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SOLVABILITY OF A CLASS OF CONVOLUTION INTEGRAL EQUATIONS WITH SINGULAR INTEGRAL–DIFFERENTIAL OPERATORS

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Abstract In this paper, we consider a class of convolution integral equations with singular integral-differential operators. First, we establish the relation between Fourier analysis theory and Riemann boundary value problems, and investigate the theory of Noether solvability and some properties of Cauchy integral operators. Via using Fourier transform, we convert such equations into complex boundary value problems. By means of the regularity theory of the classical Riemann-Hilbert problems and of the theory of complex analysis, we obtain the conditions of Noether solvability and analytical solutions. In addition, we also study the analytical property of solution near nodes. Thus, this article is significant for the study of developing complex analysis, functional analysis, integral equations and complex boundary value problems, and it also provides theoretical support to quantum field theory and Ising model.

Keywords Convolution integral equations, Riemann-Hilbert problems, singular integral-differential operators, Noether theory, Fourier transform.

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1. Introduction

There were rather complete investigations on the method of solution for integral equations of convolution type and equations of Cauchy type. Karapetjeantz [19] considered the invertibility of Wiener-Hopf operators with discontinuous coefficients. For operators containing both Cauchy principal value integral and convolution, Duduchava [10, 11] discussed the conditions of their Noethericity in more general cases. Litvinchuk [40] studied first the singular integral-differential equations, in which the class of differentiable functions was extended to the Hölder continuous function class, next the singular integral-differential equations which the coefficients contain a discontinuity point of the first kind were also studied, and he later again considered the integral-differential equations with convolution kernel. Subsequently, many mathematicians proposed a general method to solve some classes of singular integral equation with a mixture of convolution kernel and Cauchy kernel, in which the convolution kernel has discontinuous property, that is to transform this kind of

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integral equation to Riemann boundary value problem by using Fourier transform (see [12, 44] and references therein). Particularly, Li and Ren [37] dealt with the solvability of one class of singular integro-differential equations in the case of non-normal type, and the explicit solutions and conditions of solvability were obtained, which extended the results of [12, 44] to the classes of continuous functions and the classes of discontinuous coefficients. Later on, Li [25, 27–29, 31–34, 36, 38, 39] further considered the problem of finding solutions for convolution integral equations with singular kernels, and gave the conditions of solvability and the general solutions by Riemann-Hilbert approach.

In the practical problems, such as atomic diffusion theory, heat conduction, transport and nuclear collision, and mathematics physics and so on, these problems are closely related to singular integral-differential equations of Cauchy type and integral-differential equations of convolution type [3, 7, 9, 22, 30, 45]. Especially, the above practical problems also occur more general singular integral-differential equations with a mixture of convolution kernel and Cauchy kernel. The presentation and method of this class of equations rich the theory of singular integral equations, and the method of solution mentioned in the paper is still effective for solving other singular integral-differential equations.

This paper is devoted to the study of a class of singular integral-differential equations with convolution and Cauchy kernels, that is,

$$\begin{aligned}&\sum_{j=0}^{n}\{\alpha_{1,j}f_{+}^{(j)}(t)+\alpha_{2,j}f_{-}^{(j)}(t)+\frac{\beta_{1,j}}{\pi i}\int_{\mathbb{R}^{+}}\frac{f^{(j)}(\tau)}{\tau-t}d\tau+\frac{\beta_{2,j}}{\pi i}\int_{\mathbb{R}^{-}}\frac{f^{(j)}(\tau)}{\tau-t}d\tau\\&+\frac{\zeta_{1,j}}{\sqrt{2\pi}}\int_{\mathbb{R}^{+}}k_{1,j}(t-\tau)f^{(j)}(\tau)d\tau+\frac{\zeta_{2,j}}{\sqrt{2\pi}}\int_{\mathbb{R}^{-}}k_{2,j}(t-\tau)f^{(j)}(\tau)d\tau\}\\&=g(t),\quad t\in\mathbb{R},\end{aligned}\tag{1.1}$$

for $j=0,1,\cdots,n;\ n\in\mathbb{N}$, where $\alpha_{p,j},\ \beta_{p,j},\zeta_{p,j}$ $(p=1,2)$ are real constants with $\beta_{p,j}$ not all equal to zero simultaneously, the given functions $k_{p,j}(t),g(t)\in\{0\}$. The unknown function $f(t)\in\{0\}$, and its derivatives $f^{(j)}(t)\in\{0\}$ for any $j=1,2,\cdots,n$. In Eq. (1.1) , $f_{\pm}(t)$ are given by

$$f_{\pm}(t)=\frac{1}{2}f(t)(\mathrm{sgn}t\pm 1).$$

In this paper, we give effective methods of solution for Eq. (1.1). By using the classical Fourier analysis theory and lemmas given in this paper, we turn Eq. (1.1) into a Riemann boundary value problem. Via using the method of complex boundary value problems and the generalized principle of analytic continuation [6, 8, 17, 35, 50], we obtain the general solution and conditions of solvability in class $\{0\}$. Especially, we redefine the index formulas of coefficients in Eq. (1.1), and consider in detail the properties of the index. As its applications, the problem to find the solutions of Eq. (1.1) is also very important. Hence, Eq. (1.1) has important meaning not only in application but also in the theory of resolving the equation itself.

Our work is organized as follows. In Section 2, we introduce the concepts of classes $\{0\}$, $\{\{0\}\}$ and investigate their properties. In Section 3, we adopt the Fourier transform approach to transform Eq. (1.1) into boundary value problems for analytical functions with discontinuous coefficients. In Section 4, we use the theory of complex analysis and Sokhotski–Plemelj formula to study the Riemann-Hilbert problem obtained above. This allows us to show that Eq. (1.1) can be solved under certain conditions, so we obtain the conditions of solvability for Eq. (1.1). Finally, we give the conclusion of this article.

2. Definitions and lemmas

It is necessary for us to introduce certain new classes of functions in advance and to point out some of their properties. In this section, let us introduce some notations.

We denote $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = (0, +\infty), \mathbb{R}^- = (-\infty, 0)$, and $L^p(\mathbb{R})(p \geq 1)$ denote the spaces of Lebesgue integrable functions on $\mathbb{R}$. If $f(t)$ is continuous on $\mathbb{R}$, we denote as $f(t) \in C(\mathbb{R})$, similarly we also have $f(t) \in C(\dot{\mathbb{R}})$, where $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. H is a Hölder continuous function space. And $\mathbb{C}^+$ and $\mathbb{C}^-$ stand for the upper and lower half-planes, respectively.

The Fourier transforms used in this paper understood to be performed in $L^2(\mathbb{R})$ and the functions involved certainly belong to this space.

Definition 2.1. The Fourier transform operator of $f(t) \in L^2(\mathbb{R})$ is defined as follows

$$(\mathscr{F} f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) \exp(ixt) dt, \tag{2.1}$$

denote $(\mathscr{F} f)(x)$ as $F(x)$. And the inverse Fourier transform operator of $F(x)$ is defined by

$$(\mathscr{F}^{-1} F)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(x) \exp(-ixt) dx, \tag{2.2}$$

denote $(\mathscr{F}^{-1} F)(t)$ as $f(t)$.

Note that both the integrals of (2.1) and (2.2) exist in the sense of the Cauchy principal value. Obviously, we have

$$\mathscr{F}[f(-t)] = F(-x), \quad \mathscr{F}^{-1}[F(-x)] = f(-t). \tag{2.3}$$

Definition 2.2. Let $f(t) \in C(\mathbb{R})$, if there exist two positive real numbers A and M satisfying the following two conditions:

$$\begin{aligned} &(1)\ |f(t_1) - f(t_2)| \leq A\,|t_1 - t_2|^{\mu}, && \forall t_1, t_2 \in [-M, M]; \\ &(2)\ |f(t_1) - f(t_2)| \leq A\left|{t_1}^{-1} - {t_2}^{-1}\right|^{\mu}, && \forall t_1, t_2 \in \mathbb{R} \setminus [-M, M], \end{aligned} \tag{2.4}$$

then we say that $f(t) \in \hat{H}$, where $\mu \in (0, 1]$.

The concepts of classes $\{0\}$ and $\{\{0\}\}$ are introduced as follows.

Definition 2.3. A function $F(x)$ belongs to $\{\{0\}\}$, if the following two conditions are fulfilled:

1) $F(x) \in L^2(\mathbb{R})$; 2) $F(x) \in \hat{H}$, that is, $F(x)$ satisfies the Hölder conditions on $\dot{\mathbb{R}}$. That is,

$$\{\{0\}\} = \{F(x) | F \in \widehat{H} \cap L^2(\mathbb{R})\}.$$

Therefore, from Definition 2.3 we know that if $F(x) \in \{\{0\}\}$, then $F(x) \in C(\dot{\mathbb{R}})$ and $F(\infty) = 0$.

Definition 2.4. Let $F(x) = (\mathscr{F} f)(x)$, if $F(x) \in \{\{0\}\}$, we say that $f(t) \in \{0\}$, that is to say,

$$\{0\} = \{f(t) | \mathscr{F} f = F \in \{\{0\}\}\}.$$

Definition 2.5. The operators $\mathbb{N}$ and $\mathbb{S}$ are defined by

$$\mathbb{N}f(t) = f(-t), \quad \mathbb{S}f(t) = f(t)\mathrm{sgn}t. \tag{2.5}$$

It is easy to see that [31, 34, 38]

$$\mathbb{SN} + \mathbb{NS} = 0, \quad \mathbb{N}^2 = \mathbb{S}^2 = I, \tag{2.6}$$

where I is a unit operator.

For two functions $k(t)$ and $f(t)$, if we use the notation of convolution

$$(k * f)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(t-\tau)f(\tau)d\tau, \tag{2.7}$$

then it is well known that

$$\mathscr{F}\{(k * f)(t)\} = \mathscr{F}k(t) \cdot \mathscr{F}f(t) = K(x)F(x), \tag{2.8}$$

where K, F are the Fourier transforms of k, f respectively.

According to the properties of convolution integral (see [30,45,51]), we can easily obtain that if $k, f \in \{0\}$, then $k * f \in \{0\}$.

Definition 2.6. Let $f^{(j)}(t) \in \{0\}$ $(j = 0, 1, \cdots, n)$, we introduce the following singular integral–differential operator

$$(Tf^{(j)})(t) = \frac{1}{\pi i} \int_{\mathbb{R}} f^{(j)}(\tau) \frac{d\tau}{\tau - t}, \quad t \in \mathbb{R}, \quad j = 0, 1, \cdots, n, \tag{2.9}$$

where the integral on the right-hand side of (2.9) is also the Cauchy principal value integral, and $f^{(j)}(t)$ stands for the j-order derivatives of a function $f(t)$.

Note that we also can write (2.9) as

$$(Td^j f)(t) = \frac{1}{\pi i} \int_{\mathbb{R}} f^{(j)}(\tau) \frac{d\tau}{\tau - t} dt^j,$$

and when $j = 0$,

$$(Tf)(t) = \frac{1}{\pi i} \int_{\mathbb{R}} f(\tau) \frac{d\tau}{\tau - t}, \quad t \in \mathbb{R},$$

where $d^j f$ stands for the j-order differentiations of a function f.

From [6, 48, 51], we know that T maps $\{\{0\}\}$ into itself, and

$$T^2 = I.$$

It is evident that

$$\mathscr{F}^{-1} = \mathbb{N}\mathscr{F} = \mathscr{F}\mathbb{N}, \quad \mathscr{F}^2 = \mathbb{N}. \tag{2.10}$$

It was proved in [10, 26, 53], that when applying to functions in $\{0\}$,

$$\mathscr{F}\mathbb{S} = T\mathscr{F}. \tag{2.11}$$

The following Lemma 2.1 plays an important role in the proof of the subsequent results and can be stated as follows.

Lemma 2.1. *When applying to functions in* $\{0\}$,

$$\mathscr{F}T = -\mathbb{S}\mathscr{F}, \tag{2.12}$$

that is,

$$\mathscr{F}[\frac{1}{\pi i}\int_{\mathbb{R}} f(\tau)\frac{d\tau}{\tau - t}] = -F(x)\text{sgn}x. \tag{2.13}$$

Proof. From (2.11), we have

$$T = \mathscr{F}\mathbb{S}\mathscr{F}^{-1}. \tag{2.14}$$

Since

$$\mathscr{F}^{-1} = \mathbb{N}\mathscr{F} = \mathscr{F}\mathbb{N}, \quad \mathscr{F}^2 = \mathbb{N}, \quad T = \mathscr{F}\mathbb{S}\mathscr{F}^{-1}, \tag{2.15}$$

we get

$$\mathscr{F}T = \mathscr{F}^2\mathbb{S}\mathscr{F}^{-1} = \mathbb{N}\mathbb{S}\mathbb{N}\mathscr{F} = -\mathbb{N}^2\mathbb{S}\mathscr{F} = -\mathbb{S}\mathscr{F}, \tag{2.16}$$

therefore, (2.12) is true. □

Note that, from $f \in \{0\}$, generally we could not assure that $Tf \in \{0\}$. However, we have

Lemma 2.2. *If $f \in \{0\}$, and $(\mathscr{F}f)(0) = 0$, then $Tf \in \{0\}$.*

Proof. Owing to $f \in \{0\}$, so we have $\mathscr{F}f \in \{\{0\}\}$. From Lemma 2.1, we know that

$$\mathscr{F}Tf = -\text{sgn}x\mathscr{F}f. \tag{2.17}$$

Noting that

$$(\mathscr{F}f)(\infty) = (\mathscr{F}f)(0) = 0, \tag{2.18}$$

hence we obtain $\mathscr{F}f\text{sgn}x \in \{\{0\}\}$, i.e., $\mathscr{F}Tf \in \{\{0\}\}$, then $Tf \in \{0\}$. □

Moreover, when $f(t) \in L^1(\mathbb{R})$, we have

$$\lim_{x\to 0} F(x) = \lim_{x\to 0}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)\exp(ixt)dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)dt.$$

Since F is continuous on $\mathbb{R}$, therefore

$$(\mathscr{F}f)(0) = 0$$

if and only if

$$\int_{\mathbb{R}} f(t)dt = 0. \tag{2.19}$$

The following several lemmas are important to our paper.

Lemma 2.3. *Let $f^{(j)}(t) \in \{0\}$ $(j = 0, 1, \cdots, n)$, then*

$$\mathscr{F}[f_{\pm}^{(j)}(t)] = (-ix)^j\mathscr{F}[f_{\pm}(t)] - \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{j-1}(-ix)^m f^{(j-m-1)}(0) \tag{2.20}$$

for $j = 1, 2, \cdots, n$.

Proof. We will use mathematical induction to prove (2.20). Since

$$\begin{aligned}\mathscr{F}[f_+^{'}(t)] &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f_+^{'}(t)\exp(ixt)dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+} f^{'}(t)\exp(ixt)dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^+} \exp(ixt)df(t)\\ &= \frac{1}{\sqrt{2\pi}}[f(t)\exp(ixt)]|_0^{+\infty} - \frac{1}{\sqrt{2\pi}}ix\int_{\mathbb{R}^+} f(t)\exp(ixt)dt\\ &= -\frac{1}{\sqrt{2\pi}}f(0) + (-ix)\mathscr{F}[f_+(t)],\end{aligned} \tag{2.21}$$

hence the result is true for $j = 1$. We now suppose (2.20) holds for $j = k$, that is,

$$\mathscr{F}[f_+^{(k)}(t)] = (-ix)^k\mathscr{F}[f_+(t)] - \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{k-1}(-ix)^m f^{(k-m-1)}(0), \tag{2.22}$$

and prove it for $j = k + 1$.

By using integral subsection method and (2.22), then when $j = k + 1$, we have

$$\mathscr{F}[f_+^{(k+1)}(t)] = (-ix)^{k+1}\mathscr{F}[f_+(t)] - \frac{1}{\sqrt{2\pi}}\sum_{m=0}^{k}(-ix)^m f^{(k-m)}(0). \tag{2.23}$$

Similarly, we can prove the case $f_-^{(j)}(t)$ for $j = 1, 2, \cdots, n$. □

Lemma 2.4. *Let $f^{(j)}(t) \in \{0\}$ and $F(0) = 0$, $Tf^{(j)}(t)$ is given by (2.9), that is,*

$$(Tf^{(j)})(t) = \frac{1}{\pi i}\int_{\mathbb{R}} f^{(j)}(\tau)\frac{d\tau}{\tau - t}, \quad j = 0, 1, 2, \cdots, n,$$

then we have

$$(\mathscr{F}Tf^{(j)})(t) = -(-ix)^j \operatorname{sgn} x F(x). \tag{2.24}$$

Proof. By (2.20) in Lemma 2.3, we know that

$$\begin{aligned}(\mathscr{F}f^{(j)})(t) &= (\mathscr{F}f_+^{(j)})(t) - (\mathscr{F}f_-^{(j)})(t)\\ &= (-ix)^j(\mathscr{F}f_+)(t) - (-ix)^j(\mathscr{F}f_-)(t)\\ &= (-ix)^j F(x).\end{aligned} \tag{2.25}$$

Again using Lemma 2.2, we can obtain

$$(\mathscr{F}Tf^{(j)})(t) = -(-ix)^j \operatorname{sgn} x F(x). \tag{2.26}$$

□

The definitions of all above notations can be found in [12, 40, 44].

To transform Eq. (1.1) into a Riemann boundary value problem, in the following we need to establish the relation between Fourier transforms and Riemann boundary value problems. Let $f \in \{0\}$, we define the Cauchy type integral as follows

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(t)\frac{dt}{t - z}, \quad \forall z \in \mathbb{C}^+ \cup \mathbb{C}^-, \tag{2.27}$$

then we can obtain

$$\lim_{z\to x, z\in\mathbb{C}^+} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-z}dt = F^+(x), \quad x\in\mathbb{R}, \tag{2.28}$$

and

$$\lim_{z\to x, z\in\mathbb{C}^-} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(t)}{t-z}dt = F^-(x), \quad x\in\mathbb{R}. \tag{2.29}$$

From [40, 41], we have

$$F^+(x) = \frac{1}{2\pi i}\int_{\mathbb{R}^+} f(t)\frac{dt}{t-x} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f_+(t)\exp(ixt)dt, \tag{2.30}$$

and

$$F^-(x) = \frac{1}{2\pi i}\int_{\mathbb{R}^-} f(t)\frac{dt}{t-x} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f_-(t)\exp(ixt)dt, \tag{2.31}$$

where $F^{\pm}(x)$ are the Fourier transforms of $f_{\pm}(t)$ respectively, which are the boundary values of the sectionally analytical function $F(z)$ in the upper half plane $\mathbb{C}^+$ and the lower half plane $\mathbb{C}^-$, respectively.

At the end of this section, we give the generalized Sokhotski–Plemelj formulas of Fourier transform.

Assume that $f \in \{0\}$, the Cauchy type integral $F(z)$ is given by (2.27), then the following generalized Sokhotski–Plemelj formulas hold

$$F^{\pm}(x) = \pm\frac{1}{2}f(x) + \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-x}dt, \quad x\in\mathbb{R}, \tag{2.32}$$

i.e.,

$$F^{\pm}(x) = \pm\frac{1}{2}f(x) + F(x), \qquad x\in\mathbb{R}. \tag{2.33}$$

Moreover, when $F^{\pm} \in \{\{0\}\}$, via the extended residue theorem [37, 41, 44], we also know that

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{F(t)dt}{t-z}, \quad \forall z\in\mathbb{C}^+\cup\mathbb{C}^-. \tag{2.34}$$

It follows that $F(z)$ is also analytic in $\mathbb{C}^+\cup\mathbb{C}^-$.

3. Reducing Eq. (1.1) to a Riemann boundary value problem

In this section, via using Fourier transform, we will convert Eq. (1.1) into a Riemann boundary value problem. In order to solve Eq. (1.1), we may write it as the following form

$$\begin{aligned}&\sum_{j=0}^{n}\{\alpha_{1,j}f_+^{(j)}(t) + \alpha_{2,j}f_-^{(j)}(t) + \beta_{1,j}Tf_+^{(j)}(t) - \beta_{2,j}Tf_-^{(j)}(t)\\ &+ \zeta_{1,j}k_{1,j} * f_+^{(j)}(t) - \zeta_{2,j}k_{2,j} * f_-^{(j)}(t)\} = g(t), \quad t\in\mathbb{R}.\end{aligned} \tag{3.1}$$

In the following discussions, we will see that Eq. (3.1) has a solution under some conditions, hence we take Fourier transforms in both sides of (3.1) by using

Lemmas 2.1-2.4, then (3.1) should be simplified as the following Riemann boundary value problem with discontinuous coefficients:

$$F^{+}(x) = A(x)F^{-}(x) + C(x), \quad x \in \mathbb{R}, \tag{3.2}$$

where

$$A(x) = \frac{\sum\limits_{j=0}^{n}\{(-ix)^j[-\alpha_{2,j} - \beta_{2,j}\mathrm{sgn}x + \zeta_{2,j}K_{2,j}(x)]\}}{\sum\limits_{j=0}^{n}\{(-ix)^j[\alpha_{1,j} - \beta_{1,j}\mathrm{sgn}x + \zeta_{1,j}K_{1,j}(x)]\}},$$

$$C(x) = \frac{G(x) + \frac{1}{\sqrt{2\pi}}\sum\limits_{j=1}^{n}[\sum\limits_{m=0}^{j-1}(-ix)^m f^{(j-m-1)}(0)]E_j(x)}{\sum\limits_{j=0}^{n}\{(-ix)^j[\alpha_{1,j} - \beta_{1,j}\mathrm{sgn}x + \zeta_{1,j}K_{1,j}(x)]\}},$$

$$E_j(x) = \alpha_{1,j} + \alpha_{2,j} + \zeta_{1,j}K_{1,j}(x) - \zeta_{2,j}K_{2,j}(x),$$

$$F^{\pm}(x) = (\mathscr{F}f_{\pm})(x), \quad G(x) = (\mathscr{F}g)(x), \quad K_{p,j} = (\mathscr{F}k_{p,j})(x)$$

for $p = 1, 2$; $j = 0, 1, 2, \cdots, n$.

It is not difficult to prove that (3.1) and (3.2) are equivalent, that is, they have the same solutions. From previous discussions, we also know that (3.2) is a Riemann boundary value problem with nodes $x = 0, \infty$ on $\mathbb{R}$, and it can be directly solved by the method of [23, 24, 41].

In this paper, we shall take another method to solve Eq. (3.2). Applying linear transform [2, 47]

$$z = \frac{\xi}{-1 + i\xi}, \tag{3.3}$$

this transform maps the real axis X on the plane Z onto the unit circle Γ on the complex plane ξ

$$\Gamma = \{\xi \in \mathbb{C} : |2\xi + i| = 1\},$$

which surrounds an interior region Σ^{+} and an exterior region Σ^{-}, and maps the upper half-plane $\mathbb{C}^{+}$ and the lower half-plane $\mathbb{C}^{-}$ onto the Σ^{+} and Σ^{-} respectively, where

$$\Sigma^{+} = \{\xi \in \mathbb{C} : |2\xi + i| < 1\}, \quad \Sigma^{-} = \{\xi \in \mathbb{C} : |2\xi + i| > 1\}.$$

Again let

$$F(z) = \Psi(\xi), \quad G(z) = U(\xi), \quad K_{p,j}(z) = B_{p,j}(\xi)$$

for $p = 1, 2$; $j = 0, 1, 2, \cdots, n$, then (3.2) is readily reduced to the following Riemann boundary value problem on the plane ξ:

$$\Psi^{+}(\tau) = N(\tau)\Psi^{-}(\tau) + M(\tau), \quad \tau \in \Gamma, \tag{3.4}$$

where

$$M(\tau)=\frac{U(\tau)+\frac{1}{\sqrt{2\pi}}\sum\limits_{j=1}^{n}[\sum\limits_{m=0}^{j-1}(-\frac{\tau}{\tau+i})^m f^{(j-m-1)}(0)]L_j(\tau)}{\sum\limits_{j=0}^{n}\{[\alpha_{1,j}-\beta_{1,j}\delta(\tau)+\zeta_{1,j}B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j\}},$$

$$N(\tau)=\frac{\sum\limits_{j=0}^{n}\{[-\alpha_{2,j}-\beta_{2,j}\delta(\tau)+\zeta_{2,j}B_{2,j}(\tau)](-\frac{\tau}{\tau+i})^j\}}{\sum\limits_{j=0}^{n}\{[\alpha_{1,j}-\beta_{1,j}\delta(\tau)+\zeta_{1,j}B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^j\}},$$

$$L_j(\tau)=\alpha_{1,j}+\alpha_{2,j}+\zeta_{1,j}B_{1,j}(\tau)-\zeta_{2,j}B_{2,j}(\tau),\ j=0,1,2,\cdots,n$$

and

$$\delta(\tau)=\begin{cases}1, & \tau\in\Gamma_1,\\ -1, & \tau\in\Gamma_2,\end{cases}$$

where Γ_1,Γ_2 are the left half circles and the right half circles of Γ, respectively.

Note that transform (3.3) maps the real axis X on the plane Z onto the unit circle Γ on the complex plane ξ. Then, for any $x\in X$, by (3.3) we can find the unique $\tau\in\Gamma$, such that $\tau=\frac{x}{-1+ix}$, thus we have

$$E_j(x)=L_j(\tau),\quad j=0,1,2,\cdots,n.$$

To solve a Riemann boundary value problem (3.4), in the following section 4 we will consider the two cases: the normal type and the non-normal type.

4. The solving method of problem (3.4)

In Section 3, we have transformed Eq. (1.1) into a Riemann boundary value problem (3.4). Some special kinds of Riemann boundary value problems with discontinuous coefficients appear in the course of solving equation (1.1), which are solved in the same time. Now we will consider the conditions of Noether solvability and the general solution of problem (3.4) in class $\{0\}$ by means of the principle of analytic continuation and of the theory of complex analysis.

4.1. Solution of problem (3.4) in the case of normal type

First, let us solve the Riemann boundary value problem (3.4) in the case of normal type.

If $\sum\limits_{j=0}^{n}\{[(-1)^{p-1}\alpha_{p,j}-\beta_{p,j}\delta(\tau)+\zeta_{p,j}B_{p,j}(\tau)](-\frac{\tau}{\tau+i})^j\}\neq 0$ $(\tau\in\Gamma;p=1,2)$, then $\tau=0$ is the discontinuity points of $M(\tau)$ and $N(\tau)$. Therefore $\tau=0$ is also node of problem (3.4), in this case we say that (3.4) is the boundary value problem of normal type with node $\tau=0$. Let

$$\gamma=\sigma_*+i\iota_*=\frac{1}{2\pi i}\log\frac{N(-0)}{N(+0)},\tag{4.1}$$

where the definitions of $N(\pm 0)$ can be found from [31,37], namely,

$$N(-0)=\lim_{\tau\to 0,\tau\in\Gamma_2}N(\tau),\quad N(+0)=\lim_{\tau\to 0,\tau\in\Gamma_1}N(\tau).$$

Because $\log N(\tau)$ has infinite number continuous branches, we take a continuous branch of $\log N(\tau)$ such as

$$\log N(-i) = 0.$$

Then choose an integer χ such as

$$0 \leq \sigma_* - \chi < 1,$$

and again denote

$$\sigma = \sigma_* - \chi, \quad \lambda = \gamma - \chi = \sigma + i\iota_*, \tag{4.2}$$

then we call χ the index of the problem (3.4). Note that, we can also define the formula of index in the following method

$$\chi = \frac{1}{2\pi i}\int_{\mathbb{R}} d\log N(s). \tag{4.3}$$

Define the following piece-wise function

$$X(\xi) = \begin{cases} e^{V(\xi)}, & \xi \in \Sigma^+, \\ (\xi + \frac{i}{2})^{-\chi} e^{V(\xi)}, & \xi \in \Sigma^-, \end{cases} \tag{4.4}$$

where

$$V(\xi) = \frac{1}{2\pi i}\int_\Gamma \log[(\tau + \frac{i}{2})^{-\chi} N(\tau)]\frac{d\tau}{\tau - \xi}, \quad \xi \notin \Gamma. \tag{4.5}$$

Via using the Sokhotski-Plemelj formula to (4.4) and (4.5), we obtain

$$\begin{aligned} &X^+(\tau) = e^{V^+(\tau)}, \quad X^-(\tau) = e^{V^-(\tau)}(\tau + \frac{i}{2})^{-\chi}, \\ &V^\pm(\tau) = \pm\frac{1}{2}\log[(\tau + \frac{i}{2})^{-\chi}N(\tau)] + \frac{1}{2\pi i}\int_\Gamma \log[(t + \frac{i}{2})^{-\chi}N(t)]\frac{dt}{t - \tau}, \end{aligned}$$

therefore, from our previous discussions, we know that

$$\begin{aligned} \frac{X^+(\tau)}{X^-(\tau)} &= \frac{e^{V^+(\tau)}}{e^{V^-(\tau)}(\tau + \frac{i}{2})^{-\chi}} \\ &= e^{V^+(\tau) - V^-(\tau)}(\tau + \frac{i}{2})^{\chi} \\ &= e^{\log[N(\tau)(\tau + \frac{i}{2})^{-\chi}]}(\tau + \frac{i}{2})^{\chi} \\ &= N(\tau)(\tau + \frac{i}{2})^{-\chi}(\tau + \frac{i}{2})^{\chi} \\ &= N(\tau), \end{aligned}$$

that is,

$$X^+(\tau) = N(\tau)X^-(\tau), \quad \tau \in \Gamma. \tag{4.6}$$

Next, we want to study the solvability of problem (3.4) from a more general point of view. For the time being, we do not consider the property of $\Psi(\xi)$ at a neighborhood of $\xi = 0$. We know that $\Psi(\xi)$ is bounded at $\xi = \infty$, therefore (3.4) has a solution in R_0. Similar to the solving method in [7,31,41], again by means of the principle of analytic continuation and of the theory of complex analysis [41,44], we can obtain the general solutions of (3.4) as follows

$$\Psi(\xi) = X(\xi)P_\chi(\xi) + X(\xi)D(\xi), \tag{4.7}$$

where

$$D(\xi) = \frac{1}{2\pi i}\int_\Gamma \frac{M(\tau)}{X^+(\tau)}\frac{1}{\tau-\xi}d\tau, \quad \xi \notin \Gamma, \tag{4.8}$$

and when $\chi \geq 0$, $P_\chi(\xi)$ is a polynomial with the degree χ; when $\chi < 0$, $P_\chi(\xi) \equiv 0$; and when $\chi < -1$, the following $-\chi - 1$ conditions of Noether solvability should be satisfied

$$\int_\Gamma \frac{M(\tau)}{X^+(\tau)}\tau^j d\tau = 0 \tag{4.9}$$

for $j = 0, 1, \cdots, -\chi - 2$. Therefore, when $\chi < 0$, (3.4) has only solution and its solution is still (4.7), in this case, $P_\chi(\xi) \equiv 0$.

In the following, we shall consider the case near $\tau = 0$. By Lemma 2.2, we have $(\mathscr{F}f)(t) \in \{\{0\}\}$ and $(\mathscr{F}f)(0) = 0$.

If $\tau = 0$ is an ordinary node, then $0 < \sigma < 1$, and $\lambda \neq 0$, because $\Psi(\xi)$ is a continuous function near $\tau = 0$, and $(\mathscr{F}f)(0) = 0$, we may obtain

$$\Psi(0) = (\mathscr{F}f)(0) = 0,$$

hence one has the following condition of Noether solvability:

$$\sqrt{2\pi}U(0) + \sum_{j=1}^{n} f^{(j-1)}(0)L_j(0) = 0. \tag{4.10}$$

If $\tau = 0$ is a special node, that is, $\sigma = 0, \lambda = i\iota_0$.

(1) If $\iota_0 \neq 0$, in this case, besides (4.9), the following condition of Noether solvability must be fulfilled:

$$2\pi i c + \int_\Gamma \frac{M(\tau)}{X^+(\tau)} \cdot \frac{1}{\tau}d\tau = 0, \tag{4.11}$$

where c is a constant term of $P_\chi(\xi)$. Once (4.11) is fulfilled, then (4.9) must be also satisfied, and $D(\xi) \in H$ near $\tau = 0$. Therefore, in order that $\Psi^\pm(\xi)$ is continuous at $\tau = 0$, the constant term c of $P_\chi(\xi)$ must satisfy (4.11).

(2) If $\iota_0 = 0$, then $\lambda = 0$, hence $\Psi(0) = 0$ if and only if (4.11) holds.

In summary above cases, we may obtain that the necessary conditions of the existence of solution for (3.4) are (4.9) and (4.11).

From our previous discussions, we have the following

Theorem 4.1. *Under the case of the normal type, the necessary conditions of the existence of solution for Eq.* (3.1) *are* (4.9) *and* (4.11). *Assume that* (4.9) *and* (4.11) *hold.*

(1) If $\tau = 0$ is an ordinary node, when $\chi \geq -1$, Eq. (3.1) *is solvable; when $\chi < -1$, and* (4.10) *satisfies,* (3.1) *has only solution* (4.7), *in this case, $P_\chi(z) \equiv 0$.*

(2) If $\tau = 0$ is a special node, then (4.9) *and* (4.11) *hold. When $\chi \geq -1$,* (3.1) *has always a solution; when $\chi < -1$ and* (4.10) *satisfies, then* (3.1) *has a solution.*

Therefore, the solution of (3.4) *is given by*

$$f(t) = \mathscr{F}^{-1}F(x), \tag{4.12}$$

where

$$F(x) = F^{+}(x) - F^{-}(x). \tag{4.13}$$

From $F(x) \in \{\{0\}\}$, *we have* $f(t) \in \{0\}$, *and the uncertain constants* $f^{(j)}(0)(j = 0, 1, \cdots, n-1)$ *are determined by* (4.27)–(4.29).

4.2. Solution of problem (3.4) in the case of non-normal type

Now, we assume that $N(\tau)$ has some zero-points and pole-points on Γ, then problem (3.4) is called the non-normal type case. Let

$$\sum_{j=0}^{n}[\alpha_{1,j} - \beta_{1,j}\delta(\tau) + \zeta_{1,j}B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^{j}$$

and

$$\sum_{j=0}^{n}[-\alpha_{2,j} - \beta_{2,j}\delta(\tau) + \zeta_{2,j}B_{2,j}(\tau)](-\frac{\tau}{\tau+i})^{j}$$

have common and the same order zero-points $a_1, a_2, \cdots, a_q$ with the orders $\gamma_1, \gamma_2, \cdots, \gamma_q$ respectively on Γ;

$$\sum_{j=0}^{n}[\alpha_{1,j} - \beta_{1,j}\delta(\tau) + \zeta_{1,j}B_{1,j}(\tau)](-\frac{\tau}{\tau+i})^{j}$$

has some zero-points $b_1, b_2, \cdots, b_s$ with the orders $\alpha_1, \alpha_2, \cdots, \alpha_s$ respectively;

$$\sum_{j=0}^{n}[-\alpha_{2,j} - \beta_{2,j}\delta(\tau) + \zeta_{2,j}B_{2,j}(\tau)](-\frac{\tau}{\tau+i})^{j}$$

has some zero-points $c_1, c_2, \cdots, c_l$ with the orders $\beta_1, \beta_2, \cdots, \beta_l$ respectively, and $\alpha_j, \beta_j, \gamma_j$ are positive integers. Again let

$$\Pi_1(\tau) = \prod_{j=1}^{s}(\tau - b_j)^{\alpha_j}, \quad \Pi_2(\tau) = \prod_{j=1}^{l}(\tau - c_j)^{\beta_j},$$

$$\sum_{j=1}^{s}\alpha_j = N_1, \quad \sum_{j=1}^{l}\beta_j = N_2, \quad \sum_{j=1}^{q}\gamma_j = N_3.$$

Then, (3.4) is reduced to

$$\Psi^{+}(\tau) = \frac{\Pi_2(\tau)}{\Pi_1(\tau)}N_0(\tau)\Psi^{-}(\tau) + M(\tau), \quad \tau \in \Gamma. \tag{4.14}$$

Since $F(x) \in \{\{0\}\}$, so $\Psi(\tau)$ is continuous on Γ. In order that $\Psi(\tau)$ satisfy the conditions at $a_j(1 \leq j \leq q)$, then the following conditions of solvability must be fulfilled

$$\sqrt{2\pi}U^{(k)}(a_j) + \{\sum_{j=1}^{n}\sum_{m=0}^{j-1}(-\frac{\tau}{\tau+i})^{m}f^{(j-m-1)}(0)L_j(\tau)\}^{(k)}|_{\tau=a_j} = 0 \tag{4.15}$$

for $k = 0, 1, \cdots, \gamma_j - 1$; $j = 1, 2, \cdots, q$.

In order that computation of (4.15) is feasible, $U(\tau), B_{1,j}(\tau), B_{2,j}(\tau)$ must exist derivatives until order $\gamma_j - 1$ at the neighborhood of a_j, and all order derivatives satisfy Hölder conditions.

Here, according to the values of $\alpha_{1,j} \pm \beta_{1,j}, \alpha_{2,j} \pm \beta_{2,j} (j = 0, 1, \cdots, n)$, we have the following four cases

(1) $\alpha_{1,j} \pm \beta_{1,j} \neq 0, \quad \alpha_{2,j} \pm \beta_{2,j} \neq 0$; (2) $\alpha_{1,j} \pm \beta_{1,j} = 0, \quad \alpha_{2,j} \pm \beta_{2,j} \neq 0$;
(3) $\alpha_{1,j} \pm \beta_{1,j} \neq 0, \quad \alpha_{2,j} \pm \beta_{2,j} = 0$; (4) $\alpha_{1,j} \pm \beta_{1,j} = 0, \quad \alpha_{2,j} \pm \beta_{2,j} = 0$
for $j = 0, 1, \cdots, n$.

Without loss of generality, we discuss only the case (1) in this paper. On other cases, similar to the statement in the case (1), here we do not discuss. Under satisfying the above conditions, (3.4) is a Riemann boundary value problem with the discontinuous coefficients, and $\tau = 0$ is a node of (3.4). $X(\xi)$ take also (4.4), in which $\Gamma(\xi)$ take the following value

$$\Gamma(\xi) = \frac{1}{2\pi i} \int_\Gamma \log N_0(\tau) \frac{d\tau}{\tau - \xi}, \quad \xi \notin \Gamma. \tag{4.16}$$

Moreover, we can easily see that the homogeneous problem of (4.14) is

$$\Pi_1(\tau)\Psi^+(\tau) = \Pi_2(\tau) N_0(\tau) \Psi^-(\tau). \tag{4.17}$$

Using the extended Liouville theory and the regularity theory of the classical Riemann boundary value problems [9, 22, 44], we may obtain the general solutions of (4.17) as follows

$$\Psi_*(\xi) = \begin{cases} \Pi_2(\xi) X(\xi) P_{\chi - N_1}(\xi), & \xi \in \Sigma^+, \\ \Pi_1(\xi) X(\xi) P_{\chi - N_1}(\xi), & \xi \in \Sigma^-. \end{cases} \tag{4.18}$$

Next, we discuss the non-homogeneous case of (4.14). Construct the following function

$$\Phi(\xi) = \frac{1}{2\pi i} \int_\Gamma \frac{\Pi_1(\tau) M(\tau)}{X^+(\tau)} \cdot \frac{1}{\tau - \xi} d\tau, \quad \xi \notin \Gamma. \tag{4.19}$$

For the time being, we do not consider $\Psi(\xi)$ at case of neighborhood of $\tau = 0$. Since $F(x) \in \{\{0\}\}$, hence $\Psi(\xi)$ is bounded on Γ, and $\Psi(\xi)$ has no singularity at $\tau = b_j, c_j$. Via Sokhotski-Plemelj formula and the extended Liouville theory [3,41,44], and again using Hermite interpolation polynomial $\Omega_\rho(\xi) (\rho = N_1 + N_2 - 1)$, hence we can define the following function:

$$S(\xi) = \begin{cases} \dfrac{\Phi(\xi) - \Omega_\rho(\xi)}{\Pi_1(\xi)} X(\xi), & \xi \in \Sigma^+, \\[2ex] \dfrac{\Phi(\xi) - \Omega_\rho(\xi)}{\Pi_2(\xi)} X(\xi), & \xi \in \Sigma^-. \end{cases} \tag{4.20}$$

In (4.20), $\Omega_\rho(\xi)$ is a Hermite interpolation polynomial with the zero-points of order α_j, β_j at b_j, c_j, respectively. When the solvable conditions (4.15) are satisfied, $S(\xi)$ is a special solution of non-homogeneous Riemann boundary value problem (4.14). Via using the structure theorem of solution, we obtain the general solutions of the problem (4.14) as follows

$$\Psi(\xi) = \Psi_*(\xi) + S(\xi). \tag{4.21}$$

Taking the boundary values to $\Psi(\xi)$ in (4.21), we can obtain

$$\Psi(\xi) = \begin{cases} \Psi^+(\xi), & \xi \in \Sigma^+, \\ \Psi^-(\xi), & \xi \in \Sigma^-, \end{cases} \tag{4.22}$$

where

$$\Psi^+(\xi) = \frac{X^+(\xi)}{\Pi_1(\xi)}[\Phi(\xi) - \Omega_\rho(\xi)] + \frac{1}{2}M(\xi) + X^+(\xi)\Pi_2(\xi)P_{\chi-N_1}(\xi),$$
$$\Psi^-(\xi) = \frac{X^-(\xi)}{\Pi_2(\xi)}[\Phi(\xi) - \Omega_\rho(\xi)] - \frac{1}{2}\frac{M(\xi)}{N(\xi)} + X^-(\xi)\Pi_1(\xi)P_{\chi-N_1}(\xi).$$

In the following, we shall discuss the analytical property of solutions near $\tau = 0$ and ∞.

Firstly, we consider the case near $\tau = 0$. When $\tau = 0$, on the case of the solution (4.21), similar to the above discussion.

If $\tau = 0$ is an ordinary node if and only if (4.9) fulfills.

If $\tau = 0$ is a special node, besides (4.9) holds, the constant term c of $P_\chi(\xi)$ is determined by the following equality:

$$c = \frac{1}{\Pi_1(0)\Pi_2(0)}(\Omega_\rho(0) - \Phi(0)). \tag{4.23}$$

Secondly, we consider the case near $\tau = \infty$, by (4.18) and (4.20), we know that if $N_1 - \chi - 1 > 0$, then $S(\xi)$ has a pole point with the order $N_1 - \chi - 1$ at ∞. In order that $\Psi(\xi)$ is bounded at $\xi = \infty$, suppose $\Omega_\rho(\xi)$ is written as

$$\Omega_\rho(\xi) = e_0\xi^\rho + e_1\xi^{\rho-1} + \cdots + e_\rho, \tag{4.24}$$

where e_j are constants, then one must have

$$e_0 = e_1 = \cdots = e_{N_1-\chi-2} = 0, \tag{4.25}$$

here $\Omega_\rho(\xi)$ is a polynomial with the degree $\rho - (N_1 - \chi - 1)$. Note that

$$\rho - (N_1 - \chi - 1) = N_2 + \chi.$$

Therefore, from previous discussions, we also know that

(1) When $N_2 + \chi = -1$, we have $\Omega_\rho(\xi) \equiv 0$.

(2) When $N_2 + \chi < -1$, besides $\Omega_\rho(\xi) \equiv 0$, the following conditions of Noether solvability must be satisfied

$$\begin{aligned} &\int_\Gamma \frac{\Pi_1(\tau)M(\tau)}{X^+(\tau)}\frac{1}{(\tau - b_j)^p}d\tau = 0, \quad j = 1, 2, \cdots, s, \quad p = 0, 1, 2, \cdots, \alpha_j, \\ &\int_\Gamma \frac{\Pi_1(\tau)M(\tau)}{X^+(\tau)}\frac{1}{(\tau - c_r)^q}d\tau = 0, \quad r = 1, 2, \cdots, l, \quad p = 0, 1, 2, \cdots, \beta_r, \\ &\int_\Gamma \frac{\Pi_1(\tau)M(\tau)}{X^+(\tau)}\tau^{k-1}d\tau = 0, \qquad k = 0, 1, \cdots, N_1 - \chi - \rho. \end{aligned} \tag{4.26}$$

Since we require that $f^{(j)}(t) \in \{0\}$, therefore $(F^\pm(x))^{(j)} \in \{\{0\}\}(j = 0, 1, 2, \cdots, n)$. Here we will use the following method to solve the undetermined constants

$f(0)$,$f'(0), \cdots, f^{(n-1)}(0)$. Suppose that $F^+(x)$ may be expanded the Laurent series in the hollow neighborhood of $x = \infty$ [41, 44], we only take former finite number terms, namely,

$$\begin{aligned} &F^+(x) = \frac{\upsilon_1}{x} + O(\frac{1}{\mid x \mid}), \\ &F^+(x) = \frac{\upsilon_1}{x} + \frac{\upsilon_2}{x^2} + O(\frac{1}{\mid x \mid^2}), \\ &\cdots\cdots \\ &F^+(x) = \frac{\upsilon_1}{x} + \cdots + \frac{\upsilon_n}{x^n} + O(\frac{1}{\mid x \mid^n}). \end{aligned} \tag{4.27}$$

Owing to

$$\upsilon_j = \frac{f^{(j-1)}(0)}{\sqrt{2\pi} i}, \quad j = 1, 2, \cdots, n, \tag{4.28}$$

therefore, we may obtain $f(0), f'(0), \cdots, f^{(n-1)}(0)$ by solving Eq. (4.27).

Moreover, we can also apply the following method, because

$$\lim_{x \to +\infty} \{(-ix)^j F^+(x) - \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{j-1} (-ix)^m f^{(j-m-1)}(0)\} = 0, \tag{4.29}$$

the undetermined constants $f(0), f'(0), \cdots, f^{(n-1)}(0)$ can also be determined by solving Eq. (4.29).

Through the above method of solution, we have the following theorem.

Theorem 4.2 (Theory of Noether solvability). *Under the conditions*

$$\alpha_{p,j} \pm \beta_{p,j} \neq 0, \quad j = 0, 1, \cdots, n, \quad p = 1, 2,$$

the necessary conditions of the existence of solution for Eq. (3.1) *are* (4.9) *and* (4.11). *Assume that the above conditions hold.*

(1) When $\tau = 0$ is an ordinary node, if $\chi - N_1 \geq 0$, Eq. (3.1) *is solvable, and contains $\chi - N_1$ linearly independent solutions; if $\chi - N_1 = -1$,* (3.1) *has only solution; if $\chi - N_1 < -1$, we require also that* (4.15) *satisfies, and if $N_2 + \chi < -1$, one require also that* (4.26) *holds, then* (3.1) *has just only solution, and its solution is given by* (4.21)*, and when $\chi - N_1 \leq -1$, $\Omega_\rho(\xi) \equiv 0$.*

(2) When $\tau = 0$ is a special node, the above corresponding conditions and (4.23) *should be fulfilled. Then the solution of Eq.* (3.1) *is also given by* (4.12)*, where $F(x)$ is determined by* (4.13)*. And $F^\pm(x)$ (or $\Psi(\xi)$) are given by* (4.22).

Therefore, the uncertain constants $f^{(j)}(0) (j = 0, 1, 2, \cdots, n-1)$ are determined by (4.27)–(4.29).

5. Conclusions

In this article, we consider Noether solvability and analytical solutions for a class of convolution integral equations with singular integral-differential operators of Cauchy type. From the previous work, we know that other class of singular integral-different equation of convolution type can be also solved by the method of this paper, such as Wiener-Hopf type equations, dual type and so on. It is of great significance for improving and developing integral equations and boundary value theory, Clifford analysis theory, complex analysis and other disciplines. For other convolution integral equations with singular integral-differential operators, our method is very appropriate, which is very helpful for solving other equations and applications.

Moreover, we can also study the stability of solution as well as solvability for Eq. (1.1) in Clifford analysis (see [4, 5, 8, 14, 20, 35, 46]). With respect to the following some partial differential equations: the modified short pulse equation, the Hirota equation, and the nonlinear Schrödinger equation, it is well known that these equations play important parts in actual applications. Therefore, solving PDEs by integral equation method has important meaning not only in application but also in the theory of resolving the equation itself. Especially, our result also provides theoretical support to quantum field theory and Ising model [16,21,49,50,52]. Currently, there appear the so-called hype singular integral-differential equations with the order of a singular point higher than the special dimension in the fields of airmechanics, electron optics, fracture mechanics and others, and its solvability can also be solved by the method proposed in this paper (see [1, 13, 15, 17, 18, 42, 43]).

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