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ON THE EQUIVALENCE OF THE EFFECTIVE DEGREE NETWORK MODEL AND DYNAMICAL SURVIVAL ANALYSIS MODEL*

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Abstract We delve into the existing effective degree model and dynamical survival analysis model for network epidemic dynamics. By employing the integrating factor method, we elaborate on the mutual derivation process between the two models, demonstrating their equivalence. Leveraging this result, the effective degree model is simplified to an equation that only involves susceptible individuals.

Keywords Effective degree model, dynamical survival analysis model, equivalence.

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1. Introduction

Infectious diseases, which are caused by a variety of pathogens and capable of being transmitted between humans and animals, have attracted considerable attention from scholars who have employed mathematical models to study these diseases [1, 4, 24]. Due to the profound application of network transmission dynamics in the context of infectious diseases, various network-based infectious disease models have emerged [6, 11, 13, 21], and these models represent contact as a random graph of N nodes formed using a configuration model [18, 20]. Rand [22] and Keeling's pairwise model [7] takes the binary and ternary groups formed by adjacent nodes as the basic variables of the model, and adopts a pairwise approximation method to close the model. Keeling's model provides a basic reproduction number that differs from those presented by [19] and [3], as it employs a rough approximation of the degree of a given edge node. Volz's edge-based compartmental model [23], on the other hand, utilizes the probability of a susceptible node remaining susceptible at a given time and the probability generating function, making it suitable for any configuration network, and Decreusefond et al. [2] demonstrate that it is the large N limit of a stochastic SIR epidemic on a configuration model network. Unlike the first two models, the effective degree model established by Lindquist et al. [15] focuses on the degree and state of nodes, as well as the states of their neighboring

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nodes, tracking the changes in neighbor states to investigate the dynamic properties. Notably, Miller [17] has demonstrated that these three models are equivalent under certain assumptions, implying that different models may possess similar predictive capabilities and effects when describing the same infectious disease transmission process.

Jacobsen and colleagues [5] successfully derived the large graph limit system for the stochastic SIR model in multilayer networks using a statistical inference approach. KhudaBukhsh and others [8] applied this system to a single-layer network to study the prevalence of COVID-19, referring to it as dynamical survival analysis (DSA) [9, 14]. In their recent work, Kiss et al. [12] further named the single-layer network form of the Jacobsen model as the DSA model and proved that under precise closure, the pairwise model is equivalent to the DSA model. Additionally, they demonstrated that the Volz model is also equivalent to the DSA model under any distribution. Unfortunately, there are no relevant research on the relationship between the effective degree model and the DSA model. In this paper, we focus on the equivalence between the effective degree model and the DSA model, leveraging an integral factor approach and variable relationships to prove this point, providing a detailed mutual transformation process between the two models.

The remainder of this paper is organized as follows. In Section 2, we present the formulations of the effective degree model and the DSA model and prove two relationships regarding the variables of the DSA model, which will assist in the subsequent proof of equivalence. Section 3 first describes the process of transforming the effective degree model into the DSA model and then provides the reverse process from the DSA model to the effective degree model. In Section 4, we demonstrate the advantages of the equivalence between the DSA model and the effective degree model, and derive an equation for susceptible individuals from the effective degree model. In Section 5, we elaborate on some details regarding the equivalence of the models and conclude with a closing remark.

2. Network-based infectious disease model

The model considered in this paper is the SIR model, where individuals in the network have three states: susceptible (S), infected (I), and recovered (R). It is assumed that the infectious disease spreads in a static network of size N with a configuration model structure, which can be generated by a specific algorithm. Infected individuals are assumed to transmit the disease to each of their partners independently at a rate β according to a Poisson process and recover independently at a rate γ according to a Poisson process. The infectious period and the Poisson processes are assumed to be independent.

2.1. Effective degree model

We define "ineffective" partnerships as those through which we know infection will never be transmitted. The definition of "effective" partnerships hinges on the assumption of which partnerships we know will disseminate infection. If a partnership has not transmitted infection and neither individual is recovered, then it is an effective partnership. However, the effective degree model presented in this paper does not require tracking the number of partners an infected individual has; thus, we augment our definition by stating that an effective partnership must involve at least one susceptible individual. With this definition in place, there is no need to track partnerships among infected individuals. To investigate the effective degree, we define x_j as the number of susceptible nodes with an effective degree of j, and y_j as the number of infected nodes with an effective degree of j. S, I and R represent the number of susceptible nodes, infected nodes, and recovered nodes, respectively. $\langle I\rangle$ is the probability that a randomly selected effective partner is infected, and ν is the total number of effective links between susceptible and infected nodes. The following effective degree model [17] is established:

$$
\begin{aligned}
\dot{x_j} &= \gamma\langle I\rangle\left[(j+1)x_{j+1} - jx_j\right] - \beta\langle I\rangle jx_j,\\
\dot{\nu} &= -(\beta+\gamma)\,\nu + \beta\langle I\rangle(1-2\langle I\rangle)\sum_j j(j-1)x_j,\\
\langle I\rangle &= \frac{\nu}{\sum_j jx_j},\\
S &= \sum_j x_j,\\
I &= N - S - R,\\
\dot{R} &= \gamma I.
\end{aligned}
\tag{2.1}
$$

Note that the model implicitly includes y_j since $\nu = \sum_j jy_j$. Assuming that the maximum degree of nodes in the network is M, it is not difficult to observe that the number of differential equations in the effective degree model is $M+3$, which is far fewer than the number of differential equations in the effective degree model proposed by Lindquist et al. [15]. Therefore, model (2.1) is also referred to as the reduced effective degree model. The initial conditions are:

$$
\begin{aligned}
x_j(0) &= \varepsilon_j,\\
\nu(0) &= N\mu\rho,\\
\langle I\rangle(0) &= \frac{\rho}{1+\rho},\\
S(0) &= N(1-\rho),\\
I(0) &= N\rho,\\
R(0) &= 0,
\end{aligned}
\tag{2.2}
$$

where $0<\rho\ll 1$, $\varepsilon_j \geq 0$, $\mu>0$ and $\sum_j \varepsilon_j = N(1-\rho)$.

2.2. Dynamical survival analysis model

In Jacobsen et al.'s work [5], the derived stochastic process was difficult to analyze. Subsequently, using a limiting theorem, when the number of nodes approaches infinity, the stochastic process converges to a system of ordinary differential equations. This limiting system, under the context of a single-layer network, is referred to as the dynamical survival analysis model. Define x_θ as the probability that an initially degree-1 susceptible node remains susceptible at time t in an infinite network. $[A]$ represents the number of nodes in state A , $[AB]$ represents the number of pairs formed by nodes in state A and nodes in state B, x_A denotes $\lim\limits_{N\to\infty}\frac{[A]}{N}$, and x_{AB} denotes $\lim\limits_{N\to\infty}\frac{[AB]}{N}$, where $A,\ B\in\{S\ ,I\ ,R\}$. The following dynamical survival analysis model [12] is established:

$$
\begin{aligned}
\dot{x}_\theta &= -\beta \frac{x_{SI}}{\psi'(x_\theta)}, \\
\dot{x}_{SS} &= -2\beta x_{SI} x_{SS} \frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}, \\
\dot{x}_{SI} &= x_{SI}\left[\beta\big(x_{SS}-x_{SI}\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} - (\beta+\gamma)\right], \\
\dot{x}_S &= -\beta x_{SI}, \\
\dot{x}_I &= \beta x_{SI} - \gamma x_I.
\end{aligned}
\tag{2.3}
$$

Here, $\psi(x_\theta) = \sum_{k=0}^{\infty} p_k x_\theta^k$ is the probability generating function [16], representing the probability that a randomly selected node remains susceptible at time t, p_k represents the degree distribution, which is the probability that a randomly selected node in the network has a degree of k. x_θ^k denotes the probability that an initial susceptible node with degree k remains susceptible at time t. The variables x_S, x_I and x_R satisfy the constraint $x_S + x_I + x_R = 1$. The initial conditions are:

$$
\begin{aligned}
&x_S(0) = x_\theta(0) = 1-\rho, \\
&x_I(0) = \rho, \\
&x_{SS}(0) = \mu, \\
&x_{SI}(0) = \mu\rho,
\end{aligned}
\tag{2.4}
$$

where $0 < \rho \ll 1$ and $\mu > 0$.

We will now prove that $(x_{SS} + x_{SI} + x_{SR})/\psi'(x_\theta) = x_\theta$ and $x_S = \psi(x_\theta)$. Define $[S_k]$ as the number of degree-k susceptible nodes at time t, then we have the following:

$$\lim_{N\to\infty} \frac{[S_k]}{N} = p_k x_\theta^k, \tag{2.5}$$

since the series $\sum_k \frac{k[S_k]}{N} = \frac{[SS]+[SI]+[SR]}{N}$ is convergent, then

$$\lim_{N\to\infty} \sum_k \frac{k[S_k]}{N} = \sum_k \lim_{N\to\infty} \frac{k[S_k]}{N} = \sum_k k p_k x_\theta^k = x_\theta \sum_k k p_k x_\theta^{k-1} = x_\theta \psi'(x_\theta), \tag{2.6}$$

so

$$\lim_{N\to\infty} \frac{[SS]+[SI]+[SR]}{N} = x_{SS} + x_{SI} + x_{SR} = x_\theta \psi'(x_\theta), \tag{2.7}$$

namely

$$\frac{x_{SS} + x_{SI} + x_{SR}}{\psi'(x_\theta)} = x_\theta. \tag{2.8}$$

Similarly, since the series $\sum_k \frac{[S_k]}{N} = \frac{[S]}{N}$ is convergent, then

$$\lim_{N\to\infty} \sum_k \frac{[S_k]}{N} = \sum_k \lim_{N\to\infty} \frac{[S_k]}{N} = \sum_k p_k x_\theta^k = \psi(x_\theta), \tag{2.9}$$

so

$$x_S = \psi(x_\theta). \tag{2.10}$$

These two results will play a role in the derivation of the effective degree model from the DSA model in the subsequent sections.

3. Equivalence of models

3.1. From effective degree model to DSA model

During the derivation of the DSA model from the effective degree model, the variables in the DSA model need to be assumed to appear "for the first time." Therefore, the form of each variable in the DSA model will be gradually defined. The key question is how the forms of these variables should be related to the variables in the effective degree model. From the definition of ν, it is proportional to x_{SI}. Similarly, $\sum_j jx_j$ is proportional to $x_{SS} + x_{SI}$. Therefore, to define x_{SI}, it is necessary to study $\dot{\nu}$. According to the second and third equations in model (2.1), we have

$$\begin{aligned}\dot{\nu} &= -(\beta+\gamma)\nu + \beta\langle I\rangle(1-2\langle I\rangle)\sum_j j(j-1)x_j \\ &= -(\beta+\gamma)\nu + \beta\langle I\rangle(1-\langle I\rangle)\sum_j j(j-1)x_j - \beta\langle I\rangle^2\sum_j j(j-1)x_j \\ &= -(\beta+\gamma)\nu + \beta\langle I\rangle(1-\langle I\rangle)\sum_j j(j-1)x_j - \beta\langle I\rangle\frac{\sum_j j(j-1)x_j}{\sum_j jx_j}\nu.\end{aligned} \tag{3.1}$$

Here, an integrating factor method is employed to eliminate the term $-\beta\langle I\rangle \times \frac{\sum_j j(j-1)x_j}{\sum_j jx_j}\nu$. Define an integrating factor $F(t)$ such that $F'(t) = \beta\langle I\rangle\frac{\sum_j j(j-1)x_j}{\sum_j jx_j}$. Introduce a variable x_θ and define $\dot{x}_\theta = -\beta\frac{x_{SI}}{\sum_k k(\frac{x_k(0)}{N})x_\theta^{k-1}}$. Simultaneously, define $\psi(x_\theta) = \sum_k \frac{x_k(0)}{N}x_\theta^k$. It is clear that $\dot{x}_\theta = -\beta\frac{x_{SI}}{\psi'(x_\theta)}$. Furthermore, define $\frac{x_{SI}}{\psi'(x_\theta)} = \nu e^{F(t)}$ and $\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} = e^{F(t)}\sum_j jx_j$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SI}}{\psi'(x_\theta)} &= -(\beta+\gamma)\nu e^{F(t)} + \beta\langle I\rangle(1-\langle I\rangle)\sum_j j(j-1)x_j e^{F(t)} \\ &= -(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)} + \beta\langle I\rangle\left(1-\frac{\nu}{\sum_j jx_j}\right)\sum_j j(j-1)x_j e^{F(t)} \\ &= -(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)} + \beta\langle I\rangle\frac{\sum_j jx_j - \nu}{\sum_j jx_j}e^{F(t)}\sum_j j(j-1)x_j \\ &= -(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)} + \beta\langle I\rangle\frac{x_{SS}}{\psi'(x_\theta)}\frac{\sum_j j(j-1)x_j}{\sum_j jx_j} \\ &= -(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)} + \frac{x_{SS}}{\psi'(x_\theta)}F'(t).\end{aligned} \tag{3.2}$$

Next, we consider $\frac{\mathrm{d}}{\mathrm{d}t}\sum_j jx_j$. According to the first equation in model (2.1), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sum_j jx_j &= \sum_j j\dot{x}_j \\
&= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \sum_j j^2 x_j \\
&= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \sum_j (j^2 x_j - jx_j + jx_j) \\
&= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \sum_j \left[j(j-1)x_j + jx_j\right] \\
&= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \sum_j j(j-1)x_j - \beta\langle I\rangle \sum_j jx_j \\
&= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \frac{\sum_j j(j-1)x_j}{\sum_j jx_j} \sum_j jx_j \\
&\quad - \beta\langle I\rangle \sum_j jx_j,
\end{aligned}
\tag{3.3}
$$

using the same integrating factor $F(t)$ to eliminate $-\beta\langle I\rangle \frac{\sum_j j(j-1)x_j}{\sum_j jx_j}\sum_j jx_j$, it was previously defined that $\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} = e^{F(t)}\sum_j jx_j$, then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} &= \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] e^{F(t)} - \beta\langle I\rangle \sum_j jx_j e^{F(t)} \\
&= \gamma\langle I\rangle \left\{ \sum_j \left[(j+1)^2 - (j+1)\right] x_{j+1} - \sum_j j^2 x_j \right\} e^{F(t)} \\
&\quad - \beta\langle I\rangle \frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} \\
&= -\gamma\langle I\rangle \sum_j (j+1)x_{j+1} e^{F(t)} - \beta\langle I\rangle \frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} \\
&= -\gamma\langle I\rangle \frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} - \beta\langle I\rangle \frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}.
\end{aligned}
\tag{3.4}
$$

From $\frac{x_{SI}}{\psi'(x_\theta)} = \nu e^{F(t)}$ and $\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} = e^{F(t)}\sum_j jx_j$ along with the third equation in model (2.1), we can derive $\langle I\rangle = \frac{x_{SI}}{\psi'(x_\theta)} / \frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}$. Substituting this expression into equation (3.4) yields

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)} = -(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)}, \tag{3.5}
$$

combining with equation (3.2), we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SS}}{\psi'(x_\theta)} = -\frac{x_{SS}}{\psi'(x_\theta)}F'(t). \tag{3.6}
$$

Next, we introduce a new variable: x_{SR} and define it as $x_{SR} = (x_\theta - \frac{x_{SS}-x_{SI}}{\psi'(x_\theta)}) \cdot \psi'(x_\theta)$. Using the previous results, it is straightforward to derive $\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SR}}{\psi'(x_\theta)} = \gamma\frac{x_{SI}}{\psi'(x_\theta)}$. We now claim that $x_j = \sum_{k\geq j} x_k(0)C_k^j(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)})^j(\frac{x_{SR}}{\psi'(x_\theta)})^{k-j}$. We will verify the correctness of this claim by taking the derivative of x_j:

$$\begin{aligned}
\dot{x}_j &= \sum_{k\geq j} x_k(0)C_k^j \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right] \\
&= \sum_{k\geq j} x_k(0)C_k^j \left[-j(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)}\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j-1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\quad \left.+\gamma(k-j)\frac{x_{SI}}{\psi'(x_\theta)}\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
&= \frac{\left(\frac{x_{SI}}{\psi'(x_\theta)}\right)}{\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)}\sum_{k\geq j} x_k(0)C_k^j \left[-j(\beta+\gamma)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\quad \left.+\gamma(k-j)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j+1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
&= \frac{\left(\frac{x_{SI}}{\psi'(x_\theta)}\right)}{\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)}\sum_{k\geq j} x_k(0)\left[-jC_k^j(\beta+\gamma)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\quad \left.+\gamma C_k^j(k-j)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j+1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
&= \langle I\rangle\left[-(\beta+\gamma)jx_j+\gamma x_{j+1}\right].
\end{aligned} \tag{3.7}$$

This is consistent with the first equation in model (2.1), thus our claim is verified. Therefore

$$\begin{aligned}
\sum_j x_j &= \sum_k\sum_j x_k(0)C_k^j\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j} \\
&= \sum_k x_k(0)\left(\frac{x_{SS}+x_{SI}+x_{SR}}{\psi'(x_\theta)}\right)^k \\
&= N\psi(x_\theta).
\end{aligned} \tag{3.8}$$

Now, we need to express $F'(t)$ in terms of the variables in the DSA model. To achieve this, we introduce two equations:

$$\begin{aligned}
\sum_{j=0}^{\infty} C_k^j jp^jq^{k-j} &= p\sum_{j=0}^{\infty} C_k^j jp^{j-1}q^{k-j} \\
&= p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{j=0}^{\infty} C_k^j p^jq^{k-j} \\
&= p\frac{\mathrm{d}}{\mathrm{d}p}(p+q)^k \\
&= kp(p+q)^{k-1},
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\sum_{j=0}^{\infty} C_k^j j(j-1)p^j q^{k-j} &= p^2 \sum_{j=0}^{\infty} C_k^j j(j-1)p^{j-2} q^{k-j} \\ &= p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} \sum_{j=0}^{\infty} C_k^j p^j q^{k-j} \\ &= p^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2} (p+q)^k \\ &= k(k-1)p^2(p+q)^{k-2}.\end{aligned} \tag{3.10}$$

Using these two equations, we can derive that

$$\begin{aligned}F'(t) &= \beta\langle I\rangle \frac{\sum_j j(j-1)x_j}{\sum_j jx_j} \\ &= \beta\langle I\rangle \frac{\sum_k \sum_j j(j-1)x_k(0)C_k^j (\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)})^j (\frac{x_{SR}}{\psi'(x_\theta)})^{k-j}}{\sum_k \sum_j jx_k(0)C_k^j (\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)})^j (\frac{x_{SR}}{\psi'(x_\theta)})^{k-j}} \\ &= \beta\langle I\rangle \frac{(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)})^2 \sum_k k(k-1)x_k(0)x_\theta^{k-2}}{(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}) \sum_k kx_k(0)x_\theta^{k-1}} \\ &= \beta\langle I\rangle \left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right) \frac{\psi''(x_\theta)}{\psi'(x_\theta)} \\ &= \beta \frac{x_{SI}}{\psi'(x_\theta)} \frac{\psi''(x_\theta)}{\psi'(x_\theta)} \\ &= \beta x_{SI} \frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}.\end{aligned} \tag{3.11}$$

Next, we need to find $\dot{x}_{SS}$ and $\dot{x}_{SI}$. The derivative of $\frac{x_{SS}}{\psi'(x_\theta)}$ is obtained as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SS}}{\psi'(x_\theta)} &= \frac{\dot{x}_{SS}\psi'(x_\theta) - x_{SS}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\psi'(x_\theta)}}{\psi'(x_\theta)^2} \\ &= \frac{\dot{x}_{SS}(\psi'(x_\theta))^2 + \beta x_{SS}x_{SI}\psi''(x_\theta)}{\psi'(x_\theta)^3},\end{aligned} \tag{3.12}$$

so

$$\dot{x}_{SS} = \frac{1}{\psi'(x_\theta)^2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SS}}{\psi'(x_\theta)}\right)\psi'(x_\theta)^3 - \beta x_{SS}x_{SI}\psi''(x_\theta)\right], \tag{3.13}$$

using equations (3.6) and (3.11), we can obtain

$$\begin{aligned}\dot{x}_{SS} &= \frac{1}{\psi'(x_\theta)^2}\left(-\frac{x_{SS}}{\psi'(x_\theta)}F'(t)\psi'(x_\theta)^3 - \beta x_{SS}x_{SI}\psi''(x_\theta)\right) \\ &= -x_{SS}F'(t) - \beta x_{SS}x_{SI}\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\ &= -\beta x_{SS}x_{SI}\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} - \beta x_{SS}x_{SI}\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\ &= -2\beta x_{SI}x_{SS}\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}.\end{aligned} \tag{3.14}$$

Similarly, The derivative of $\frac{x_{SI}}{\psi'(x_\theta)}$ is obtained as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SI}}{\psi'(x_\theta)} &= \frac{\dot{x}_{SI}\psi'(x_\theta) - x_{SI}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\psi'(x_\theta)}}{\psi'(x_\theta)^2} \\ &= \frac{\dot{x}_{SI}(\psi'(x_\theta))^2 + \beta x_{SI}^2\psi''(x_\theta)}{\psi'(x_\theta)^3}, \end{aligned} \tag{3.15}$$

so

$$\dot{x}_{SI} = \frac{1}{\psi'(x_\theta)^2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_{SI}}{\psi'(x_\theta)}\right)\psi'(x_\theta)^3 - \beta x_{SI}^2\psi''(x_\theta)\right], \tag{3.16}$$

using equations (3.2) and (3.11), we can obtain

$$\begin{aligned} \dot{x}_{SI} &= \frac{1}{\psi'(x_\theta)^2}\left\{\left[-(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)} + \frac{x_{SS}}{\psi'(x_\theta)}F'(t)\right]\psi'(x_\theta)^3 - \beta x_{SI}^2\psi''(x_\theta)\right\} \\ &= -(\beta+\gamma)x_{SI} + x_{SS}F'(t) - \beta x_{SI}^2\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\ &= -(\beta+\gamma)x_{SI} + \beta x_{SI}x_{SS}\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} - \beta x_{SI}^2\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\ &= x_{SI}\left[\beta\big(x_{SS} - x_{SI}\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} - (\beta+\gamma)\right]. \end{aligned} \tag{3.17}$$

Finally, we define x_S such that $\dot{x}_S = -\beta x_{SI}$ and x_I such that $\dot{x}_I = \beta x_{SI} - \gamma x_I$. Therefore, the DSA model (2.3) is derived from the effective degree model (2.1).

3.2. From DSA model to effective degree model

Now we proceed to derive the effective degree model from the DSA model. Similarly, during the derivation process, the variables in the effective degree model need to be assumed to appear for the "first time". Here are the definitions:

$$\begin{aligned} &x_j = \sum_{k\geq j} N p_k C_k^j \left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j \left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}, \\ &\nu = N x_{SI}, \\ &\sum_j j x_j = N(x_{SS}+x_{SI}), \\ &\langle I\rangle = \frac{\nu}{\sum_j j x_j} = \frac{x_{SI}}{x_{SS}+x_{SI}}, \\ &S = N x_S, \\ &I = N x_I, \\ &R = N x_R. \end{aligned} \tag{3.18}$$

Based on the above definitions, we first take the derivative of x_j:

$$
\begin{aligned}
\dot{x}_j =& \sum_{k\geq j} N p_k C_k^j \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j \left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right] \\
=& \sum_{k\geq j} N p_k C_k^j \left[-j(\beta+\gamma)\frac{x_{SI}}{\psi'(x_\theta)}\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j-1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\left. +\gamma(k-j)\frac{x_{SI}}{\psi'(x_\theta)}\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
=& \frac{\left(\frac{x_{SI}}{\psi'(x_\theta)}\right)}{\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)} \sum_{k\geq j} N p_k C_k^j \left[-j(\beta+\gamma)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\left. +\gamma(k-j)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j+1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
=& \frac{\left(\frac{x_{SI}}{\psi'(x_\theta)}\right)}{\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)} \sum_{k\geq j} N p_k \left[-C_k^j j(\beta+\gamma)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\right. \\
&\left. +C_k^j(k-j)\gamma\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^{j+1}\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j-1}\right] \\
=& \langle I\rangle\big[-(\beta+\gamma)jx_j+\gamma x_{j+1}\big],
\end{aligned}
\tag{3.19}
$$

and then the derivative of ν:

$$
\begin{aligned}
\dot{\nu} &= N\dot{x}_{SI} \\
&= N x_{SI}\left[\beta\big(x_{SS}-x_{SI}\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}-(\beta+\gamma)\right] \\
&= \beta N x_{SI}\big(x_{SS}+x_{SI}-2x_{SI}\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}-(\beta+\gamma)Nx_{SI} \\
&= \beta N\frac{x_{SI}}{x_{SS}+x_{SI}}\big(x_{SS}+x_{SI}\big)\big(x_{SS}+x_{SI}-2x_{SI}\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2}-(\beta+\gamma)\nu \\
&= -(\beta+\gamma)\nu+\beta N\frac{x_{SI}}{x_{SS}+x_{SI}}(x_{SS}+x_{SI})^2\left(1-2\frac{x_{SI}}{x_{SS}+x_{SI}}\right)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\
&= -(\beta+\gamma)\nu+\beta N\langle I\rangle(x_{SS}+x_{SI})^2\big(1-2\langle I\rangle\big)\frac{\psi''(x_\theta)}{\psi'(x_\theta)^2} \\
&= -(\beta+\gamma)\nu+\beta N\langle I\rangle\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^2\big(1-2\langle I\rangle\big)\psi''(x_\theta) \\
&= -(\beta+\gamma)\nu+\beta\langle I\rangle\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^2\big(1-2\langle I\rangle\big)\sum_k k(k-1)Np_k x_\theta^{k-2}.
\end{aligned}
\tag{3.20}
$$

As mentioned earlier when introducing the DSA model, we have already derived the conclusion that $(x_{SS}+x_{SI}+x_{SR})/\psi^{'}(x_\theta)=x_\theta$. Using this conclusion and equation

(3.10), we can infer that

$$k(k-1)Np_k\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^2 x_\theta^{k-2}=\sum_j Np_kC_k^j j(j-1)\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}, \tag{3.21}$$

given the expression $x_j=\sum_{k\geq j} Np_kC_k^j(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)})^j(\frac{x_{SR}}{\psi'(x_\theta)})^{k-j}$, equation (3.20) transforms into

$$\dot{\nu}=-(\beta+\gamma)\nu+\beta\langle I\rangle(1-2\langle I\rangle)\sum_j j(j-1)x_j. \tag{3.22}$$

Finally, we perform transformations on the variables S, I and R. For $\sum_j x_j$, we have

$$\begin{aligned}\sum_j x_j &= \sum_k\sum_j Np_kC_k^j\left(\frac{x_{SS}+x_{SI}}{\psi'(x_\theta)}\right)^j\left(\frac{x_{SR}}{\psi'(x_\theta)}\right)^{k-j}\\ &=\sum_k Np_k\left(\frac{x_{SS}+x_{SI}+x_{SR}}{\psi'(x_\theta)}\right)^k\\ &=N\psi(x_\theta).\end{aligned} \tag{3.23}$$

Using the conclusion $x_S=\psi(x_\theta)$, we have $\sum_j x_j=Nx_S$, which implies that

$$S=\sum_j x_j. \tag{3.24}$$

Given that x_S, x_I and x_R satisfy $x_S+x_I+x_R=1$, it follows that $Nx_S+Nx_I+Nx_R=N$, which means

$$I=N-S-R. \tag{3.25}$$

Also means that

$$\begin{aligned}\dot{R}&=-\dot{S}-\dot{I}\\ &=-N\dot{x}_S-N\dot{x}_I\\ &=N\beta x_{SI}-N(\beta x_{SI}-\gamma x_I)\\ &=\gamma Nx_I\\ &=\gamma I.\end{aligned} \tag{3.26}$$

So far, the effective degree model (2.1) has been derived from the DSA model (2.3).

4. Survival analysis perspective of effective degree model

The significant advantage of proving the equivalence between the effective degree model and the DSA model is that the effective degree model can inherit the statistical interpretation of the DSA model. In [12], the notation $S_t := x_S$ is employed to represent the survival probability of susceptible nodes (i.e., the probability that a node that is susceptible at $t=0$ remains susceptible at $t>0$), and it is demonstrated how to derive a single autonomous differential equation for $S_t := x_S$ from the DSA model. This equation allows for numerical calculations of the survival probability for any $t \in [0, \infty)$ solely based on the parameters of the network model. In this section, we will proceed to show how a single autonomous differential equation for $s = \frac{S}{N}$ can be derived from the effective degree model. In the effective degree model, the average degree of the network is given by $\frac{\sum_j jx_j}{\sum_j x_j}$, and the average excess degree is $\frac{\sum_j j(j-1)x_j}{\sum_j jx_j}$. Let the ratio of the average excess degree to the average degree be denoted as $\kappa = \frac{\sum_j j(j-1)x_j \sum_j x_j}{(\sum_j jx_j)^2} = \frac{\sum_j j(j-1)x_j}{(\sum_j jx_j)^2} S$. Although S varies with time t, κ remains a constant due to the static nature of the network. Furthermore, let $D = \frac{\nu}{S}$, $B = \sum_j jx_j - \nu$, with initial conditions $D(0) = \frac{\mu\rho}{1-\rho}$ and $B(0) = \mu N$, hence

$$
\begin{aligned}
\dot{B} =& \sum_j j\dot{x}_j - \dot{\nu} \\
=& \gamma\langle I\rangle \sum_j \left[(j+1)jx_{j+1} - j^2 x_j\right] - \beta\langle I\rangle \sum_j j(j-1)x_j - \beta\langle I\rangle \sum_j jx_j \\
& - \left[-(\beta+\gamma)\nu + \beta\langle I\rangle(1-\langle I\rangle) \sum_j j(j-1)x_j - \beta\langle I\rangle^2 \sum_j j(j-1)x_j \right] \\
=& \gamma\langle I\rangle \left\{ \sum_j \left[(j+1)^2 - (j+1)\right] x_{j+1} - \sum_j j^2 x_j \right\} \\
& - \beta\langle I\rangle \sum_j j(j-1)x_j - \beta\langle I\rangle \sum_j jx_j \\
& + (\beta+\gamma)\nu - \beta\langle I\rangle(1-\langle I\rangle) \sum_j j(j-1)x_j + \beta\langle I\rangle^2 \sum_j j(j-1)x_j \\
=& -\gamma\langle I\rangle \sum_j (j+1)x_{j+1} - \beta\langle I\rangle \sum_j jx_j + (\beta+\gamma)\nu \\
& - 2\beta\langle I\rangle(1-\langle I\rangle) \sum_j j(j-1)x_j \\
=& -2\beta\langle I\rangle(1-\langle I\rangle) \sum_j j(j-1)x_j \\
=& -2\beta\nu B \frac{\sum_j j(j-1)x_j}{(\sum_j jx_j)^2}.
\end{aligned}
\tag{4.1}
$$

Derived from (2.1), we obtain

$$
\begin{aligned}
\dot{S} &= \sum_j \dot{x_j} \\
&= \sum_j \gamma\langle I\rangle \left[(j+1)x_{j+1} - jx_j\right] - \sum_j \beta\langle I\rangle jx_j \\
&= -\sum_j \beta\langle I\rangle jx_j \\
&= -\frac{\sum_j \beta\nu jx_j}{\sum_j jx_j} \\
&= -\beta\nu,
\end{aligned}
\tag{4.2}
$$

combining (4.1) and (4.2) leads to

$$\frac{\dot{B}}{\dot{S}} = 2B\frac{\sum_j j(j-1)x_j}{(\sum_j jx_j)^2} = 2\kappa\frac{B}{S}, \tag{4.3}$$

integrating it with the initial conditions $B(0) = \mu N$ and $S(0) = N(1-\rho)$ leads to

$$B(t) = \frac{\mu}{1-\rho}S^{2\kappa}. \tag{4.4}$$

So

$$\begin{aligned}
\dot{D} &= \frac{\dot{\nu}S - \nu\dot{S}}{S^2} \\
&= \frac{-(\beta+\gamma)\nu S + \beta\langle I\rangle(1-\langle I\rangle)\sum_j j(j-1)x_j S - \beta\langle I\rangle^2\sum_j j(j-1)x_j S + \beta\nu^2}{S^2} \\
&= \frac{-(\beta+\gamma)\nu S + \beta\kappa\nu B - \beta\kappa\nu^2 + \beta\nu^2}{S^2} \\
&= -(\beta+\gamma)\frac{\nu}{S} + \beta\kappa\frac{\mu}{1-\rho}S^{2\kappa-1}\frac{\nu}{S} - \beta\kappa(\frac{\nu}{S})^2 + \beta(\frac{\nu}{S})^2 \\
&= \beta(1-\kappa)D^2 + \left[\frac{\beta\kappa\mu}{1-\rho}S^{2\kappa-1} - (\beta+\gamma)\right]D.
\end{aligned} \tag{4.5}$$

By appropriately transforming the expressions regarding S and I in (2.1), we obtain

$$\begin{aligned}
\dot{S} &= -\beta DS, \\
\dot{I} &= \beta DS - \gamma I, \\
\dot{D} &= \beta(1-\kappa)D^2 + \left[\frac{\beta\kappa\mu}{1-\rho}S^{2\kappa-1} - (\beta+\gamma)\right]D.
\end{aligned} \tag{4.6}$$

Further processing the Eq. (4.6) leads to

$$\frac{\dot{D}}{\dot{S}} + (1-\kappa)\frac{D}{S} = -\frac{\kappa\mu}{1-\rho}S^{2\kappa-2} + \frac{\beta+\gamma}{\beta}\frac{1}{S}. \tag{4.7}$$

When $\kappa \neq 1$, the differential equation (4.7) is solved to obtain

$$D = C_1S^{\kappa-1} - \frac{\mu}{1-\rho}S^{2\kappa-1} + \frac{\beta+\gamma}{\beta(1-\kappa)}S, \tag{4.8}$$

where $C_1 = \mu\rho N^{1-\kappa}(1-\rho)^{-\kappa} + \mu N^{\kappa}(1-\rho)^{\kappa-1} - \frac{\beta+\gamma}{\beta(1-\kappa)}N^{-\kappa}(1-\rho)^{-\kappa}$. Substituting equation (4.8) into the first Eq.(4.6) leads to

$$\dot{S} = -C_2S^{\kappa} + \frac{\beta\mu}{1-\rho}S^{2\kappa} - \frac{\beta+\gamma}{1-\kappa}S^2, \tag{4.9}$$

where $C_2 = \beta\mu\rho N^{1-\kappa}(1-\rho)^{-\kappa} + \beta\mu N^{\kappa}(1-\rho)^{\kappa-1} - \frac{\beta+\gamma}{1-\kappa}N^{-\kappa}(1-\rho)^{-\kappa}$. When $\kappa = 1$, the differential equation (4.7) is solved to obtain

$$D = \frac{\beta+\gamma}{\beta}\ln S - \frac{\mu}{1-\rho}S + C_3, \tag{4.10}$$

where $C_3 = \frac{\beta+\gamma}{\beta}\ln[N(1-\rho)] - N\mu - \frac{\mu\rho}{1-\rho}$. Substituting equation (4.10) into the first equation of Eq. (4.6) leads to

$$\dot{S} = -(\beta+\gamma)S\ln S + \frac{\beta\mu}{1-\rho}S^2 - C_4 S, \tag{4.11}$$

where $C_4 = (\beta+\gamma)\ln[N(1-\rho)] - \beta N\mu - \frac{\beta\mu\rho}{1-\rho}$. Regarding $s = \frac{S}{N}$ as the survival probability of susceptible nodes, it is easy to derive that

$$\dot{s} = \begin{cases} -C_2 s^{\kappa} N^{\kappa-1} + \dfrac{\beta\mu}{1-\rho}s^{2\kappa}N^{2\kappa-1} - \dfrac{\beta+\gamma}{1-\kappa}s^2 N, & \kappa \neq 1, \\ -(\beta+\gamma)s\ln(Ns) + \dfrac{\beta\mu}{1-\rho}s^2 N - C_4 s, & \kappa = 1. \end{cases} \tag{4.12}$$

Since it is evident that $\dot{s}(\infty) = 0$, Eq.(4.12) implies that the condition $s(\infty) > 0$ has to satisfy

$$C_2 s^{\kappa-2}N^{\kappa-2} = -\frac{\beta\mu}{1-\rho}s^{2\kappa-2}N^{2\kappa-2} + \frac{\beta+\gamma}{1-\kappa}, \quad \kappa \neq 1, \tag{4.13}$$

$$(\beta+\gamma)\ln(Ns) = -\frac{\beta\mu}{1-\rho}sN + C_4, \quad \kappa = 1. \tag{4.14}$$

Given that we observe the infection times $(t_1, \ldots, t_k)$ for a randomly chosen subset of k initially susceptible nodes within a time interval $[0, T]$, where T is less than or equal to infinity, we can formulate the approximate log-likelihood function as follows:

$$\ell(\beta, \gamma, \mu, \rho, N | t_1, \ldots, t_k) = \sum_{i=1}^{k} \ln s(t_i) - k\ln(1 - s(T)). \tag{4.15}$$

Equation (4.12) indicates that we only need a few parameters to obtain the value of s, without relying on other variables, which is very convenient. This approach is similar to the single equation regarding S_t obtained through the DSA model in [12]. It is worth mentioning that to obtain quantities other than s, evaluation of additional ODEs is needed [10].

5. Discussion

In the dynamics of network-based infectious diseases, three types of complex network models are widely employed: the pairwise model [7], the edge-based compartmental model [23], and the effective degree model [15]. Miller et al. [17] demonstrated the equivalence of these three models under certain conditions. Recently, KhudaBukhsh et al. [8] derived the DSA model based on the large graph limit system studied by Jacobsen et al. [5] Subsequently, Kiss et al. [12] further proved the equivalence between the pairwise model, the edge-based compartmental model, and the DSA model.

This paper demonstrates the equivalence between the network effective degree model and the DSA model, strengthening the connections between network models. The greatest benefit of this result is that the effective degree model can share the statistical interpretation of the DSA model, especially in terms of statistical inference from data. We also simplify the effective degree model into a differential equation regarding susceptible individuals, which can be represented by certain parameters. Based on this, the effective degree model can be better applied to address practical infectious disease issues in the future, such as effectively tracking infectious diseases and taking corresponding measures.

As a future research direction, we propose to apply dynamical survival analysis to analyze data arriving from the effective degree model. This approach, in addition to enabling the derivation of the likelihood function presented in this paper, has the potential to yield insights into basic reproduction number, dropout rates, recovery rates, the final epidemic size, and other key epidemiological metrics.

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