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THE STUDY ON THE CYCLIC GENERALIZED ANTI-PERIODIC BOUNDARY VALUE PROBLEMS OF THE TRIPLED FRACTIONAL LANGEVIN DIFFERENTIAL SYSTEMS

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Abstract The purpose of this paper is to deal with the cyclic generalized anti-periodic boundary value problems of the tripled fractional Langevin differential systems. By using some fixed theorems, the existence and uniqueness of solutions to the problem have been obtained. Moreover, the Ulam-Hyers stability of the problem has also been presented. Furthermore, some examples are supplied to verify our main results.

Keywords Fractional Langevin differential system, boundary value problem, well-posedness, Ulam-Hyers stability, fixed point.

MSC(2010) 26A33, 34A08, 34B15.

1. Introduction

In this paper, we are concerned with the following generalized anti-periodic boundary value problems of the tripled fractional Langevin differential systems.

$$\begin{cases} {}^{C}D_{0+}^{\beta}\left({}^{C}D_{0+}^{\alpha}+\lambda\right)x_i\left(t\right)=g_i\left(t,x_1\left(t\right),x_2\left(t\right),x_3\left(t\right)\right),t\in(0,1),\ i=1,2,3,\\ a\left(x_1(0)+x_2(0)\right)=-\left(x_2(1)+x_3(1)\right),\\ {}^{C}D_{0+}^{\alpha}x_1\left(0\right)+{}^{C}D_{0+}^{\alpha}x_2\left(0\right)=-\left({}^{C}D_{0+}^{\alpha}x_2\left(1\right)+{}^{C}D_{0+}^{\alpha}x_3\left(1\right)\right),\\ a\left(x_2(0)+x_3(0)\right)=-\left(x_3(1)+x_1(1)\right),\\ {}^{C}D_{0+}^{\alpha}x_2\left(0\right)+{}^{C}D_{0+}^{\alpha}x_3\left(0\right)=-\left({}^{C}D_{0+}^{\alpha}x_3\left(1\right)+{}^{C}D_{0+}^{\alpha}x_1\left(1\right)\right),\\ a\left(x_3(0)+x_1(0)\right)=-\left(x_1(1)+x_2(1)\right),\\ {}^{C}D_{0+}^{\alpha}x_3\left(0\right)+{}^{C}D_{0+}^{\alpha}x_1\left(0\right)=-\left({}^{C}D_{0+}^{\alpha}x_1\left(1\right)+{}^{C}D_{0+}^{\alpha}x_2\left(1\right)\right), \end{cases} \tag{1.1}$$

where ${}^{C}D_{0+}^{\alpha}$ and ${}^{C}D_{0+}^{\beta}$ stand for the Caputo fractional derivative of order α and β with $0<\alpha<1,\ 0<\beta<1,\ 1<\alpha+\beta<2$, $g_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}, i=1,2,3$ represent continuous functions, $a,\ \lambda\in(0,+\infty)$.

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With the continuous development of fractional calculus, the research on the basic theory of fractional differential equation has become more and more popular. The main reason is that the problems of fractional differential equations can greatly describe the real world and has been applied in many practical research fields such as biology, physics, fluid mechanics (see [16, 22, 23, 28]). Therefore, it is meaningful to consider the well-posedness and Ulam-Hyers stability of boundary value problems for fractional differential equations.

It is well-known that the Caputo fractional differential equation is an important part of fractional differential equations. Its boundary value problems have been extensively investigated by many scholars (see [1, 2, 4, 6, 10, 14, 15, 18, 21, 24] and references therein). For example, Ahmad and Nieto [4] considered the existence of solutions for the following fractional anti-periodic boundary value problem via Leray-Schauder degree theory.

$$\begin{cases} {}^{C}D_{0+}^{q}x(t) = f\left(t, x(t)\right), t \in [0, T], \\ x(0) = -x(T),\ x'(0) = -x'(T), \end{cases} \tag{1.2}$$

where $q \in (1, 2]$, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Since the Langevin equation has strong physical significance, which was established by Langevin in 1908 according to Newton's laws (see [11]), its fractional boundary value problems have consequently attracted many scholars' attention. Yu, Deng and Luo [26] investigated the solvability of a class of initial value problem for fractinal Langevin equation via the Leray-Schauder nonlinear alternative as follows.

$$\begin{cases} {}_{0}^{C}D_{t}^{\beta}\left({}_{0}^{C}D_{t}^{\alpha} + \gamma\right) x\left(t\right) = f\left(t, x\left(t\right)\right),\ t \in \left(0, 1\right), \\ x^{k}(0) = \mu_{k},\ 0 \le k < l, \\ x^{\alpha+k}(0) = \nu_{k},\ 0 \le k < n, \end{cases} \tag{1.3}$$

where ${}_{0}^{C}D_{t}^{\beta}$ and ${}_{0}^{C}D_{t}^{\alpha}$ stand for the Caputo fractional derivative of order β and α with $\beta \in (n-1, n], \alpha \in (m-1, m],\ m, n \in \mathbb{N}^{+},\ l = \max\{n, m\},\ f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\gamma \in \mathbb{R}$. Subsequently, Baghani, Alzabut and Nieto [8] dealt with the existence and uniqueness of solutions to the anti-periodic boundary value problems for a coupled system of factional Langevin equation by Banach fixed point theorem.

$$\begin{cases} D^{\beta_1}\left(D^{\alpha_1} + \chi_1\right) x\left(t\right) = f\left(t, x\left(t\right), y\left(t\right)\right),\ t \in \left(0, 1\right), \\ D^{\beta_2}\left(D^{\alpha_2} + \chi_2\right) y\left(t\right) = g\left(t, x\left(t\right), y\left(t\right)\right),\ t \in \left(0, 1\right), \\ x(0) + x(1) = 0,\ D^{\alpha_1}x(0) + D^{\alpha_1}x(1) = 0,\ D^{2\alpha_1}x(0) + D^{2\alpha_1}x(1) = 0, \\ y(0) + y(1) = 0,\ D^{\alpha_2}y(0) + D^{\alpha_2}y(1) = 0,\ D^{2\alpha_2}y(0) + D^{2\alpha_2}y(1) = 0, \end{cases} \tag{1.4}$$

where D^{α_i} and D^{β_i} stand for the Caputo fractional derivative of order α_i and β_i with $\alpha_i \in (0, 1], \beta_i \in (1, 2], i = 1, 2,\ f, g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous, $\chi_1, \chi_2 \in \mathbb{R}$. Furthermore, for more papers related to boundary value problems of fractional Langevin equation, please refer to [5, 12, 13, 17] and references therein.

The cyclic boundary value problem has a far-reaching influence and is widely used in many research fields [3, 7, 9, 20, 25]. The cyclic boundary conditions are particularly prominent in the study of channel flow and has been used to construct railway track coupling dynamics models [7]. Moreover, it can effectively describe the repeated behavior of the fluid on the boundary surface. In general, these boundary conditions help to approximate regions of infinite length to smaller regions. Furthemore, the cyclic boundary conditions also play an important role in the study of lattice particles [3]. In addition, the cyclic boundary value problems are also widely used in the variational principle of Hamiltonian systems [25] and in solving the problems of Schrödinger operator [9].

Recently, the cyclic boundary value problems of fractional differential system have become a hot research topic. Its characteristic is that the equations and boundary conditions are coupled. Hence, compared to decoupling boundary conditions with coupling equation, these types of problems are more complex and challenging(see [3, 19, 27]). For example, Zhang and Ni [27] dealt with a class of the tripled system of fractional Langevin equations with the cyclic anti-periodic boundary value conditions as follows.

$$\begin{cases} {}^CD^{\beta}_{0+}\left({}^CD^{\alpha}_{0+}+\lambda\right)x_i\left(t\right)=f_i\left(t,x_1\left(t\right),x_2\left(t\right),x_3\left(t\right)\right),\ t\in(0,1), i=1,2,3,\\ x_1(0)+x_2(1)=0,\quad {}^CD^{\alpha}_{0+}x_1(0)+{}^CD^{\alpha}_{0+}x_2(1)=0,\\ x_2(0)+x_3(1)=0,\quad {}^CD^{\alpha}_{0+}x_2(0)+{}^CD^{\alpha}_{0+}x_3(1)=0,\\ x_3(0)+x_1(1)=0,\quad {}^CD^{\alpha}_{0+}x_3(0)+{}^CD^{\alpha}_{0+}x_1(1)=0, \end{cases} \tag{1.5}$$

where ${}^CD^{\alpha}_{0+}$ and ${}^CD^{\beta}_{0+}$ stand for the Caputo fractional derivative of order α and β with $0<\alpha<1,\ 0<\beta<1,\ 1<\alpha+\beta<2,\ f_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}, i=1,2,3$ are continuous, $\lambda\in(0,+\infty)$. Based on some fixed point theorems, the well-posedness of solutions to (1.5) are acquired. Furthermore, the Ulam-Hyers and Ulam-Hyers-Rassias stabilities for the problem are also obtained.

Motivated by the works mentioned above, we are concerned with the generalized cyclic anti-periodic boundary conditions to the tripled fractional Langevin differential system (1.1). By Krasnoselskii fixed point theorem and Banach contraction mapping theorem, the well-posedness of solutions to (1.1) has been obtained. Moreover, the Ulam-Hyers stability of (1.1) has also been presented. Let's describe the contributions of this paper as follows.

• Our model includes the special case (1.5). Thus, our main results extend the conclusions of [27].

• Since the equations and boundary conditions are coupled. Therefore, it is more complex and challenging than the case of decoupling boundary conditions with coupling equation.

• Studying the anti-periodic boundary value problem itself is very meaningful. Moreover, there are few papers considering the cyclic boundary value problems of tripled fractional Langevin differential system. Our main results enhance and upon some previous results.

2. Preliminaries

For the classical definitions and properties of Riemann-Liouville fractional integrals and Caputo fractional derivatives, one can refer to [16]. So, we won't repeat it here.

Lemma 2.1 (Krasnoselskii fixed point theorem, [28]). *Assume that X is a Banach space and the nonempty subset $\Omega \subset X$ is bounded, convex and closed. Let $\mathcal{A}$ and $\mathcal{B}$ be two operators satisfying*

(i) $\mathcal{A}x + \mathcal{B}y \in \Omega$ *for all* $x, y \in \Omega$*;*

(ii) $\mathcal{A}$ *is an operator of complete continuity;*

(iii) $\mathcal{B}$ *is a contraction mapping.*

Then there exists $z \in \Omega$ *such that* $z = \mathcal{A}z + \mathcal{B}z$.

Considering the Banach space $\mathbb{X} = C[0,1]$, with the norm defined by $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Let $X = \mathbb{X} \times \mathbb{X} \times \mathbb{X}$ be equipped with the following normal

$$\|(x_1, x_2, x_3)\|_X = \|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty,$$

where $(x_1, x_2, x_3) \in X$. Clearly, $(X, \|\cdot\|_X)$ is also a Banach space.

Lemma 2.2. *For* $i = 1, 2, 3$, *let* $1 < \alpha + \beta < 2$, $\Upsilon_i \in AC[0,1]$. *Then,* $x = (x_1, x_2, x_3) \in X$ *is a solution of the following linear system of integral equations*

$$^{C}D^{\beta}_{0+}(^{C}D^{\alpha}_{0+} + \lambda)x_i(t) = \Upsilon_i(t) \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} &a\left(x_1(0) + x_2(0)\right) = -\left(x_2(1) + x_3(1)\right), \\ &^{C}D^{\alpha}_{0+}x_1\left(0\right) + {^{C}D^{\alpha}_{0+}}x_2\left(0\right) = -\left(^{C}D^{\alpha}_{0+}x_2\left(1\right) + {^{C}D^{\alpha}_{0+}}x_3\left(1\right)\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &a\left(x_2(0) + x_3(0)\right) = -\left(x_3(1) + x_1(1)\right), \\ &^{C}D^{\alpha}_{0+}x_2\left(0\right) + {^{C}D^{\alpha}_{0+}}x_3\left(0\right) = -\left(^{C}D^{\alpha}_{0+}x_3\left(1\right) + {^{C}D^{\alpha}_{0+}}x_1\left(1\right)\right), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &a\left(x_3(0) + x_1(0)\right) = -\left(x_1(1) + x_2(1)\right), \\ &^{C}D^{\alpha}_{0+}x_3\left(0\right) + {^{C}D^{\alpha}_{0+}}x_1\left(0\right) = -\left(^{C}D^{\alpha}_{0+}x_1\left(1\right) + {^{C}D^{\alpha}_{0+}}x_2\left(1\right)\right). \end{aligned} \tag{2.4}$$

The form of x_i *is given by*

$$\begin{aligned} x_i\left(t\right) = &\frac{1}{\Gamma\left(\alpha+\beta\right)} \int_0^t \left(t-s\right)^{\alpha+\beta-1} h_i\left(s\right) ds \\ &- \frac{\lambda}{\Gamma\left(\alpha\right)} \int_0^t \left(t-s\right)^{\alpha-1} x_i\left(s\right) ds \\ &+ \frac{-M_{i1}(t) + M_{i4}(t)\lambda}{\Gamma\left(\alpha+\beta\right)} \int_0^1 \left(1-s\right)^{\alpha+\beta-1} \left(\Upsilon_2\left(s\right) + \Upsilon_3\left(s\right)\right) ds \\ &+ \frac{-M_{i2}(t) + M_{i5}(t)\lambda}{\Gamma\left(\alpha+\beta\right)} \int_0^1 \left(1-s\right)^{\alpha+\beta-1} \left(\Upsilon_3\left(s\right) + \Upsilon_1\left(s\right)\right) ds \\ &+ \frac{-M_{i3}(t) + M_{i6}(t)\lambda}{\Gamma\left(\alpha+\beta\right)} \int_0^1 \left(1-s\right)^{\alpha+\beta-1} \left(\Upsilon_1\left(s\right) + \Upsilon_2\left(s\right)\right) ds \\ &+ \frac{M_{i1}(t)\lambda - M_{i4}(t)\lambda^2}{\Gamma\left(\alpha\right)} \int_0^1 \left(1-s\right)^{\alpha-1} \left(x_2\left(s\right) + x_3\left(s\right)\right) ds \\ &+ \frac{M_{i2}(t)\lambda - M_{i5}(t)\lambda^2}{\Gamma\left(\alpha\right)} \int_0^1 \left(1-s\right)^{\alpha-1} \left(x_3\left(s\right) + x_1\left(s\right)\right) ds \\ &+ \frac{M_{i3}(t)\lambda - M_{i6}(t)\lambda^2}{\Gamma\left(\alpha\right)} \int_0^1 \left(1-s\right)^{\alpha-1} \left(x_1\left(s\right) + x_2\left(s\right)\right) ds \end{aligned} \tag{2.5}$$

$$-\frac{M_{i4}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_2(s)+\Upsilon_3(s)\right)ds$$
$$-\frac{M_{i5}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_3(s)+\Upsilon_1(s)\right)ds$$
$$-\frac{M_{i6}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_1(s)+\Upsilon_2(s)\right)ds, \quad i=1,2,3,$$

where

$$M_{1j}(t)=\frac{t^\alpha}{\Gamma(\alpha+1)}E_{1j}+E_{4j}, \quad M_{2j}(t)=\frac{t^\alpha}{\Gamma(\alpha+1)}E_{2j}+E_{5j},$$
$$M_{3j}(t)=\frac{t^\alpha}{\Gamma(\alpha+1)}E_{3j}+E_{6j}, \quad E_{\tau j}=\frac{m_{\tau j}}{e}, \ \tau,j=1,2,3,4,5,6,$$
$$m_{11}=\frac{4\lambda^2(-a+1)}{\Gamma(\alpha)}+4a\lambda(-a+1)+4\lambda,$$
$$m_{12}=\frac{4a\lambda^2(-a+1)}{\Gamma(\alpha)}+4a\lambda(a+1)-4\lambda,$$
$$m_{13}=\frac{4\lambda^3\left(-a^2+2a-1\right)}{(\Gamma(\alpha))^2}+\frac{8\lambda^2\left(a^2-1\right)}{\Gamma(\alpha)}-4a\lambda(a+1)-4\lambda,$$
$$m_{14}=\frac{2\lambda^2(a^3-3a^2+3a-1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(-3a^3+3a^2+a-1)}{\Gamma(\alpha)}+2a^3+2,$$
$$m_{15}=\frac{2\lambda^2(-a^3+3a^2-3a+1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(a^3-a^2-3a+3)}{\Gamma(\alpha)}+2a^3+2,$$
$$m_{16}=\frac{2\lambda^2(-a^3+a^2+a-1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(3a^3-a^2+a-3)}{\Gamma(\alpha)}-6(a^3+1),$$
$$m_{21}=\frac{4\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^2}+\frac{8\lambda^2(a^2-1)}{\Gamma(\alpha)}-4\lambda(a^2+a+1),$$
$$m_{22}=\frac{4\lambda^2(-a+1)}{\Gamma(\alpha)}+4\lambda(-a^2+a+1),$$
$$m_{23}=\frac{4a\lambda^2(-a+1)}{\Gamma(\alpha)}+4\lambda(a^2+a-1),$$
$$m_{24}=\frac{2\lambda^2(-a^3+a^2+a-1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(3a^3-a^2+a-3)}{\Gamma(\alpha)}-6(a^3+1),$$
$$m_{25}=\frac{2\lambda^2(a^3-3a^2+3a-1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(-3a^3+3a^2+a-1)}{\Gamma(\alpha)}+2(a^3+1),$$
$$m_{26}=\frac{2\lambda^2(-a^3+3a^2-3a+1)}{(\Gamma(\alpha))^2}+\frac{2\lambda(a^3-a^2-3a+3)}{\Gamma(\alpha)}+2(a^3+1),$$
$$m_{31}=\frac{4a\lambda^2(-a+1)}{\Gamma(\alpha)}+4\lambda(a^2+a-1),$$
$$m_{32}=\frac{4\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^2}+\frac{8\lambda^2(a^2-1)}{\Gamma(\alpha)}-4\lambda(a^2+a+1),$$
$$m_{33}=\frac{4\lambda^2(-a+1)}{\Gamma(\alpha)}+4\lambda(-a^2+a+1),$$

$$m_{34} = \frac{2\lambda^2(-a^3+3a^2-3a+1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(a^3-a^2-3a+3)}{\Gamma(\alpha)} + 2(a^3+1),$$

$$m_{35} = \frac{2\lambda^2(-a^3+a^2+a-1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(3a^3-a^2+a+3)}{\Gamma(\alpha)} - 6 - 6a^3,$$

$$m_{36} = \frac{2\lambda^2(a^3-3a^2+3a-1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(-3a^3+3a^2+a-1)}{\Gamma(\alpha)} + 2 + 2a^3,$$

$$m_{41} = \frac{2\lambda^3(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda^2(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(3a^2-a+1)}{\Gamma(\alpha)} + 4(-a^2+a+1),$$

$$m_{42} = \frac{2\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^3} + \frac{6\lambda^2(a^2-1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(-3a^2-a-1)}{\Gamma(\alpha)} + 4(a^2+a-1),$$

$$m_{43} = \frac{2\lambda^3(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda^2(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(3a^2+a-1)}{\Gamma(\alpha)} - 4(a^2+a+1),$$

$$m_{44} = \frac{2\lambda^2(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{6a^2+2a-2}{\Gamma(\alpha)},$$

$$m_{45} = \frac{2\lambda^2(-a^2+2a-1)}{(\Gamma(\alpha))^3} + \frac{6\lambda(a^2-1)}{(\Gamma(\alpha))^2} - \frac{2a^2+6a+2}{\Gamma(\alpha)},$$

$$m_{46} = \frac{2\lambda^2(a^2-2a+1)}{(\Gamma(\alpha)^3} + \frac{2\lambda(-a^2-2a+3)}{(\Gamma(\alpha))^2} + \frac{-2a^2+2a+6}{\Gamma(\alpha)},$$

$$m_{51} = \frac{2\lambda^3(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda^2(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(3a^2+a-1)}{\Gamma(\alpha)} - 4(a^2+a+1),$$

$$m_{52} = \frac{2\lambda^3(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda^2(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{2\lambda(3a^2-a+1)}{\Gamma(\alpha)} + 4(-a^2+a+1),$$

$$m_{53} = \frac{2\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^3} + \frac{6\lambda^2(a^2-1)}{(\Gamma(\alpha))^2} - \frac{2\lambda(3a^2+a+1)}{\Gamma(\alpha)} + 4(a^2+a-1),$$

$$m_{54} = \frac{2\lambda^2(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda(-a^2-2a+3)}{(\Gamma(\alpha))^2} + \frac{-2a^2+2a+6}{\Gamma(\alpha)},$$

$$m_{55} = \frac{2\lambda^2(a^2-2a+1)}{(\Gamma(\alpha))^3} + \frac{2\lambda(-3a^2+2a+1)}{(\Gamma(\alpha))^2} + \frac{6a^2+2a-2}{\Gamma(\alpha)},$$

$$m_{56} = \frac{2\lambda^2(-a^2+2a-1)}{(\Gamma(\alpha))^3} + \frac{6\lambda(a^2-1)}{(\Gamma(\alpha))^2} - \frac{2a^2+6a+2}{\Gamma(\alpha)},$$

$$m_{61} = \frac{-2\lambda^3(a^2-2a+1)}{(\Gamma(\alpha))^3} - \frac{2\lambda^2(-3a^2+3)}{(\Gamma(\alpha))^2} - \frac{2\lambda(3a^2+a+1)}{\Gamma(\alpha)} - 4(-a^2-a+1),$$

$$m_{62} = \frac{-2\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^3} - \frac{2\lambda^2(3a^2-2a-1)}{(\Gamma(\alpha))^2} - \frac{2\lambda(-3a^2-a+1)}{\Gamma(\alpha)} - 4(a^2+a+1),$$

$$m_{63} = \frac{-2\lambda^3(-a^2+2a-1)}{(\Gamma(\alpha))^3} - \frac{2\lambda^2(3a^2-2a-1)}{(\Gamma(\alpha))^2} - \frac{2\lambda(-3a^2+a-1)}{\Gamma(\alpha)}$$

$$
\begin{aligned}
&- 4(a^2 - a + 1),\\
m_{64} &= \frac{-2\lambda^2(a^2-2a+1)}{\left(\Gamma(\alpha)\right)^3} - \frac{2\lambda(-3a^2+3)}{\left(\Gamma(\alpha)\right)^2} - \frac{2a^2+6a+2}{\Gamma(\alpha)},\\
m_{65} &= \frac{-2\lambda^2(-a^2+2a-1)}{\left(\Gamma(\alpha)\right)^3} - \frac{2\lambda(a^2+2a-3)}{\left(\Gamma(\alpha)\right)^2} - \frac{2a^2-2a-6}{\Gamma(\alpha)},\\
m_{66} &= \frac{-2\lambda^2(-a^2+2a-1)}{\left(\Gamma(\alpha)\right)^3} - \frac{2\lambda(3a^2-2a-1)}{\left(\Gamma(\alpha)\right)^2} - \frac{-6a^2-2a+2}{\Gamma(\alpha)},\\
e &= 4\left(\frac{\lambda(a-1)}{\Gamma(\alpha)} - 2(a+1)\right)\left(\frac{\lambda^2(a^2-2a+1)}{\left(\Gamma(\alpha)\right)^2} + \frac{\lambda(-a^2+1)}{\Gamma(\alpha)} + a^2 - a + 1\right).
\end{aligned}
$$

Proof. Applying the operator I_{0+}^{β} to both sides of (2.1), we can deduce

$$
{}^{C}D_{0+}^{\alpha}x_i(t) = I_{0+}^{\beta}\Upsilon_i(t) - \lambda x_i(t) + c_{i1}, c_{i1} \in \mathbb{R}, \quad i = 1,2,3. \tag{2.6}
$$

Using the operator I_{0+}^{α} to act on both sides of (2.6), it follows

$$
x_i(t) = I_{0+}^{\alpha+\beta}\Upsilon_i(t) - \lambda I_{0+}^{\alpha}x_i(t) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}c_{i1} + c_{i2}, \quad c_{i1}, c_{i2} \in \mathbb{R},\ i = 1,2,3. \tag{2.7}
$$

Next, by (2.6) and (2.7), we can derive

$$
x_i(0) = c_{i2}, \quad x_i(1) = I_{0+}^{\alpha+\beta}\Upsilon_i(t)\Big|_{t=1} - \lambda I_{0+}^{\alpha}x_i(t)\big|_{t=1} + \frac{1}{\Gamma(\alpha+1)}c_{i1} + c_{i2}, \tag{2.8}
$$

$$
{}^{C}D_{0+}^{\alpha}x_i(0) = -\lambda x_i(0) + c_{i1}, \quad {}^{C}D_{0+}^{\alpha}x_i(1) = I_{0+}^{\beta}\Upsilon_i(t)\Big|_{t=1} - \lambda x_i(1) + c_{i1}, \tag{2.9}
$$

which together with the cyclic anti-periodic boundary conditions (2.2)-(2.4) yield that

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}c_{21} + \frac{1}{\Gamma(\alpha+1)}c_{31} + ac_{12} + (a+1)c_{22} + c_{32}\\
=&- I_{0+}^{\alpha+\beta}\left(\Upsilon_2(1) + \Upsilon_3(1)\right) + \lambda I_{0+}^{\alpha}\left(x_2(1) + x_3(1)\right),\\
&\frac{1}{\Gamma(\alpha+1)}c_{11} + \frac{1}{\Gamma(\alpha+1)}c_{31} + c_{12} + ac_{22} + (a+1)c_{32}\\
=&- I_{0+}^{\alpha+\beta}\left(\Upsilon_1(1) + \Upsilon_3(1)\right) + \lambda I_{0+}^{\alpha}\left(x_1(1) + x_3(1)\right),\\
&\frac{1}{\Gamma(\alpha+1)}c_{11} + \frac{1}{\Gamma(\alpha+1)}c_{21} + (a+1)c_{12} + c_{22} + ac_{32}\\
=&- I_{0+}^{\alpha+\beta}\left(\Upsilon_1(1) + \Upsilon_2(1)\right) + \lambda I_{0+}^{\alpha}\left(x_1(1) + x_2(1)\right),\\
&c_{11} + \left(2 - \frac{\lambda}{\Gamma(\alpha+1)}\right)c_{21} + \left(1 - \frac{\lambda}{\Gamma(\alpha+1)}\right)c_{31} - \lambda c_{12} - 2\lambda c_{22} - \lambda c_{32}\\
=&\lambda I_{0+}^{\alpha+\beta}\left(\Upsilon_2(1) + \Upsilon_3(1)\right) - \lambda^2 I_{0+}^{\alpha}\left(x_2(1) + x_3(1)\right) - I_{0+}^{\beta}\left(\Upsilon_2(1) + \Upsilon_3(1)\right),\\
&\left(1 - \frac{\lambda}{\Gamma(\alpha+1)}\right)c_{11} + c_{21} + \left(2 - \frac{\lambda}{\Gamma(\alpha+1)}\right)c_{31} - \lambda c_{12} - \lambda c_{22} - 2\lambda c_{32}\\
=&\lambda I_{0+}^{\alpha+\beta}\left(\Upsilon_3(1) + \Upsilon_1(1)\right) - \lambda^2 I_{0+}^{\alpha}\left(x_3(1) + x_1(1)\right) - I_{0+}^{\beta}\left(\Upsilon_3(1) + \Upsilon_1(1)\right),
\end{aligned}
$$

$$\left(2-\frac{\lambda}{\Gamma(\alpha+1)}\right)c_{11}+\left(1-\frac{\lambda}{\Gamma(\alpha+1)}\right)c_{21}+c_{31}-2\lambda c_{12}-\lambda c_{22}-\lambda c_{32}$$
$$=\lambda I_{0+}^{\alpha+\beta}\left(\Upsilon_1(1)+\Upsilon_2(1)\right)-\lambda^2 I_{0+}^{\alpha}\left(x_1(1)+x_2(1)\right)-I_{0+}^{\beta}\left(\Upsilon_1(1)+\Upsilon_2(1)\right).$$

So, we only need to solve the following system of linear equations to get the values of $c_{i1}, c_{i2}, i=1,2,3,$

$$\begin{pmatrix} 0 & \frac{1}{\Gamma(\alpha+1)} & \frac{1}{\Gamma(\alpha+1)} & a & a+1 & 1 \\ \frac{1}{\Gamma(\alpha+1)} & 0 & \frac{1}{\Gamma(\alpha+1)} & 1 & a & a+1 \\ \frac{1}{\Gamma(\alpha+1)} & \frac{1}{\Gamma(\alpha+1)} & 0 & a+1 & 1 & a \\ 1 & 2-\frac{\lambda}{\Gamma(\alpha+1)} & 1-\frac{\lambda}{\Gamma(\alpha+1)} & -\lambda & -2\lambda & -\lambda \\ 1-\frac{\lambda}{\Gamma(\alpha+1)} & 1 & 2-\frac{\lambda}{\Gamma(\alpha+1)} & -\lambda & -\lambda & -2\lambda \\ 2-\frac{\lambda}{\Gamma(\alpha+1)} & 1-\frac{\lambda}{\Gamma(\alpha+1)} & 1 & -2\lambda & -\lambda & -\lambda \end{pmatrix}\begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{12} \\ c_{22} \\ c_{33} \end{pmatrix}$$
$$=\begin{pmatrix} -I_{0+}^{\alpha+\beta}\left(\Upsilon_2(1)+\Upsilon_3(1)\right)+\lambda I_{0+}^{\alpha}\left(x_2(1)+x_3(1)\right) \\ -I_{0+}^{\alpha+\beta}\left(\Upsilon_3(1)+\Upsilon_1(1)\right)+\lambda I_{0+}^{\alpha}\left(x_3(1)+x_1(1)\right) \\ -I_{0+}^{\alpha+\beta}\left(\Upsilon_1(1)+\Upsilon_2(1)\right)+\lambda I_{0+}^{\alpha}\left(x_1(1)+x_2(1)\right) \\ \lambda I_{0+}^{\alpha+\beta}(\Upsilon_2(1)+\Upsilon_3(1))-\lambda^2 I_{0+}^{\alpha}(x_2(1)+x_3(1))-I_{0+}^{\beta}(\Upsilon_2(1)+\Upsilon_3(1)) \\ \lambda I_{0+}^{\alpha+\beta}(\Upsilon_3(1)+\Upsilon_1(1))-\lambda^2 I_{0+}^{\alpha}(x_3(1)+x_1(1))-I_{0+}^{\beta}(\Upsilon_3(1)+\Upsilon_1(1)) \\ \lambda I_{0+}^{\alpha+\beta}(\Upsilon_2(1)+\Upsilon_1(1))-\lambda^2 I_{0+}^{\alpha}(x_1(1)+x_2(1))-I_{0+}^{\beta}(\Upsilon_1(1)+\Upsilon_2(1)) \end{pmatrix}. \tag{2.10}$$

It's simple to demonstrate that the determinant of the coefficient matrix associated with the linear system (2.10) is not zero. Therefore, (2.10) admits the unique solution.

$$c_{11}=\frac{m_{11}\varsigma_1+m_{12}\varsigma_2+m_{13}\varsigma_3+m_{14}\varsigma_4+m_{15}\varsigma_5+m_{16}\varsigma_6}{e},$$
$$c_{21}=\frac{m_{21}\varsigma_1+m_{22}\varsigma_2+m_{23}\varsigma_3+m_{24}\varsigma_4+m_{25}\varsigma_5+m_{26}\varsigma_6}{e},$$
$$c_{31}=\frac{m_{31}\varsigma_1+m_{32}\varsigma_2+m_{33}\varsigma_3+m_{34}\varsigma_4+m_{35}\varsigma_5+m_{36}\varsigma_6}{e},$$
$$c_{12}=\frac{m_{41}\varsigma_1+m_{42}\varsigma_2+m_{43}\varsigma_3+m_{44}\varsigma_4+m_{45}\varsigma_5+m_{46}\varsigma_6}{e},$$
$$c_{22}=\frac{m_{51}\varsigma_1+m_{52}\varsigma_2+m_{53}\varsigma_3+m_{54}\varsigma_4+m_{55}\varsigma_5+m_{56}\varsigma_6}{e},$$
$$c_{32}=\frac{m_{61}\varsigma_1+m_{62}\varsigma_2+m_{63}\varsigma_3+m_{64}\varsigma_4+m_{65}\varsigma_5+m_{66}\varsigma_6}{e},$$

where

$$\varsigma_1=-\frac{1}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha-1+\beta}\left(\Upsilon_2(1)+\Upsilon_3(1)\right)ds$$

$$
\begin{aligned}
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_2(1)+x_3(1)\right)ds,\\
\varsigma_2 &= -\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left(\Upsilon_3(1)+\Upsilon_1(1)\right)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_3(1)+x_1(1)\right)ds,\\
\varsigma_3 &= -\frac{1}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left(\Upsilon_1(1)+\Upsilon_2(1)\right)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_1(1)+x_2(1)\right)ds,\\
\varsigma_4 &= \frac{\lambda}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left(\Upsilon_2(1)+\Upsilon_3(1)\right)ds\\
&-\frac{\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_2(1)+x_3(1)\right)ds\\
&-\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_2(1)+\Upsilon_3(1)\right)ds,\\
\varsigma_5 &= \frac{\lambda}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left(\Upsilon_3(1)+\Upsilon_1(1)\right)ds\\
&-\frac{\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_3(1)+x_1(1)\right)ds\\
&-\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_3(1)+\Upsilon_1(1)\right)ds,\\
\varsigma_6 &= \frac{\lambda}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}\left(\Upsilon_1(1)+\Upsilon_2(1)\right)ds\\
&-\frac{\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(x_1(1)+x_2(1)\right)ds\\
&-\frac{1}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left(\Upsilon_1(1)+\Upsilon_2(1)\right)ds.
\end{aligned}
$$

So, putting the values of $c_{i1}, c_{i2}, i = 1, 2, 3$ into (2.7), we get the desired solution (2.5).

On the contrary, it is easy to verify that $(x_1, x_2, x_3) \in X$ given by (2.5) satisfies the system (2.1) and the boundary conditions (2.2)-(2.4). □

3. The well-posedness of (1.1)

For convenience, let $\phi_i(t) = g_i(t, x_1(t), x_2(t), x_3(t)),\ i = 1, 2, 3,\ t \in [0, 1]$. According to Lemma 2.2, define the operator $T : X \to X$ by

$$
\begin{aligned}
(Tx)(t) :&= ((T_1x)(t), (T_2x)(t), (T_3x)(t))\\
&= (T_1(x_1, x_2, x_3)(t), T_2(x_1, x_2, x_3)(t), T_3(x_1, x_2, x_3)(t)), \qquad (3.1)
\end{aligned}
$$

where

$$
(T_ix)(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\phi_i(s)\,ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_i(s)\,ds
$$

$$
\begin{aligned}
&+\frac{-M_{i1}(t)+M_{i4}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left(\phi_2\left(s\right)+\phi_3\left(s\right)\right)ds\\
&+\frac{-M_{i2}(t)+M_{i5}(t)\lambda}{\Gamma\left(\alpha+\beta\right)}\int_0^1(1-s)^{\alpha+\beta-1}\left(\phi_3\left(s\right)+\phi_1\left(s\right)\right)ds\\
&+\frac{-M_{i3}(t)+M_{i6}(t)\lambda}{\Gamma\left(\alpha+\beta\right)}\int_0^1(1-s)^{\alpha+\beta-1}\left(\phi_1\left(s\right)+\phi_2\left(s\right)\right)ds\\
&+\frac{M_{i1}(t)\lambda-M_{i4}(t)\lambda^2}{\Gamma\left(\alpha\right)}\int_0^1(1-s)^{\alpha-1}\left(x_2\left(s\right)+x_3\left(s\right)\right)ds\\
&+\frac{M_{i2}(t)\lambda-M_{i5}(t)\lambda^2}{\Gamma\left(\alpha\right)}\int_0^1(1-s)^{\alpha-1}\left(x_3\left(s\right)+x_1\left(s\right)\right)ds\\
&+\frac{M_{i3}(t)\lambda-M_{i6}(t)\lambda^2}{\Gamma\left(\alpha\right)}\int_0^1(1-s)^{\alpha-1}\left(x_1\left(s\right)+x_2\left(s\right)\right)ds\\
&-\frac{M_{i4}(t)}{\Gamma\left(\beta\right)}\int_0^1(1-s)^{\beta-1}\left(\phi_2\left(s\right)+\phi_3\left(s\right)\right)ds\\
&-\frac{M_{i5}(t)}{\Gamma\left(\beta\right)}\int_0^1(1-s)^{\beta-1}\left(\phi_3\left(s\right)+\phi_1\left(s\right)\right)ds\\
&-\frac{M_{i6}(t)}{\Gamma\left(\beta\right)}\int_0^1(1-s)^{\beta-1}\left(\phi_1\left(s\right)+\phi_2\left(s\right)\right)ds,\quad i=1,2,3.
\end{aligned}
$$

Therefore, the solution to problem (1.1) corresponds to the function $x=(x_1,x_2,x_3)$, given that $x=(x_1,x_2,x_3)$ is a fixed point of the operator T. By the Krasnoselskii's fixed point theorem, we proceed to establish the existence of solutions for the system (1.1).

Theorem 3.1. *Assume that the following conditions hold.*

(H_1) $g_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}, i=1,2,3$ *are continuous;*

(H_2) *For all* $(t,u,v,w)\in[0,1]\times\mathbb{R}^3$, *there exist nonnegative functions* $k_i^1,k_i^2,k_i^3,k_i^4\in C[0,1], i=1,2,3$ *satisfying*

$$|g_i(t,u,v,w)|\le k_i^1(t)+k_i^2(t)|u|+k_i^3(t)|v|+k_i^4(t)|w|.$$

Then the system (1.1) admits at least one solution with the condition that

$$\Gamma(\alpha+1)>\Gamma(\alpha+1)(\xi+\eta+r)\sum_{i=1}^{3}\ell_i+(3N_1+1)\lambda+3N_2\lambda^2, \tag{3.2}$$

where

$$
\begin{aligned}
\xi_1&=\frac{1+\|M_{12}\|_\infty+\|M_{15}\|_\infty\lambda+\|M_{13}\|_\infty+\|M_{16}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)}+\frac{\|M_{15}\|_\infty+\|M_{16}\|_\infty}{\Gamma(\beta+1)},\\
\xi_2&=\frac{1+\|M_{21}\|_\infty+\|M_{24}\|_\infty\lambda+\|M_{23}\|_\infty+\|M_{26}\|_\infty\lambda}{\Gamma\left(\alpha+\beta+1\right)}+\frac{\|M_{24}\|_\infty+\|M_{26}\|_\infty}{\Gamma(\beta+1)},\\
\xi_3&=\frac{1+\|M_{31}\|_\infty+\|M_{34}\|_\infty\lambda+\|M_{32}\|_\infty+\|M_{35}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)}+\frac{\|M_{34}\|_\infty+\|M_{35}\|_\infty}{\Gamma(\beta+1)},\\
\eta_1&=\frac{\|M_{11}\|_\infty+\|M_{14}\|_\infty\lambda+\|M_{13}\|_\infty+\|M_{16}\|_\infty\lambda}{\Gamma\left(\alpha+\beta+1\right)}+\frac{\|M_{14}\|_\infty+\|M_{16}\|_\infty}{\Gamma(\beta+1)},
\end{aligned}
$$

$$
\begin{aligned}
&\eta_2 = \frac{\|M_{21}\|_\infty + \|M_{24}\|_\infty\lambda + \|M_{22}\|_\infty + \|M_{25}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)} + \frac{\|M_{24}\|_\infty + \|M_{25}\|_\infty}{\Gamma(\beta+1)},\\
&\eta_3 = \frac{\|M_{32}\|_\infty + \|M_{35}\|_\infty\lambda + \|M_{33}\|_\infty + \|M_{36}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)} + \frac{\|M_{35}\|_\infty + \|M_{36}\|_\infty}{\Gamma(\beta+1)},\\
&\gamma_1 = \frac{\|M_{11}\|_\infty + \|M_{14}\|_\infty\lambda + \|M_{12}\|_\infty + \|M_{15}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)} + \frac{\|M_{14}\|_\infty + \|M_{15}\|_\infty}{\Gamma(\beta+1)},\\
&\gamma_2 = \frac{\|M_{22}\|_\infty + \|M_{25}\|_\infty\lambda + \|M_{23}\|_\infty + \|M_{26}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)} + \frac{\|M_{25}\|_\infty + \|M_{26}\|_\infty}{\Gamma(\beta+1)},\\
&\gamma_3 = \frac{\|M_{31}\|_\infty + \|M_{34}\|_\infty\lambda + \|M_{33}\|_\infty + \|M_{36}\|_\infty\lambda}{\Gamma(\alpha+\beta+1)} + \frac{\|M_{34}\|_\infty + \|M_{36}\|_\infty}{\Gamma(\beta+1)},\\
&\xi = \max\{\xi_1, \xi_2, \xi_3\},\\
&\eta = \max\{\eta_1, \eta_2, \eta_3\},\\
&\gamma = \max\{\gamma_3, \gamma_3, \gamma_3\},\\
&k_i^1 = \max_{t\in[0,1]} |k_i^1(t)|, \quad k_i^2 = \max_{t\in[0,1]} |k_i^2(t)|, \quad k_i^3 = \max_{t\in[0,1]} |k_i^3(t)|,\\
&k_i^4 = \max_{t\in[0,1]} |k_i^4(t)|, \quad \ell_i = k_i^2 + k_i^3 + k_i^4, \quad i = 1,2,3.
\end{aligned}
$$

Proof. Fix $\delta > 0$ such that

$$
\delta \geq \frac{(\xi+\eta+\gamma)\Gamma(\alpha+1)\sum_{i=1}^{3} k_i^1}{\Gamma(\alpha+1) - \Gamma(\alpha+1)(\xi+\eta+\gamma)\sum_{i=1}^{3}\ell_i - [(3N_1+1)\lambda + 3N_2\lambda^2]}.
$$

Consider the set

$$
\Omega_\delta = \{x = (x_1, x_2, x_3) \in \mathbb{X}^3 : \|x\|_X \leq \delta\}.
$$

Define the operators $F, G : \Omega_\delta \to X$ by

$$
\begin{aligned}
(Fx)(t) &= ((F_1x)(t), (F_2x)(t), (F_3x)(t))\\
&= (F_1(x_1,x_2,x_3)(t), F_2(x_1,x_2,x_3)(t), F_3(x_1,x_2,x_3)(t)),\\
(Gx)(t) &= ((G_1x)(t), (G_2x)(t), (G_3x)(t))\\
&= (G_1(x_1,x_2,x_3)(t), G_2(x_1,x_2,x_3)(t), G_3(x_1,x_2,x_3)(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
(F_ix)(t) = &-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} x_i(s)ds\\
&+\frac{M_{i2}(t)\lambda - M_{i5}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_3(s)+x_1(s))\,ds\\
&+\frac{M_{i3}(t)\lambda - M_{i6}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_1(s)+x_2(s))\,ds\\
&+\frac{M_{i1}(t)\lambda - M_{i4}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}(x_2(s)+x_3(s))\,ds, \; i=1,2,3,\\
(G_ix)(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\phi_i(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-M_{i1}(t)+M_{i4}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\phi_2(s)+\phi_3(s))\,ds\\
&+\frac{-M_{i2}(t)+M_{i5}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\phi_3(s)+\phi_1(s))\,ds\\
&+\frac{-M_{i3}(t)+M_{i6}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(\phi_1(s)+\phi_2(s))\,ds\\
&-\frac{M_{i4}(t)}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(\phi_2(s)+\phi_3(s))\,ds\\
&-\frac{M_{i5}(t)}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(\phi_3(s)+\phi_1(s))\,ds\\
&-\frac{M_{i6}(t)}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(\phi_1(s)+\phi_2(s))\,ds,\ \ i=1,2,3.
\end{aligned}
$$

Now, in terms of Krasnoselskii's fixed point theorem, our proof can be divided into three steps.

(i) The following property will be proved.

$$Gx+Fy\in\Omega_\delta \text{ for any } x=(x_1,x_2,x_3)\in\Omega_\delta \text{ and } y=(y_1,y_2,y_3)\in\Omega_\delta.$$

As a matter of fact, for any $x,y\in\Omega_\delta$, it follow $\|x\|_X\le\delta,\|y\|_X\le\delta$. Then, from (H_2), we can get that

$$
\begin{aligned}
|(G_1x)(t)|\le{}&\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}|\phi_1(s)|\,ds\\
&+\frac{|M_{11}(t)|+|M_{14}(t)|\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\phi_2(s)|+|\phi_3(s)|)\,ds\\
&+\frac{|M_{12}(t)|+|M_{15}(t)|\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\phi_3(s)|+|\phi_1(s)|)\,ds\\
&+\frac{|M_{13}(t)|+|M_{16}(t)|\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}(|\phi_1(s)|+|\phi_2(s)|)\,ds\\
&+\frac{|M_{14}(t)|}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(|\phi_2(s)|+|\phi_3(s)|)\,ds\\
&+\frac{|M_{15}(t)|}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(|\phi_3(s)|+|\phi_1(s)|)\,ds\\
&+\frac{|M_{16(t)}|}{\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}(|\phi_1(s)|+|\phi_2(s)|)\,ds\\
\le{}&\frac{k_1^1+\ell_1\parallel x\|_X}{\Gamma(\alpha+\beta+1)}+\frac{|M_{11}(t)|+|M_{14}(t)|\lambda}{\Gamma(\alpha+\beta+1)}\left(k_2^1+k_3^1+\ell_2\|x\|_X+\ell_3\|x\|_X\right)\\
&+\frac{|M_{12}(t)|+|M_{15}(t)|\lambda}{\Gamma(\alpha+\beta+1)}\left(k_1^1+k_3^1+\ell_1\|x\|_X+\ell_3\|x\|_X\right)\\
&+\frac{|M_{13}(t)|+|M_{16}(t)|\lambda}{\Gamma(\alpha+\beta+1)}\left(k_1^1+k_2^1+\ell_1\|x\|_X+\ell_2\|x\|_X\right)\\
&+\frac{|M_{14}(t)|}{\Gamma(\beta+1)}\left(k_2^1+k_3^1+\ell_2\|x\|_X+\ell_3\|x\|_X\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|M_{15}(t)|}{\Gamma(\beta+1)}\left(k_1^1+k_3^1+\ell_1\|x\|_X+\ell_3\|x\|_X\right)\\
&+\frac{|M_{16}(t)|}{\Gamma(\beta+1)}\left(k_1^1+k_2^1+\ell_1\|x\|_X+\ell_2\|x\|_X\right)\\
\leq\ &\xi k_1^1+\eta k_2^1+\gamma k_3^1+(\xi\ell_1+\eta\ell_2+\gamma\ell_3)\,\delta.
\end{aligned}
$$

Similarly, we also find

$$
\begin{aligned}
|(G_2x)(t)| &\leq \xi k_2^1+\eta k_3^1+\gamma k_1^1+(\xi\ell_2+\eta\ell_3+\gamma\ell_1)\,\delta,\\
|(G_3x)(t)| &\leq \xi k_3^1+\eta k_1^1+\gamma k_2^1+(\xi\ell_3+\eta\ell_1+\gamma\ell_2)\,\delta.
\end{aligned}
$$

Moreover, we can get the following inequality

$$
\begin{aligned}
&|(F_1y)(t)|\\
\leq\ &\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|y_1(s)|\,ds\\
&+\frac{|M_{12}(t)|\,\lambda+|M_{15}(t)|\,\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(|y_3(s)|+|y_1(s)|\right)ds\\
&+\frac{|M_{13}(t)|\,\lambda+|M_{16}(t)|\,\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(|y_1(s)|+|y_2(s)|\right)ds\\
&+\frac{|M_{11}(t)|\,\lambda+|M_{14}(t)|\,\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left(|y_2(s)|+|\,y_3(s)\,|\right)ds\\
\leq\ &\frac{\lambda}{\Gamma(\alpha+1)}\|y_1\|_\infty+\frac{|M_{12}(t)|\,\lambda+|M_{15}(t)|\,\lambda^2}{\Gamma(\alpha+1)}\left(\|y_3\|_\infty+\|y_1\|_\infty\right)\\
&+\frac{|M_{13}(t)|\,\lambda+|M_{16}(t)|\,\lambda^2}{\Gamma(\alpha+1)}\left(\|y_1\|_\infty+\|y_2\|_\infty\right)\\
&+\frac{|M_{11}(t)|\,\lambda+|M_{14}(t)|\,\lambda^2}{\Gamma(\alpha+1)}\left(\|y_2\|_\infty+\|y_3\|_\infty\right)\\
\leq\ &\frac{[(|M_{12}(t)|+|M_{13}(t)|+|M_{11}(t)|)\lambda+(|M_{15}(t)|+|M_{16}(t)|+|M_{14}(t)|)\lambda^2]\delta}{\Gamma(\alpha+1)}\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\|y_1\|_\infty.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&|(F_2y)(t)|\\
\leq&\frac{\left[(|M_{22}(t)|+|M_{23}(t)|+|M_{21}(t)|)\,\lambda+(|M_{25}(t)|+|M_{26}(t)|+|M_{24}(t)|)\,\lambda^2\right]\delta}{\Gamma(\alpha+1)}\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\|y_2\|_\infty,\\
&|(F_3y)(t)|\\
\leq&\frac{\left[(|M_{31}(t)|+|M_{32}(t)|+|M_{33}(t)|)\,\lambda+(|M_{34}(t)|+|M_{35}(t)|+|M_{36}(t)|)\,\lambda^2\right]\delta}{\Gamma(\alpha+1)}\\
&+\frac{\lambda}{\Gamma(\alpha+1)}\|y_3\|_\infty.
\end{aligned}
$$

For convenience, we introduce the following constants

$$\begin{aligned}
N_{11} &= \|M_{12}\|_\infty + \|M_{13}\|_\infty + \|M_{11}\|_\infty, \quad N_{21} = \|M_{14}\|_\infty + \|M_{15}\|_\infty + \|M_{16}\|_\infty, \\
N_{12} &= \|M_{22}\|_\infty + \|M_{23}\|_\infty + \|M_{21}\|_\infty, \quad N_{22} = \|M_{24}\|_\infty + \|M_{25}\|_\infty + \|M_{26}\|_\infty, \\
N_{13} &= \|M_{32}\|_\infty + \|M_{33}\|_\infty + \|M_{31}\|_\infty, \quad N_{23} = \|M_{34}\|_\infty + \|M_{35}\|_\infty + \|M_{36}\|_\infty,
\end{aligned}$$

where

$$N_1 = \max\{N_{11}, N_{12}, N_{13}\}, N_2 = \max\{N_{21}, N_{22}, N_{23}\}.$$

Therefore, we can obtain

$$\begin{aligned}
|(F_1y)(t)| &\le \frac{\lambda}{\Gamma(\alpha+1)}\|y_1\|_\infty + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta, \\
|(F_2y)(t)| &\le \frac{\lambda}{\Gamma(\alpha+1)}\|y_2\|_\infty + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta, \\
|(F_3y)(t)| &\le \frac{\lambda}{\Gamma(\alpha+1)}\|y_3\|_\infty + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta.
\end{aligned}$$

According to the above results, we can obtain the following estimates immediately.

$$\begin{aligned}
|(G_1x)(t) + (F_1y)(t)| &\le \xi k_1^1 + \eta k_2^1 + \gamma k_3^1 + (\xi\ell_1 + \eta\ell_2 + \gamma\ell_3)\,\delta + \frac{\lambda\|y_1\|_\infty}{\Gamma(\alpha+1)} \\
&\quad + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta, \\
|(G_2x)(t) + (F_2y)(t)| &\le \xi k_2^1 + \eta k_3^1 + \gamma k_1^1 + (\xi\ell_2 + \eta\ell_3 + \gamma\ell_1)\,\delta + \frac{\lambda\|y_2\|_\infty}{\Gamma(\alpha+1)} \\
&\quad + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta, \\
|(G_3x)(t) + (F_3y)(t)| &\le \xi k_3^1 + \eta k_1^1 + \gamma k_2^1 + (\xi\ell_3 + \eta\ell_1 + \gamma\ell_2)\,\delta + \frac{\lambda\|y_3\|_\infty}{\Gamma(\alpha+1)} \\
&\quad + \frac{N_1\lambda + N_2\lambda^2}{\Gamma(\alpha+1)}\delta.
\end{aligned}$$

Taking the norm for $Gx + Fy$ on X, one has

$$\begin{aligned}
\|Gx + Fy\|_X &= \|G_1x + F_1y\|_\infty + \|G_2x + F_2y\|_\infty + \|G_3x + F_3y\|_\infty \\
&\le (\xi + \eta + \gamma)\sum_{i=1}^{3}\left(k_i^1 + \delta\ell_i\right) + \frac{(3N_1 + 1)\,\lambda + 3N_2\lambda^2}{\Gamma(\alpha+1)}\delta \\
&\le \delta.
\end{aligned}$$

Hence, $Gx + Fy \in \Omega_\delta$ for all $x, y \in \Omega_\delta$.

(ii) The operator F is a contraction on Ω_δ will be shown. In fact, for any $x = (x_1, x_2, x_3) \in \Omega_\delta$ and $y = (y_1, y_2, y_3) \in \Omega_\delta$, it follows

$$\begin{aligned}
&|(F_1x)(t) - (F_1y)(t)| \\
&\le \frac{\left(N_1\lambda + N_2\lambda^2\right)\left(\|x_1 - y_1\|_\infty + \|x_2 - y_2\|_\infty + \|x_3 - y_3\|_\infty\right)}{\Gamma(\alpha+1)}
\end{aligned}$$

$$+\frac{\lambda\|x_1-y_1\|_\infty}{\Gamma(\alpha+1)}$$
$$\leq \frac{\lambda}{\Gamma(\alpha+1)}\|x_1-y_1\|_\infty+\frac{\left(N_1\lambda+N_2\lambda^2\right)}{\Gamma(\alpha+1)}\|x-y\|_X.$$

Similarly, we have

$$|(F_2x)(t)-(F_2y)(t)|\leq \frac{\lambda}{\Gamma(\alpha+1)}\|x_2-y_2\|_\infty+\frac{\left(N_1\lambda+N_2\lambda^2\right)}{\Gamma(\alpha+1)}\|x-y\|_X,$$
$$|(F_3x)(t)-(F_3y)(t)|\leq \frac{\lambda}{\Gamma(\alpha+1)}\|x_3-y_3\|_\infty+\frac{\left(N_1\lambda+N_2\lambda^2\right)}{\Gamma(\alpha+1)}\|x-y\|_X.$$

Taking the norm $Fx-Fy$ on X, we get

$$\|Fx-Fy\|_X\leq \frac{(3N_1+1)\,\lambda+3N_2\lambda^2}{\Gamma(\alpha+1)}\|x-y\|_X.$$

By (3.2), we can get that F is a contraction.

(iii) The G is equicontinuous on Ω_δ will be obtained. As a matter of fact, since the functions g_1, g_2, g_3 are continuous, this means that the operator G is continuous on Ω_δ. Therefore, we need to prove that G is relatively compact on Ω_δ. In fact, for any $x\in\Omega_\delta$, by using (i), we obtain G is uniformly bounded on Ω_δ. For convenience, we have import the following constants.

$$\widetilde{L}_i=\left[\frac{|m_{i1}|+|m_{i2}|+|m_{i3}|+\left(|m_{i4}|+|\ m_{i5}|+|m_{i6}|\right)\lambda}{\Gamma\left(\alpha+\beta+1\right)}+\frac{|m_{i4}|+|m_{i5}|+|m_{i6}|}{\Gamma\left(\beta+1\right)}\right]$$
$$\times\sum_{i=1}^{3}\left(k_i^1+\ell_i\delta\right).$$

Next, for any $x=(x_1,x_2,x_3)\in\Omega_\delta$ and $t_1,t_2\in[0,1]$ with $0\leq t_1\leq t_2\leq 1$, we can obtain

$$|(G_1x)(t_2)-(G_1x)(t_1)|$$
$$\leq \frac{k_1^1+(k_1^2+k_1^3+k_1^4)\|x\|_X}{\Gamma(\alpha+\beta)}\{\int_0^{t_1}[(t_2-s)^{\alpha+\beta-1}-(t_1-s)^{\alpha+\beta-1}]ds$$
$$+\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}ds\}$$
$$+\frac{(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\left[\frac{\left(|E_{11}|+|E_{14}|\,\lambda\right)\left(k_2^1+k_3^1+\ell_2\|x\|_X+\ell_3\|x\|_X\right)}{\Gamma(\alpha+\beta+1)}\right]$$
$$+\frac{(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\left[\frac{\left(|E_{12}|+|E_{15}|\,\lambda\right)\left(k_1^1+k_3^1+\ell_1\|x\|_X+\ell_3\|x\|_X\right)}{\Gamma(\alpha+\beta+1)}\right]$$
$$+\frac{(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\left[\frac{\left(|E_{13}|+|E_{16}|\,\lambda\right)\left(k_1^1+k_2^1+\ell_1\|x\|_X+\ell_2\|x\|_X\right)}{\Gamma(\alpha+\beta+1)}\right]$$
$$+\frac{(t_2^\alpha-t_1^\alpha)}{\Gamma(\alpha+1)}\left[\frac{|E_{14}|\left(k_3^1+k_2^1+\ell_2\|x\|_X+\ell_3\|x\|_X\right)}{\Gamma(\beta+1)}\right]$$

$$
\begin{aligned}
&+\frac{(t_2^{\alpha}-t_1^{\alpha})}{\Gamma(\alpha+1)}\left[\frac{|E_{15}|\left(k_1^1+k_3^1+\ell_1\|x\|_X+\ell_3\|x\|_X\right)}{\Gamma(\beta+1)}\right]\\
&+\frac{(t_2^{\alpha}-t_1^{\alpha})}{\Gamma(\alpha+1)}\left[\frac{|E_{16}|\left(k_1^1+k_2^1+\ell_1\|x\|_X+\ell_2\|x\|_X\right)}{\Gamma(\beta+1)}\right]\\
\leq\ & \frac{k_1^1+\left(k_1^2+k_1^3+k_1^4\right)\delta}{\Gamma(\alpha+\beta+1)}\left(t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right)+\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}\widetilde{L}_1.
\end{aligned}
$$

Similarly, the following conclusions can also been obtained.

$$
\begin{aligned}
|(G_2x)(t_2)-(G_2x)(t_1)| &\leq \frac{k_2^1+\left(k_2^2+k_2^3+k_2^4\right)\delta}{\Gamma(\alpha+\beta+1)}\left(t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right)+\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}\widetilde{L}_2,\\
|(G_3x)(t_2)-(G_3x)(t_1)| &\leq \frac{k_3^1+\left(k_3^2+k_3^3+k_3^4\right)\delta}{\Gamma(\alpha+\beta+1)}\left(t_2^{\alpha+\beta}-t_1^{\alpha+\beta}\right)+\frac{t_2^{\alpha}-t_1^{\alpha}}{\Gamma(\alpha+1)}\widetilde{L}_3.
\end{aligned}
$$

Based on the facts that $t^{\alpha+\beta}$ and t^{α} are uniformly continuous on $[0,1]$, we can get

$$
|(G_ix)(t_2)-(G_ix)(t_1)|\to 0,\ \textit{as}\ t_2\to t_1 \text{ independent of } x,\quad i=1,2,3.
$$

Thus, the operator G is equicontinuous on Ω_δ. Therefore, by the Arzelá-Ascoli theorem, we obtain that G is a relatively compact on Ω_δ. Hence, all the conditions of Lemma 2.1 hold, then the operator $G+F$ has a fixed point, which means that it is a solution of the system (1.1). □

In the results below, the uniqueness of solution to the system (1.1) has been established by the Banach's contraction mapping theorem.

Theorem 3.2. *Assume that the condition* (H_1) *and the following conditions hold.*

(H_3) *For for any* $t\in[0,1], x_i, y_i\in\mathbb{R}, i=1,2,3$*, there exist constants* $L_i>0, i=1,2,3$ *satisfying*

$$
\begin{aligned}
&|g_i(t,x_1,x_2,x_3)-g_i(t,y_1,y_2,y_3)|\\
&\leq L_i(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|),\ i=1,2,3.
\end{aligned}
$$

Then the system (1.1) admits the unique solution with the condition that

$$
(\Lambda_1+\Lambda_2+\Lambda_3)\,\Gamma(\alpha+1)+(3N_1+1)\,\lambda+3N_2\lambda^2<\Gamma(\alpha+1), \tag{3.3}
$$

where

$$
\Lambda_1=\xi L_1+\eta L_2+\gamma L_3, \Lambda_2=\xi L_2+\eta L_3+\gamma L_1, \Lambda_3=\xi L_3+\eta L_1+\gamma L_2.
$$

Proof. Fix $\rho>0$ such that

$$
\frac{\Gamma(\alpha+1)(\xi+\gamma+\eta)\,(w_1+w_2+w_3)}{\Gamma(\alpha+1)-3\left[(N_1+1)\,\lambda+N_2\lambda^2\right]-\Gamma(\alpha+1)(\xi+\gamma+\eta)\,(L_1+L_2+L_3)}\leq\rho,
$$

where

$$
w_1=\max_{t\in[0,1]}|g_1(t,0,0,0)|\,, w_2=\max_{t\in[0,1]}|g_2(t,0,0,0)|\,, w_3=\max_{t\in[0,1]}|g_3(t,0,0,0)|\,.
$$

To begin with, consider the set

$$\Omega_\rho = \{(x_1, x_2, x_3) \in X : \|x\|_X \le \rho\},$$

and show that $T\Omega_\rho \subset \Omega_\rho$. In fact, for any $x = (x_1, x_2, x_3) \in \Omega_\rho$, from (H_3), it follows

$$\begin{aligned}|g_1(t, x_1, x_2, x_3)| &\le |g_1(t, x_1, x_2, x_3) - g_1(t, 0, 0, 0)| + |g_1(t, 0, 0, 0)| \\ &\le L_1 \left(\|x_1\|_\infty + \|x_2\|_\infty + \|x_3\|_\infty\right) + w_1 \\ &= L_1 \|x\|_X + w_1 \\ &\le L_1 \rho + w_1.\end{aligned}$$

Similarly, by (H_3), we can derive

$$\begin{aligned}|g_2\left(t, x_1, x_2, x_3\right)| \le L_2 \|x\|_X + w_2 \le L_2 \rho + w_2, \\ |g_3\left(t, x_1, x_2, x_3\right)| \le L_3 \|x\|_X + w_3 \le L_3 \rho + w_3.\end{aligned}$$

Thus, it follows

$$\begin{aligned}&|\left(T_1 x\right)(t)| \\ \le\ & \frac{L_1 p + w_1}{\Gamma(\alpha+\beta+1)} + \frac{\lambda\rho}{\Gamma(\alpha+1)} + \frac{\left(|M_{11}(t)| + |M_{14}(t)|\,\lambda\right)\left(L_2 p + w_2 + L_3 p + w_3\right)}{\Gamma(\alpha+\beta+1)} \\ & + \frac{\left[\left(|M_{11}(t)| + |M_{12}(t)| + |M_{13}(t)|\right)\lambda + \left(|M_{14}(t)| + |M_{15}(t)| + |M_{16}(t)|\right)\lambda^2\right]\rho}{\Gamma(\alpha+1)} \\ & + \frac{\left(|M_{12}(t)| + |M_{15}(t)|\,\lambda\right)\left(L_3 p + w_3 + L_1 p + w_1\right)}{\Gamma(\alpha+\beta+1)} \\ & + \frac{\left(|M_{13}(t)| + |M_{16}(t)|\,\lambda\right)\left(L_1 p + w_1 + L_2 p + w_2\right)}{\Gamma(\alpha+\beta+1)} \\ & + \frac{|M_{14}(t)(t)|\left(L_2\rho + w_2 + L_3\rho + w_3\right)}{\Gamma(\beta+1)} \\ & + \frac{|M_{15}(t)|\left(L_3\rho + w_3 + L_1\rho + w_1\right)}{\Gamma(\beta+1)} \\ & + \frac{|M_{16}(t)|\left(L_1\rho + w_1 + L_2\rho + w_2\right)}{\Gamma(\beta+1)} \\ \le\ & \frac{\left[\left(N_1+1\right)\lambda + N_2\lambda^2\right]\rho}{\Gamma(\alpha+1)} + \xi\left(L_1\rho + w_1\right) + \eta\left(L_2\rho + w_2\right) + \gamma\left(L_3\rho + w_3\right).\end{aligned}$$

Similarly, we also find

$$\begin{aligned}&|\left(T_2 x\right)(t)| \\ \le\ & \frac{\left[\left(N_1+1\right)\lambda + N_2\lambda^2\right]\rho}{\Gamma(\alpha+1)} + \gamma\left(L_1\rho + w_1\right) + \xi\left(L_2\rho + w_2\right) + \eta\left(L_3\rho + w_3\right), \\ &|\left(T_3 x\right)(t)| \\ \le\ & \frac{\left[\left(N_1+1\right)\lambda + N_2\lambda^2\right]\rho}{\Gamma(\alpha+1)} + \eta\left(L_1\rho + w_1\right) + \gamma\left(L_2\rho + w_2\right) + \xi\left(L_3\rho + w_3\right).\end{aligned}$$

Thus, we can get

$$\|Tx\|_X \le \frac{3\left[\left(N_1+1\right)\lambda + N_2\lambda^2\right]\rho}{\Gamma(\alpha+1)} + (\xi+\gamma+\eta)\left(L_1\rho + w_1 + L_2\rho + w_2 + L_3\rho + w_3\right) \le \rho.$$

This means $T\Omega_\rho \subset \Omega_\rho$. For convenience, let

$$\phi_{ix}(t) = g_i(t, x_1(t), x_2(t), x_3(t)),$$
$$\phi_{iy}(t) = g_i(t, y_1(t), y_2(t), y_3(t)),\ \ t \in [0,1],\ \ i = 1,2,3.$$

Now, we show that T is a contraction mapping on Ω_ρ. As a matter of fact, for any $x = (x_1, x_2, x_3) \in X$ and $y = (y_1, y_2, y_3) \in X$, we have

$$\begin{aligned}
&|(T_1x)(t) - (T_1y)(t)| \\
&\le \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |\phi_{1x}(s) - \phi_{1y}(s)|\, ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x_1(s) - y_1(s)|\, ds + \frac{|M_{11}(t)| + |M_{14}(t)|\lambda}{\Gamma(\alpha+\beta)} \\
&\quad \times \int_0^1 (1-s)^{\alpha+\beta-1} (|\phi_{2x}(s) - \phi_2 y(s)| + |\phi_{3x}(s) - \phi_3 y(s)|) ds \\
&\quad + \frac{|M_{12}(t)| + |M_{15}(t)|\lambda}{\Gamma(\alpha+\beta)} \\
&\quad \times \int_0^1 (1-s)^{\alpha+\beta-1} (|\phi_{3x}(s) - \phi_{3y}(s)| + |\phi_{1x}(s) - \phi_{1y}(s)|)\, ds \\
&\quad + \frac{|M_{13}(t)| + |M_{16}(t)|\lambda}{\Gamma(\alpha+\beta)} \\
&\quad \times \int_0^1 (1-s)^{\alpha+\beta-1} (|\phi_{1x}(s) - \phi_{1y}(s)| + |\phi_{2x}(s) - \phi_{2y}(s)|)\, ds \\
&\quad + \frac{|M_{11}(t)|\lambda + |M_{14}(t)|\lambda^2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (|x_2(s) - y_2(s)| + |x_3(s) - y_3(s)|) ds \\
&\quad + \frac{|M_{12}(t)|\lambda + |M_{15}(t)|\lambda^2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (|x_3(s) - y_3(s)| + |x_1(s) - y_1(s)|)\, ds \\
&\quad + \frac{|M_{13}(t)|\lambda + |M_{16}(t)|\lambda^2}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (|x_1(s) - y_1(s)| + |x_2(s) - y_2(s)|)\, ds \\
&\quad + \frac{|M_{14}(t)|}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (|\phi_{2x}(s) - \phi_{2y}(s)| + |\phi_{3x}(s) - \phi_{3y}(s)|)\, ds \\
&\quad + \frac{|M_{15}(t)|}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (|\phi_{3x}(s) - \phi_{3y}(s)| + |\phi_{1x}(s) - \phi_{1y}(s)|)\, ds \\
&\quad + \frac{|M_{16}(t)|}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} (|\phi_{1x}(s) - \phi_{1y}(s)| + |\phi_{2x}(s) - \phi_{2y}(s)|)\, ds \\
&\le \frac{\left(|M_{11}(t)|\lambda + |M_{14}(t)|\lambda^2\right)(\|x_2 - y_2\|_\infty + \|x_3 - y_3\|_\infty)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\left(|M_{12}(t)|\lambda + |M_{15}(t)|\lambda^2\right)(\|x_3 - y_3\|_\infty + \|x_1 - y_1\|_\infty)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\left(|M_{13}(t)|\lambda + |M_{16}(t)|\lambda^2\right)(\|x_1 - y_1\|_\infty + \|x_2 - y_2\|_\infty)}{\Gamma(\alpha+1)} \\
&\quad + \frac{L_1}{\Gamma(\alpha+\beta+1)} \|x - y\|_x + \frac{\lambda}{\Gamma(\alpha+1)} \|x_1 - y_1\|_\infty
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\left(\left|M_{11}(t)\right|+\left|M_{14}(t)\right|\lambda\right)\left(L_2+L_3\right)}{\Gamma(\alpha+\beta+1)}\|x-y\|_X \\
&+\frac{\left(\left|M_{12}(t)\right|+\left|M_{15}(t)\right|\lambda\right)\left(L_3+L_1\right)}{\Gamma(\alpha+\beta+1)}\|x-y\|_X \\
&+\frac{\left(\left|M_{13}(t)\right|+\left|M_{16}(t)\right|\lambda\right)\left(L_1+L_2\right)}{\Gamma(\alpha+\beta+1)}\|x-y\|_X \\
&+\frac{\left|M_{14}(t)\right|\left(L_2+L_3\right)\|x-y\|_X}{\Gamma(\beta+1)}+\frac{\left|M_{15}(t)\right|\left(L_3+L_1\right)\|x-y\|_X}{\Gamma(\beta+1)} \\
&+\frac{\left|M_{16}(t)\right|\left(L_1+L_2\right)\|x-y\|_X}{\Gamma(\beta+1)} \\
\leq\ & \Lambda_1\|x-y\|_X+\frac{N_1\lambda+N_2\lambda^2}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda}{\Gamma(\alpha+1)}\|x_1-y_1\|_\infty .
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
&\left|(T_2x)(t)-(T_2y)(t)\right| \\
\leq\ & \Lambda_2\|x-y\|_X+\frac{N_1\lambda+N_2\lambda^2}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda}{\Gamma(\alpha+1)}\|x_2-y_2\|_\infty, \\
&\left|(T_3x)(t)-(T_3y)(t)\right| \\
\leq\ & \Lambda_3\|x-y\|_X+\frac{N_1\lambda+N_2\lambda^2}{\Gamma(\alpha+1)}\|x-y\|_X+\frac{\lambda}{\Gamma(\alpha+1)}\|x_3-y_3\|_\infty .
\end{aligned}
$$

According to the above inequality, we have

$$
\|Tx-Ty\|_X \leq \left[\Lambda_1+\Lambda_2+\Lambda_3+\frac{(3N_1+1)\lambda+3N_2\lambda^2}{\Gamma(\alpha+1)}\right]\|x-y\|_X . \tag{3.4}
$$

By (3.3), it follows that T is a contraction. Then the operator T has the unique fixed point $x\in\Omega_\rho$, which is the unique solution of the system (1.1). □

4. Ulam-Hyers stability analysis of (1.1)

In this part, the Ulam-Hyers stability of the system (1.1) will be shown. For this purpose, we first present the concept of stability related to our problem. For $(i=1,2,3)$, assume that $\epsilon_i>0, g_i:[0,1]\times\mathbb{R}^3\to\mathbb{R}$ are continuous functions and $\Psi_i:[0,1]\to\mathbb{R}^+$ are non-increase continuous functions. Now, let us show the following two inequalities.

$$
\begin{aligned}
&\left|{}^C D_{0+}^{\beta}\left({}^C D_{0+}^{\alpha}+\lambda\right)x_i(t)-g_i\left(t,x_1(t),x_2(t),x_3(t)\right)\right| \\
\leq\ & \epsilon_i,\ \ t\in[0,1],\, i=1,2,3, \qquad (4.1)\\
&\left|{}^C D_{0+}^{\beta}\left({}^C D_{0+}^{\alpha}+\lambda\right)x_i(t)-g_i\left(t,x_1(t),x_2(t),x_3(t)\right)\right| \\
\leq\ & \Psi_i(t)\,\epsilon_i,\ \ t\in[0,1],\, i=1,2,3. \qquad (4.2)
\end{aligned}
$$

Definition 4.1. If there is a constant $c_{g_1,g_2,g_3}>0$ such that for each $\epsilon=\epsilon\left(\epsilon_1,\epsilon_2,\epsilon_3\right)>0$ and for each $v=(v_1,v_2,v_3)\in X$ satisfying the inequalities (4.1) and conditions (2.2)-(2.4), there exists a solution $u=(u_1,u_2,u_3)\in X$ of (1.1) meeting

$$
\|\,u-v\|_X\leq c_{g_1,g_2,g_3}\epsilon .
$$

Then, the system (1.1) is called Ulam-Hyers stable.

Remark 4.1. The fuction $v=(v_1,v_2,v_3)\in X$ is called a solution of (4.1), for $i=1,2,3$, if there exist functions $\Phi_i\in C[0,1]$ that depend on v_i respectively such that the following conditions hold.

(i) $|\Phi_i(t)|\leq \epsilon_i, t\in[0,1]$;

(ii) ${}^CD_{0+}^{\beta}\left({}^CD_{0+}^{\alpha}+\lambda\right)v_i(t)=g_i(t,v_1(t),v_2(t),v_3(t))+\Phi_i(t), t\in[0,1]$.

Next, the sufficient conditions of Ulam-Hiers stability for the system (1.1) is provided.

Theorem 4.1. *Assume that $(H_1),(H_3)$ and (3.14) are satisfied. If $u=(u_1,u_2,u_3)\in X$ is the solution of the system (1.1) and $v=(v_1,v_2,v_3)\in X$ is the solution of the inequality problem (4.1) and (2.2)-(2.4). Then, there exists a constant $c_{g_1,g_2,g_3}>0$ such that for each $\epsilon=\epsilon(\epsilon_1,\epsilon_2,\epsilon_3)>0$,*

$$\| u-v\|_X\leq c_{g_1,g_2,g_3}\epsilon,$$

which means that the system (1.1) is Ulam-Hyers stable.

Proof. Based on the fact that v is the solution of (4.1) and (2.2)-(2.4), in view of Remark 4.1, we get v_i is the solution of the following problem.

$$\begin{cases} {}^CD_{0+}^{\beta}\left({}^CD_{0+}^{\alpha}+\lambda\right)v_i(t)=g_i(t,v_1(t),v_2(t),v_3(t))+\Phi_i(t), t\in(0,1), i=1,2,3,\\ a(v_1(0)+v_2(0))=-(v_2(1)+v_3(1)),\\ {}^CD_{0+}^{\alpha}v_1(0)+{}^CD_{0+}^{\alpha}v_2(0)=-\left({}^CD_{0+}^{\alpha}v_2(1)+{}^CD_{0+}^{\alpha}v_3(1)\right),\\ a(v_2(0)+v_3(0))=-(v_3(1)+v_1(1)),\\ {}^CD_{0+}^{\alpha}v_2(0)+{}^CD_{0+}^{\alpha}v_3(0)=-\left({}^CD_{0+}^{\alpha}v_3(1)+{}^CD_{0+}^{\alpha}v_1(1)\right),\\ a(v_3(0)+v_1(0))=-(v_1(1)+v_2(1)),\\ {}^CD_{0+}^{\alpha}v_3(0)+{}^CD_{0+}^{\alpha}v_1(0)=-\left({}^CD_{0+}^{\alpha}v_1(1)+{}^CD_{0+}^{\alpha}v_2(1)\right). \end{cases} \tag{4.3}$$

From Lemma 3.1, the solution $v=(v_1,v_2,v_3)\in X$ of system (4.3) is presented as follows.

$$\begin{aligned} & v_i(t)\\ =& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\left[\tilde{\phi}_i(s)+\Phi_i(s)\right]ds-\frac{\lambda}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v_i(s)\,ds\\ &+\frac{-M_{i1}(t)+M_{i4}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left[\tilde{\phi}_2(s)+\tilde{\phi}_3(s)+\Phi_2(s)+\Phi_3(s)\right]ds\\ &+\frac{-M_{i2}(t)+M_{i5}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left[\tilde{\phi}_3(s)+\tilde{\phi}_1(s)+\Phi_3(s)+\Phi_1(s)\right]ds\\ &+\frac{-M_{i3}(t)+M_{i6}(t)\lambda}{\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta-1}\left[\tilde{\phi}_1(s)+\tilde{\phi}_2(s)+\Phi_1(s)+\Phi_2(s)\right]ds\\ &+\frac{M_{i1}(t)\lambda-M_{i4}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left[v_2(s)+v_3(s)\right]ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{M_{i2}(t)\lambda - M_{i5}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left[v_3(s)+v_1(s)\right]ds \\
&+\frac{M_{i3}(t)\lambda - M_{i6}(t)\lambda^2}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left[v_1(s)+v_2(s)\right]ds \\
&-\frac{M_{i4}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\tilde{\phi}_2(s)+\tilde{\phi}_3(s)+\Phi_2(s)+\Phi_3(s)\right]ds \\
&-\frac{M_{i5}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\tilde{\phi}_3(s)+\tilde{\phi}_1(s)+\Phi_3(s)+\Phi_1(s)\right]ds \\
&-\frac{M_{i6}(t)}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\tilde{\phi}_1(s)+\tilde{\phi}_2(s)+\Phi_1(s)+\Phi_2(s)\right]ds,\ t\in[0,1],
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{\phi}_1(s) &= g_1(s, v_1(s), v_2(s), v_3(s)), \\
\tilde{\phi}_2(s) &= g_2(s, v_1(s), v_2(s), v_3(s)), \\
\tilde{\phi}_3(s) &= g_3(s, v_1(s), v_2(s), v_3(s)).
\end{aligned}
$$

Under current conditions, review the operator T that defined in (3.11), it follows that T is a contraction operator. Thus, the system (1.1) has the unique solution $u=(u_1,u_2,u_3)\in X$ that is the fixed point of T. From (3.4), we have

$$
\begin{aligned}
\| Tu - Tv\|_X &= \| u - Tv\|_X \\
&\le \left[\Lambda_1+\Lambda_2+\Lambda_3+\frac{\left[(3N_1+1)\lambda+3N_2\lambda^2\right]}{\Gamma(\alpha+1)}\right]\| u-v\|_X,
\end{aligned}
$$

which means

$$
\| u-v\|_X \le \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)\left[1-(\Lambda_1+\Lambda_2+\Lambda_3)\right]-\left[(3N_1+1)\lambda+3N_2\lambda^2\right]}\| Tv-v\|_X. \tag{4.4}
$$

Moreover, the following estimate can be obtained.

$$
\begin{aligned}
&|(T_1v)(t)-v_1(t)| \\
\le& \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}|\Phi_1(s)|ds \\
&+\frac{|M_{11}(t)|+|M_{14}\lambda|}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|\Phi_2(s)+\Phi_3(s)|ds \\
&+\frac{|M_{12}(t)|+|M_{15}\lambda|}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|\Phi_3(s)+\Phi_1(s)|ds \\
&+\frac{|M_{13}(t)|+|M_{16}\lambda|}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}|\Phi_1(s)+\Phi_2(s)|ds \\
&+\frac{|M_{14}(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\Phi_2(s)+\Phi_3(s)\right]ds \\
&+\frac{|M_{15}(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\Phi_3(s)+\Phi_1(s)\right]ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|M_{16}(t)|}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\left[\Phi_1(s)+\Phi_2(s)\right]ds\\
&\le \frac{(|M_{15}(t)|+|M_{16}(t)|)\,\epsilon_1+(|M_{14}(t)|+|M_{16}(t)|)\,\epsilon_2+(|M_{15}(t)|+|M_{14}(t)|)\,\epsilon_3}{\Gamma(\beta+1)}\\
&+\frac{(1+|M_{12}(t)|+|M_{13}(t)|+|M_{15}(t)|\,\lambda+|M_{16}(t)|\,\lambda)\,\epsilon_1}{\Gamma(\alpha+\beta+1)}\\
&+\frac{(|M_{11}(t)|+|M_{13}(t)|+|M_{14}(t)|\,\lambda+|M_{16}(t)|\,\lambda)\,\epsilon_2}{\Gamma(\alpha+\beta+1)}\\
&+\frac{(|M_{1}(t)|+|M_{12}(t)|+|M_{14}(t)|\,\lambda+|M_{15}(t)|\,\lambda)\,\epsilon_3}{\Gamma(\alpha+\beta+1)}\\
&\le \xi\epsilon_1+\eta\epsilon_2+\gamma\epsilon_3.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|(T_2v)(t)-v_2(t)| &\le \xi\epsilon_2+\eta\epsilon_3+\gamma\epsilon_1,\\
|(T_3v)(t)-v_3(t)| &\le \xi\epsilon_3+\eta\epsilon_1+\gamma\epsilon_2.
\end{aligned}
$$

Thus, it follows

$$
\begin{aligned}
\|\,Tv-v\|_X &= \|T_1v-v_1\|_\infty+\|T_2v-v_2\|_\infty+\|T_3v-v_3\|_\infty\\
&\le (\xi+\eta+\gamma)\sum_{i=1}^{3}\epsilon_i.
\end{aligned}
$$

Setting $\epsilon=\max\{\epsilon_1,\epsilon_2,\epsilon_3\}$, by (4.4), we obtain

$$
\|\,u-v\|_X \le \frac{3\Gamma(\alpha+1)(\xi+\eta+\gamma)\,\epsilon}{\Gamma(\alpha+1)\left[1-(\Lambda_1+\Lambda_2+\Lambda_3)\right]-\left[(3N_1+1)\,\lambda+3N_2\lambda^2\right]}.
$$

Consequently, the system (1.1) is Ulam-Hyers stable. □

5. Example

Example 5.1. Let $\alpha=0.1,\ \beta=0.2,\ \lambda=0.001\ \ a=0.8$. The following tripled system has been considered.

$$
\begin{cases}
{}^{C}D_{0+}^{1/5}\left({}^{C}D_{0+}^{1/10}+(1/1000)\right)x_i(t)=g_i(t,x_1(t),x_2(t),x_3(t)),\ i=1,2,3,\\
\frac{4}{5}(x_1(0)+x_2(0))=-(x_2(1)+x_3(1)),\\
{}^{C}D_{0+}^{1/10}x_1(0)+{}^{C}D_{0+}^{1/10}x_2(0)=-\left({}^{C}D_{0+}^{1/10}x_2(1)+{}^{C}D_{0+}^{1/10}x_3(1)\right),\\
\frac{4}{5}(x_2(0)+x_3(0))=-(x_3(1)+x_1(1)),\\
{}^{C}D_{0+}^{1/10}x_2(0)+{}^{C}D_{0+}^{1/10}x_3(0)=-\left({}^{C}D_{0+}^{1/10}x_3(1)+{}^{C}D_{0+}^{1/10}x_1(1)\right),\\
\frac{4}{5}(x_3(0)+x_1(0))=-(x_1(1)+x_2(1)),\\
{}^{C}D_{0+}^{1/10}x_3(0)+{}^{C}D_{0+}^{1/10}x_1(0)=-\left({}^{C}D_{0+}^{1/10}x_1(1)+{}^{C}D_{0+}^{1/10}x_2(1)\right),
\end{cases}
\tag{5.1}
$$

where

$$g_1(t, x_1(t), x_2(t), x_3(t)) = t^{3/4} + \frac{1}{80}\sin x_1(t) + \frac{1}{80}x_2(t) + \frac{1}{80}x_3(t),$$
$$g_2(t, x_1(t), x_2(t), x_3(t)) = t^{5/8} + \frac{1}{280}x_1(t) + \frac{1}{90}x_2(t) + \frac{1}{50}\sin x_3(t),$$
$$g_3(t, x_1(t), x_2(t), x_3(t)) = t^{7/10} + \frac{1}{80}x_1(t) + \frac{1}{70}\sin x_2(t) + \frac{1}{80}x_3(t).$$

Choosing

$$k_1^1(t) = t^{3/4}, k_2^1(t) = t^{5/8}, k_3^1(t) = t^{7/10},$$
$$k_1^2(t) = \frac{1}{80}, k_2^2(t) = \frac{1}{280}, k_3^2(t) = \frac{1}{80},$$
$$k_1^3(t) = \frac{1}{80}, k_2^3(t) = \frac{1}{90}, k_3^3(t) = \frac{1}{70},$$
$$k_1^4(t) = \frac{1}{80}, k_2^4(t) = \frac{1}{50}, k_3^4(t) = \frac{1}{80},$$

then the assumptions (H_1) and (H_2) hold. Furthermore, we can figure it out as follows.

$$\xi + \eta + \gamma < 8.1, \quad N_1 < 2, \quad N_2 < 1.5, \quad \sum_{i=1}^{3} \ell_i \approx 0.1115.$$

Thus, we get

$$(\xi + \eta + \gamma)\sum_{i=1}^{3} \ell_i + \frac{(3N_1 + 1)\lambda + 3N_2\lambda^2}{\Gamma(\alpha + 1)} \approx 0.9026 < 1.$$

So, the condition (2.2) is satisfied. Consequently, it follows that the system (5.1) has at least one solution.

Example 5.2. Let $\alpha = 0.1, \beta = 0.15, \lambda = 0.002, a = 0.8$. The following tripled system has been considered.

$$\begin{cases} {}^{C}D_{0+}^{3/20}\left({}^{C}D_{0+}^{1/10} + (1/500)\right)x_i(t) = g_i(t, x_1(t), x_2(t), x_3(t)), \ i = 1, 2, 3, \\ \frac{4}{5}(x_1(0) + x_2(0)) = -(x_2(1) + x_3(1)), \\ {}^{C}D_{0+}^{1/10}x_1(0) + {}^{C}D_{0+}^{1/10}x_2(0) = -\left({}^{C}D_{0+}^{1/10}x_2(1) + {}^{C}D_{0+}^{1/10}x_3(1)\right), \\ \frac{4}{5}(x_2(0) + x_3(0)) = -(x_3(1) + x_1(1)), \\ {}^{C}D_{0+}^{1/10}x_2(0) + {}^{C}D_{0+}^{1/10}x_3(0) = -\left({}^{C}D_{0+}^{1/10}x_3(1) + {}^{C}D_{0+}^{1/10}x_1(1)\right), \\ \frac{4}{5}(x_3(0) + x_1(0)) = -(x_1(1) + x_2(1)), \\ {}^{C}D_{0+}^{1/10}x_3(0) + {}^{C}D_{0+}^{1/10}x_1(0) = -\left({}^{C}D_{0+}^{1/10}x_1(1) + {}^{C}D_{0+}^{1/10}x_2(1)\right), \end{cases} \tag{5.2}$$

where

$$f_1(t, x_1(t), x_2(t), x_3(t)) = \frac{1}{40}\left[\frac{|x_1(t)|}{4 + |x_1(t)|} + \sin|x_2(t)| + \frac{|x_3(t)|}{1 + |x_3(t)|}\right],$$

$$f_2(t,x_1(t),x_2(t),x_3(t)) = \frac{2}{55}\left[\sin|x_1(t)| + \frac{|x_2(t)|}{1+|x_2(t)|} + \frac{|x_3(t)|}{4+|x_3(t)|}\right],$$
$$f_3(t,x_1(t),x_2(t),x_3(t)) = \frac{1}{20}\left[\frac{|x_1(t)|}{1+|x_1(t)|} + \frac{|x_2(t)|}{4+|x_2(t)|} + \sin|x_3(t)|\right].$$

Choosing

$$L_1 = \frac{1}{40}, \quad L_2 = \frac{2}{55}, \quad L_3 = \frac{1}{20},$$

then the assumption (H_1) and (H_3) hold. Furthermore, we can figure it out as follows.

$$\xi + \eta + \gamma < 8, \quad N_1 < 2, \quad N_2 < 1.5, \quad \Lambda_1 + \Lambda_2 + \Lambda_3 \approx 0.8845.$$

So, we obtain

$$\Lambda_1 + \Lambda_2 + \Lambda_3 + \frac{\left[(3N_1+1)\lambda + 3N_2\lambda^2\right]}{\Gamma(\alpha+1)} \approx 0.8992 < 1.$$

Thus, the condition (3.3) is also satisfied. Then the system (5.2) has the unique solution.

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Conffícts of interest

The authors declare no conffícts of interest.

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