Journal of Applied Analysis and Computation
Volume 15, Number 5, October 2025, 2508–2527

Website:http://www.jaac-online.com
DOI:10.11948/20240020

SYNCHRONIZATION OF A NOVEL COMPLEX SYSTEM

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Abstract In this paper, we introduce a new hyperchaotic system which is a four-dimensional system of nonlinear differential equations. This system exhibits very rich chaotic dynamical behaviors. The dynamical characteristics of this system are analysed theoretically and numerically, including the dissipation, the quilibrium points and their stability, Lyapunov exponents, the Lyapunov dimension of the attractors, the global exponential attractive set. In order to achieve synchronization fast, global exponential synchronization is adopted. Suitable linear and nonlinear controllers have been designed to achieve global exponential synchronization between two identical chaotic systems by using the Lyapunov stability theory and Dini derivative. The innovation of this paper is that firstly we get the globally exponential attractive set of this system. Secondly, the result of globally exponential attractive set of the chaotic system is applied to chaos synchronization. Thirdly, we can get the precise lower bound of the coefficient of linear feedback controller k_1, k_2, k_3 and k_4. Finally, chaos synchronization is studied numerically. Numerical simulations are in excellent agreement with the theoretical study.

Keywords Hyperchaotic system, Dini derivative, stability theory, bifurcation behavior, Hamilton energy function, global exponential synchronization.

MSC(2010) 34A08, 34D06, 34H10, 34H15.

1. Introduction

Henri Poincaré, a distinguished scientist and mathematician [24], was the first person to discover the unpredictability in his research on the three-body problem. Then unpredictability was developed into chaos theory as we know it nowadays. In 1975, Li and Yorke [13] coined the mathematical, physical concept of "chaos", with the relevant feature that a minimal variation in the initial conditions of a dynamical system has strong consequences in its dynamics. However, Poincaré's results were

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neglected and did not receive the deserved attention. It was not until Edward N. Lorenz [16], in 1963, discovered the first chaotic system when he worked on some numerical experiments of meteorology, the scientific gateway to chaos research was reopened. Since then, chaotic systems and chaos phenomena have attracted continuous attention from scientists. In 1976, the Rössler chaotic system was discovered [25]. In 1986, the Chua circuit chaotic system was found by Leon Chua in a physical experiment [3, 9]. In 1996, the Swedish physicist Stenflo [28] found a generalized Lorenz equations for acoustic-gravity waves in the atmosphere, namely, the Lorenz-Stenflo system. In 1999, Chen and Ueta found another new chaotic system, namely, Chen chaotic system [2]. In 2002, Lu and Chen reported a new chaotic system which connected the Lorenz attractor and the Chen's attractor [17]. In 2002, Lu et al. introduced the unified chaotic system [18]. Since then, more chaotic systems have been discovered and studied [1,5,6,8,10–12,14,19,22,27,29,31,34,35]. Many papers are devoted to the study of the mechanism of chaos and the discovery of the new chaotic systems [5, 6, 8, 10–12, 19, 22, 27, 29, 31, 34, 35]. Chaotic systems and chaos phenomena have been found and studied in many fields, including meteorology, physics, engineering, economics, biology, astronomy, neural network, and fluid mechanics [1, 5, 6, 8–12, 19, 22, 27, 29, 31, 34, 35].

A chaotic system is a deterministic system that displays complex and unpredictable behaviors. Because of the sensibility of the initial value, people used to think that the chaotic systems could not be controlled and could not be synchronized. In 1990, OGY method was proposed [21]. In the same year, the American Navy Laboratory made a secure communication system via chaotic synchronization. It changed the old idea, and people realized the feasibility of chaos control. Chaos control generally has two purposes, one is to stabilize the chaotic system, and the other is to track another chaotic system, that is, chaotic synchronization. Synchronization can also be observed in nature, such as the synchronous flashing phenomenon caused by a large number of fireflies gathering in trees, the migration of birds in groups, and the synchronization between the physiological rhythm of life system and the environmental rhythm. Synchronization also exists in physics and chemistry. For example, when two pendulums have the same length, they will tend to be synchronized with time even if the initial swinging directions are different, and synchronous vibration will also occur when two atoms interact at the resonant frequency. Even the neurons in the human brain will have rhythmic synchronization, so synchronization is closely related to real life. After the concept of complete synchronization of chaotic systems was put forward by Louis Pecora and Thomas Caroll in 1990 [23], people applied the theory of chaos synchronization to more fields, such as acoustics, current, oscillator design, secure communication and complex networks [4, 14, 15, 30, 32, 37, 38].

The globally exponential attractive set is an important concept in the study of chaotic dynamical systems. If we can conclude that a chaotic system has a globally attractive set, one can confirm that there are no equilibrium, periodic solutions, almost periodic motions, wandering motions or other chaotic attractors outside the global attractive set. The strange attractors of a chaotic system are located in the global attractive set only which greatly simplifies the dynamical analysis of a chaotic system. Compared with a chaotic system, hyperchaotic systems generally have more complex dynamical behaviors and the uncertainty of the hyperchaotic system are greatly increased, which can greatly increase the confidentiality of information. Therefore, using the hyperchaotic systems to realize chaos synchronization and secure communication has stronger security [4,14,15,32,33,38].

2. Complex dynamics

2.1. System model and hyperchaotic attractor

The famous Lorenz chaotic system is as follows

$$\begin{cases} \dfrac{dx}{dt} = a(y-x), \\ \dfrac{dy}{dt} = cx - xz - y, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{2.1}$$

where the parameters a, b, c are real constants of the Lorenz system.

In this section, we will introduce a new hyperchaotic system as follows on the basis of the Lorenz system

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx - xz - cy + w, \\ \dot{z} = xy - dz, \\ \dot{w} = -ex - rw, \end{cases} \tag{2.2}$$

where the parameters a, b, c, d, e, r are real constants of system (2.2). When the parameters $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$, the Lyapunov exponents of the system (2.2) are λ_{L_1}=0.3278, λ_{L_2}=0.3249, λ_{L_3}= − 1.5621, λ_{L_4}= − 36.7166. System (2.2) shows hyperchaotic behaviour for parameters $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$. When the initial value of system (2.2) is selected as $(x_0, y_0, z_0, w_0) = (0.5, 0.1, 0.5, 0.1)$, then the three-dimensional hyperchaotic attractor of system (2.2) can be obtained, as shown in Figure 1.

Remark 2.1. For the circuit implementation of this hyperchaotic system and the calculation of the field energy in the capacitors used in the corresponding circuits, interested readers can refer to the references [15,38] for the detailed discussion.

2.2. Dissipation

Let us denote the vector field of system as

$$F(x,y,z,w) = \begin{pmatrix} f_1(x,y,z,w) \\ f_2(x,y,z,w) \\ f_3(x,y,z,w) \\ f_4(x,y,z,w) \end{pmatrix} = \begin{pmatrix} a(y-x) \\ bx - xz - cy + w \\ xy - dz \\ -ex - rw \end{pmatrix}.$$

System (2.2) is dissipative under the condition $a + c + d + r > 0$, since we have

$$\nabla V = \frac{\partial f_1(x,y,z,w)}{\partial x} + \frac{\partial f_2(x,y,z,w)}{\partial y} + \frac{\partial f_3(x,y,z,w)}{\partial z} + \frac{\partial f_4(x,y,z,w)}{\partial w}$$

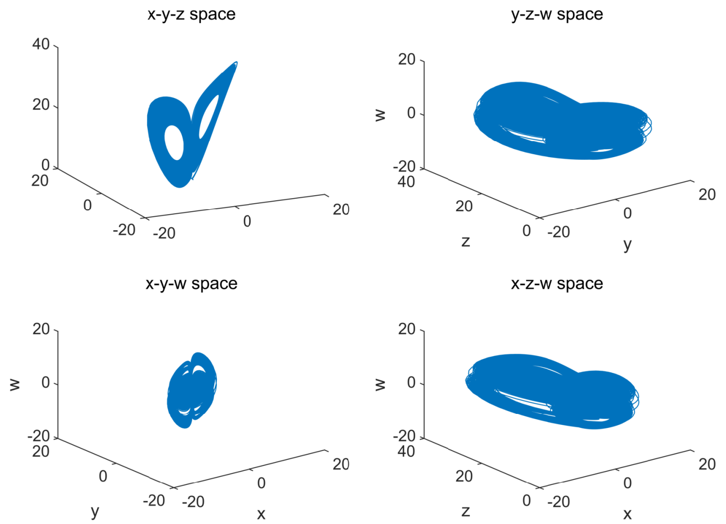

Figure 1. The hyperchaotic attractors of system (2.2) in the 3D space.

$$= -\left(a + c + d + r\right).$$

That is to say, $\nabla V < 0$ holds when $a + c + d + r > 0$. It shows that system (2.2) is a dissipative system and it converges with the exponential rate $e^{-(a+c+d+r)}$.

2.3. Fixed points and their stability

The fixed points of system (2.2) are determined by solving the following equations

$$\begin{cases} a(y-x) = 0, \\ bx - xz - cy + w = 0, \\ xy - dz = 0, \\ -ex - rw = 0. \end{cases} \tag{2.3}$$

Solving the above equation (2.3), the real equilibrium points of system (2.2) can be obtained as the following cases:

(i) If $d = 0, r \neq 0,$, there is only one real fixed point $S_0 = (0,0,0,0)$.

(ii) If $d \neq 0, r = 0, e \neq 0$, there is only one real fixed point $S_0 = (0,0,0,0)$.

(iii) If $d \neq 0, r = 0, e = 0$, there are an infinite number of real fixed points.

(iv) If $d = 0, r = 0$, there are an infinite number of real fixed points.

(v) If $d \neq 0, r \neq 0$, then system (2.2) has only one real fixed point $S_0 = (0,0,0,0)$ when $p = bd - cd - \frac{de}{r} \leq 0$. When $p = bd - cd - \frac{de}{r} \geq 0$, system (2.2) has the following three fixed points:

$$S_0 = (0,0,0,0)\,, S_+ = \left(\sqrt{p}\sqrt{p}\frac{p}{d} - \frac{e\sqrt{p}}{r}\right), S_- = \left(-\sqrt{p} - \sqrt{p}\frac{p}{d}\frac{e\sqrt{p}}{r}\right).$$

In the following, we will study the stability of the fixed points of system (2.2) with parameters $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$. Consider the parameters of system (2.2) when $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$, which satisfies the second category above, so the system (2.2) has three fixed points. To study the stability of $S_0 = (0,0,0,0)$, we will calculate the Jacobian matrix of system (2.2) at $S_0 = (0,0,0,0)$ as follows:

$$J|_{S_0} = \begin{pmatrix} -35 & 35 & 0 & 0 \\ 20 & -1 & 0 & 1 \\ 0 & 0 & -3 & 0 \\ -3 & 0 & 0 & -1.5 \end{pmatrix}.$$

The eigenvalues of matrix $J|_{S_0}$ are calculated as $\lambda_1 = -49.4831$, $\lambda_2 = 13.3356$, $\lambda_3 = -1.3525$, and $\lambda_4 = -3$ by using Matlab software. Since there exists positive eigenvalue of matrix $J|_{S_0}$, so $S_0 = (0,0,0,0)$ is an unstable fixed point of system (2.2). The stability analysis of S_+ and S_- by using the same method yields that S_+ and S_- are both unstable fixed point of system (2.2).

2.4. Lyapunov exponents and Lyapunov dimension of the system

When the parameters are chosen as $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$, with the initial value $(x_0, y_0, z_0, w_0) = (0.5, 0.1, 0.5, 0.1)$, the Lyapunov exponents of system (2.2) are calculated as $\lambda_{L_1} = 0.3278$, $\lambda_{L_2} = 0.3249$, $\lambda_{L_3} = -1.5621$ and $\lambda_{L_4} = -36.7166$, respectively. And the Lyapunov dimension of the attractors of system (2.2) is calculated as [19, 24, 31]

$$D_L = j + \frac{\sum_{i=1}^{j} \lambda_{L_i}}{|\lambda_{L_{j+1}}|},$$

such that j is the largest integer that guarantees the inequality $\sum_{i=1}^{j} \lambda_{L_i} > 0$. And the Lyapunov dimension of system (2.2) in this case is

$$D_L = 2 + \frac{\lambda_{L_1} + \lambda_{L_2}}{|\lambda_{L_3}|} = 2.4178.$$

The Lyapunov dimension of system (2.2) is a fractional number which ensures the presence of a strange attractor. The Lyapunov exponent of system (2.2) is shown in Figure 2.

Remark 2.2. While positive Lyapunov exponent is widely used as indication of chaos, but it is extremely difficult to prove this and in general this is not true. Rigorous consideration requires verification of additional properties of considered system (such as regularity, ergodicity), because of so-called Perron effects of Lyapunov exponents sign reversal (see the excellent papers [8, 12] for a detailed discussion of chaos in nonlinear dynamical systems).

2.5. Effects of changes for system parameters

When parameters $b = 20, c = 1, d = 3, e = 3, r = 1.5$ of system (2.2) are fixed, the value of parameter a is changed with the initial value $(x_0, y_0, z_0, w_0) = (0.5, 0.1, 0.5,$

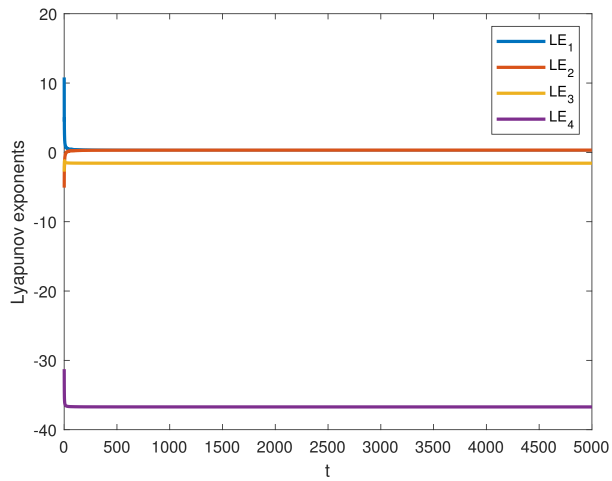

Figure 2. The Lyapunov exponent chart of system (2.2).

0.1). For $a \in [0,100]$, the Lyapunov exponents of system (2.2) with respect to parameter a can be obtained, as shown in Figure 3. The bifurcation diagram of the state variable x of system (2.2) with respect to the parameter a is presented in Figure 4.

Figure 3 reveals that the Lyapunov exponent of system (2.2) is less than 0 for $a \in [0, 4.8]$ and the solutions of system (2.2) approach fixed points. When $a \in (4.8, 44]$, the maximum Lyapunov exponent of system (2.2) consistently exceeds 0, indicating the presence of chaotic attractors. However, in certain small intervals, there exist points corresponding to two positive Lyapunov exponents, indicating the presence of hyperchaotic attractors in system (2.2). For $a \in (44, 100]$, all Lyapunov exponents of system (2.2) are less than 0 and the solutions of system (2.2) approach fixed points. We fix the parameters of system (2.2) as $a = 35, c = 1, d = 3, e = 3, r = 1.5$ with $(x_0, y_0, z_0, w_0) = (0.5, 0.1, 0.5, 0.1)$ and change the value of parameter b. When $b \in [0, 100]$, the Lyapunov exponents of system (2.2) with respect to parameter a can be obtained, as shown in Figure 5. In addition, Figure 6 displays the bifurcation diagram of the state variable x in system (2.2) with respect to parameter b.

It can be observed from Figure 5 that when $b \in [[0, 2.6), (4, 17.2]]$, the Lyapunov exponent of system (2.2) is less than 0 and the solutions of system (2.2) approach fixed points. When $b \in [2.6, 4]$, the maximum Lyapunov exponent of system (2.2) is equal to 0, while the other three Lyapunov exponents are all less than 0, and system (2.2) displays periodic solutions. When $b \in (17.2, 100]$, system (2.2) exhibits two Lyapunov exponents greater than 0 indicating the presence of hyperchaotic attractors. The Lyapunov exponent diagram of system (2.2) with $b \in [0, 100]$ can be observed in Figure 5.

Remark 2.3. Lyapunov exponent represents the exponential convergence or divergence rate between adjacent trajectories in the phase space, which reflects the intensity of chaos in the system. A positive Lyapunov exponent is a necessary condition for the system to show chaotic state [15]. By analyzing Figure 3, Figure 4, Figure 5 and Figure 6, it can be clearly seen that when the Lyapunov exponent cor-

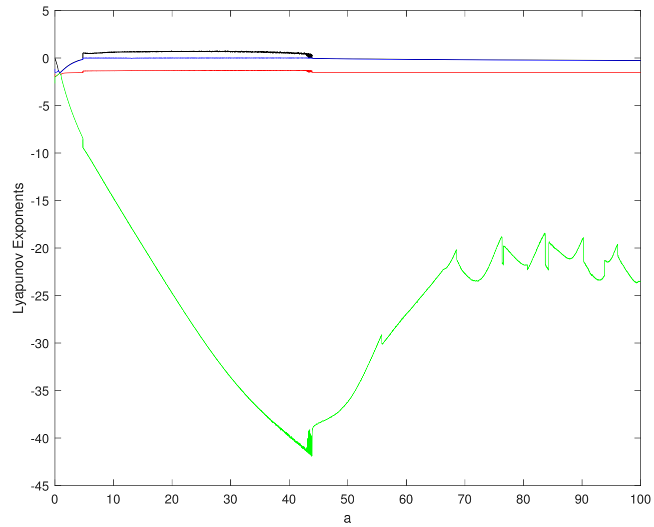

Figure 3. Lyapunov exponents diagram of system with $a \in [0, 100]$.

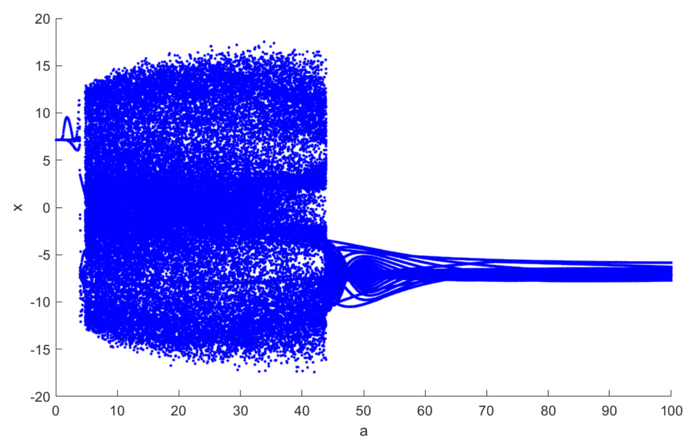

Figure 4. Bifurcation diagram of state variable x versus a.

responding to a certain parameter of the system is positive, there will be countless signal points in the bifurcation diagram of a certain state variable of the system about this parameter, which shows that the system is in a chaotic state at this time.

2.6. Hamilton energy function

The Hamilton energy plays a crucial role in the stability of the dynamical system. This section is devoted to the computation of the Hamilton energy of hyperchaotic system (2.2) based on the Helmholtz theorem [7,20,26,36]. According to the literatures [7,20,26,36], the Hamilton energy function $H(x, y, z, w)$ of system (2.2) must

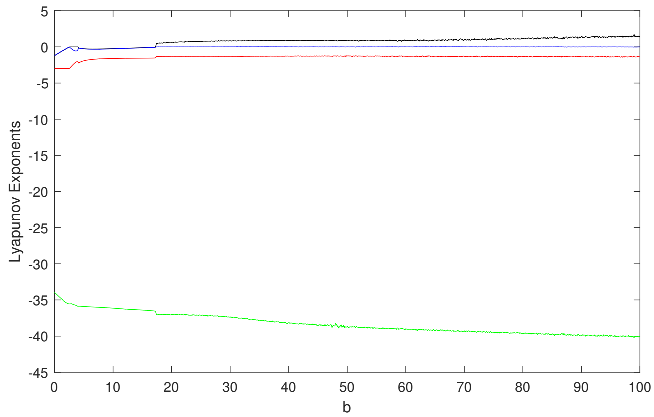

Figure 5. Lyapunov exponents diagram of system (2.2) with $b \in [0, 100]$.

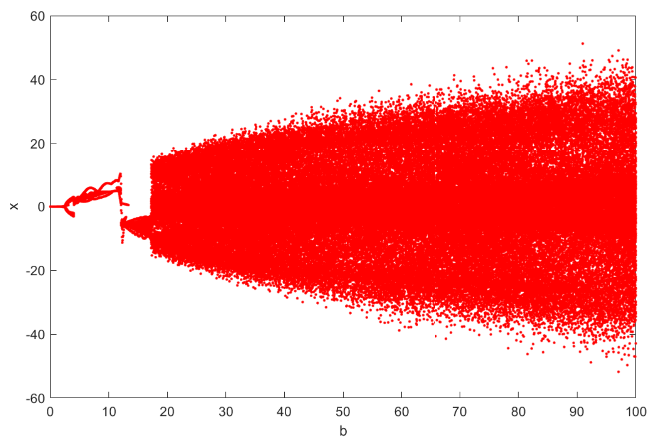

Figure 6. Bifurcation diagram of state variable x with respect to parameter b.

satisfy the following conditions:

$$\begin{cases} \dfrac{dX}{dt} = F(X) = F_c(X) + F_d(X), X = (x, y, z, w), \\ (\nabla H)^T F_c(X) = 0, \\ (\nabla H)^T F_d(X) = \dfrac{dH(x, y, z, w)}{dt}, \end{cases} \tag{2.4}$$

where $H(X)$ is the Hamilton energy function, $\nabla H(x, y, z, w) = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}, \frac{\partial H}{\partial w}\right)^T$ is the gradient vector of a smooth energy function, the symbol "T" represents the transpose of a matrix, $F_c(*)$ is a conservative field containing all rotations, and $F_d(*)$ is a dissipative field including divergence. Accounting for the physical properties of $F_c(*)$ and $F_d(*)$, the vector field can be regulated by a matrix, as shown in the following equation:

$$\begin{cases} \dfrac{dX}{dt} = \left[J\left(X\right) + R\left(X\right)\right]\nabla H, \\ \dfrac{dH}{dt} = \left(\nabla H\right)^T\left[J\left(X\right) + R\left(X\right)\right]\nabla H, \\ \left(\nabla H\right)^T J\left(X\right)\nabla H = \left(\nabla H\right)^T F_c\left(X\right), \end{cases} \tag{2.5}$$

where $J(X)$ represents a skew symmetric matrix and $R(X)$ represents a symmetric matrix.

According to the above formula, system (2.2) can be rewritten as the following formula

$$\begin{aligned} \dot{X} &= \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{w} \end{pmatrix} \\ &= \begin{pmatrix} ay - ax \\ bx - xz - cy + w \\ xy - dz \\ -ex - rw \end{pmatrix} \\ &= F_c\left(X\right) + F_d\left(X\right) \\ &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -bx \\ ay \\ z \\ w \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{bmatrix} -bx \\ ay \\ z \\ w \end{bmatrix} \\ &= J\left(X\right)\nabla H + R\left(X\right)\nabla H, \end{aligned}$$

where

$$a_{11} = \frac{ax + z}{bx}, a_{22} = \frac{-xz - cy}{ay}, a_{33} = \frac{xy - dz - bx - w}{z}, a_{44} = \frac{-ex + ay + z - rw}{w}.$$

Thus, the Hamilton energy function of system (2.2) can be obtained as follows:

$$H\left(x, y, z, w\right) = -b\frac{x^2}{2} + a\frac{y^2}{2} + \frac{z^2}{2} + \frac{w^2}{2}.$$

3. Globally exponential attractive set

Theorem 3.1. *Let* $X(t) = (x(t), y(t), z(t), w(t))$, $G = \frac{(a+b)^2 d^2}{4d-2}$. *If* $V(X(t)) > G, V(X(t_0)) > G$, *we have the following estimate of the exponential inequality with* *respect to the globally exponential attractive set of system (2.2)*

$$[V(X(t)) - G] \leq [V(X(t_0)) - G]\, \mathrm{e}^{-(t-t_0)}.$$

In particular,

$$\Omega = \{X \,|V(X) \leq G\} = \left\{ (x, y, z, w) \middle| x^2 + y^2 + (z - a - b)^2 + w^2 \leq \frac{(a+b)^2 d^2}{4d-2} \right\}$$

is the globally exponential attractive set of system (2.2).

Proof. Construct the Lyapunov-like function

$$V(X) = V(x, y, z, w) = \frac{1}{2}[x^2 + y^2 + (z - a - b)^2 + w^2].$$

Let

$$F(x, y, z, w) = \left(\frac{1}{2} - a\right) x^2 + \left(\frac{1}{2} - c\right) y^2 + \left(\frac{1}{2} - d\right) z^2 + \left(\frac{1}{2} - r\right) w^2 + yw - exw + (a+b)(d-1)z + \frac{(a+b)^2}{2}.$$

And

$$\begin{aligned}
&\left.\frac{dV(X)}{dt}\right|_{(2.2)} \\
=&x\frac{dx}{dt} + y\frac{dy}{dt} + (z - a - b)\frac{dz}{dt} + w\frac{dw}{dt} \\
=&x[a(y - x)] + y(bx - xz - cy + w) + (z - a - b)(xy - dz) + w(-ex - rw) \\
=& - ax^2 - cy^2 - dz^2 - rw^2 + yw + (a+b)dz - exw \\
=& - V(X(t)) + \left(\frac{1}{2} - a\right) x^2 + \left(\frac{1}{2} - c\right) y^2 + \left(\frac{1}{2} - d\right) z^2 + \left(\frac{1}{2} - r\right) w^2 \\
&+ yw - exw + (a+b)(d-1)z + \frac{(a+b)^2}{2} \\
=& - V(X(t)) + F(X).
\end{aligned}$$

Let

$$\begin{cases}
\dfrac{\partial F}{\partial x} = (1 - 2a)\, x - ew = 0, \\
\dfrac{\partial F}{\partial y} = (1 - 2c)y + w = 0, \\
\dfrac{\partial F}{\partial z} = (1 - 2d)\, z + (a+b)(d-1) = 0, \\
\dfrac{\partial F}{\partial w} = (1 - 2r)w - ex + y = 0.
\end{cases} \tag{3.1}$$

We can get the solution of the equation (3.1) as

$$x = x_* = 0, y = y_* = 0, z = z_* = \frac{(a+b)(d-1)}{2d-1}, w = w_* = 0.$$

To find the maximum value of the function $F(x, y, z, w)$, the Hessian matrix of $F(x, y, z, w)$ at $P_0 = (x_*, y_*, z_*, w_*)$ can be obtained as

$$
\begin{aligned}
\mathrm{H}_F(P_0) &= \left.\begin{pmatrix}
\dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x\partial y} & \dfrac{\partial^2 F}{\partial x\partial \mathrm{z}} & \dfrac{\partial^2 F}{\partial x\partial w} \\
\dfrac{\partial^2 F}{\partial y\partial x} & \dfrac{\partial^2 F}{\partial y^2} & \dfrac{\partial^2 F}{\partial y\partial z} & \dfrac{\partial^2 F}{\partial y\partial w} \\
\dfrac{\partial^2 F}{\partial z\partial x} & \dfrac{\partial^2 F}{\partial z\partial y} & \dfrac{\partial^2 F}{\partial z^2} & \dfrac{\partial^2 F}{\partial z\partial w} \\
\dfrac{\partial^2 F}{\partial w\partial x} & \dfrac{\partial^2 F}{\partial w\partial y} & \dfrac{\partial^2 F}{\partial w\partial z} & \dfrac{\partial^2 F}{\partial w^2}
\end{pmatrix}\right|_{P(x,y,z,w)=P_0} \\
&= \begin{pmatrix}
1-2a & 0 & 0 & -e \\
0 & 1-2c & 0 & 1 \\
0 & 0 & 1-2d & 0 \\
-e & 1 & 0 & 1-2r
\end{pmatrix}.
\end{aligned}
$$

According to the extreme value theory of multivariate functions, $F(x, y, z, w)$ can obtain a maximum value at P_0 when the matrix $\mathrm{H}_F(P_0)$ is a negative definite matrix. When the parameters of system (2.2) satisfy the following condition (3.2), the matrix is a negative definite matrix. When the parameters of system (2.2) satisfy the following condition (3.2), the matrix $\mathrm{H}_F(P_0)$ is a negative definite matrix.

$$
\begin{cases}
a > \dfrac{1}{2}, \\
c > \dfrac{1}{2}, \\
d > \dfrac{1}{2}, \\
r > -\dfrac{[(1-2a) + e^2(1-2c)]}{2(1-2a)(1-2c)} + \dfrac{1}{2}.
\end{cases} \tag{3.2}
$$

Since $F(x, y, z, w)$ is quadratic and its local maximum is the global maximum, so

$$
\sup_{\mathrm{X}\in\mathrm{R}^4} F(X) = F(X)|_{x=x_*, y=y_*, z=z_*, w=w_*} = \frac{(a+b)^2 d^2}{4d-2} = G. \tag{3.3}
$$

Therefore,

$$
\left.\frac{\mathrm{d}V(X(t))}{\mathrm{d}t}\right|_{(2.2)} \le -V(X(t)) + G, \tag{3.4}
$$

can be derived from the above calculation. From the exponential inequality (3.4), we can get

$$
[V(X(t)) - G] \le [V(X(t_0)) - G]\,\mathrm{e}^{-(t-t_0)}. \tag{3.5}
$$

So, we can get

$$
\overline{\lim_{t\to+\infty}} V(X(t)) \le G,
$$

which indicate that

$$\Omega = \{X \,|V(X) \le G\} = \left\{ (x, y, z, w) \middle| x^2 + y^2 + (z - a - b)^2 + w^2 \le \frac{(a+b)^2 d^2}{4d-2} \right\}$$

is the globally exponential attractive set of system (2.2). □

4. Synchronization of hyperchaotic attractors

In the following, we firstly apply the linear and nonlinear control method to achieve global exponential synchronization with two identical hyperchaotic attractors of system (2.2).

Assume the drive system is

$$\begin{cases} \dot{x}_1 = -ax_1 + ay_1, \\ \dot{y}_1 = bx_1 - x_1 z_1 - cy_1 + w_1, \\ \dot{z}_1 = x_1 y_1 - dz_1, \\ \dot{w}_1 = -ex_1 - rw_1. \end{cases} \tag{4.1}$$

And the response system is

$$\begin{cases} \dot{x}_2 = -ax_2 + ay_2 - u_1, \\ \dot{y}_2 = bx_2 - x_2 z_2 - cy_2 + w_2 - u_2, \\ \dot{z}_2 = x_2 y_2 - dz_2 - u_3, \\ \dot{w}_2 = -ex_2 - rw_2 - u_4. \end{cases} \tag{4.2}$$

Let $e_x = x_2 - x_1, e_y = y_2 - y_1, e_z = z_2 - z_1, e_w = w_2 - w_1$, then the error dynamical system can be obtained

$$\begin{cases} \dot{e}_x = -ae_x + ae_y - u_1(e_x, e_y, e_z, e_w), \\ \dot{e}_y = be_x - x_2 z_2 + x_1 z_1 - ce_y + e_w - u_2(e_x, e_y, e_z, e_w), \\ \dot{e}_z = x_2 y_2 - x_1 y_1 - de_z - u_3(e_x, e_y, e_z, e_w), \\ \dot{e}_w = -ee_x - re_w - u_4(e_x, e_y, e_z, e_w), \end{cases} \tag{4.3}$$

where $u_i = u_i(e_x, e_y, e_z, e_w)$ is the controller which satisfies $u_i(0,0,0,0) = 0, (i = 1, 2, 3, 4)$.

Theorem 4.1. *The linear feedback control law*

$$u_1 = k_1 e_x, u_2 = k_2 e_y, u_3 = k_3 e_z, u_4 = k_4 e_w, k_i \ge 0\,(i = 1, 2, 3, 4)$$

can always be chosen such that the zero solution of system (4.3) is globally exponential stable, so that systems (4.1) and (4.2) achieve globally exponential synchronization.

Proof. Define the radial unbounded vector Lyapunov function

$$V(X) = (|e_x|, |e_y|, |e_z|, |e_w|)^{\mathrm{T}}$$

for the system (4.3) and then its Dini derivative along the trajectory of system (4.3) is

$$\begin{cases} D^+|e_x| \le -(a+k_1)|e_x| + a|e_y|, \\ D^+|e_y| \le (b+|z_1|)|e_x| - (c+k_2)|e_y| + |x_2||e_z| + |e_w|, \\ D^+|e_z| \le |y_1||e_x| + |x_2||e_y| - (d+k_3)|e_z|, \\ D^+|e_w| \le -e|e_x| - (r+k_4)|e_w|. \end{cases}$$

The above inequality can be written as the following matrix

$$\begin{pmatrix} D^+|e_x| \\ D^+|e_y| \\ D^+|e_z| \\ D^+|e_w| \end{pmatrix} \le \begin{pmatrix} -a-k_1 & a & 0 & 0 \\ b+|z_1| & -c-k_2 & |x_2| & 1 \\ |y_1| & |x_2| & -d-k_3 & 0 \\ -e & 0 & 0 & -r-k_4 \end{pmatrix} \begin{pmatrix} |e_x| \\ |e_y| \\ |e_z| \\ |e_w| \end{pmatrix} = B \bullet \begin{pmatrix} |e_x| \\ |e_y| \\ |e_z| \\ |e_w| \end{pmatrix}.$$

Let $B_i (i = 1,2,3,4)$ be the i-order principal minor determinant of matrix B, and if matrix B is a negative definite matrix, the following condition (4.4) should be satisfied.

$$\begin{cases} B_1 = -a-k_1 < 0, \\ B_2 = \begin{vmatrix} -a-k_1 & a \\ b+|z_1| & -c-k_2 \end{vmatrix} > 0, \\ B_3 = \begin{vmatrix} -a-k_1 & a & 0 \\ b+|z_1| & -c-k_2 & |x_2| \\ |y_1| & |x_2| & -d-k_3 \end{vmatrix} < 0, \\ B_4 = |B| > 0. \end{cases} \tag{4.4}$$

Combined with Theorem 3.1., it can be seen that when the parameters a, b, c, d, e, r satisfies condition (4.4), so there is the exponential estimate of (4.1) and (4.2)

$$\frac{1}{2}[{x_i}^2 + {y_i}^2 + (z_i - a - b)^2 + {w_i}^2] \le G, i = 1,2.$$

Therefore, substituting the maximum $\max|x_i| = \sqrt{2G}, \max|y_i| = \sqrt{2G}, \max|z_i|$

$=\sqrt{2G}+a+b, i=1,2$, into (4.4), we can obtain

$$\begin{cases} k_1 \geq -a, \\ k_2 > \dfrac{a(|z_1|+b)}{a+k_1} - c, \\ k_3 > \dfrac{(a+k_1)|x_2|^2 + a\,|y_1|\,|x_2|}{|B_2|} - d, \\ k_4 > \dfrac{ea(d+k_3)}{-B_3} - r. \end{cases} \tag{4.5}$$

Therefore, when $k_i (i=1,2,3,4)$ satisfy the condition (4.5), the matrix B can be guaranteed to be a negative definite matrix. Moreover, it can be seen from (4.5) that there exist $k_i (i=1,2,3,4)$ such that (4.5) holds. Hence, we have

$$D^+(|e_x|,|e_y|,|e_z|,|e_w|)^{\mathrm{T}} \leq B(|e_x|,|e_y|,|e_z|,|e_w|)^{\mathrm{T}}. \tag{4.6}$$

Consider the comparing equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^{\mathrm{T}} = B(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^{\mathrm{T}}.$$

From the above differential inequality (4.6), we can obtain

$$(\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t))^{\mathrm{T}} = e^{B(t-t_0)}(\alpha_1(t_0),\alpha_2(t_0),\alpha_3(t_0),\alpha_4(t_0))^{\mathrm{T}},\ t \geq t_0.$$

Since the matrix B is a negative definite matrix, there exist $M \geq 1$ and $\alpha > 0$ such that

$$\left|e^{B(t-t_0)}\right| \leq Me^{-\alpha(t-t_0)},\ t \geq t_0.$$

And since

$$(|e_x(t_0)|,|e_y(t_0)|,|e_z(t_0)|\,|e_w(t_0)|)^{\mathrm{T}} = (\alpha_1(t_0),\alpha_2(t_0),\alpha_3(t_0),\alpha_4(t_0))^{\mathrm{T}},\ t \geq t_0.$$

So, we have

$$\begin{aligned} &\left\|(|e_x(t)|,|e_y(t)|,|e_z(t)|\,|e_w(t)|)^{\mathrm{T}}\right\| \\ \leq &\left\|(\alpha_1(t),\alpha_2(t),\alpha_3(t),\alpha_4(t))^{\mathrm{T}}\right\| \\ \leq &\left\|(\alpha_1(t_0),\alpha_2(t_0),\alpha_3(t_0),\alpha_4(t_0))^{\mathrm{T}}\right\| \bullet Me^{-\alpha(t-t_0)}. \end{aligned} \tag{4.7}$$

Since $(0,0,0,0)$ is the zero solution of the error system (4.3), and (4.7) can show that the zero solution of the error system (4.3) can achieve globally exponential stability when $t \to +\infty$, so that system (4.1) and system (4.2) achieve globally exponential synchronization. □

Remark 4.1. For synchronization approach, it is important to estimate the range or average value for the controllers by changing the coupling intensity k_1, k_2, k_3 and k_4. We have obtained the precise lower bound of the coefficient of linear feedback controller k_1, k_2, k_3 and k_4 in the above theorem.

Theorem 4.2. *For the error system (4.3), when the nonlinear feedback controller is designed as*

$$\begin{cases} u_1 = l_1 e_x, \\ u_2 = (b - z_1)e_x - x_2 e_z + l_2 e_y, \\ u_3 = y_1 e_x + x_2 e_y + l_3 e_z, \\ u_4 = l_4 e_y, \end{cases} \tag{4.8}$$

by choosing the appropriate control parameters l_1, l_2, l_3, l_4, then the zero solution of the error system (4.3) can be made globally exponential stable, so that the drive system (4.1) and the response system (4.2) can achieve globally exponential synchronization.

Proof. If the nonlinear feedback control is shown in (4.8), the error system becomes

$$\begin{cases} \dot{e}_x = (-a - l_1)e_x + a e_y, \\ \dot{e}_y = (-c - l_2)e_y + e_w, \\ \dot{e}_z = (-d - l_3)e_z, \\ \dot{e}_w = -e e_x + (-r - l_2)e_w. \end{cases} \tag{4.9}$$

Construct the following Lyapunov function

$$V_1 = e_x^2 + e_y^2 + e_z^2 + e_w^2.$$

The derivative of V_1 with respect to the time t along the trajectory of the system (4.3) is

$$\begin{aligned} \frac{\mathrm{d}V_1}{\mathrm{d}t} &= 2e_x \cdot \dot{e}_x + 2e_y \cdot \dot{e}_y + 2e_z \cdot \dot{e}_z + 2e_w \cdot \dot{e}_w \\ &= \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -2a - 2l_1 & 2a & 0 & 0 \\ 0 & -2c - 2l_2 & 0 & 1 \\ 0 & 0 & -2d - 2l_3 & 0 \\ -e & 0 & 0 & -2r - 2l_4 \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix} \\ &= \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}^{\mathrm{T}} C \begin{pmatrix} e_x \\ e_y \\ e_z \\ e_w \end{pmatrix}. \end{aligned}$$

When the feedback control parameters satisfy

$$l_1 > -a, l_2 > -c, l_3 > -d, l_4 > \frac{ea}{(a + l_1)(c + l_2)} - r, \tag{4.10}$$

the matrix C can be guaranteed to be a negative definite matrix, so that $\frac{\mathrm{d}V_1}{\mathrm{d}t}$ is negative definite. There exists the largest negative eigenvalue $\lambda_{\max}(C)$ of negative definite matrix C so that the following equation holds

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} \leq \lambda_{\max}(C) \cdot ({e_x}^2 + {e_y}^2 + {e_z}^2 + {e_w}^2) = \lambda_{\max}(C) \cdot V_1.$$

Hence,

$${e_x}^2 + {e_y}^2 + {e_z}^2 + {e_w}^2 = V_1(X(t)) \leq V_1(X(t_0)) e^{\lambda_{\max}(C)(t-t_0)}, \quad t > t_0.$$

And since $\lambda_{\max}(C) < 0$, so $V_1(X(t)) \to 0$ when $t \to +\infty$. So, the zero solution of the error system (4.9) is globally exponentially stable which indicates that the systems (4.1) and (4.2) can achieve globally exponential synchronization. □

5. Numerical simulations

In this section, we will give numerical simulation of global exponential synchronization for $a = 35, b = 20, c = 1, d = 3, e = 3, r = 1.5$. The initial conditions of the drive system and the response system at $t_0 = 0$ are chosen as

$$\begin{aligned}(x_1(0), y_1(0), z_1(0), w_1(0)) = (0.2, 6, 0.4, 8), &(x_2(0), y_2(0), z_2(0), w_2(0)) \\ &= (5, 0.3, 7, 0.5).\end{aligned}$$

Let us choose $k_1 = 38, k_2 = 72, k_4 = 5, k_3 = 5160, l_1 = -33, l_2 = 1, l_3 = -1, l_4 = 1$, then the conditions for the above theorems can be satisfied. The linear synchronization error is shown in Figure 7. and the nonlinear synchronization error is shown in Figure 8.

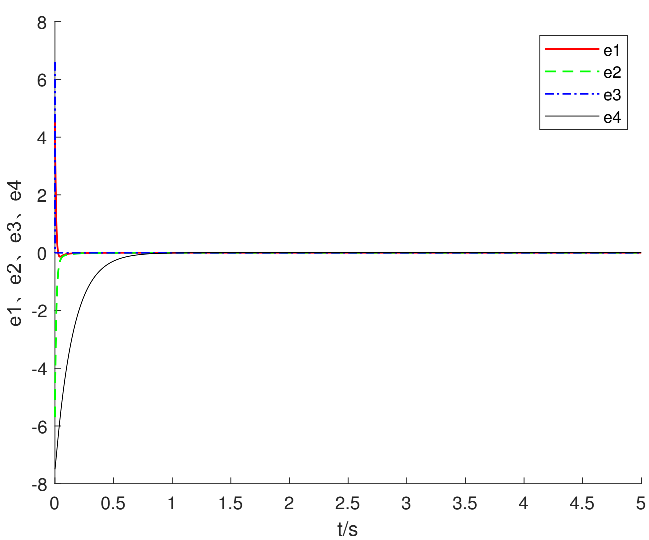

Figure 7. Synchronization error of linear feedback control is illustrated when $k_1 = 38, k_2 = 72, k_3 = 5160, k_4 = 5$.

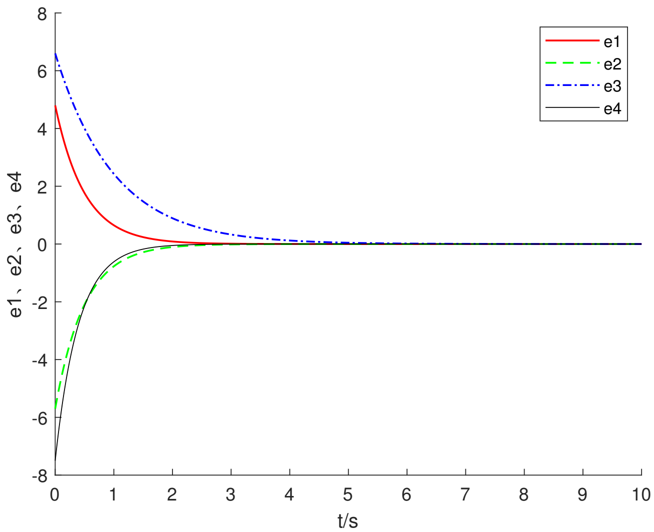

Figure 8. Synchronization error of nonlinear feedback control is illustrated when $l_1 = -33, l_2 = 1, l_3 = -1, l_4 = 1$.

From Figure 7 and Figure 8, we can see that the oscillations of the drive system and the response system rapidly become totally indistinguishable which indicate that synchronization is achieved very quickly. The above simulation results show that the two feedback control methods can both make systems (4.1) and (4.2) achieve global exponential synchronization very quickly, which confirms that two control methods are very effective.

6. Conclusions

As everyone knows, the Lorenz system is a simplified chaotic model to describe the Rayleigh-Bénard convection with a variety of examples seen throughout nature. The study of nonlinear dynamics of the new chaotic system contributes to a better understanding of nonlinear dynamics and chaos theory. In this paper, we have constructed and analyzed a new hyperchaotic system which is a four-dimensional system of nonlinear differential equations. Dynamical properties are analyzed theoretically and numerically, including the dissipation, the equilibrium points and their stability, Lyapunov exponents, the Lyapunov dimension of the attractors, the global exponential attractive set, global exponential synchronization. Numerical simulations are also given in order to verify the feasibility of the theoretical results of this paper. It is hoped that the results reported in this paper contribute to a better understanding of nonlinear dynamics and chaos theory.

Acknowledgments

We will thank professor Weinian Zhang in Sichuan University, professor Shutang Liu in Shandong University, professor Jinde Cao in Southeast University, Jinhu Lu in Beihang University (also at Chinese Academy of Sciences), professor Xiaofeng

Liao in Chongqing University and professor Guanrong Chen in the City University of Hong Kong for their help with us. We also thank professor Miguel A. F. Sanjuan in the Spanish Royal Academy of Sciences (also at Universidad Rey Juan Carlos of Spain) and professor Nikolay V. Kuznetsov in Russian Academy of Science for their support. Finally, the authors will thank the anonymous reviewers for their careful reading and useful suggestions, which greatly improved the presentation of this paper. Without the help of the reviewers, this paper would not be at this level.

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Received January 2024; Accepted February 2025; Available online April 2025.