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CONTROL OF JULIA SETS OF COURNOT-BERTRAND DUOPOLY GAME MODEL*

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Abstract Cournot-Bertrand duopoly game model is a very important model in the economic field. Based on the idea of feedback control, three controllers are designed to control Julia sets of Cournot-Bertrand duopoly game model in this study. The first method is the feedback control by use of the fixed point, the second is based on feedback control and cumulative error and the third is designed based on the feedback control and difference. The efficacy of three control methods is illustrated in simulations.

Keywords Julia set, feedback control, Cournot-Bertrand duopoly game model.

MSC(2010) 34F10, 70K20.

1. Introduction

In economics, oligopoly mainly refers to an oligopoly market, which refers to a market organization in which a few manufacturers control the production and sales of certain products in the entire market. Oligopoly, relying on their absolute advantage in certain products, can have a significant impact on the economic activity of the entire market [28]. When studying oligopoly competition, the Cournot model and the Bertrand model are often used. In the Cournot model, firms control their own production levels and select the output that maximizes their profits based on predictions of the output of other firms, while in the Bertrand model, firms do not influence market prices by controlling production levels, but directly choose the price of a unit of product to influence market demand [10, 20]. At present, there is a large amount of literature related to Cournot models or Bertrand models in oligopolistic markets [1–3, 6, 13, 17, 26]. Overall, both models are important tools for analyzing market structure and corporate behaviours, revealing the behavioral patterns of firms in competitive markets from different perspectives.

The Cournot-Bertrand duopoly game model combines the characteristics of these two models, where one firm considers production and the other firm considers price. It describes many situations of oligopoly competition in the real world and has significant application value in economics. Many scholars have also studied this

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model. To the best of our knowledge, Zanchetti [31] and Arya et al. [4] pointed out that Cournot-Bertrand competition may be optimal under certain circumstances. Lately, Tremblay et al. [25] provided examples of small cars where Honda and Sabau set quantities and Saturn and Scion set prices. Semenov and Tondji [18] examined mixed competition where both firms invest in R&D and compare the profits with the Cournot and Bertrand competition. Dalla [7] presented a Cournot-Bertrand model with differentiated loans and deposits. Quadir [16] considered a Cournot-Bertrand competition with uncertain demand where firms receive private information about it. Mukherjee et al. [15] provided a new reason for Bertrand-Cournot profit reversal. Zhu et al. [33] considerd the Cournot-Bertrand duopoly mixed competition model and analyzed the general stability conditions of the four equilibria with eigenvalues and Jury criterion. Tremblay and Tremblay [24] considered a duopoly model with a Cournot-type firm and a Bertrand-type firm and found that the equilibrium is stable when there is sufficient product differentiation. Xiang and Cao [30] studied the two-dimensional game of Cournot-Bertrand model with incomplete information and its equilibrium when the two companies have some of the old and the new alternative products according to the theory of multidimensional game.

Researchers are devoted to study the dynamical behaviors of the system from many aspects, such as the stability, bifurcation and chaos. Stability and bifurcation of a discrete-time prey-predator system with the Allee effect on the prey population are discussed [11]. And the OGY method and hybrid control method are used to control the chaotic behavior that results from Neimark-Sacker bifurcation. In [8], the authors studied dynamics of a new discrete large-sub-center system, such as stabilities, bifurcations, chaotic dynamics, global asymptotical stability etc. In addition, researchers have discussed the dynamical properties from the viewpoint of fractal. The Julia set was proposed by French mathematician Gaston Julia [12] and has been applied to fields such as biology [14], physics [5] and cryptography [23]. It is an important feature of nonlinear systems. By controlling Julia sets of nonlinear systems, we are able to have a deeper understanding of the behaviours of nonlinear systems. This provides us with help in solving some difficult problems about nonlinear systems in reality. At present, there has been a lot of research on the control problem of Julia sets. For example, Zhang and Liu [32] controlled Julia sets for multiple nonlinear systems using feedback control and gradient control. Sun and Zhang [22] used feedback control to control Julia sets in the Brusselator model. Wang and Liu [29] extended the existing fractional-order Lotka-Volterra model and used optimal control and gradient control to control the Julia sets of the system. Shu and Zhang [19] fuzzified the Lotka-Volterra model based on the T-S fuzzy model, and then designed a controller using the distributed compensation method to control Julia sets of this model. Wang et al. [27] considered a class of rational time-delay complex mappings and designed an effective hybrid controller, obtaining escape radius of Julia sets of the controlled system and achieving boundary control of Julia sets of the controlled system. In [21], two new methods are presented to realize synchronization control and parameter identification for two kinds of sine-function.

The trajectory of a point in the filled Julia set is bounded, which is important to analyze the dynamical properties of the Cournot-Bertrand duopoly game model. In this paper, control of Julia sets of the Cournot-Bertrand duopoly game model is studied. In section 2, the Julia set of Cournot-Bertrand duopoly game model is defined. In Section 3, three controllers are established to control the Julia sets of Cournot-Bertrand duopoly game model and the variations of Julia sets as the parameters change are considered.

2. Preliminaries

Suppose there are two firms in the market, firm i produces goods x_i, $i = 1, 2$. Firm 1 competes in terms of output, following a Cournot strategy, while Firm 2 competes in terms of price, following a Bertrand strategy. And suppose that firms make strategic choices at the same time and each firm knows the output and price of the other firm. Then, the Cournot-Bertrand duopoly game model [28] can be described by the nonlinear difference equation:

$$\begin{cases} q_{n+1} = q_n + \alpha q_n \left(1 - c - d + dp_n - 2q_n + 2d^2 q_n\right), \\ p_{n+1} = p_n + \beta p_n \left(1 + c - 2p_n - dq_n\right), \end{cases} \tag{2.1}$$

where q_n denotes the output of goods produced by firm 1, p_n denotes the price of goods produced by firm 2, c denotes the marginal costs of two firms (assuming the marginal costs of the two firms are the same), d denotes the index of product differentiation or product substitution, α and β denote the adjustment speed of two firms.

After calculation, we obtain four fixed points, which are

$$E_0 = (0,0), \quad E_1\left(0, \frac{1+c}{2}\right), \quad E_2\left(\frac{1-c-d}{2\left(1-d^2\right)}, 0\right), \quad E^*\left(q^*, p^*\right),$$

where $q^* = (2-2c-d+cd)/(4-3d^2)$, $p^* = (2+2c-d+cd-d^2-2cd^2)/(4-3d^2)$.

Let J be the Jacobian matrix of system (2.1) corresponding to the state variables (q,p), then

$$J\left(q,p\right) = \begin{pmatrix} J_{11} & \alpha dq \\ -\beta dp & J_{22} \end{pmatrix},$$

where $J_{11} = 1 + \alpha(1-c-d+dp+4(d^2-1)q)$, $J_{22} = 1+\beta(1+c-4p-dq)$.

Analogous to the classic definition of Julia set [9], the Julia set of Cournot-Bertrand duopoly game model can be described as follows.

Definition 2.1. The filled Julia set of model (2.1), represented as K, is defined as the following set: $K = \left\{(q_1,p_1) \in \mathbb{R}^2 \mid \{(q_n,p_n)\}_{n=1}^{\infty}\right\}$ remains bounded with the initial points $(q_1,p_1)\}$.

The Julia set of model (2.1), denoted as J, refers to the boundary of the filled Julia set K, i.e., $J = \partial K$.

Figure 1 shows the Julia set of model (2.1) when the model parameters are taken to be as follows: α=2.05, β =0.2, c=2.7 and d= 0.8. In this paper, the graphics of the Julia sets uniformly use the coordinate range [-10,10]×[-10,10].

3. Control of Julia sets of Cournot-Bertrand duopoly game model

In this section, based on feedback control, we design three controllers to control Julia sets of the model while ensuring the stability of the system at the fixed point

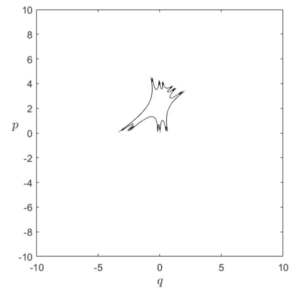


Figure 1. Julia sets for model (2.1) when $\alpha = 2.05$, $\beta = 0.2$, $c = 2.7$ and $d = 0.8$.

(q^*, p^*).

The controlled Cournot-Bertrand duopoly game model is

$$\begin{cases} q_{n+1} = q_n + \alpha q_n\big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\big) + u_n, \\ p_{n+1} = p_n + \beta p_n\big(1 + c - 2p_n - dq_n\big) + v_n, \end{cases} \tag{3.1}$$

where u_n and v_n are control items.

3.1. Feedback control by use of the fixed point

Based on simple linear feedback control, the controller (I) takes the following form:

$$\begin{cases} u_n = -k(q_n - q^*), \\ v_n = -k(p_n - p^*), \end{cases} \tag{3.2}$$

where k is the control parameter.

Substitute the controller into the model (3.1), then the controlled model is obtained as follows:

$$\begin{cases} q_{n+1} = q_n + \alpha q_n\big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\big) - k(q_n - q^*), \\ p_{n+1} = p_n + \beta p_n\big(1 + c - 2p_n - dq_n\big) - k(p_n - p^*). \end{cases} \tag{3.3}$$

Obviously, the fixed point of the controlled model (3.3) is (q^*, p^*). Let J_1 be the Jacobian matrix of system (3.3) at point (q^*, p^*), then

$$J_1 = \begin{pmatrix} 1 + \alpha(1 - c - d + dp^* + 4(d^2 - 1)q^*) - k & \alpha dq^* \\ -\beta dp^* & 1 + \beta(1 + c - 4p^* - dq^*) - k \end{pmatrix}.$$

Assuming that λ_1 and λ_2 are eigenvalues of the matrix, and by selecting an appropriate k, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are satisfied, then the fixed point (q^*, p^*) of the controlled model is stable in this case.

Next, we will explore the changes in the Julia sets of the controlled model (3.3) as the parameter k changes when $\alpha = 1.9$, $\beta = 0.2$, $c = 0.2$ and $d = 0.8$ are set.

So, in this case, we can obtain

$$J_1 = \begin{pmatrix} 0.3686 - k & 0.7015 \\ -0.0665 & 0.8338 - k \end{pmatrix}.$$

Our aim is to find the range of values of k such that in this case moduli of all eigenvalues of the Jacobian matrix J_1 are less than 1, so that the fixed point is stable.

Here, we consider using computer-aided solution. Through symbolic calculation in matlab, we can get one eigenvalue is $\lambda_1 = 0.5147 - k$, and the other is $\lambda_2 = 0.6877 - k$. Let $|\lambda_1| < 1$, $|\lambda_2| < 1$. Finally, we obtain the range of values of k is the interval $(-0.3123, 1.5147)$.

In fact, when the size of the Jacobian matrix is small, this method can still be operated. As in the above case, we can easily calculate the range of values of k by simple calculation. However, when the size of the Jacobian matrix is large, the number of times of variable k in the eigenvalues obtained by symbolic computation will become larger, and it becomes very difficult to compute the range of values of k in this case. Therefore, we propose a simpler numerical solution to estimate the range of the main distribution of values of k, and we prefer to use this method in the future.

Algorithm 1. Estimation of the distribution range of values of k that stabilize the fixed point.

Step 1: Give the range of k to be searched in advance. It can be set to [-6, 6].

Step 2: Give the step size. It can be set to 0.0001 here, but it can also be smaller.

Step 3: Traverse all values of k which are given. Also output the k that makes the moduli of all eigenvalues of the Jacobian matrix of the controlled model at the fixed point less than 1.

Step 4: Graphic display. The x-axis represents the values of k, and on the y-axis, the corresponding values of k satisfying the condition that the moduli of all eigenvalues of the Jacobian matrix of the controlled model at the fixed point are less than 1 are marked as 1, and the corresponding values of k that do not meet the condition are marked as 0.

Step 5: Based on the graph and actual output, estimate the main distribution range of values of k.

Then, with the help of this algorithm, we get Figure 2.

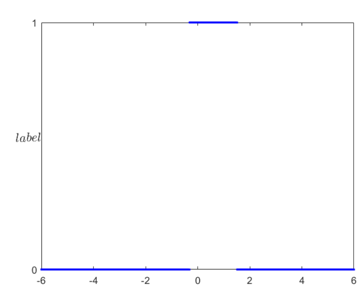
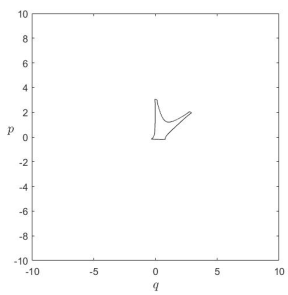


Figure 2. Estimation of the distribution range of values of k that stabilize the fixed point.

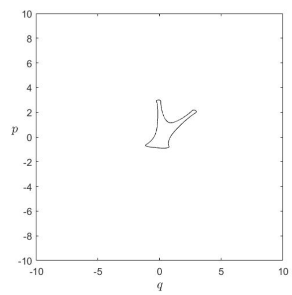
Based on the actual output of k and Figure 2, we estimate that the values of k that stabilize the fixed point are mainly distributed in the interval $(-0.3122, 1.5147)$. This indicates that using this estimation method is effective and simpler.

Remark 3.1. By setting a fixed step size to traverse a range of values, we can estimate the values of k that meet the condition within this range. However, it is evident that we cannot make the step size zero. We can only make the step size as small as possible, such as 0.0001. That is to say, we cannot truly provide an extremely accurate range of values of k because we only traverse a portion of the values of k within the set numerical range. However, we can still estimate the approximate distribution range of values of k that meet the condition through output results and graphical display, which can provide great convenience for finding the exact values of k that satisfy the condition. This method is simple, fast and can greatly reduce computational costs and time consumption when facing large amounts of data or complex problems.

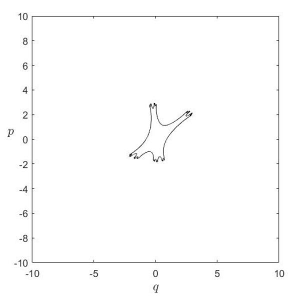
Next, we change the values of k and observe the changes of the Julia sets, as shown in Figure 3.



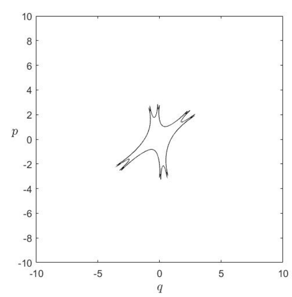
(a) k=0.1



(b) k=0.4



(c) k=0.8



(d) k=1.4

Figure 3. Variation of Julia sets of the controlled system (3.3) with control parameter k.

Figure 3 shows the variation of the Julia sets of the controlled model (3.3) with parameter k when $\alpha = 1.9$, $\beta = 0.2$, $c = 0.2$ and $d = 0.8$. It can be seen that by changing the values of k, the sizes of the Julia sets of the controlled model also change. The sizes of the Julia sets of the controlled model are relatively large for some values of k, and the sizes of the Julia sets of the controlled model are relatively small for certain values of k. That is, we are able to control the sizes of Julia sets of the controlled system by controlling the values of k. It is worth noting that we control the Julia sets of the system while ensuring the stability of the fixed point (q^*, p^*). Therefore, the selection of the values of k here must ensure that the fixed point (q^*, p^*) of the controlled system (3.3) is a stable point.

3.2. Hybrid control based on feedback method and cumulative error

In the previous section, we designed a simple feedback controller and verified the effectiveness of the controller in simulation. In this section, a cumulative error term is added to the simple feedback controller and the controller is designed as follows.

$$\begin{cases} u_n = -k(q_n - q^* + h_n), \\ v_n = -k(p_n - p^* + w_n), \end{cases} \tag{3.4}$$

where k is the control parameter, $h_n = h_{n-1} + (q_n - q^*)$, $w_n = w_{n-1} + (p_n - p^*)$. Here, the role of h_n is to accumulate the difference between q_n and the target value q^*, and the same for w_n. Substituting the controller into the model (3.1), the controlled model is obtained as follows:

$$\begin{cases} q_{n+1} = q_n + \alpha q_n \Big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\Big) - k(q_n - q^* + h_n), \\ p_{n+1} = p_n + \beta p_n \Big(1 + c - 2p_n - dq_n\Big) - k(p_n - p^* + w_n). \end{cases} \tag{3.5}$$

Concatenating all the equations, it can be written as

$$\begin{cases} q_{n+1} = q_n + \alpha q_n \big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\big) - k(q_n - q^* + h_n), \\ p_{n+1} = p_n + \beta p_n \big(1 + c - 2p_n - dq_n\big) - k(p_n - p^* + w_n), \\ h_{n+1} = q_{n+1} - q^* + h_n, \\ w_{n+1} = p_{n+1} - p^* + w_n. \end{cases} \tag{3.6}$$

It can be seen that the point $(q^*, p^*, 0, 0)$ is the fixed point of the controlled model (3.6).

Let J_2 be the Jacobian matrix of the controlled model (3.6) at the fixed point $(q^*, p^*, 0, 0)$, then

$$J_2 = \begin{pmatrix} r_1 & \alpha dq^* & -k & 0 \\ -\beta dp^* & r_2 & 0 & -k \\ r_1 & \alpha dq^* & 1-k & 0 \\ -\beta dp^* & r_2 & 0 & 1-k \end{pmatrix},$$

where $r_1 = 1 + \alpha(1 - c - d + dp^* + 4(d^2 - 1)q^*) - k$, $r_2 = 1 + \beta(1 + c - 4p^* - dq^*) - k$.

Let λ_1, λ_2, λ_3 and λ_4 be eigenvalues of the matrix, by choosing a suitable k, as long as $|\lambda_1| < 1$, $|\lambda_2| < 1$, $|\lambda_3| < 1$ and $|\lambda_4| < 1$ are satisfied, then the fixed point $(q^*, p^*, 0, 0)$ of the controlled model (3.6) are stable in this case.

With the help of the previously proposed algorithm, Figure 4 is obtained as follows.

Combining the values of k output by the algorithm, we finally estimate that the values of k that stabilize the fixed point are mainly distributed in the interval (0.0001, 1.0098).

Next, we explore how the Julia sets of the controlled model (3.5) change as the values of the parameter k are changed. This is shown in the following Figure 5.

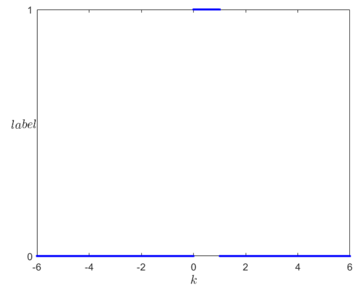
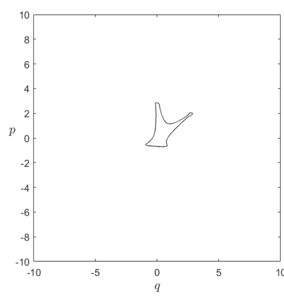
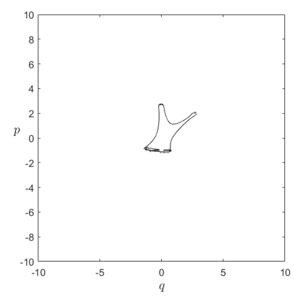


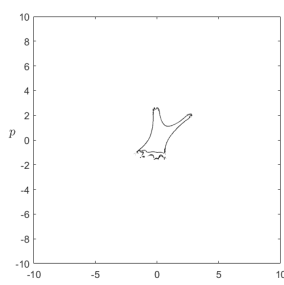
Figure 4. Estimation of the distribution range of values of k that make the fixed point $(q^*, p^*, 0, 0)$ be stable.



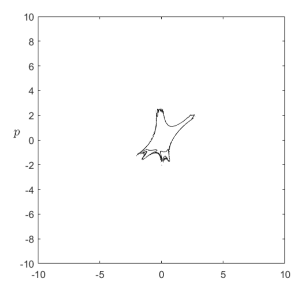
(a) k=0.1



(b) k=0.2



(c) k=0.3



(d) k=0.4

Figure 5. Variation of Julia sets of the controlled system (3.5) with control parameter k.

Figure 5 shows the variation of Julia sets of the controlled model (3.5) with parameter k when $\alpha = 1.9$, $\beta = 0.2$, $c = 0.2$ and $d = 0.8$. It can be seen that the sizes of Julia sets of the model can be controlled by controlling the values of parameter k. However, it is important to note that the values of k must be chosen in such a way that the fixed point of the controlled model is stable.

Remark 3.2. Using a single control factor k can greatly simplify the controller design and debugging process. Since only one parameter needs to be adjusted, this method can reduce the complexity and the difficulty of regulating the controller, which is suitable for some application scenarios that do not require a high level of control accuracy or relatively simple system dynamic characteristics.

3.3. Hybrid control based on feedback method and difference

This section introduces a new control term based on the simple feedback controller, and the controller adopts the following form:

$$\begin{cases} u_n = -k(q_n - q^* + h_n), \\ v_n = -k(p_n - p^* + w_n), \end{cases} \tag{3.7}$$

where k is the control parameter, $h_n = q_n - q_{n-1}$ and $w_n = p_n - p_{n-1}$. Substituting the controller into the model (3.1), the controlled model is obtained as follows:

$$\begin{cases} q_{n+1} = q_n + \alpha q_n \Big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\Big) - k(q_n - q^* + h_n), \\ p_{n+1} = p_n + \beta p_n \Big(1 + c - 2p_n - dq_n\Big) - k(p_n - p^* + w_n). \end{cases} \tag{3.8}$$

Concatenating all the equations, it can be written as

$$\begin{cases} q_{n+1} = q_n + \alpha q_n \big(1 - c - d + dp_n - 2q_n + 2d^2 q_n\big) - k(q_n - q^* + h_n), \\ p_{n+1} = p_n + \beta p_n \big(1 + c - 2p_n - dq_n\big) - k(p_n - p^* + w_n), \\ h_{n+1} = q_{n+1} - q_n, \\ w_{n+1} = p_{n+1} - p_n. \end{cases} \tag{3.9}$$

It can be seen that the point $(q^*, p^*, 0, 0)$ is the fixed point of the controlled model (3.9).

Let J_3 be the Jacobian matrix of the controlled model (3.9) at point $(q^*, p^*, 0, 0)$, then

$$J_3 = \begin{pmatrix} r_3 & \alpha d q^* & -k & 0 \\ -\beta d p^* & r_4 & 0 & -k \\ r_3 - 1 & \alpha d q^* & -k & 0 \\ -\beta d p^* & r_4 - 1 & 0 & -k \end{pmatrix},$$

where $r_3 = 1 + \alpha(1 - c - d + dp^* + 4(d^2 - 1)q^*) - k$, $r_4 = 1 + \beta(1 + c - 4p^* - dq^*) - k$.

Let λ_1, λ_2, λ_3 and λ_4 be eigenvalues of the matrix, by choosing a suitable k, as long as $|\lambda_1| < 1$, $|\lambda_2| < 1$, $|\lambda_3| < 1$ and $|\lambda_4| < 1$ are satisfied, then the fixed point $(q^*, p^*, 0, 0)$ of the controlled model (3.9) are stable in this case.

With the help of the previously proposed algorithm, Figure 6 is obtained as follows.

Combining the values of k output by the algorithm, we finally estimate that the values of k that stabilize the fixed point are mainly distributed in the interval $(-0.3122, 0.5049)$.

Next, we explore how the Julia sets of the controlled model (3.8) change as the values of the parameter k are changed. This is shown in the following Figure 7.

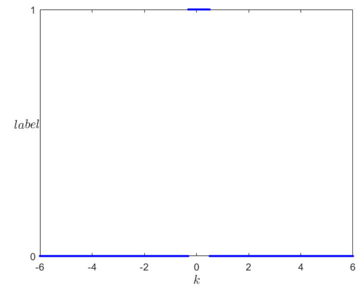
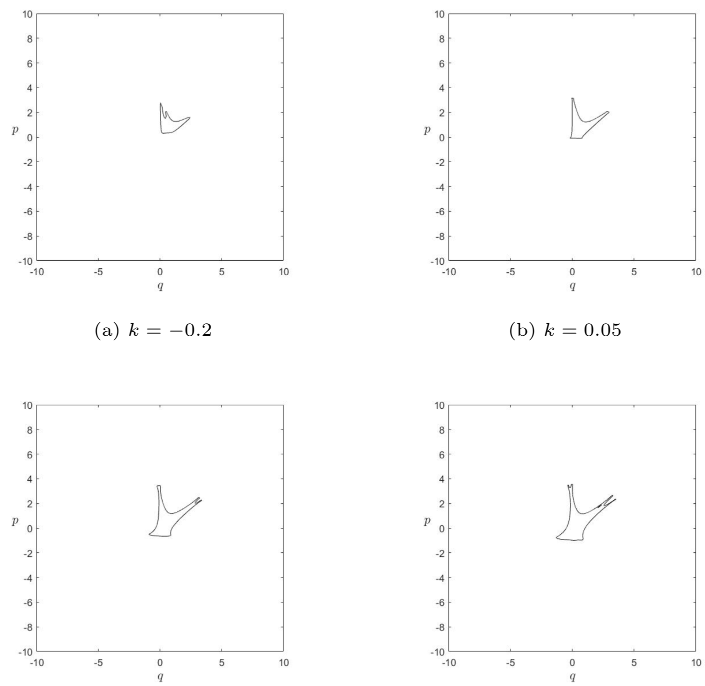


Figure 6. Estimation of the distribution range of values of k that make the fixed point be stable.



(a) $k = -0.2$

(b) $k = 0.05$

(c) $k = 0.3$

(d) $k = 0.45$

Figure 7. Variation of Julia sets of the controlled system (3.8) with control parameter k.

Figure 7 shows the variation of Julia sets with parameter k for the controlled model when $\alpha = 1.9$, $\beta = 0.2$, $c = 0.2$ and $d = 0.8$. By changing the values of k, it can be clearly found that sizes of the Julia sets of the controlled model change with the change of the values of k. The effect of controlling the Julia sets is achieved.

Remark 3.3. In fact, we cannot mathematically analyse the exact relationship between the sizes of the Julia sets and the parameter k, but we can change the sizes of the Julia sets by changing the values of k. When k is taken some values, the sizes of Julia sets are relatively large, and similarly, when k is taken some values, the sizes of Julia sets are relatively small, which has achieved our purpose of controlling the Julia sets.

3.4. Comparison of three control methods

It is evident from the experiments that all the three methods are effective in achieving control of Julia sets. In terms of the form of the three controllers themselves, the first controller is significantly simpler compared to the other two controllers, while the second controller introduces a cumulative error term and the third controller introduces a difference term. In terms of the main distribution range of the values of k that ensure the stabilization of the fixed point, the effective distribution range of the values of k is significantly larger when the first controller is in action.

4. Conclusion

In this paper, we propose three kinds of controllers. And it can be seen that by changing the values of the control parameters, we can effectively control the Julia sets of the system. Of course, sometimes the controller is not the more complex the better. In the actual situation, we need to choose different controllers with corresponding control parameters to achieve the desired control effect on Julia sets of the system.

The Cournot-Bertrand duopoly game model itself usually exhibits complex nonlinear behaviors, such as fluctuations in market prices and output levels, instability and even chaos, difficulty for making long-term predictions. By controlling the Julia sets of the model, we could analyze its dynamic behaviors from the initial stations of the model, which provides a new perspective to study the model.

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