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ORBITAL STABILITY OF SOLITARY WAVES TO THE COUPLED SCHRÖDINGER-BBM EQUATION

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Abstract This paper investigated the orbital stability of the solitary waves of the coupled Schrödinger-BBM equation through abstract theoretical results and detailed spectral analysis. First, we derived the explicit exact solitary wave solutions of the coupled Schrödinger-BBM equation. Then, using the orbital stability theory developed by Grillakis et al., we established general criteria for assessing the orbital stability of these solitary waves.

Keywords Orbital stability, solitary waves, coupled Schrödinger-BBM equation.

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1. Introduction

Nonlinear evolution equations (NLEEs), as an important research topic in mathematics, physics, and engineering, play a crucial role in describing various complex phenomena such as nonlinear waves, fluid dynamics, optics, and quantum mechanics. The dynamic behavior of many systems depends on the interaction of multiple factors. In order to study the evolution and stability of these systems, many combined equations and coupled equations are introduced such as the Benjamin-Ono-Burgers equation [17], the coupled Schrödinger-KdV equation [5] and the coupled KdV [9]. In this paper, we investigate the coupled Schrödinger–Benjamin–Bona–Mahony (cS-BBM) equation, which describes nonlinear wave phenomena and interactions across multiple physical fields. This system is widely applied in shallow water waves, plasma physics, optics, and other fields to model the interactions between long waves (described by the BBM equation) and short waves (described by the Schrödinger equation),

$$\begin{cases} iu_t + u_{xx} + \beta_1 vu + \beta_2 q(|u|^2)u = 0, \ (t,x) \in \mathbb{R}, \\ v_t - v_{xxt} + (\beta_3 |u|^2 + \beta_4 f(v))_x = 0, \ (t,x) \in \mathbb{R}, \\ u(0) = u_0(x), \ x \in \mathbb{R}, \\ v(0) = v_0(x), \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $u(x,t)$ and $v(x,t)$ represent complex and real functions respectively, β_i $(1 \leq i \leq 4)$ are real constants. The well-posedness problem of (1.1) in $H^2 \times H^2$ when $\beta_1 = -1,\ \beta_3 = \beta_4 = 1$ was studied [11].

When $\beta_1 = \beta_3 = \beta_4 = -1,\ \beta_2 = -\alpha,\ q(|u|^2) = |u|^{p-1},\ f(v) = v^2$, we have

$$\begin{cases} iu_t + u_{xx} = vu + \alpha|u|^{p-1}u, & (x,t) \in \mathbb{R}, \\ v_t - v_{xxt} = (|u|^2 + v^2)_x, & (x,t) \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \\ v(x,0) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $\alpha \in \mathbb{R},\ 1 < p < \infty$. Guo [8] proved the global solvability in space $L^\infty(0,T;H^m) \times L^\infty(0,T;H^m)$ for $m \geq 2$ via integral estimation method and the fixed point theorem. Guo and Miao [10] proved the global well-posedness in space $L^2 \times L^2$ when $1 < p < 5$ and $(u_0, v_0) \in L^2 \times L^2$ and the global well-posedness in space $H^1 \times H^1$ when $1 < p < \infty$ and $(u_0, v_0) \in H^1 \times H^1$ by Strichartz type estimates.

Orbital stability is an important concept in dynamical systems and partial differential equations, concerning the long-term behavior of solutions as they evolve over time, particularly the stability of solutions in nonlinear systems. In the NLEEs, orbital stability is a key tool for determining whether these solutions can maintain their structures under small perturbations. Recently, Luo [14] proved the orbital stability in $H^1(\mathbb{R})$ of the modified Camassa-Holm-Novikov equation by conservation laws and the monotonicity property of energy functional. Zheng [19] considered the orbital stability of the solitary wave of a two-component Novikov system via the method proposed by [2,3]. Xiao [16] studied the orbital stability of multi-solitons for the Hirota equation via the method proposed by [1]. The orbital stability of NLEEs has been studied extensively through the method proposed by [6,7], such as the generalized Boussinesq equation [18], the EK equation [13], the coupled BBM [4] and the Schrödinger-KdV system [12], etc.

In this paper, we mainly give the orbital stability of solitary waves to the cS-BBM equation under the condition that $u(x,t)$ and $v(x,t)$ as well as their derivatives decay to zero as $|x| \to \infty$. In section 2, we will study the explicit exact solitary wave solutions of the coupled Schrödinger-BBM equation. In section 3 and 4, we will give the proof of the main result of this paper in detail.

2. Solitary waves of the cS-BBM equation

We only consider the solitary wave solutions of the case of $p = 3$ in (1.2)

$$\begin{cases} iu_t + u_{xx} = vu + \alpha|u|^2u, \\ v_t - v_{xxt} = (|u|^2 + v^2)_x. \end{cases} \tag{2.1}$$

Assume that (2.1) has solitary wave solutions in the form of

$$u(x,t) = \mathrm{e}^{-i\omega t}a(\xi) = \mathrm{e}^{-i\omega t}\mathrm{e}^{iq\xi}\hat{a}(\xi),\ v(t,x) = b(\xi),\ \xi = x - ct, \tag{2.2}$$

where $\hat{a}(\xi),\ b(\xi)$ are real functions, $\omega,\ q,\ c$ are constants to be determined later. $a(\xi),\ b(\xi)$ as well as their derivatives decay to zero as $|\xi| \to \infty$. Substituting (2.2) into equation (2.1), we obtain

$$\begin{cases} \omega a(\xi) - ica'(\xi) + a''(\xi) - a(\xi)b(\xi) - \alpha|a(\xi)|^2a(\xi) = 0, \\ -cb'(\xi) + cb'''(\xi) - (|a(\xi)|^2 + b^2(\xi))' = 0. \end{cases} \tag{2.3}$$

Set both the real part and the imaginary part of the first equation in (2.3) to be zero, and integrate the second equation, we have

$$\left(\omega+\frac{c^2}{4}\right)\hat{a}+\hat{a}''-\hat{a}b-\alpha\hat{a}^3=0, \tag{2.4}$$

$$-cb+cb''-\hat{a}^2-b^2=0. \tag{2.5}$$

Let $\hat{a}(\xi)=\mathrm{AsechB}\xi$, where A and B are constants to be determined later. Substituting it into (2.4), we obtain

$$\left(\omega+\frac{c^2}{4}\right)\hat{a}+B^2\hat{a}-\frac{2B^2}{A^2}\hat{a}^3-\hat{a}b-\alpha\hat{a}^3=0. \tag{2.6}$$

By comparing the coefficients of terms with the same power, we obtain

$$B^2=-\omega-\frac{c^2}{4},\ b=\left(-\alpha-\frac{2B^2}{A^2}\right)\hat{a}^2. \tag{2.7}$$

Inserting (2.7) into (2.5), one has

$$\begin{aligned}&\left((-c+4cB^2)\left(-\alpha-\frac{2B^2}{A^2}\right)-1\right)\hat{a}^2\\&+\left(6c\frac{B^2}{A^2}-\alpha-\frac{2B^2}{A^2}\right)\left(\alpha+\frac{2B^2}{A^2}\right)\hat{a}^4=0.\end{aligned} \tag{2.8}$$

Thus, we get

$$\alpha=6c\frac{B^2}{A^2}-\frac{2B^2}{A^2},\ A^2=6c^2B^2-24c^2B^4. \tag{2.9}$$

We only consider the case of $A>0,\ B>0$, from (2.2), (2.7) and (2.9), it yields

$$\hat{a}(\xi)=\left(6c^2\left(-\omega-\frac{c^2}{4}\right)(1+4\omega+c^2)\right)^{\frac{1}{2}}\operatorname{sech}\left(\left(-\omega-\frac{c^2}{4}\right)^{\frac{1}{2}}\xi\right), \tag{2.10}$$

$$b(\xi)=-6c\left(-\omega-\frac{c^2}{4}\right)\operatorname{sech}^2\left(\left(-\omega-\frac{c^2}{4}\right)^{\frac{1}{2}}\xi\right). \tag{2.11}$$

Therefore, the following theorem is obtained.

Theorem 2.1. *If $c>0$ and $-\frac{1}{4}(c^2+1)<\omega<-\frac{c^2}{4}$, there exists solitary wave solutions of equation (2.1) in the form of (2.2) with $\hat{a}$, b satisfying (2.10) and (2.11), respectively.*

3. Orbital stability of solitary waves for the cS-BBM equation

Setting $\vec{q}=(u,v)$ and defining the work space $\mathbb{X}=\mathbb{H}^1_{complex}(\mathbb{R})\times\mathbb{H}^1_{real}(\mathbb{R})$ with the inner product

$$(\vec{q}_1,\vec{q}_2)=\mathrm{Re}\int_{\mathbb{R}}u_1\bar{u}_2+u_{1x}\bar{u}_{2x}+v_1v_2+v_{1x}v_{2x}dx,\ \forall\ q_1,q_2\in\mathbb{X}, \tag{3.1}$$

$\mathbb{X}^* = \mathbb{H}^{-1}(\mathbb{R})$ is viewed as the dual space of $\mathbb{X}$. Then, there is a nature isomorphism $I: \mathbb{X} \to \mathbb{X}^*$ write as $\langle I\vec{q}_1, \vec{q}_2\rangle = (q_1, q_2)$, where $\langle \vec{f}, \vec{q}\rangle = \mathrm{Re}\int_{\mathbb{R}} f_1\bar{u} + f_2 v dx$. Therefore, $I = \begin{pmatrix} 1-\partial_x^2 & 0 \\ 0 & 1-\partial_x^2 \end{pmatrix}$.

Let $T_1(m_1)\vec{q}(\cdot) = \vec{q}(\cdot - m_1)$, $T_2(m_2)\vec{q}(\cdot) = \big(e^{im_2}u(\cdot), v(\cdot)\big)$ where $\vec{q} \in \mathbb{X}$ and $m_1,\ m_2 \in \mathbb{R}$. Denote $\vec{\psi}(x) = (a(x), b(x))$, then $T_1(ct)T_2(\omega t)\vec{\psi}(x)$ can be viewed as the solitary wave solutions of equation (2.1) in Theorem 2.1. By computation, we have

$$T_1'(0) = \begin{pmatrix} -\partial_x & \\ & -\partial_x \end{pmatrix},\ T_2'(0) = \begin{pmatrix} -i & \\ & 0 \end{pmatrix}. \tag{3.2}$$

The orbital stability of solitary waves $T_1(ct)T_2(wt)\vec{\psi}(x)$ is defined as follows.

Definition 3.1. The solitary wave $T_1(ct)T_2(wt)\vec{\psi}(x)$ is orbitally stable: $\forall\ \epsilon > 0$, there exists $\tau > 0$, if $||\vec{q}_0 - \vec{\psi}||_{\mathbb{X}} < \tau$ and $\vec{q}(t)$ is the solution of (2.1) on $[0, t_0)$ with $\vec{q}(0) = \vec{q}_0$, then $\vec{q}(t)$ can be continued to a solution in $0 \le t < +\infty$, and

$$\sup_{0<t<\infty}\inf_{m_1\in\mathbb{R}}\inf_{m_2\in\mathbb{R}} ||\vec{q}(t) - T_1(m_1)T_2(m_2)\vec{\psi}(x)||_{\mathbb{X}} < \epsilon. \tag{3.3}$$

Otherwise, $T_1(ct)T_2(wt)\vec{\psi}(x)$ is called orbitally unstable [12].

Now, we study the orbital stability of solitary waves of the cS-BBM equation (2.1). System (2.1) can be transformed into a Hamiltonian form

$$\frac{d\vec{q}}{dt} = JE'(\vec{q}), \tag{3.4}$$

where

$$E(\vec{q}) = \int |u_x|^2 + \frac{\alpha}{2}|u|^4 + v|u|^2 + \frac{1}{3}v^3 dx, \tag{3.5}$$

$$J = \begin{pmatrix} -\frac{i}{2} & \\ & (1-\partial_x^2)^{-1}\partial_x \end{pmatrix}, \tag{3.6}$$

$$E'(\vec{q}) = \begin{pmatrix} -2u_{xx} + 2\alpha|u|^2u + 2vu \\ |u|^2 + v^2 \end{pmatrix}. \tag{3.7}$$

According to $T_1'(0) = JM_1,\ T_2'(0) = JM_2$ [6,7], we have

$$M_1 = \begin{pmatrix} -2i\partial_x & \\ & \partial_x^2 - 1 \end{pmatrix},\quad M_2 = \begin{pmatrix} 2 & \\ & 0 \end{pmatrix}.$$

Defining the following functional

$$Q_1(\vec{q}) = \frac{1}{2}\langle M_1\vec{q}, \vec{q}\rangle = \frac{1}{2}\mathrm{Re}\int -2iu_x\bar{u} + (v_{xx} - v)v dx, \tag{3.8}$$

$$Q_2(\vec{q}) = \frac{1}{2}\langle M_2\vec{q}, \vec{q}\rangle = \mathrm{Re}\int |u|^2 dx. \tag{3.9}$$

The corresponding Fréchet derivatives of Q_1 and Q_2 are $Q_1'(\vec{q}) = \begin{pmatrix} -2iu_x \\ v_{xx} - v \end{pmatrix}$ and $Q_2'(\vec{q}) = \begin{pmatrix} 2u \\ 0 \end{pmatrix}$. By simple computation, it can be proved that $E(\vec{q})$, $Q_1(\vec{q})$ and $Q_2(\vec{q})$ are invariants under T_1 and T_2.

Proposition 3.1. *For any $m_1,\ m_2 \in \mathbb{R}$, we have*

$$\begin{aligned} &E(T_1(m_1)T_2(m_2)\vec{q}) = E(\vec{q}),\\ &Q_1(T_1(m_1)T_2(m_2)\vec{q}) = Q_1(\vec{q}),\\ &Q_2(T_1(m_1)T_2(m_2)\vec{q}) = Q_2(\vec{q}), \end{aligned} \tag{3.10}$$

and $\forall\ t \in \mathbb{R}$, $E(\vec{q}(t)) = E(\vec{q}(0))$, $Q_1(\vec{q}(t)) = Q_1(\vec{q}(0))$, $Q_2(\vec{q}(t)) = Q_2(\vec{q}(0))$.

Proof.

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\int |u_x|^2 + \frac{\alpha}{2}|u|^4 + v|u|^2 + \frac{1}{3}v^3 dx\\ &= \int -\bar{u}_{xx}u_t - u_{xx}\bar{u}_t + \alpha|u|^2\bar{u}u_t + \alpha|u|^2u\bar{u}_t + |u|^2v_t + v\bar{u}u_t + vu\bar{u}_t + v^2v_t dx\\ &= 0,\\ \frac{dQ_1}{dt} &= \frac{1}{2}\frac{d}{dt}\mathrm{Re}\int -2iu_x\bar{u} + (v_{xx} - v)v dx = 0,\\ \frac{dQ_2}{dt} &= \frac{d}{dt}\mathrm{Re}\int |u|^2 dx = 0. \end{aligned}$$

□

For $\vec{\psi}$, by using (2.3), we deduce the following equation holds

$$\begin{aligned} E'(\vec{\psi}) - cQ_1'(\vec{\psi}) - \omega Q_2'(\vec{\psi}) &= \begin{pmatrix} -2a'' + 2ba + 2\alpha|a|^2a + 2ica' - 2\omega a \\ |a|^2 + b^2 + cb - cb'' \end{pmatrix}\\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.11}$$

For $\vec{\psi} = (a, b), \vec{\phi} = (\phi_1, \phi_2) \in \mathbb{X}$, we have

$$\begin{aligned} E''(\vec{\psi})\vec{\phi} &= \begin{pmatrix} -2\phi_{1xx} + 2b\phi_1 + 2\alpha a^2\bar{\phi_1} + 4\alpha|a|^2\phi_1 + 2a\phi_2 \\ a\bar{\phi_1} + \bar{a}\phi_1 + 2b\phi_2 \end{pmatrix},\\ Q_1''(\vec{\psi})\vec{\phi} &= \begin{pmatrix} -2i\phi_{1x} \\ \phi_{2xx} - \phi_2 \end{pmatrix},\\ Q_2''(\vec{\psi})\vec{\phi} &= \begin{pmatrix} 2\phi_1 \\ 0 \end{pmatrix}. \end{aligned}$$

Let us define the operator $H_{\omega,c}: \mathbb{X} \to \mathbb{X}^*$,

$$H_{\omega,c} = E''(\vec{\psi}) - cQ_1''(\vec{\psi}) - \omega Q_2''(\vec{\psi}). \tag{3.12}$$

Since $T_1'(0)\vec{\psi} = \begin{pmatrix} -a' \\ -b' \end{pmatrix}$ and $T_2'(0)\vec{\psi} = \begin{pmatrix} -ia \\ 0 \end{pmatrix}$, we have

$$H_{\omega,c}T_1'(0)\vec{\psi}(x) = 0,\ H_{\omega,c}T_2'(0)\vec{\psi}(x) = 0. \tag{3.13}$$

$H_{\omega,c}$ is a self-adjoint operator, that is, $I^{-1}H_{\omega,c}$ is a bounded self-adjoint operator. The operator $H_{\omega,c}$ has a spectrum composed entirely by real numbers λ such that $H_{\omega,c} - \lambda\mathbf{I}$ is non-invertible ($\mathbf{I}$ is identity operator). Set $\mathbb{K} = \{k_1T_1'(0)\vec{\psi} + k_2T_2'(0)\vec{\psi} | k_1, k_2 \in \mathbb{R}\}$, then $\mathbb{K}$ is the kernel of $H_{\omega,c}$.

In order to prove the orbital stability of solitary waves, we introduce the following assumption.

Assumption 3.1. *(Spectral decomposition of operator $H_{\omega,c}$ [6, 7]). The space $\mathbb{X}$ can be decomposed as a direct sum*

$$\mathbb{X} = \mathbb{F} + \mathbb{K} + \mathbb{P}, \tag{3.14}$$

where $\mathbb{F}$ is a finite-dimensional negative subspace of $\mathbb{X}$ such that

$$\langle H_{\omega,c}\vec{q}, \vec{q}\rangle < 0,\ \vec{q} \neq 0 \in \mathbb{F}, \tag{3.15}$$

$\mathbb{K} = \{k_1T_1'(0)\vec{\psi} + k_2T_2'(0)\vec{\psi} | k_1, k_2 \in \mathbb{R}\}$ *is a subspace of $\mathbb{X}$ such that*

$$\langle H_{\omega,c}\vec{q}, \vec{q}\rangle = 0,\ \vec{q} \in \mathbb{K}, \tag{3.16}$$

and $\mathbb{P}$ is a closed subspace of $\mathbb{X}$, there exists a constant $\delta > 0$ such that

$$\langle H_{\omega,c}\vec{q}, \vec{q}\rangle \geq \delta||\vec{q}||^2_{\mathbb{X}},\ \vec{q} \in \mathbb{P}. \tag{3.17}$$

We define a mapping $d(\omega, c) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in the form of

$$d(\omega, c) = E(\vec{\psi}) - cQ_1(\vec{\psi}) - \omega Q_2(\vec{\psi}), \tag{3.18}$$

and let $d''(\omega, c)$ be the Hessian matrix of $d(\omega, c)$.

Furthermore, $p(d'')$ represents the positive-eigenvalue's number of d'' while $n(H_{\omega,c})$ is the negative-eigenvalue's number of $H_{\omega,c}$.

Theorem 3.1. *(Abstract stability theorem [6, 7]) Assume that there exists three functional $E(\vec{\psi})$, $Q_1(\vec{\psi})$, $Q_2(\vec{\psi})$ satisfying Proposition 3.1, solitary waves $T_1(ct) \times T_2(\omega t)\vec{\psi}(x)$ satisfying (3.12) and operator $H_{\omega,c}$ satisfying Assumption 3.1. If $d(\omega, c)$ is non-singular and $p(d'') = n(H_{\omega,c})$, the solitary waves $T_1(ct)T_2(\omega t)\vec{\psi}(x)$ of system (2.1) are orbital stable.*

Now, the main result about the orbital stability of solitary waves of system (2.1) is as follow.

Theorem 3.2. *(Main theorem). Under the condition of Theorem 2.1, if $\omega > -\frac{1}{c} - \frac{c^2}{4}$ the solitary waves $T_1(ct)T_2(\omega t)\vec{\psi}(x)$ of system (2.1) are orbital stable.*

4. The proof of Theorem 3.2

In order to prove the orbital stability of system (2.1), according to Theorem 3.1, we only need to prove Assumption 3.1 holds and $p(d'') = n(H)$.

4.1. Proof of Assumption 3.1

For arbitrary $\vec{\phi} = (\phi_1, \phi_2) = (e^{i\frac{c}{2}x} z_1, z_2) \in \mathbb{X}$, where $z_1 = r_1 + ir_2,\ r_1 = \text{Re}z_1,\ r_2 = \text{Im}z_1$ and $r_1,\ r_2,\ z_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\langle H_{\omega,c}\vec{\phi}, \vec{\phi}\rangle =& \text{Re}\int_{\mathbb{R}} \left(-2\phi_{1xx} + 2b\phi_1 + 2\alpha a^2\bar{\phi}_1 + 4\alpha|a|^2\phi_1 + 2ic\phi_{1x} - 2\omega\phi_1\right)\bar{\phi}_1 dx \\
&+ \text{Re}\int_{\mathbb{R}} \left(a\bar{\phi}_1 + \bar{a}\phi_1 + 2b\phi_2 - c\phi_{2xx} + c\phi_2\right)\phi_2 + 2a\phi_2\bar{\phi}_1 dx \\
=& \text{Re}\int_{\mathbb{R}} 2\left(-z_{1xx} - \omega z_1 - \frac{c^2}{4}z_1 + bz_1 + 2\alpha\hat{a}^2 z_1\right)\bar{z}_1 + 2\alpha\hat{a}^2\bar{z}_1^2 dx \\
&+ \text{Re}\int_{\mathbb{R}} \left(-cz_{2xx} + cz_2 + 2bz_2\right)z_2 + 2\hat{a}z_2\bar{z}_1 + \hat{a}\bar{z}_1 z_2 + \hat{a}z_1 z_2 dx \\
=& \langle \tilde{L}_1 r_1, r_1\rangle + \langle L_2 r_2, r_2\rangle + \langle \tilde{L}_3 z_2, z_2\rangle + 4\text{Re}\int_{\mathbb{R}} \hat{a}r_1 z_2 dx \\
=& \langle L_1 r_1, r_1\rangle + \langle L_2 r_2, r_2\rangle + \langle L_3 z_2, z_2\rangle \\
&+ \int_{\mathbb{R}} \left(\frac{2\hat{a}r_1}{(c-4cB^2)^{\frac{1}{2}}} + \left(c - 4cB^2\right)^{\frac{1}{2}} z_2\right)^2 dx,
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
&\tilde{L}_1 = 2\left(-\frac{\partial^2}{\partial x^2} - \frac{c^2}{4} - \omega + b + 3\alpha\hat{a}^2\right), \quad L_1 = \tilde{L}_1 - \frac{4}{c - 4cB^2}\hat{a}^2, \\
&L_2 = 2\left(-\frac{\partial^2}{\partial x^2} - \frac{c^2}{4} - \omega + b + \alpha\hat{a}^2\right), \\
&\tilde{L}_3 = -c\frac{\partial^2}{\partial x^2} + c + 2b, \quad L_3 = \tilde{L}_3 - (c - 4cB^2) = -c\frac{\partial^2}{\partial x^2} + 4cB^2 + 2b.
\end{aligned}$$

By computation, we have $L_1(\hat{a}') = 0,\ L_2(\hat{a}) = 0$ and $L_3(b') = 0$ while $x = 0$ is the simple zero of $\hat{a}'$ and b'. Then, according to the Sturm-Liouville theorem, 0 is the second eigenvalue of L_1 and L_3 and 0 is the first simple eigenvalue of L_2. Therefore, there only exists one negative eigenvalue of L_1 and L_3, respectively, that is, $L_1(\hat{a}^2) = -6B^2\hat{a}^2$ and $L_3(-2b\hat{a}) = -5B^2(-2b\hat{a})$.

In addition, $L_1 = 2\left(-\partial_x^2 - \omega - \frac{c^2}{4}\right) + M_1(x)$, $L_2 = 2\left(-\partial_x^2 - \omega - \frac{c^2}{4}\right) + M_2(x)$ and $L_3 = -c\partial_x^2 + 4cB^2 + M_3(x)$ where $M_i(x) \to 0$ $(i = 1,\ 2,\ 3)$ as $|x| \to \infty$. Thus, thanks to the Weyl's essential spectral theorem [15], we get $\sigma_{ess}(L_1) = \sigma_{ess}(L_2) = \left[2B^2, +\infty\right)$ and $\sigma_{ess}(L_3) = \left[4cB^2, +\infty\right)$.

Next, we will prove that $n(H_{\omega,c}) = 1$ and Assumption 3.1 holds.

We denote $\vec{\phi}^- = \left(e^{i\frac{c}{2}x}(r_1^- + ir_2^-), z_2^-\right)$, where $r_1^- = \hat{a}^2,\ r_2^- = 0$ and $z_2^- = -2b\hat{a}$.

$$\langle H_{\omega,c}\vec{\phi}^-, \vec{\phi}^-\rangle = -6B^2\langle \hat{a}^2, \hat{a}^2\rangle - 5B^2\langle -2b\hat{a}, -2b\hat{a}\rangle < 0. \tag{4.2}$$

Define $\vec{\phi}_{0,1} = (\hat{a}', 0, b')$ and $\vec{\phi}_{0,2} = (0, \hat{a}, 0)$, then they are the kernel of $H_{\omega,c}$. Let

$$\mathbb{F} = \{k\vec{\phi}^- | k \in \mathbb{R}\}, \tag{4.3}$$

$$\mathbb{K} = \{k_1\vec{\phi}_{0,1} + k_2\vec{\phi}_{0,2} | k_1, k_2 \in \mathbb{R}\}, \tag{4.4}$$

$$\begin{aligned}\mathbb{P} = \{\vec{p} \in \mathbb{X} | \vec{p} = (p_1, p_2, p_3), \langle p_1, \hat{a}^2\rangle + \langle p_3, -2b\hat{a}\rangle = 0,\\ \langle p_1, \hat{a}'\rangle + \langle p_3, b'\rangle = 0, \langle p_2, \hat{a}\rangle = 0\}.\end{aligned} \tag{4.5}$$

For arbitrary $\vec{q} = \left(\mathrm{e}^{i\frac{c}{2}}(r_1 + ir_2), z_2\right) \in \mathbb{X}$, let $\chi = \frac{\langle r_1, \hat{a}^2\rangle + \langle z_2, -2\hat{a}b\rangle}{\langle \hat{a}^2, \hat{a}^2\rangle + \langle -2\hat{a}b, -2\hat{a}b\rangle}$, $\chi_1 = \frac{\langle r_1, \hat{a}'\rangle + \langle z_2, b'\rangle}{\langle \hat{a}', \hat{a}'\rangle + \langle b', b'\rangle}$, $\chi_2 = \frac{\langle r_2, \hat{a}\rangle}{\langle \hat{a}, \hat{a}\rangle}$, then we can represent $\vec{q}$ as

$$\vec{q} = \chi\vec{\phi}^- + \chi_1\vec{\phi}_{0,1} + \chi_2\vec{\phi}_{0,2} + \vec{p},\ \vec{p} \in \mathbb{P}. \tag{4.6}$$

By the above analysis, the following lemmas are given in [12].

Lemma 4.1. *For any real functions $r_1 \in \mathbb{H}^1(\mathbb{R})$, there are positive numbers $\tilde{\delta}_1$ and δ_1 when $\langle r_1, \hat{a}^2\rangle = \langle r_1, \hat{a}'\rangle = 0$ such that $\langle L_1 r_1, r_1\rangle \geq \tilde{\delta}_1 ||r_1||^2_{\mathbb{L}^2}$ and $\langle L_1 r_1, r_1\rangle \geq \delta_1 ||r_1||^2_{\mathbb{H}^1}$.*

Lemma 4.2. *For any real functions $r_2 \in \mathbb{H}^1(\mathbb{R})$, there is a positive number δ_2 when $\langle r_2, \hat{a}\rangle = 0$ such that $\langle L_2 r_2, r_2\rangle \geq \delta_2 ||r_2||^2_{\mathbb{H}^1}$.*

Lemma 4.3. *For any real functions $z_2 \in \mathbb{H}^1(\mathbb{R})$, there are positive numbers $\tilde{\delta}_3$ and δ_3 when $\langle z_2, -2b\hat{a}\rangle = \langle z_2, b'\rangle = 0$ such that $\langle L_3 z_2, z_2\rangle \geq \tilde{\delta}_3 ||z_2||^2_{\mathbb{L}^2}$ and $\langle L_3 z_2, z_2\rangle \geq \delta_3 ||z_2||^2_{\mathbb{H}^1}$.*

Lemma 4.4. *For any $\vec{p} \in \mathbb{P}$ given in (4.6), there is a positive number δ independent of $\vec{p}$ satisfying $\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta ||\vec{p}||_{\mathbb{X}}$.*

Proof. For any $\vec{p} \in \mathbb{P}$ combining with Lemmas 4.1-4.3, we have

$$\begin{aligned}\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq{}& \delta_1 ||p_1||^2_{\mathbb{H}^1} + \delta_2 ||p_2||^2_{\mathbb{H}^1} + \delta_3 ||p_3||^2_{\mathbb{H}^1}\\ &+ \int_{\mathbb{R}} \left(\frac{2\hat{a}p_1}{\sqrt{(c - 4cB^2)}} + \sqrt{(c - 4cB^2)}p_3\right)^2 dx.\end{aligned}$$

(i) If $||p_3||_{\mathbb{L}^2} \leq 2C_0 ||p_1||_{\mathbb{L}^2}$, $C_0 = \frac{2||\hat{a}||_{\mathbb{L}^\infty}}{c - 4cB^2}$, then $\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta_1 ||p_1||^2_{\mathbb{H}^1} \geq \frac{\delta_1}{2} ||p_1||^2_{\mathbb{H}^1} + \frac{\delta_1}{2C_0^2} ||p_3||^2_{\mathbb{H}^1}$.

(ii) If $||p_3||_{\mathbb{L}^2} \geq 2C_0 ||p_1||_{\mathbb{L}^2}$, then

$$\begin{aligned}\int_{\mathbb{R}} \left(\frac{2\hat{a}p_1}{\sqrt{(c - 4cB^2)}} + \sqrt{(c - 4cB^2)}p_3\right)^2 dx \geq{}& \frac{(c - 4cB^2)||p_3||^2_{\mathbb{L}^2}}{2} - \frac{4||\hat{a}||^2_{\mathbb{L}^\infty} ||p_1||^2_{\mathbb{L}^2}}{c - 4cB^2}\\ \geq{}& \frac{c - 4cB^2}{4} ||p_3||^2_{\mathbb{L}^2}.\end{aligned}$$

To sum up, for any $\vec{p} \in \mathbb{P}$, $\langle H_{\omega,c}\vec{p}, \vec{p}\rangle \geq \delta ||\vec{p}||_{\mathbb{X}}$. □

Therefore, by above analysis, we obtain that $n(H_{\omega,c}) = 1$ and Assumption 3.1 holds.

4.2. Proof of $p(d'') = 1$

Now we prove $p(d'') = 1$, that is $det(d'') < 0$. Since $d(\omega, c) = E(\vec{\psi}) - cQ_1(\vec{\psi}) - \omega Q_2(\vec{\psi})$, we have $d''(\omega, c) = \begin{pmatrix} d_{\omega\omega}(\omega, c) & d_{\omega c}(\omega, c) \\ d_{c\omega}(\omega, c) & d_{cc}(\omega, c) \end{pmatrix}$, and

$$
\begin{aligned}
d_\omega(\omega, c) &= -\int \hat{a}^2(\xi)d\xi = -12c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}}(1 + 4\omega + c^2),\\
d_c(\omega, c) &= -\int \frac{c}{2}\hat{a}^2(\xi)d\xi + \int 18c^2B^4\text{sech}^4(B\xi)\\
&\quad - 72c^2B^6\left(\text{sech}^4(B\xi) - \frac{3}{2}\text{sech}^6(B\xi)\right)d\xi\\
&= -6c^3\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}}(1 + 4\omega + c^2) + 24c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{3}{2}}\\
&\quad + \frac{96}{5}c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{5}{2}},\\
d_{\omega c}(\omega, c) &= -24c\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}}(1 + 4\omega + c^2) + 3c^3\left(-\omega - \frac{c^2}{4}\right)^{-\frac{1}{2}}(1 + 4\omega + c^2)\\
&\quad - 24c^3\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}},\\
d_{\omega\omega}(\omega, c) &= 6c^2\left(-\omega - \frac{c^2}{4}\right)^{-\frac{1}{2}}(1 + 4\omega + c^2) - 48c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}},\\
d_{c\omega}(\omega, c) &= 3c^3\left(-\omega - \frac{c^2}{4}\right)^{-\frac{1}{2}}(1 + 4\omega + c^2) - 48c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{3}{2}}\\
&\quad - (24c^3 + 36c^2)\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}},\\
d_{cc}(\omega, c) &= -18c^2\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}}(1 + 4\omega + c^2) + \frac{3}{2}c^4\left(-\omega - \frac{c^2}{4}\right)^{-\frac{1}{2}}(1 + 4\omega + c^2)\\
&\quad - 12c^4\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}} + 48c\left(-\omega - \frac{c^2}{4}\right)^{\frac{3}{2}} - 18c^3\left(-\omega - \frac{c^2}{4}\right)^{\frac{1}{2}}\\
&\quad + \frac{192}{5}c\left(-\omega - \frac{c^2}{4}\right)^{\frac{5}{2}} - 24c^3\left(-\omega - \frac{c^2}{4}\right)^{\frac{3}{2}},\\
det(d'') &= d_{\omega\omega}d_{cc} - d_{\omega c}d_{c\omega}\\
&= -36c^4(1 + 4\omega + c^2)^2 - 576c^3B^2(1 + 4\omega + c^2) + 288c^4B^2(1 + 4\omega + c^2)\\
&\quad - \frac{4608}{5}c^3B^4(1 + 4\omega + c^2) - 2304c^3B^2 - \frac{9216}{5}c^3B^6\\
&= -36c^3\Big(c((1 + 4\omega + c^2) - 4B^2)^2 + \frac{128}{5}B^4((1 + 4\omega + c^2) + 2B^2)\\
&\quad + 16B^2((1 + 4\omega + c^2) + 4B^2 - cB^2)\Big).
\end{aligned}
$$

Thus, we obtain $det(d'') < 0$ under the conditions $\omega > -\frac{1}{c} - \frac{c^2}{4}$ and $c > 0$. Moreover, we have $p(d'') = 1$, that is, $p(d'') = n(H_{\omega,c})$ and we complete the proof of Theorem 3.2.

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