_0 = \phi + g(x) \in C([-r,0],H), \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{2}, 1)$, $D^{\alpha,\psi}$ is an α-order ψ-Riemann-Liouville type fractional derivative operator involving a function $\psi(s)$, A is a generator of an analytic semigroup $\{T(s)\}_{s\geq 0}$, $F : [0,b] \times C([-r,0],H) \to H$, $g : C_{\alpha,\psi}([0,b],H) \to C([-r,0],H)$, $B \in \mathscr{L}(U,H)$. In addition

$$\widetilde{x}_s(\theta) = \widetilde{x}(s+\theta) = \begin{cases} \Gamma(\alpha)\psi^{1-\alpha}(s+\theta)x(s+\theta), \; s+\theta \in [0,b], \\ \phi(s+\theta) + g(x)(s+\theta), \quad s+\theta \in [-r,0], \end{cases}$$

for $s \in [0,b]$ and $\theta \in [-r,0]$.

If $\alpha = 1$, $\psi(t) = t$, $A = \Delta$, system (1.1) becomes the classical parabolic system, which can serve as important models to describe various physical phenomena in many fields, such as heat conduction, diffusion and seepage, see [40].

If $\psi(t) = t$, we explored the approximate controllability problem of this system in [40] by employing the topological structure of the solution set and the resolvent method without the Lipschitz condition.

We, in this paper, will develop the approximation solvability method to tackle the exact null controllability problem of system (1.1). The main difficulty is how to construct the control-state sequence pair (u, x) without incorporating the compactness assumption and the Lipschitz restriction condition or the noncompactness measure condition. We propose the following idea to deal with the difficulty: by employing the orthogonal projector operator $\mathbb{P}_m$ and the Yosida approximation of A, we begin by constructing the approximation system to yield the first control-state approximation sequence pair $(u_{x^{mn}}, x^{mn})$. Then, by the approximation theory of semigroup or resolvent, we set up the second control-state approximation sequence pair (u_{x^n}, x^n). Finally, by utilizing the approximation theory of semigroup or resolvent, we garner the required control-state sequence pair (u, x).

Compared to the existing literature, this work enjoys the highlighted features:

(i) To avoid the shortcoming of singularity of ψ-Riemann-Liouville type fractional systems, we adopt the weighted delay initial condition and the weighted Banach space.

(ii) By the approximation solvability method, we construct control-state approximation sequence pairs twice to explore the exact null controllability problem without incorporating the compactness assumption and the Lipschitz restriction condition or the noncompactness measure condition.

Finally, we give a breakdown of this work's structure. Section 2 contains some helpful notations and preliminary facts. Section 3 provides a comprehensive investigation of the exact null controllability problem. Section 4 presents a significant application model. We end the work in Section 5 with the conclusion.

2. Preliminaries

We here start by some helpful notations. Let $\psi(t)$ be a special function satisfying that $\psi \in C^1([0,b],\mathbb{R})$ with the norm $\|\psi\|_{C^1} = \sup\limits_{s\in[0,b]} (|\psi(t)| + |\psi'(t)|)$, $\psi'(t) > 0$ and $\psi(0) = 0$. Let $C([c,d],H) = \{y : [c,d] \to H : y \text{ is continuous}\}$. With the norm $\|y\|_{[c,d]} = \sup\limits_{t\in[c,d]} \|y(t)\|$, $C([c,d],H)$ is a Banach space. Set $J' = (0,b]$, $J = [0,b]$ and

$$\widetilde{x}(s) = \begin{cases} \Gamma(\alpha)\psi^{1-\alpha}(s)x(s), \ s \in J, \\ \phi(s) + g(x)(s), \qquad s \in [-r,0]. \end{cases}$$

With the weighted norm $\|x\|_{\alpha,\psi} = \sup\limits_{t\in J} \|\widetilde{x}(t)\|$, we receive an important weighted Banach space

$$C_{\alpha,\psi}(J,H) = \left\{ x \in C(J',H) \ |\widetilde{x}(0) = \lim_{t\to 0^+} x(t),\ \widetilde{x} \in C(J,H) \right\}.$$

Let $\mathscr{L}(H,U) = \{f : H \to U|\ f \text{ is linear and bounded}\}$ and write $\mathscr{L}(H,H)$ as $\mathscr{L}(H)$. The notation $\mathbb{P}_m : H \to H_m$ means the orthogonal projector from H to its m-dimensional subspace H_m. Moreover, for $R > 0$, set

$$Q_R = \{x \in C_{\alpha,\psi}(J,H) | \|x\|_{\alpha,\psi} < R\} \text{ and } Q^{(m)} = Q_R \bigcap C_{\alpha,\psi}(J,H_m).$$

Next, we review some required notions involving the function ψ from fractional calculus.

Definition 2.1. [18] The α-order ψ-fractional integral operator $I^{\alpha,\psi}$ is described as

$$I^{\alpha,\psi} f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (\psi(s) - \psi(\tau))^{\alpha-1} \psi'(\tau) f(\tau) \mathrm{d}\tau,\ s > 0,\ \alpha > 0.$$

Definition 2.2. [18] The α-order ψ-Riemann-Liouville type fractional derivative operator $D^{\alpha,\psi}$ is depicted as

$$D^{\alpha,\psi} f(s) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{\psi'(s)} \frac{\mathrm{d}}{\mathrm{d}s} \right) \int_0^s (\psi(s) - \psi(\tau))^{-\alpha} \psi'(\tau) f(\tau) \mathrm{d}\tau,\ s > 0,\ 0 < \alpha < 1.$$

Remark 2.1. When $\psi(t) = t$, the operators $I^{\alpha,\psi}$ and $D^{\alpha,\psi}$ turn into the fractional integral operator I^{α} and the Riemann-Liouville type fractional derivative operator D^{α} introduced in [27], respectively.

Then, we compile the concepts of fraction resolvent and ψ-Laplace transform, respectively.

Definition 2.3. [3] The fractional resolvent $\{S_\alpha(t)\}_{t>0} \subseteq \mathscr{L}(H)$ is the family satisfying that $S_\alpha(\cdot)x \in C(\mathbb{R}^+, H)$ for $x \in H$ and there exists $\omega > 0$ to guarantee that for $x \in H$,

$$(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\mathrm{d}t, \ Re\lambda > \omega.$$

Definition 2.4. [21] The ψ-Laplace transform of a function f is

$$\mathcal{L}_\psi[f](\lambda) = \int_0^\infty e^{-\lambda\psi(s)}\psi'(s)f(s)\mathrm{d}s.$$

By utilizing a well-known one-side stable probability density $\omega_\alpha(s)$ [42] and an important Wright type function $\Psi_\alpha(s) = \frac{1}{\alpha}s^{-1-1/\alpha}\omega_\alpha(s^{-1/\alpha})$ [23], we can easily frame the following fractional resolvent:

Lemma 2.1. [38, 39] *Let A generate an analytic semigroup $\{T(s)\}_{s\geq 0}$. Then $\{S_\alpha(s)\}_{s>0}$ is a fractional resolvent satisfying that $\{s^{1-\alpha}S_\alpha(s)\}_{s\geq 0}$ is equicontinuous, where $S_\alpha(s) = s^{\alpha-1}\int_0^\infty \alpha\tau\Psi_\alpha(\tau)T(s^\alpha\tau)\mathrm{d}\tau$.*

Afterward, by exploring the ensuing auxiliary abstract linear system, we propose the notion of mild solution to (1.1) and the definition of exact null controllability.

$$\begin{cases} D^{\alpha,\psi}x(t) = Ax(t) + f(t) + Bu(t), \ t \in J', \\ I^{1-\alpha,\psi}x(t)|_{t=0} = x_0, \end{cases} \tag{2.1}$$

where $f \in L^2(J, H)$.

Due to (2.1), we acquire

$$x(t) = \frac{\psi^{\alpha-1}(t)}{\Gamma(\alpha)}x_0 + I^{\alpha,\psi}Ax(t) + I^{\alpha,\psi}f(t) + I^{\alpha,\psi}Bu(t).$$

An application of the ψ-Laplace transform approach gives

$$\mathcal{L}_\psi[x](\lambda) = (\lambda^\alpha I - A)^{-1}x_0 + (\lambda^\alpha I - A)^{-1}(\mathcal{L}_\psi[f](\lambda) + \mathcal{L}_\psi[Bu](\lambda)).$$

Thereby, by utilizing the inverse ψ-Laplace transform method, we receive

$$x(t) = S_\alpha(\psi(t))x_0 + \int_0^t S_\alpha(\psi(t) - \psi(\tau))\psi'(\tau)(f(\tau) + Bu(\tau))\mathrm{d}\tau, \ t \in J'.$$

Since $\lim\limits_{s\to 0^+} \widetilde{x}(s) = x_0$ implies $I^{1-\alpha,\psi}x(s)|_{s=0} = x_0$, we thus can formulate the following concept:

Definition 2.5. The mild solution to system (1.1) related to u is the function x satisfying $\widetilde{x} \in C([-r, b], H)$, $x|_J \in C_{\alpha,\psi}(J, H)$, $\widetilde{x}(s) = \phi(s) + g(x)(s), \ s \in [-r, 0]$, and for $s \in J'$,

$$x(s) = S_\alpha(\psi(s))(\phi(0) + g(x)(0)) + \int_0^s S_\alpha(\psi(s) - \psi(\tau))\psi'(\tau)(F(\tau, \widetilde{x}_\tau) + Bu(\tau))\mathrm{d}\tau.$$

Remark 2.2. The reason of utilizing the weighted delay initial condition lies in the fact that Riemann-Liouville systems possess the singularity and the advantage of employing it is to guarantee the continuity of $\widetilde{x}$ at zero. Moreover, for convenience, we set

$$S(u) = \{x : x \text{ is the solution to system (1.1) related to } u\}.$$

Now, we designate

$$L_0^b u = \int_0^b S_\alpha(\psi(b) - \psi(\tau))\psi'(\tau)Bu(\tau)\mathrm{d}\tau,\ u \in L^2(J,U),$$

$$N_0^b(x_0, f) = S_\alpha(\psi(b))x_0 + \int_0^b S_\alpha(\psi(b) - \psi(\tau))\psi'(\tau)f(\tau)\mathrm{d}\tau,\ (x_0, f) \in H \times L^2(J,H).$$

Definition 2.6. [8, 14] System (2.1) is exactly null controllable on J, if for all $x \in H$, there is a $\gamma > 0$ to ensure that $\|(L_0^b)^* x\|^2 \geq \gamma \|(N_0^b)^* x\|^2$ or $\mathrm{Im} L_0^b \supset \mathrm{Im} N_0^b$.

Lemma 2.2. *Let L_0 be the restriction of L_0^b to $[\ker L_0^b]^\perp$ and set $W = (L_0)^{-1} N_0^b$: $H \times L^2(J,H) \to L^2(J,U)$. If system (2.1) is exactly null controllable, then H is a bounded linear operator and the control function*

$$u(t) = -W(x_0, f)(t) = -(L_0)^{-1}\left(S_\alpha(\psi(b))x_0 + \int_0^b S_\alpha(\psi(b) - \psi(\tau))\psi'(\tau)f(\tau)\mathrm{d}\tau\right)(t)$$

drives $x(t)$ from x_0 to 0.

Proof. By the similar analysis of [9], we can easily establish this criterion related to the control function. □

Definition 2.7. [9] System (1.1) is exactly null controllable on J, if there is a state-control pair (x, u) related to $u \in L^2(J,U)$ and $x \in S(u)$ to guarantee that $x(b) = 0$.

In the end, to fulfil our aim, we require the following crucial preparations:

Theorem 2.1. [39] *Let B be a bounded equicontinuous subset of $C_{\alpha,\psi}(J,H)$. Then B is relatively weak sequentially compact in $C_{\alpha,\psi}(J,H)$.*

Remark 2.3. $x_n \rightharpoonup x$ in $C_{\alpha,\psi}(J,H)$ means $\widetilde{x_n} \rightharpoonup \widetilde{x}$ in $C(J,H)$, which implies that $\psi^{1-\alpha}(t)x_n(t) \rightharpoonup \psi^{1-\alpha}(t)x(t)$ for $t \in J$ and $x_n(t) \rightharpoonup x(t)$ for $t \in J'$.

Theorem 2.2. [2] *Let E be a convex closed subset of H, $T : [0,1] \times E \to H$ be a compact map with a closed graph and $T(0, E) \subset \mathring{E}$. If $T(\lambda, \cdot)$ does not admit fixed points on ∂E for all $\lambda \in [0, 1)$, then we possess $y \in E$ satisfying $T(1, y) = y$.*

3. Exact null controllability

We here address the exact null controllability problem of system (1.1) without involving the compactness assumption and the Lipschitz restriction condition or the noncompactness measure condition. To deal with it, we need the following prerequisites:

(HA) $\{T(s)\}_{s\geq 0}$ is analytic with $\|T(s)\| \leq M < +\infty$.

(HF) $F : J \times C([-r, 0], H) \to H$ satisfies

(i) For every $\phi \in C([-r,0],H)$, $F(\cdot,\phi) : J \to H$ is measurable;
(ii) For a.e. $\tau \in J$, $F(\tau,\cdot) : C([-r,0],H) \to H$ is weak-to-weak continuous;
(iii) For every $\phi \in C([-r,0],H)$ and $\tau \in J$, $\|F(\tau,\phi)\| \leq \rho(\tau)(1+\|\phi\|_{[-r,0]})$ with $\rho \in L^2(J,\mathbb{R}^+)$.

(Hg) $g \in \mathscr{L}(C_{\alpha,\psi}(J,H), C([-r,0],H))$ and $\|g(y)\|_{[-r,0]} \leq L\|y\|_{\alpha,\psi}$ with $L \geq 0$, for every $y \in C_{\alpha,\psi}(J,H)$.

(Hl) The linear system (2.1) is exactly null controllable.

Remark 3.1. To facilitate our subsequent exploration, we firstly emphasize the following important facts:

(a) By (HA) and Lemma 2.1, a fractional resolvent $\{S_\alpha(s)\}_{s>0}$ can be generated by A. Moreover, for any $s \in J$, $\|\Gamma(\alpha)\psi^{1-\alpha}(s)S_\alpha(\psi(s))\| \leq M$ and $\{s^{1-\alpha}S_\alpha(s)\}_{s>0}$ is equicontinuous.

(b) $A^{(n)}$, the Yosida approximation of A, can generate a contraction semigroup $\{e^{sA^{(n)}}\}_{s>0}$. Thus, a fractional resolvent (denoted by $\{S^n_\alpha(s)\}_{s>0}$) can also be generated by $A^{(n)}$.

(c) Put $A^{(n)}_m = \mathbb{P}_m A^{(n)} : H_m \to H_m$. Thanks to the boundedness of $A^{(n)}_m$, $A^{(n)}_m$ can generate a semigroup. Thus, a fractional resolvent (written $\{S^{mn}_\alpha(s)\}_{s>0}$) can also be generated by $A^{(n)}_m$.

(d) As is known to all, A^*, the adjoint of A, can generate a semigroup. Let $A^{(n)*} = nA^*(nI-A^*)^{-1}$. $A^{(n)*}$ can also generate a semigroup. Thus, $A^{(n)*}$ and A^* can, respectively, generate resolvents $\{S^{n*}_\alpha(s)\}_{s>0}$ and $\{S^*_\alpha(s)\}_{s>0}$. Furthermore, we have $\|S^{n*}_\alpha(s)x - S^*_\alpha(s)x\| \to 0$ for $x \in H$.

(e) Due to Lemma 2.1, one can suppose that $\|\Gamma(\alpha)\psi^{1-\alpha}(s)S^n_\alpha(\psi(s))\| \leq M$ and $\|\Gamma(\alpha)\psi^{1-\alpha}(s)S^{mn}_\alpha(\psi(s))\| \leq M$, for any $s \in J$.

Lemma 3.1. *The resolvents $\{S_\alpha(s)\}_{s>0}$, $\{S^n_\alpha(s)\}_{s>0}$, $\{S^{mn}_\alpha(s)\}_{s>0}$ established in Remark 3.1 possess the following features:*

(a) $\widetilde{S^n_\alpha}(\psi(s))x \to \widetilde{S_\alpha}(\psi(s))x,\ n \to \infty$*, for any $x \in H$ and $s \in J$.*

(b) $\|\widetilde{S^{mn}_\alpha}(\psi(s))\mathbb{P}_m - \widetilde{S^n_\alpha}(\psi(s))\| \to 0,\ m \to \infty$*, uniformly for $s \in J$.*

Proof. (a) Owing to Remark 3.1, we can receive $e^{sA^{(n)}}x \to T(s)x$, for any $x \in H$. We thus can conclude from Lemma 2.1 that

$$\Gamma(\alpha)\psi^{1-\alpha}(s)S^n_\alpha(\psi(s))x \to \Gamma(\alpha)\psi^{1-\alpha}(s)S_\alpha(\psi(s))x.$$

(b) By the discussion in Lemma 3.4 of [38], one can easily receive this assertion. □

Lemma 3.2. *Let (HA) hold, $h \in L^2(J,H)$ and $\Lambda : L^2(J,H) \to C_{\alpha,\varphi}(J,H)$ be a map described by*

$$(\Lambda h)(\cdot) = \int_0^{\cdot} S_\alpha(\psi(\cdot)-\psi(s))\psi'(s)h(s)\mathrm{d}s.$$

Then Λ is equicontinuous.

Proof. Let $t_1, t_2 \in J$ with $0 \leq t_1 < t_2 \leq b$. If $t_1 = 0$, in light of the condition of ψ, we get

$$\begin{aligned}&\Gamma(\alpha)\psi^{1-\alpha}(t_2)\|(\Lambda h)(t_2)\|\\ &\leq M\psi^{1-\alpha}(t_2)\int_0^{t_2}(\psi(t_2)-\psi(s))^{\alpha-1}\psi'(s)\|h(s)\|\mathrm{d}s\end{aligned}$$

$$\leq M\psi^{1-\alpha}(t_2)\sqrt{\|\psi\|_{C^1}}\left(\int_0^{t_2}(\psi(t_2)-\psi(s))^{2\alpha-2}\psi'(s)\mathrm{d}s\right)^{1/2}\|h\|_{L^2}$$
$$\leq \sqrt{\frac{\psi(t_2)\|\psi\|_{C^1}}{2\alpha-1}}M\|h\|_{L^2}.$$

We thus arrive at $\lim\limits_{t_2\to 0^+}\Gamma(\alpha)\psi^{1-\alpha}(t_2)\|(\Lambda h)(t_2)\|=0$.

If $t_1>0$, for abbreviation, we set

$$\widetilde{S_\alpha}(\psi(t)-\psi(s))=\Gamma(\alpha)(\psi(t)-\psi(s))^{1-\alpha}S_\alpha(\psi(t)-\psi(s)),$$
$$\Phi(t_2,t_1)=\widetilde{S}_\alpha(\psi(t_2)-\psi(s))-\widetilde{S}_\alpha(\psi(t_1)-\psi(s)),$$
$$g(t_2,t_1)=(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}.$$

Let $0<\varepsilon<t_1$. We derive

$$\begin{aligned}
&\|\Gamma(\alpha)\psi^{1-\alpha}(t_2)(\Lambda h)(t_2)-\Gamma(\alpha)\psi^{1-\alpha}(t_1)(\Lambda h)(t_1)\|\\
\leq{}&\sqrt{\frac{\psi^{2\alpha-1}(b)\|\psi\|_{C^1}}{2\alpha-1}}M\|h\|_{L^2}\Big(\psi^{1-\alpha}(t_2)-\psi^{1-\alpha}(t_1)\Big)\\
&+\Gamma(\alpha)\psi^{1-\alpha}(b)\int_{t_1}^{t_2}\|S_\alpha(\psi(t_2)-\psi(s))\psi'(s)h(s)\|\mathrm{d}s\\
&+\psi^{1-\alpha}(b)\left\|\int_0^{t_1-\varepsilon}\Phi(t_2,t_1)(\psi(t_2)-\psi(s))^{\alpha-1}\psi'(s)h(s)\mathrm{d}s\right\|\\
&+\psi^{1-\alpha}(b)\left\|\int_{t_1-\varepsilon}^{t_1}\Phi(t_2,t_1)(\psi(t_2)-\psi(s))^{\alpha-1}\psi'(s)h(s)\mathrm{d}s\right\|\\
&+\psi^{1-\alpha}(b)\left\|\int_0^{t_1}\widetilde{S_\alpha}(\psi(t_1)-\psi(s))g(t_2,t_1)\psi'(s)h(s)\mathrm{d}s\right\|\\
\leq{}&\sqrt{\frac{\psi^{2\alpha-1}(b)\|\psi\|_{C^1}}{2\alpha-1}}M\|h\|_{L^2}\Big(\psi^{1-\alpha}(t_2)-\psi^{1-\alpha}(t_1)\Big)\\
&+\sqrt{\frac{(\psi(t_2)-\psi(t_1))^{2\alpha-1}\|\psi\|_{C^1}}{2\alpha-1}}M\|h\|_{L^2}\psi^{1-\alpha}(b)\\
&+\sup_{s\in[0,t_1-\varepsilon]}\Big\|\Phi(t_2,t_1)\Big\|\sqrt{\frac{\psi(b)\|\psi\|_{C^1}}{2\alpha-1}}\|h\|_{L^2}\\
&+2M\psi^{1-\alpha}(b)\|h\|_{L^2}\sqrt{\frac{(\psi(t_2)-\psi(t_1-\varepsilon))^{2\alpha-1}\|\psi\|_{C^1}}{2\alpha-1}}\\
&+\psi^{1-\alpha}(b)M\|h\|_{L^2}\left(\int_0^{t_1}\big(g(t_2,t_1)\psi'(s)\big)^2\mathrm{d}s\right)^{1/2}.
\end{aligned}$$

Hence, a joint combination of Remark 3.1, the arbitrariness of ε and 2-mean continuity (see Problem 23.9 in [35]) enables us to acquire the equicontinuity of Λ. □

We now focus on the exact null controllability problem.

Theorem 3.1. *Under assumptions (HA), (HF), (Hg) and (Hl), system (1.1) is exactly null controllable if*

$$ML+\frac{M\|\psi\|_{C^1}}{\sqrt{2\alpha-1}}\big(\|B\|\|W\|(L+\|\rho\|_{L^2}(L+1))+\|\rho\|_{L^2}(L+1)\big)<1. \tag{3.1}$$

Proof. The proof mainly consists in the construction of $u \in L^2(J,U)$ and $x \in S(u)$ to guarantee that $x(b)=0$. Below, we shall construct the control function u and the mild solution $x \in S(u)$ by the approximation solvability trick. For clarity, we proceed in four procedures.

Step 1. Let $R>0$, $q \in Q^{(m)}$ and $\lambda \in [0,1]$. We start with an auxiliary operator $\Sigma : Q^{(m)} \times [0,1] \to C_{\alpha,\psi}(J,H_m)$ described by

$$\begin{aligned}\Sigma(q,\lambda)(s) &= \lambda S_\alpha^{mn}(\psi(s))\mathbb{P}_m(\phi(0)+g(q)(0))\\ &\quad + \lambda \int_0^s S_\alpha^{mn}(\psi(s)-\psi(\tau))\psi'(\tau)\mathbb{P}_m(Bu_q(\tau)+F(\tau,\widetilde{q}_\tau))\mathrm{d}\tau,\\ u_q(s) &= -W(\phi(0)+g(q)(0),F(\tau,\widetilde{q}_\tau))(s)\\ &= -(L_0)^{-1}\bigg(S_\alpha(\psi(b))(\phi(0)+g(q)(0))\\ &\quad + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{q}_\tau)\mathrm{d}\tau\bigg)(s).\end{aligned}$$

We shall verify that Σ meets the restriction conditions of Theorem 2.2.

Firstly, it is evident that $\widetilde{\Sigma}(q,0)=0 \in \mathring{Q}^{(m)}$.

We then examine the compactness of Σ. Put

$$\begin{aligned}\widetilde{\Sigma}(Q^{(m)}\times[0,1]) = \bigcup_{q\in Q^{(m)},\lambda\in[0,1]} &\Gamma(\alpha)\psi^{1-\alpha}(\cdot)\lambda\bigg(S_\alpha^{mn}(\psi(\cdot))\mathbb{P}_m(\phi(0)+g(q)(0))\\ &+\int_0^{\cdot} S_\alpha^{mn}(\psi(\cdot)-\psi(\tau))\psi'(\tau)\mathbb{P}_m(Bu_q(\tau)+F(\tau,\widetilde{q}_\tau))\mathrm{d}\tau\bigg).\end{aligned}$$

Because of (HF) and (Hg), we acquire that

$$\begin{aligned}\|\widetilde{q}_\tau\|_{[-r,0]} &= \sup_{\theta\in[-r,0]}\|\widetilde{q}_\tau(\theta)\|\\ &\le \sup_{s\in[-r,0]}\|\widetilde{q}(s)\| + \sup_{s\in J}\|\widetilde{q}(s)\|\\ &\le \|\phi\|_{[-r,0]} + LR + R\end{aligned}$$

and

$$\begin{aligned}\|u_q\|_{L^2} &= \bigg(\int_0^b \|W(\phi(0)+g(q)(0),F(\tau,\widetilde{q}_\tau))\|^2\mathrm{d}\tau\bigg)^{1/2}\\ &\le \|W\|\big(\|\phi(0)\|+\|g(q)(0)\|+\|F(\tau,\widetilde{q}_\tau)\|_{L^2}\big)\\ &\le \|W\|\big(\|\phi(0)\|+LR+\|\rho\|_{L^2}(1+\|\phi\|_{[-r,0]}+LR+R)\big)\\ &:= k.\end{aligned}$$

Accordingly, we have

$$\begin{aligned}&\|\widetilde{\Sigma}(q,\lambda)(t)\|\\ &\le M(\|\phi(0)\|+LR) + M\psi^{1-\alpha}(b)\int_0^t(\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\mathbb{P}_mBu_q(\tau)\mathrm{d}\tau\\ &\quad + M\psi^{1-\alpha}(b)\int_0^t(\psi(t)-\psi(\tau))^{\alpha-1}\psi'(\tau)\mathbb{P}_mF(\tau,\widetilde{q}_\tau)\mathrm{d}\tau\end{aligned}$$

$$\leq M\bigg(\|\phi(0)\| + LR + \frac{\|\psi\|_{C^1}}{\sqrt{2\alpha-1}}\big(\|B\|k + \|\rho\|_{L^2}(1+\|\phi\|_{[-r,0]} + (L+1)R)\big)\bigg). \quad (3.2)$$

This yields the boundedness of $\widetilde{\Sigma}(Q^{(m)} \times [0,1])$. In addition, Lemma 3.2 can yield the equicontinuity of $\widetilde{\Sigma}(Q^{(m)} \times [0,1])$. Hence, we obtain the compactness of Σ.

Next, we check that Σ possesses a closed graph. Let $\{q_k\} \subseteq Q^{(m)}$ with $q_k \to q$ and $\lambda_k \subseteq [0,1]$ with $\lambda_k \to \lambda$. Due to $q_k \to q$ and (Hg), we derive $g(q_k)(s) \to g(q)(s)$ for any $s \in [-r,0]$ and $\lim\limits_{k\to\infty} \widetilde{q}_k(s) \to \widetilde{q}(s)$ for any $s \in J$. For any $s \in [-r,0]$, we thus have

$$\widetilde{q}_k(s) = \phi(s) + g(q_k)(s) \to \phi(s) + g(q)(s) = \widetilde{q}(s).$$

As a result, we receive $\widetilde{q}_k(s) \to \widetilde{q}(s)$ for any $s \in [-r,b]$, which indicates that $\widetilde{q}_{k\tau} \to \widetilde{q}_\tau$ for $\tau \in [0,s],\ s \in J'$. As such, (HF) forces that

$$F(\tau, \widetilde{q}_{k\tau}) \rightharpoonup F(\tau, \widetilde{q}_\tau) \text{ and } \mathbb{P}_m F(\tau, \widetilde{q}_{k\tau}) \to \mathbb{P}_m F(\tau, \widetilde{q}_\tau).$$

Thereby, for any $(z,h) \in (H \times L^2(J,H))^* = H^* \times (L^2(J,H))^*$, we obtain

$$\begin{aligned}
&(z,h)(\phi(0) + g(q_k)(0), F(\tau, \widetilde{q}_{k\tau}))\\
=\ & z(\phi(0) + g(q_k)(0)) + h(F(\tau, \widetilde{q}_{k\tau}))\\
=\ & z(\phi(0)) + z(g(q_k)(0)) + h(F(\tau, \widetilde{q}_{k\tau}))\\
\to\ & z(\phi(0)) + z(g(q)(0)) + h(F(\tau, \widetilde{q}_\tau))\\
=\ & (z,h)(\phi(0) + g(q)(0), F(\tau, \widetilde{q}_\tau)),
\end{aligned}$$

which shows that $(\phi(0) + g(q_k)(0), F(\tau, \widetilde{q}_{k\tau})) \rightharpoonup (\phi(0) + g(q)(0), F(\tau, \widetilde{q}_\tau))$. Since W is linear and bounded, W is weakly continuous. Therefore we can assert that

$$H(\phi(0) + g(q_k)(0), F(\tau, \widetilde{q}_{k\tau})) \rightharpoonup H(\phi(0) + g(q)(0), F(\tau, \widetilde{q}_\tau)),$$

that is $u_{q_k}(s) \rightharpoonup u_q(s)$. Consequently, we acquire

$$\begin{aligned}
&\big\|\widetilde{\Sigma}(q_k,\lambda_k)(s) - \widetilde{\Sigma}(q,\lambda)(s)\big\|\\
=\ & \Gamma(\alpha)\psi^{1-\alpha}(s)\big\|\Sigma(q_k,\lambda_k)(s) - \Sigma(q,\lambda)(s)\big\|\\
\leq\ & |\lambda_k - \lambda|\|\widetilde{\Sigma}(q,1)(s)\| + M\|\mathbb{P}_m g(q_k)(0) - \mathbb{P}_m g(q)(0)\|\\
&+ M\psi^{1-\alpha}(b)\int_0^s (\psi(s)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|\mathbb{P}_m(F(\tau,q_{k\tau}) - F(\tau,q_\tau))\|\mathrm{d}\tau\\
&+ M\psi^{1-\alpha}(b)\int_0^s (\psi(s)-\psi(\tau))^{\alpha-1}\psi'(\tau)\|\mathbb{P}_m(u_{q_k}(\tau) - u_q(\tau))\|\mathrm{d}\tau,
\end{aligned}$$

which indicates that Σ admits a closed graph.

In the end, we show that $\Sigma(\cdot,\lambda)$ does not possess fixed points on $\partial Q^{(m)}$. Let $q = \Sigma(q,\lambda)$ and $\lambda \in (0,1)$. Due to (3.1) and (3.2), we can choose some R satisfying

$$M\bigg(\|\phi(0)\| + LR + \frac{\|\psi\|_{C^1}}{\sqrt{2\alpha-1}}\big(\|B\|k + \|\rho\|_{L^2}(1+\|\phi\|_{[-r,0]} + (L+1)R)\big)\bigg) < R,$$

which enables us to conclude that there is no $q \in \partial Q^{(m)}$ such that $q = \Sigma(q,\lambda)$.

Thereby, Theorem 2.2 can force that $q = \Sigma(q, 1)$ possesses a fixed point x^{mn}, that is

$$\begin{aligned}
x^{mn}(s) &= S_\alpha^{mn}(\psi(s))\mathbb{P}_m(\phi(0) + g(x^{mn})(0)) \\
&\quad + \int_0^s S_\alpha^{mn}(\psi(s) - \psi(\tau))\psi'(\tau)\mathbb{P}_m(Bu_{x^{mn}}(\tau) + F(\tau, \widetilde{x_\tau^{mn}}))\mathrm{d}\tau, \\
u_{x^{mn}}(s) &= -W(\phi(0) + g(x^{mn})(0), F(\tau, \widetilde{x_\tau^{mn}}))(s) \\
&= -(L_0)^{-1}\bigg(S_\alpha(\psi(b))(\phi(0) + g(x^{mn})(0)) \\
&\quad + \int_0^b S_\alpha(\psi(b) - \psi(\tau))\psi'(\tau)F(\tau, \widetilde{x_\tau^{mn}})\mathrm{d}\tau\bigg)(s).
\end{aligned}$$

Step 2. We claim that $\{x^{mn}\}_{m>0}$ established in Step 1 admits a subsequence (still relabeled as $\{x^{mn}\}_{m>0}$) satisfying that $x^{mn} \rightharpoonup x^n$, $m \to \infty$, where

$$\begin{aligned}
x^n(s) &= S_\alpha^n(\psi(s))(\phi(0) + g(x^n)(0)) \\
&\quad + \int_0^s S_\alpha^n(\psi(s) - \psi(\tau))\psi'(\tau)(Bu_{x^n}(\tau) + F(\tau, \widetilde{x_\tau^n}))\mathrm{d}\tau, \\
u_{x^n}(s) &= -W(\phi(0) + g(x^n)(0), F(\tau, \widetilde{x_\tau^n}))(s) \\
&= -(L_0)^{-1}\bigg(S_\alpha(\psi(b))(\phi(0) + g(x^n)(0)) \\
&\quad + \int_0^b S_\alpha(\psi(b) - \psi(\tau))\psi'(\tau)F(\tau, \widetilde{x_\tau^n})\mathrm{d}\tau\bigg)(s).
\end{aligned} \tag{3.3}$$

Since Step 1 gives the equicontinuity and boundedness of $\{x^{mn}\}_{m>0}$, we can assume, from Theorem 2.1, that $x^{mn} \rightharpoonup x^n$, $m \to \infty$, in $C_{\alpha,\psi}(J, H)$. Thus, we have $\widetilde{x^{mn}}(s) \rightharpoonup \widetilde{x^n}(s)$ for $s \in J$. Because of (Hg), g is weak continuous. Hence, we can derive $g(x^{mn})(s) \rightharpoonup g(x^n)(s)$ and

$$\widetilde{x^{mn}}(s) = \phi(s) + g(x^{mn})(s) \rightharpoonup \phi(s) + g(x^n)(s) = \widetilde{x^n}(s)$$

for $s \in [-r, 0]$. Thereby, we receive $\widetilde{x^{mn}}(s) \rightharpoonup \widetilde{x^n}(s)$ for $s \in [-r, b]$, which implies that $\widetilde{x_\tau^{mn}}(\theta) \rightharpoonup \widetilde{x_\tau^n}(\theta)$ for $\theta \in [-r, 0]$. Due to (3.2) and

$$\begin{aligned}
\|\widetilde{x_\tau^{mn}}\|_{[-r,0]} &= \sup_{\theta\in[-r,0]} \|\widetilde{x^{mn}}(\tau + \theta)\| \\
&\le \sup_{s\in[-r,0]} \|\widetilde{x^{mn}}(s)\| + \sup_{s\in[0,\tau]} \|\widetilde{x^{mn}}(s)\| \\
&\le \|\phi\|_{[-r,0]} + LR + R,
\end{aligned}$$

we get $\widetilde{x_\tau^{mn}} \rightharpoonup \widetilde{x_\tau^n}$. Applying (HF) leads to $F(\tau, \widetilde{x_\tau^{mn}}) \rightharpoonup F(\tau, \widetilde{x_\tau^n})$. Arguments similar to that in Step 1 can give

$$W\big(\phi(0) + g(x^{mn})(0), F(\tau, \widetilde{x_\tau^{mn}})\big) \rightharpoonup W\big(\phi(0) + g(x^n)(0), F(\tau, \widetilde{x_\tau^n})\big),$$

that is, $u_{x^{mn}}(s) \rightharpoonup u_{x^n}(s)$. Therefore, we can assert from Lemma 3.1 that

$$\|\widetilde{S_\alpha^{mn}}(\psi(s))\mathbb{P}_m(\phi(0) + g(x^{mn})(0)) - \widetilde{S_\alpha^n}(\psi(s))(\phi(0) + g(x^n)(0))\|$$

$$
\begin{aligned}
&\le \|\widetilde{S_\alpha^{mn}}(\psi(s))\big(\mathbb{P}_m(\phi(0)+g(x^{mn})(0))-\mathbb{P}_m(\phi(0)+g(x^n)(0))\big)\| \\
&\quad+\|\widetilde{S_\alpha^{mn}}(\psi(s))\mathbb{P}_m(\phi(0)+g(x^n)(0))-\widetilde{S_\alpha^n}(\psi(s))(\phi(0)+g(x^n)(0))\| \\
&\le M\|\mathbb{P}_m(\phi(0)+g(x^{mn})(0))-\mathbb{P}_m(\phi(0)+g(x^n)(0))\| \\
&\quad+\|\widetilde{S_\alpha^{mn}}(\psi(s))\mathbb{P}_m(\phi(0)+g(x^n)(0))-\widetilde{S_\alpha^n}(\psi(s))(\phi(0)+g(x^n)(0))\| \\
&\to 0,\ m\to\infty.
\end{aligned}
$$

Likewise, as $m\to\infty$, we can derive

$$
\|\widetilde{S_\alpha^{mn}}(\psi(s)-\psi(\tau))\mathbb{P}_m F(\tau,\widetilde{x_\tau^{mn}})-\widetilde{S_\alpha^n}(\psi(s)-\psi(\tau))F(\tau,\widetilde{x_\tau^n})\|\to 0
$$

and

$$
\|\widetilde{S_\alpha^{mn}}(\psi(s)-\psi(\tau))\mathbb{P}_m Bu_{x^{mn}}(s)-\widetilde{S_\alpha^n}(\psi(s)-\psi(\tau))Bu_{x^n}(s)\|\to 0.
$$

Therefore, the uniqueness of the weak limit and the dominated convergence theorem enable us to acquire (3.3).

Step 3. We examine that $\{x^n\}_{n>0}$ constructed in Step 2 possesses a subsequence (still written $\{x^n\}_{n>0}$) satisfying that $x^n\rightharpoonup x$, $n\to\infty$, where

$$
\begin{aligned}
x(s)&=S_\alpha(\psi(s))(\phi(0)+g(x)(0)) \\
&\quad+\int_0^s S_\alpha(\psi(s)-\psi(\tau))\psi'(\tau)(Bu_x(\tau)+F(\tau,\widetilde{x_\tau}))\mathrm{d}\tau,
\end{aligned}\tag{3.4}
$$

$$
\begin{aligned}
u_x(s)&=-W(\phi(0)+g(x)(0),F(\tau,\widetilde{x_\tau}))(s) \\
&=-(L_0)^{-1}\bigg(S_\alpha(\psi(b))(\phi(0)+g(x)(0)) \\
&\qquad+\int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{x_\tau})\mathrm{d}\tau\bigg)(s).
\end{aligned}\tag{3.5}
$$

Thanks to (3.2) and Lemma 3.2, we can receive the equicontinuity and boundedness of $\{x^n\}_{n>0}$. we can suppose, from Theorem 2.1, that, $x^n\rightharpoonup x,\ n\to\infty$. Utilizing similar reasoning as in Step 2 can give $g(x^n)(0)\rightharpoonup g(x)(0)$, $u_{x^n}(\tau)\rightharpoonup u_x(\tau)$ and $F(\tau,\widetilde{x_\tau^n})\rightharpoonup F(\tau,\widetilde{x_\tau})$. Based on Remark 3.1 and Lemma 3.1, for any $z\in H$, we receive

$$
\begin{aligned}
&\big\langle z,\widetilde{S_\alpha^n}(\psi(s))(\phi(0)+g(x^n)(0))-\widetilde{S_\alpha}(\psi(s))(\phi(0)+g(x)(0))\big\rangle \\
=&\ \big\langle z,\widetilde{S_\alpha^n}(\psi(s))\phi(0)-\widetilde{S_\alpha}(\psi(s))\phi(0)\big\rangle \\
&+\big\langle \widetilde{S_\alpha^{n*}}(\psi(s))z-\widetilde{S_\alpha^*}(\psi(s))z,g(x^n)(0)-g(x)(0)\big\rangle \\
&+\big\langle \widetilde{S_\alpha^*}(\psi(s))z,g(x^n)(0)-g(x)(0)\big\rangle \\
&+\big\langle z,\widetilde{S_\alpha^n}(\psi(s))g(x)(0)-\widetilde{S_\alpha}(\psi(s))g(x)(0)\big\rangle \\
\to&\ 0,
\end{aligned}
$$

which yields $\widetilde{S_\alpha^n}(\psi(s))(\phi(0)+g(x^n)(0))\rightharpoonup\widetilde{S_\alpha}(\psi(s))(\phi(0)+g(x)(0))$. Similarly, we can assert that

$$
\int_0^s S_\alpha^n(\psi(s)-\psi(\tau))\psi'(\tau)Bu_{x^n}(\tau)\mathrm{d}\tau\rightharpoonup\int_0^s S_\alpha(\psi(s)-\psi(\tau))\psi'(\tau)Bu_x(\tau)\mathrm{d}\tau,
$$

$$\int_0^s S_\alpha^n(\psi(s)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{x_\tau^n})\mathrm{d}\tau \rightharpoonup \int_0^s S_\alpha(\psi(s)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{x_\tau})\mathrm{d}\tau.$$

Hence, the uniqueness of the weak limit forces (3.4), that is $x \in S(u_x)$.

Step 4. What is left is to demonstrate $x(b)=0$. Utilizing (3.4) and (3.5) yields

$$\begin{aligned}
x(b) &= S_\alpha(\psi(b))(\phi(0)+g(x)(0)) + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)(Bu_x(\tau)+F(\tau,\widetilde{x_\tau}))\mathrm{d}\tau \\
&= S_\alpha(\psi(b))(\phi(0)+g(x)(0)) + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{x_\tau}))\mathrm{d}\tau \\
&\quad - \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)B(L_0)^{-1}\Big(S_\alpha(\psi(b))(\phi(0)+g(x)(0)) \\
&\quad + \int_0^b S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)F(\tau,\widetilde{x_\tau})\mathrm{d}\tau\Big)(\tau)\mathrm{d}\tau \\
&= 0.
\end{aligned}$$

We thus achieve the exact null controllability of (1.1). □

Remark 3.2. So far, we have offered the construction of the required control function u and the solution $x \in S(u)$ by the approximation solvability trick. Emphasis here is that our result does not involve the compactness and Lipschitz restriction condition.

4. An application

We here take up a significant diffusion model to show the applicability of our abstract outcomes. We opt for the following nonlocal delay system:

$$\begin{cases}
D^{\frac{2}{3},\ln(1+s)}x(s,z) = \Delta x(s,z) + u(s,z) + F(s,z,\widetilde{x}_s(z))), \text{ on } (0,b]\times\Omega, \\
x(s,z) = 0, \text{ on } (0,b)\times\partial\Omega, \\
\widetilde{x}(s,z) = \sum_{i=1}^N c_i\widetilde{x}(s+s_i,z) + \phi(s,z), \quad (s,z)\in[-r,0]\times\Omega,
\end{cases} \tag{4.1}$$

where $D^{\frac{2}{3},\ln(1+s)}$ is a $\frac{2}{3}$-order $\ln(1+s)$-Riemann-Liouville type fractional derivative operator, $s_i \in [r,b]$, $c_i \in \mathbb{R}$, $k=1,2,\cdots,N$, Ω is a bounded region in $\mathbb{R}^n$ $(n \geq 2)$ with a Lipschitz boundary $\partial\Omega$ and for $\theta \in [-r,0]$,

$$\widetilde{x}_s(\theta,z) = \widetilde{x}(s+\theta,z) = \begin{cases}
\Gamma(\alpha)\ln^{\frac{1}{3}}(1+s+\theta)x(s+\theta,z), & s+\theta\in[0,b], \\
\sum_{i=1}^N c_i\widetilde{x}(s+\theta+s_i,z), & s+\theta\in[-r,0].
\end{cases}$$

Let $H=U=L^2(\Omega)$. Designate an operator A:

$$Ax = \Delta x, \ x \in D(A) = H_0^1(\Omega)\cap H^2(\Omega).$$

Then, A can generate a contractive analytic self-adjoint semigroup $\{T(s)\}_{s\geq 0}$ (see [26]):

$$T(s)x = \sum_{n=1}^\infty e^{-\lambda_n s}\langle x,\xi_n\rangle\xi_n,$$

where $\{-\lambda_n, \xi_n\}_{n=1}^{\infty}$ is the eigensystem of A and $\lambda_n > 0$. Then (HA) holds. Moreover, $M = 1$. Due to Lemma 2.1, A can also generate a self-adjoint resolvent $\{S_\alpha(s)\}_{s>0}$:

$$S_\alpha(s)x = \sum_{n=1}^{\infty} s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha)\langle x, \xi_n\rangle \xi_n.$$

Let $\alpha = \frac{2}{3}$, $\psi(s) = \ln(1+s)$, $x(s)(z) = x(s,z)$, $B = I$, $u(s)(z) = u(s,z)$, $\phi(s)(z) = \phi(s,z)$, $u \in L^2([0,b],H)$, $L = \sum\limits_{i=1}^{N} c_i$. Delineate functions

$$f : J \times C([-r,0],H) \to H \text{ and } g : C_{\alpha,\psi}([0,b],H) \to C([-r,0],H)$$

as $F(s,\widetilde{x}_s)(z) = F(s,z,\widetilde{x}_s(z))$ and $g(x)(s)(z) = \sum\limits_{i=1}^{N} c_i \widetilde{x}(s+s_i,z)$, respectively. Then (Hg) holds and system (4.1) can be adapted to the abstract form of (1.1).

Consider the auxiliary linear control system:

$$\begin{cases} D^{\alpha,\psi}x(s,z) = \Delta x(s,z) + u(s,z) + f(s,z), \text{ on } (0,b] \times \Omega, \\ x(s,z) = 0, \text{ on } (0,b) \times \partial\Omega, \\ \widetilde{x}(0,z) = x_0(z), \ z \in \Omega. \end{cases} \tag{4.2}$$

Based on [29], we have

$$\begin{aligned}
&\int_0^b \|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2 \mathrm{d}\tau \\
&= \sum_{k=1}^{\infty} \int_0^b (\psi(b)-\psi(\tau))^{2\alpha-2} E^2_{\alpha,\alpha}(-\lambda_k(\psi(b)-\psi(\tau))^\alpha)\psi'^2(\tau)\mathrm{d}\tau \langle x, \xi_k\rangle^2 \\
&\geq \sum_{k=1}^{\infty} \psi^{2\alpha-2}(b) E^2_{\alpha,\alpha}(-\lambda_k \psi^\alpha(b)) \int_0^b \psi'^2(\tau)\mathrm{d}\tau \langle x, \xi_k\rangle^2 \\
&\geq \frac{\psi^2(b)}{b} \sum_{k=1}^{\infty} \psi^{2\alpha-2}(b) E^2_{\alpha,\alpha}(-\lambda_k \psi^\alpha(b))\langle x, \xi_k\rangle^2 \\
&= \frac{\psi^2(b)}{b} \|S_\alpha(\psi(b))x\|^2,
\end{aligned}$$

which forces

$$\begin{aligned}
&\int_0^b \|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2 \mathrm{d}\tau \\
&\geq \frac{\psi^2(b)}{b+\psi^2(b)} \left(\|S_\alpha(\psi(b))x\|^2 + \int_0^b \|S_\alpha(\psi(b)-\psi(\tau))\psi'(\tau)x\|^2 \mathrm{d}\tau \right).
\end{aligned}$$

Thus, system (4.2) is exactly null controllable. Hence, (Hl) holds.

Hence, by Theorem 3.1, if (HF) holds and

$$L + \frac{\|\psi\|_{C^1}}{\sqrt{2\alpha-1}} \big(\|B\|\|W\|(L + \|\rho\|_{L^2}(L+1)) + \|\rho\|_{L^2}(L+1) \big) < 1,$$

we can achieve the exact null controllability of system (4.1).

5. Conclusion

In the existing findings, with the aid of the compactness of the semigroup or the noncompactness measure condition, many investigators coped with exact controllability problems and exact null controllability problems. In the present paper, by employing the approximation solvability trick and the resolvent technique, we have displayed the exact null controllability result of the ψ-Riemann-Liouville fractional nonlocal weighted delay abstract system (1.1) without involving the compactness of the semigroup or the noncompactness measure condition. Moreover, with the help of Yosida approximation of semigroup, our methods can enable us to deal with exact null controllability problems of fractional evolutional systems with Caputo type derivatives or Hilfer type derivatives.

For our future research, the following issues will continue to be focused on:

(i) We wish to analyze whether our methods are still valid for exact controllability problems of evolution systems.

(ii) We wish to seek some natural conditions to ensure that our methods are still valid for approximate controllability problems of evolution systems.

(iii) Numerical simulation about the theoretical results will be touched in the future work.

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