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POLYNOMIOGRAPHS AND CONVERGENCE: A COMPARATIVE STUDY OF ITERATION PROCESSES UNDER KANANN-SUZUKI-(C) CONDITION

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Abstract In this paper, we study generalized (C)-conditions, specifically the Kannan-Suzuki-(C) condition (abbreviated as the (KSC)-condition). We employ the M-iteration process to investigate the convergence behavior of mappings satisfying the KSC-condition and demonstrate that this approach offers improved convergence speed and computational efficiency compared to other well-known iteration schemes in the literature. To illustrate the advantages of the M-iteration process, we present new numerical examples that highlight its effectiveness. Additionally, we validate our theoretical findings by applying the method to fractional delay differential equations, showcasing its applicability in solving complex mathematical models. Furthermore, we compare the polynomiographs generated by the M-iteration process with those produced by other well-known iteration methods, demonstrating superior visualization properties and computational performance. These results establish the M-iteration process as a powerful tool for studying generalized contraction conditions.

Keywords M-iteration, (KSC)-condition, fixed point, polynomiography, convergence, numerical method, iterative method.

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1. Introduction

In many cases, it is well known that the theory of fixed points is a celebrated area of research in nonlinear analysis, providing efficient and alternative tools for approximating solutions to both linear and nonlinear problems [12, 36, 38]. In fixed-point

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theory, a sought-after solution to a linear or nonlinear problem is often expressed as a fixed point of some mapping, whose domain is typically an appropriate subset of a distance space (e.g., Hilbert or Banach space). It should be noted that any fixed-point theorem not only establishes the existence of a fixed point but, when possible, also proves its uniqueness for a given mapping in a specified domain. In the existing literature, several researchers have extensively investigated the existence of solutions for various classes of integral and differential equations by employing techniques and methods from fixed-point theory. Furthermore, once the existence of a solution for a nonlinear problem is established, attention naturally shifts to numerical methods for approximating such solutions. In this context, we introduce the concept of contractions: A mapping K defined on a subset $\mathfrak{U}$ of a Banach space V is called a contraction [6] if

$$||K\mathfrak{z} - Ky|| \leq \mathfrak{a}||\mathfrak{z} - y||, \tag{1.1}$$

where $\mathfrak{a}$ is a real number in the interval$[0, 1)$, and $\mathfrak{z}, y$ are any points in $\mathfrak{U}$. The number $\mathfrak{a}$ is sometimes called the contraction factor of K. Notice that a fixed point of a self-map $K : \mathfrak{U} \to \mathfrak{U}$ is a point $\mathfrak{p} \in \mathfrak{U}$ that satisfies the condition $K\mathfrak{p} = \mathfrak{p}$. We shall denote the set of fixed points of K by $Fix(K)$. In 1922, Banach [6] proposed a fundamental result on the existence of fixed points for contractions, stated as follows:

Theorem 1.1. *Suppose that K is a contraction on a closed subset $\mathfrak{U}$ of a Banach space, with a contraction factor $\mathfrak{a}$. Then K has a unique fixed point $\mathfrak{p}$, and the sequence of Picard iterates, defined by$\mathfrak{z}_{n+1} = K\mathfrak{z}_n$, converges strongly to $\mathfrak{p}$ for every initial guess $\mathfrak{z}_1 \in \mathfrak{U}$.*

Theorem 1.1 is a fundamental result in analysis that establishes both the existence of a fixed point and, at the same time, provides an approximation process for obtaining its value under mild conditions. Furthermore, it is now known that for a nonexpansive map K, i.e., a map satisfying the condition $\|K\mathfrak{z} - Ky\| \leq \|\mathfrak{z} - y\|, \quad \forall \mathfrak{z}, y \in \mathfrak{U}$, a fixed point of K may exist. However, the approximation method suggested in Theorem 1.1 may no longer converge to this fixed point. It is easy to see that the class of nonlinear nonexpansive maps is more general and includes all contractions as a special case. Moreover, the study of nonexpansive nonlinear maps has its origins in applications of computer science and other fields of applied sciences.

In 1965, Browder [7] and Gohde [10] independently conducted pioneering studies on the existence of fixed points for nonexpansive mappings in certain classes of Banach spaces. They established that every nonexpansive self-mapping defined on a bounded, closed, convex subset of a uniformly convex Banach space (UCBS), a special type of Banach space denoted by $\mathfrak{U}$ admits at least one fixed point, though uniqueness is not guaranteed in general. In the same year, Kirk [19] extended the Browder-Gohde result to the setting of reflexive Banach spaces (RBS). To illustrate that the Picard iteration method may fail to converge to a fixed point for a nonexpansive self-mapping, we present the following numerical example.

Definition 1.1. ([9]) Let V be a Banach space. V is called a uniformly convex Banach space (UCBS) if for each $\vartheta \in (0, 2]$ there exists $\psi > 0$ such that $s, x \in V$.

A space V is termed as uniformly convex Banach space if;

$$\left.\begin{array}{l}\|s\| \leq 1, \\ \|x\| \leq 1, \\ \|s-x\| > \psi,\end{array}\right\} \Rightarrow \left\|\frac{s+x}{2}\right\| \leq \vartheta. \tag{1.2}$$

Example 1.1. [20] Let us consider $\mathfrak{U} = [0,1]$, which is a closed, bounded, and convex subset of the uniformly convex Banach space (UCBS) $\mathbb{R}$. We define a mapping K on $\mathfrak{U}$ by $K\mathfrak{z} = 1 - \mathfrak{z}$. It is easy to verify that K is a nonexpansive mapping with a unique fixed point at $\mathfrak{z} = 0.5$.

In 2008, Suzuki [35] studied the class of nonexpansive mappings and established the following generalization: A mapping K defined on a subset $\mathfrak{U}$ of a Banach space is called a Suzuki mapping (or a mapping satisfying condition (C)) if, for all $\mathfrak{z}, y \in \mathfrak{U}$,

$$\frac{1}{2}\|\mathfrak{z} - K\mathfrak{z}\| \leq ||\mathfrak{z} - y|| \Rightarrow ||K\mathfrak{z} - Ky|| \leq ||\mathfrak{z} - y||.$$

Suzuki proved the existence of fixed points for mapping satisfying condition (C). Obviously, every nonexpansive mapping is a Suzuki mapping; however, the following example demonstrates that the converse does not hold in general.

Example 1.2. [35] Let us consider $\mathfrak{U} = [0,3]$, which is closed and bounded convex subset of UCBS $\mathbb{R}$. We define a map K on $\mathfrak{U}$ as

$$K\mathfrak{z} = \begin{cases} \dfrac{\mathfrak{z}+24}{5}, & \text{if } \mathfrak{z} \in [0,6], \\ 5, & \text{if } \mathfrak{z} \in (6,7]. \end{cases}$$

The mapping presented in Example 1.2 is a Suzuki mapping but not a nonexpansive mapping.

Motivated by Suzuki [35], Karapinar [16] introduced a generalized condition (C) for mappings. Specifically, a mapping K defined on a subset $\mathfrak{U}$ of a Banach space is said to satisfy the (KSC)-condition if

$$\frac{1}{2}\|\mathfrak{z} - K\mathfrak{z}\| \leq \|\mathfrak{z} - y\| \Rightarrow \|K\mathfrak{z} - Ky\| \leq \frac{1}{2}(\|\mathfrak{z} - Ky\| + \|y - K\mathfrak{z}\|),\ \forall \mathfrak{z}, y \in \mathfrak{U}.$$

Karapinar [16] investigated several properties of mappings with (KSC)-condition and provided some fixed-point results for these mappings. Now, we give an example to show that the (KSC)-condition is more general than the condition (C).

Example 1.3. Define a self-mapping K on the interval $[-1,1]$ as follows:

$$K\mathfrak{z} = \begin{cases} -\dfrac{\mathfrak{z}}{10}, & \text{if } \mathfrak{z} \in [-1,0), \\ -\mathfrak{z}, & \text{if } \mathfrak{z} \in [0,1]/\dfrac{1}{10}, \\ 0, & \text{if } \mathfrak{z} = \dfrac{1}{10}. \end{cases}$$

Solution: When $\mathfrak{z} = \frac{1}{10}$ and $y = 1$, it is easy to verify that K does not satisfy condition (C). Next, we aim to prove that K satisfies the (KSC)-condition. To do so, we consider the following cases.

1. For any $\mathfrak{z}, y \in [-1, 0)$, we have $K\mathfrak{z} = -\frac{\mathfrak{z}}{10}$ and $Ky = -\frac{y}{10}$. Thus

$$\begin{aligned}
\frac{1}{2}(|\mathfrak{z} - K\mathfrak{z}| + |y - Ky|) &= \frac{1}{2}\left(\left|\mathfrak{z} + \frac{\mathfrak{z}}{10}\right| + \left|y + \frac{y}{10}\right|\right) \\
&= \frac{1}{2}\left(\left|\frac{11\mathfrak{z}}{10}\right| + \left|\frac{11y}{10}\right|\right) \\
&= \frac{11}{20}(|\mathfrak{z}| + |y|) \\
&\geq \frac{11}{20}(|\mathfrak{z} - y|) \\
&\geq \frac{1}{10}(|\mathfrak{z} - y|) \\
&= |K\mathfrak{z} - Ky|.
\end{aligned}$$

2. For any $\mathfrak{z}, y \in [0, 1]/\frac{1}{10}$, we have $K\mathfrak{z} = -\mathfrak{z}$ and $Ky = -y$. Hence

$$\begin{aligned}
\frac{1}{2}(|\mathfrak{z} - K\mathfrak{z}| + |y - Ky|) &= \frac{1}{2}(|\mathfrak{z} + \mathfrak{z}| + |y + y|) \\
&= |\mathfrak{z}| + |y| \\
&\geq |\mathfrak{z} - y| \\
&= |K\mathfrak{z} - Ky|.
\end{aligned}$$

3. For any $\mathfrak{z} \in [-1, 0)$ and $y \in [0, 1]/\frac{1}{10}$, we have $K\mathfrak{z} = \frac{-\mathfrak{z}}{10}$ and $Ky = -y$. Thus

$$\begin{aligned}
\frac{1}{2}(|\mathfrak{z} - K\mathfrak{z}| + |y - Ky|) &= \frac{1}{2}\left(\left|\mathfrak{z} + \frac{\mathfrak{z}}{10}\right| + |y + y|\right) \\
&= \frac{1}{2}\left(\left|\frac{11\mathfrak{z}}{10}\right| + |2y|\right) \\
&\geq \frac{1}{2}\left(\left|\frac{2\mathfrak{z}}{10}\right| + |2y|\right) \\
&= \left(\left|\frac{\mathfrak{z}}{10}\right| + |y|\right) \\
&\geq \frac{1}{2}\left(\left|\frac{\mathfrak{z}}{10} - y\right|\right) \\
&= |K\mathfrak{z} - Ky|.
\end{aligned}$$

4. For any $\mathfrak{z} \in [-1, 0)$ and $y = \frac{1}{10}$, we have $K\mathfrak{z} = \frac{-\mathfrak{z}}{10}$ and $Ky = 0$. Thus

$$\begin{aligned}
\frac{1}{2}(|\mathfrak{z} - K\mathfrak{z}| + |y - Ky|) &= \frac{1}{2}\left(\left|\mathfrak{z} + \frac{\mathfrak{z}}{10}\right| + |y - 0|\right) \\
&= \frac{1}{2}\left(\left|\frac{11\mathfrak{z}}{10}\right| + |y|\right) \\
&\geq \frac{1}{2}\left(\left|\frac{11\mathfrak{z}}{10}\right|\right) \\
&= \frac{11}{20}|\mathfrak{z}| \\
&\geq \left|\frac{\mathfrak{z}}{10}\right| \\
&= |K\mathfrak{z} - Ky|.
\end{aligned}$$

5. For any $y \in [0,1]/\frac{1}{10}$ and $\mathfrak{z} = \frac{1}{10}$, we have $Ky = -y$ and $K\mathfrak{z} = 0$. Thus
$$\begin{aligned}\frac{1}{2}(|\mathfrak{z} - K\mathfrak{z}| + |y - Ky|) &= \frac{1}{2}(|\mathfrak{z} - 0| + |y + y|)\\ &= \frac{1}{2}(|\mathfrak{z}| + |2y|)\\ &\geq \frac{1}{2}(|2y|)\\ &= |y|\\ &= |K\mathfrak{z} - Ky|.\end{aligned}$$

From the above cases, we conclude that K satisfies the (KSC)-condition.

Numerous researchers have developed various iterative methods for approximating fixed points of different generalizations of nonexpansive mappings. The primary motivation behind the extensive study of iterative approaches to fixed-point computation stems from their widespread applications across multiple disciplines, including root-finding, game theory, and image restoration. In such applications, the need for efficient and rapidly converging methods is paramount. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequences in $(0,1]$. Following is the Mann [22] iteration process:

$$\begin{cases}\mathfrak{z}_1 \in \mathfrak{U},\\ \mathfrak{z}_{n+1} = (1-\alpha_n)\mathfrak{z}_n + \alpha_n K\mathfrak{z}_n.\end{cases} \tag{1.3}$$

Khan [17] suggested iteration process which converges faster than the Mann iteration for contraction mappings in Banach spaces:

$$\begin{cases}\mathfrak{z}_1 \in \mathfrak{U},\\ y_n = (1-\alpha_n)\mathfrak{z}_n + \alpha_n K\mathfrak{z}_n,\\ \mathfrak{z}_{n+1} = Ky_n.\end{cases} \tag{1.4}$$

Agarwal [2] introduced the following two-step iteration process:

$$\begin{cases}\mathfrak{z}_1 \in \mathfrak{U},\\ y_n = (1-\beta_n)\mathfrak{z}_n + \beta_n K\mathfrak{z}_n,\\ \mathfrak{z}_{n+1} = (1-\alpha_n)\mathfrak{z}_n + \alpha_n Ky_n.\end{cases} \tag{1.5}$$

Noor [28] extended this approach by introducing a three-step iteration process:

$$\begin{cases}\mathfrak{z}_1 \in \mathfrak{U},\\ z_n = (1-\gamma_n)\mathfrak{z}_n + \gamma_n K\mathfrak{z}_n,\\ y_n = (1-\beta_n)\mathfrak{z}_n + \beta_n Kz_n,\\ \mathfrak{z}_{n+1} = (1-\alpha_n)\mathfrak{z}_n + \alpha_n Ky_n.\end{cases} \tag{1.6}$$

Ullah and Arshad [37] further developed a three-step iteration, referred to as the M-iteration process, defined as follows:

$$\begin{cases}\mathfrak{z}_1 \in \mathfrak{U},\\ w_n = (1-\alpha_n)\mathfrak{z}_n + \alpha_n K\mathfrak{z}_n,\\ y_n = Kw_n,\\ \mathfrak{z}_{n+1} = Ky_n.\end{cases} \tag{1.7}$$

The authors in [37] studied several convergence results of the M-iteration process for mappings satisfying condition (C). In this paper, we extend their results by investigating the weak and strong convergence of the M-iteration process under a generalized condition (C). Our convergence analysis is further supported by new numerical examples and comparisons with existing iterative methods from the literature. Additionally, we validate our findings through graphical representations generated via polynomiography.

2. Preliminaries

The following results will pave the way toward the derivation of the main result.

Definition 2.1. Assume that V is a given Banach space and $\{\mathfrak{z}_n\} \subseteq V$ is any bounded sequence. Let $\emptyset \neq \mathfrak{U} \subseteq V$ be any convex and closed set. In this case, the asymptotic radius (AR, for short) associated with $\{\mathfrak{z}_n\}$ on the set $\mathfrak{U}$ is given as

$$r(\mathfrak{U}, \{\mathfrak{z}_n\}) = \inf\{\limsup_{n\to\infty} \|\mathfrak{z}_n - s\| : s \in \mathfrak{U}\}.$$

Similarly, the asymptotic center (AC, for short) associated with $\{\mathfrak{z}_n\}$ on the set $\mathfrak{U}$ is given as

$$\mathcal{A}(\mathfrak{U}, \{\mathfrak{z}_n\}) = \left\{ s \in \mathfrak{U} : \limsup_{n\to\infty} \|\mathfrak{z}_n - s\| = r(\mathfrak{U}, \{\mathfrak{z}_n\}) \right\}.$$

Definition 2.2. [29] A Banach space V is said to satisfy Opial's condition if, for any sequence $\{\mathfrak{z}_n\} \subseteq V$ that weakly converges to some $s_0 \in V$, we have

$$\limsup_{n\to\infty} \|\mathfrak{z}_n - s_0\| < \limsup_{n\to\infty} \|\mathfrak{z}_n - e_0\|, \quad \forall e_0 \neq s_0.$$

Hilbert spaces are known to satisfy Opial's condition.

Definition 2.3. [34] Condition (I) for a self-map K on a subset $\mathfrak{U}$ of a Banach space V is defined as follows: There exists a function γ such that $\gamma(0) = 0$ and $\gamma(u) > 0$ for all $u > 0$, satisfying

$$\|\mathfrak{z} - K\mathfrak{z}\| \geq \gamma(\operatorname{dist}(\mathfrak{z}, Fix(K))), \quad \forall \mathfrak{z} \in \mathfrak{U},$$

where the notation $\operatorname{dist}(\mathfrak{z}, Fix(K))$ denotes the distance of the point $\mathfrak{z}$ from the set $Fix(K)$, given by

$$\operatorname{dist}(\mathfrak{z}, Fix(K)) = \inf\{\|\mathfrak{z} - y\| : y \in Fix(K)\}.$$

Lemma 2.1. [16] *Suppose that V is a Banach space and $\emptyset \neq \mathfrak{U} \subseteq V$. If $K : \mathfrak{U} \to \mathfrak{U}$ satisfies the (KSC)-condition and $Fix(K) \neq \emptyset$, then for any $\mathfrak{z} \in \mathfrak{U}$ and any fixed point $\mathfrak{p} \in Fix(K)$, the following holds:*

$$\|K\mathfrak{z} - \mathfrak{p}\| \leq \|\mathfrak{z} - \mathfrak{p}\|.$$

Lemma 2.2. [16] *Suppose that V is a Banach space and $\emptyset \neq \mathfrak{U} \subseteq V$. If K is a self-map on $\mathfrak{U}$ satisfying the (KSC)-condition, then for any $\mathfrak{z}, y \in \mathfrak{U}$, the following inequality holds:*

$$\|\mathfrak{z} - Ky\| \leq 5\|\mathfrak{z} - K\mathfrak{z}\| + \|\mathfrak{z} - y\|.$$

Lemma 2.3. [16] *Let $\mathfrak{U} \subseteq V$ be equipped with Opial's property, and let K be a self mapping on $\mathfrak{U}$ satisfying (KSC)-condition. If $\{\mathfrak{z}_n\}$ converges weakly to $\mathfrak{z}$ and $\|K\mathfrak{z}_n - \mathfrak{z}_n\| = 0$, then $K\mathfrak{z} = \mathfrak{z}$.*

Lemma 2.4. [33] *Assume $0 < j \leq b_n \leq k < 1$ for all $n \geq 1$. Consider $\{\mathfrak{z}_n\}$ and $\{y_n\}$ in a UCBS V that satisfy $\limsup_{n\to\infty} \|\mathfrak{z}_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\limsup_{n\to\infty} \|(1 - b_n)y_n + b_n\mathfrak{z}_n\| = c$ for all $c \geq 0$, then one has $\lim_{n\to\infty} \|\mathfrak{z}_n - y_n\| = 0$.*

3. Main results

We now present our main results.

Lemma 3.1. *Let V be a UCBS and let $\emptyset \neq \mathfrak{U} \subseteq V$ be a closed and convex set. Assume that a self-map $K : \mathfrak{U} \to \mathfrak{U}$ satisfies condition (KSC) and that $Fix(K) \neq \emptyset$. If $\{\mathfrak{z}_n\}$ is a sequence generated by the M-iteration process (1.7), then for each $\mathfrak{p} \in Fix(K)$, the limit*

$$\lim_{n\to\infty} \|\mathfrak{z}_n - \mathfrak{p}\|$$

exists.

Proof. Let $\mathfrak{p} \in Fix(K)$ be an arbitrary fixed point. By applying Lemma 2.1, we obtain

$$\begin{aligned} \|w_n - \mathfrak{p}\| &= \|(1-\alpha_n)\mathfrak{z}_n + \alpha_n K\mathfrak{z}_n - \mathfrak{p}\| \\ &\leq (1-\alpha_n)\|\mathfrak{z}_n - \mathfrak{p}\| + \alpha_n\|K\mathfrak{z}_n - \mathfrak{p}\| \\ &\leq (1-\alpha_n)\|\mathfrak{z}_n - \mathfrak{p}\| + \alpha_n\|\mathfrak{z}_n - \mathfrak{p}\| \\ &= \|\mathfrak{z}_n - \mathfrak{p}\|. \end{aligned} \tag{3.1}$$

Similarly, we obtain

$$\|y_n - \mathfrak{p}\| = \|Kw_n - \mathfrak{p}\| \leq \|w_n - \mathfrak{p}\|, \tag{3.2}$$

and

$$\|\mathfrak{z}_{n+1} - \mathfrak{p}\| = \|Ky_n - \mathfrak{p}\| \leq \|y_n - \mathfrak{p}\|. \tag{3.3}$$

Consequently, we observe that

$$\|\mathfrak{z}_{n+1} - \mathfrak{p}\| \leq \|y_n - \mathfrak{p}\| \leq \|w_n - \mathfrak{p}\| \leq \|\mathfrak{z}_n - \mathfrak{p}\|.$$

From (3.1), (3.2) and (3.3), we can note that the sequence $\{\|\mathfrak{z}_n - \mathfrak{p}\|\}$ is non-increasing and bounded below. Therefore, we conclude that

$$\lim_{n\to\infty} \|\mathfrak{z}_n - \mathfrak{p}\|$$

exists for any choice of $\mathfrak{p} \in Fix(K)$. □

We now establish another elementary result as follows:

Theorem 3.1. *Let V be a UCBS, and let $\emptyset \neq \mathfrak{U} \subseteq V$ be a closed and convex subset. Assume that a self-map $\mathfrak{U} \to \mathfrak{U}$ satisfies (KSC)-condition. If $\{\mathfrak{z}_n\}$ is a sequence generated by the M-iteration process, then then $Fix(K) \neq \emptyset$ if and only if $\{\mathfrak{z}_n\}$ is bounded and satisfies $\lim_{n\to\infty} \|K\mathfrak{z}_n - \mathfrak{z}_n\| = 0$.*

Proof. Assume that $Fix(K) \neq \emptyset$. Therefore, for any $p \in Fix(K)$, Lemma 3.1 suggests that $\{\mathfrak{z}_n\}$ is bounded and $\lim_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}||$ exists. Assume that

$$\lim_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}|| = e. \tag{3.4}$$

We need to prove $\lim_{n\to\infty} ||\mathfrak{z}_n - K\mathfrak{z}_n|| = 0$. Now, from (3.1), we get

$$\begin{aligned} &||w_n - \mathfrak{p}|| \leq ||\mathfrak{z}_n - \mathfrak{p}||, \\ \Rightarrow &\limsup_{n\to\infty} ||w_n - \mathfrak{p}|| \leq \limsup_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}|| = e. \end{aligned} \tag{3.5}$$

Since $\mathfrak{p} \in Fix(K)$, we can apply Lemma 2.1 to get

$$\begin{aligned} &||K\mathfrak{z}_n - \mathfrak{p}|| \leq ||\mathfrak{z}_n - \mathfrak{p}||, \\ \Rightarrow &\limsup_{n\to\infty} ||K\mathfrak{z}_n - \mathfrak{p}|| \leq \limsup_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}||. \end{aligned} \tag{3.6}$$

Owing to Lemma 3.1, we have

$$||\mathfrak{z}_{n+1} - \mathfrak{p}|| \leq ||w_n - \mathfrak{p}||. \tag{3.7}$$

Using (3.7) together with (3.5), we obtain

$$e \leq \liminf_{n\to\infty} ||w_n - \mathfrak{p}||. \tag{3.8}$$

From (3.5) and (3.8), we obtain

$$\lim_{n\to\infty} ||w_n - \mathfrak{p}|| = e. \tag{3.9}$$

Since

$$||w_n - \mathfrak{p}|| = ||(1-\alpha_n)(\mathfrak{z}_n - \mathfrak{p}) + \alpha_n(K\mathfrak{z}_n - \mathfrak{p})||. \tag{3.10}$$

Using (3.10) together with (3.9), we get

$$e = \lim_{n\to\infty} ||(1-\alpha_n)(\mathfrak{z}_n - \mathfrak{p}) + \alpha_n(K\mathfrak{z}_n - \mathfrak{p})||. \tag{3.11}$$

Considering (3.4), (3.6) and (3.11) along with Lemma 2.4, one gets

$$\lim_{n\to\infty} ||\mathfrak{z}_n - K\mathfrak{z}_n|| = 0.$$

Conversely, we shall assume that $\{\mathfrak{z}_n\}$ is essentially bounded with the property $\lim_{n\to\infty} ||\mathfrak{z}_n - K\mathfrak{z}_n|| = 0$ and prove that $Fix(K) \neq \emptyset$. To do this, we consider any $\mathfrak{p} \in A(\mathfrak{U}, \{\mathfrak{z}_n\})$. By Lemma 2.2, we have

$$\begin{aligned} r(K\mathfrak{p}, \{\mathfrak{z}_n\}) &= \limsup_{n\to\infty} ||\mathfrak{z}_n - K\mathfrak{p}|| \\ &\leq 5 \limsup_{n\to\infty} ||\mathfrak{z}_n - K\mathfrak{z}_n|| + \limsup_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}|| \\ &= \limsup_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}|| \\ &= r(\mathfrak{p}, \{\mathfrak{z}_n\}). \end{aligned}$$

Thus $K\mathfrak{p} \in A(\mathfrak{U}, \{\mathfrak{z}_n\})$. As V is a UCBS, the set $A(\mathfrak{U}, \{\mathfrak{z}_n\})$ contains only one point, hence $K\mathfrak{p} = \mathfrak{p}$. This implies that $\mathfrak{p} \in Fix(K)$ i.e., $Fix(K) \neq \emptyset$. □

We first suggest a weak convergence result. This result is mainly based on Opial's property.

Theorem 3.2. *Let V be a UCBS and $\emptyset \neq \mathfrak{U} \subseteq V$ be closed and convex. Assume that self-map $K : \mathfrak{U} \to \mathfrak{U}$ satisfies (KSC)-condition and $Fix(K)$ is non-empty. If $\{\mathfrak{z}_n\}$ denotes a sequence of M-iteration process (1.7) and V satisfies Opial's condition, then $\{\mathfrak{z}_n\}$ converges weakly to a fixed point of K.*

Proof. Notice that V is reflexive due to the convexity of V. Now, according to Theorem 3.1, $\{\mathfrak{z}_n\}$ is bounded. It follows that there is a point, namely, $\mathfrak{z}_0 \in \mathfrak{U}$ such that a subsequence, namely, $\{\mathfrak{z}_{n_m}\}$ of $\{\mathfrak{z}_n\}$ weakly converges to it. From Theorem 3.1, it is clear that $\lim\limits_{m\to\infty} ||\mathfrak{z}_{n_m} - K\mathfrak{z}_{n_m}|| = 0$. Using Lemma 2.3, $\mathfrak{z}_0 \in Fix(K)$. We want to prove that the point u_0 is the only weak limit of $\{\mathfrak{z}_n\}$, on contrary we suppose that $\mathfrak{z}_0$ is not a weak limit for $\{\mathfrak{z}_n\}$ i.e., there exists another subsequence, namely, $\{\mathfrak{z}_{n_s}\}$ of $\{\mathfrak{z}_n\}$ with a weak limit, namely, $\mathfrak{z}_0' \neq \mathfrak{z}_0$. From Theorem 3.1, it is annotated that $\lim\limits_{s\to\infty} ||\mathfrak{z}_{n_s} - K\mathfrak{z}_{n_s}|| = 0$. Using Opial's condition of V along with Lemma 3.1, we get

$$\begin{aligned}
\lim_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{z}_0|| &= \lim_{m\to\infty} ||\mathfrak{z}_{n_m} - \mathfrak{z}_0|| \\
&< \lim_{m\to\infty} ||\mathfrak{z}_{n_m} - \mathfrak{z}_0'|| \\
&= \lim_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{z}_0'|| \\
&= \lim_{s\to\infty} ||\mathfrak{z}_{n_s} - \mathfrak{z}_0'|| \\
&< \lim_{s\to\infty} ||\mathfrak{z}_{n_s} - \mathfrak{z}_0|| \\
&= \lim_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{z}_0||.
\end{aligned}$$

As a whole, we obtain $\lim\limits_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{z}_0|| < \lim\limits_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{z}_0||$, which is a contradiction. This finishes the proof. □

The following theorem is based on the notion of compactness:

Theorem 3.3. *Let V be a UCBS and $\emptyset \neq \mathfrak{U} \subseteq V$ be closed and compact. Assume that a self-map K defined on $\mathfrak{U}$ satisfies (KSC)-condition and $Fix(K) \neq \emptyset$. If $\{\mathfrak{z}_n\}$ denotes a sequence of M-iteration process (1.7) and $\mathfrak{U}$ is compact, then $\{\mathfrak{z}_n\}$ converges strongly to a fixed point of K.*

Proof. As assumed, the set $\mathfrak{U}$ is closed and compact, the sequence of iterates $\{\mathfrak{z}_n\}$ is contained in the set $\mathfrak{U}$ and has a subsequence $\{\mathfrak{z}_{n_m}\}$ of $\{\mathfrak{z}_n\}$ that converges strongly to $\mathfrak{p} \in \mathfrak{U}$. So, in the view of Theorem 3.1, we get $\lim\limits_{n_m\to\infty} ||\mathfrak{z}_{n_m} - \mathfrak{p}|| = 0$. Hence using these facts together with Lemma 2.2, we have

$$\|\mathfrak{z}_{n_m} - K\mathfrak{p}\| \leq 5\|\mathfrak{z}_{n_m} - K\mathfrak{z}_{n_m}\| + \|\mathfrak{z}_{n_m} - \mathfrak{p}\|. \tag{3.12}$$

By Theorem 3.1, $\lim\limits_{n_m\to\infty} ||\mathfrak{z}_{n_k} - K\mathfrak{z}_{n_k}|| = 0$ and also $\lim\limits_{n_m\to\infty} ||\mathfrak{z}_{n_m} - \mathfrak{p}|| = 0$. Accordingly the equation (3.12) provides $\lim\limits_{n_m\to\infty} \mathfrak{z}_{n_m} = K\mathfrak{p}$. It follows that $\{\mathfrak{z}_{n_m}\}$ converges to $\mathfrak{p}$ and $K\mathfrak{p}$. Thus, we have $K\mathfrak{p} = \mathfrak{p}$. Appealing Lemma 3.1, one gets the existence of $\lim\limits_{n\to\infty} ||\mathfrak{z}_n - \mathfrak{p}||$. Hence, our sequence of iterates $\{\mathfrak{z}_n\}$ converges to a fixed point of K. □

For the next result, we need the following proposition:

Proposition 3.1. [8] *Let $\mathfrak{U}$ be a nonempty closed subset of a Banach space. Let $\{\mathfrak{z}_n\}$ be a Fejer-monotone sequence with respect to $\mathfrak{U}$. Then, $\{\mathfrak{z}_n\}$ converges (strongly) to the point of $\mathfrak{U}$ if and only if $\lim\limits_{n\to\infty} dist(\mathfrak{z}_n, \mathfrak{U}) = 0$.*

Theorem 3.4. *Let V be a UCBS and $\emptyset \neq \mathfrak{U} \subseteq V$ be closed and convex. Assume that a self-map K of $\mathfrak{U}$ is with (KSC)-condition and $Fix(K) \neq \emptyset$. If $\{\mathfrak{z}_n\}$ denotes a sequence of M-iteration process (1.7) and $\liminf_{n\to\infty} dist(\mathfrak{z}_n, Fix(K)) = 0$. Eventually, the sequence $\{\mathfrak{z}_n\}$ converges strongly to a fixed point of K.*

Proof. Since $||\mathfrak{z}_{n+1}-\mathfrak{p}|| \leq ||\mathfrak{z}_n-\mathfrak{p}||$ for any fixed point $\mathfrak{p}$, it follows that $\lim\limits_{n\to\infty} dist(\mathfrak{z}_n, Fix(K))$ exists. The given condition now gives that $\lim\limits_{n\to\infty} dist(\mathfrak{z}_n, Fix(K)) = 0$. But the fixed point set is closed here and $\{\mathfrak{z}_n\}$ is Fejer-monotone by Lemma 2.2. Eventually, Proposition 3.1 gives that $\{\mathfrak{z}_n\}$ is strongly convergent to a fixed point of K. $\square$

Theorem 3.5. *Let V be a UCBS and $\emptyset \neq \mathfrak{U} \subseteq V$ is closed and convex. If K be a self-mapping defined on $\mathfrak{U}$ satisfying (KSC)-condition with $Fix(K) \neq \emptyset$, then M-iteration process (1.7) converges strongly to a fixed point of K as long as K satisfies the condition (I).*

Proof. To prove this result, we shall apply Theorem 3.4. From Theorem 3.1, one has $\liminf_{n\to\infty} ||K\mathfrak{z}_n - \mathfrak{z}_n|| = 0$. The given condition (I) associated with the map K, gives $\liminf_{n\to\infty} d(\mathfrak{z}_n, Fix(K)) = 0$. Accordingly, all the conditions of Theorem 3.4 are proved, we have $\{\mathfrak{z}_n\}$ is strongly convergent to a fixed point of K. $\square$

4. Numerical example

Now, we prove that $\{\mathfrak{z}_n\}$ generated by the M-iteration process converges faster than some other well-known iterative processes.

Example 4.1. Define a mapping K on $[7, 9]$ as follows:

$$K\mathfrak{z} = \begin{cases} \dfrac{\mathfrak{z}+42}{7}, & \text{if } \mathfrak{z} \in [7,9), \\ 6, & \text{if } \mathfrak{z} = 9. \end{cases}$$

It can be seen that K fails to satisfy condition (C) at $\mathfrak{z} = 8$ and $y = 9$. For the (KSC)-condition, we proceed as follows:

Case I. When $\mathfrak{z}, y \in [7, 9)$, we have $K\mathfrak{z} = \frac{\mathfrak{z}+42}{7}$, $Ky = \frac{y+42}{7}$

$$\begin{aligned}
\frac{1}{2}\left(\|\mathfrak{z} - K\mathfrak{z}\| + \|y - Ky\|\right) &= \frac{1}{2}\left(\|\mathfrak{z} - \frac{\mathfrak{z}+42}{7}\| + \|y - \frac{y+42}{7}\|\right) \\
&= \frac{1}{2}\left(\|\frac{6\mathfrak{z}-42}{7}\| + \|\frac{6y-42}{7}\|\right) \\
&= \frac{1}{2}\left(\frac{6}{7}\|(\mathfrak{z}-7)\| + \frac{6}{7}\|(y-7)\|\right) \\
&= 3\left(\frac{1}{7}(\|(\mathfrak{z}-7)\| + \|(y-7)\|)\right) \\
&\geq 3\left(\frac{1}{7}(\|(\mathfrak{z}-7+y-7)\|)\right) \\
&= 3\left(\frac{1}{7}(\|(\mathfrak{z}+y-14)\|)\right)
\end{aligned}$$

$$\begin{aligned}&\geq \left(\frac{1}{7}(\|(\mathfrak{z}-y)\|)\right)\\&=\|K\mathfrak{z}-Ky\|.\end{aligned}$$

Case II. When $\mathfrak{z}, y = 9$, we have $K\mathfrak{z} = 6$, $Ky = 6$

$$\frac{1}{2}\left(\|\mathfrak{z}-K\mathfrak{z}\|+\|y-Ky\|\right)=\frac{1}{2}\left(\|6-6\|\right)\geq\|K\mathfrak{z}-Ky\|.$$

Case III. When $\mathfrak{z} \in [7, 9)$, $y = 9$, we have $K\mathfrak{z} = \frac{\mathfrak{z}+42}{7}$, $Ky = 6$

$$\begin{aligned}\frac{1}{2}\left(\|\mathfrak{z}-K\mathfrak{z}\|+\|y-Ky\|\right)&=\frac{1}{2}\left(\|\mathfrak{z}-\frac{\mathfrak{z}+42}{7}\|+\|9-6\|\right)\\&\geq\frac{1}{2}(\|9-6\|)\\&=\frac{3}{2}\\&>\frac{\mathfrak{z}}{7}\,\forall\,\mathfrak{z}\in[7,9]\\&=\|K\mathfrak{z}-Ky\|.\end{aligned}$$

Case IV. When $y \in [7, 9)$, $\mathfrak{z} = 9$, then we have $Ky = \frac{y+42}{7}$, $K\mathfrak{z} = 6$

$$\begin{aligned}\frac{1}{2}\left(\|\mathfrak{z}-K\mathfrak{z}\|+\|y-Ky\|\right)&=\frac{1}{2}\left(\|9-6\|+\|y-\frac{y+42}{7}\|\right)\\&\geq\frac{1}{2}\|9-6\|\\&=\frac{3}{2}\\&>\frac{y}{7}\,\forall\,y\in[7,9]\\&=\|K\mathfrak{z}-Ky\|.\end{aligned}$$

In the numerical example, we set $\alpha_n = 0.75$, $\beta_n = 0.65$, $\gamma_n = 0.65$, and the initial value $\mathfrak{z}_1 = 8$ for all the considered iteration schemes, i.e., M, Khan, Agarwal, and Noor iterations. The stopping criterion is defined as $||x_n - x_{n+1}|| < 10^{-7}$. The results are presented in Table 1 and Figure 2.

The numerical results in Table 1 illustrate the performance of the M-iteration process in comparison with the Khan, Agarwal, Noor, and Mann iterative schemes. The primary objective is to analyze the convergence behavior of each method and determine the efficiency of the M-iteration process in estimating the fixed point of K. From the table, we observe that after the first iteration, the M-iteration (7.0072886) provides a closer approximation to the fixed point (i.e., 7) than the other methods. Specifically, it converges to the fixed point at the 4th iteration, while the Khan and Agarwal methods require additional iterations to reach the same accuracy. In contrast, the Noor and Mann methods exhibit relatively slower convergence, requiring 12 and 14 iterations, respectively, to reach the fixed point.

In Table 2, we present the results obtained for various starting points $\mathfrak{z}_1$ with fixed parameter values: $\alpha_n = 0.75$, $\beta_n = 0.65$, $\gamma_n = 0.65$, $\varepsilon = 10^{-7}$. From these

Table 1. Numerical results produced by M, Khan Agarwal and Noor iterative schemes for K of the Example 4.1.

n	M iteration	Khan iteration	Agarwal iteration	Noor iteration	Mann iteration
1	8	8	8	8	8
2	7.0072886	7.0510200	7.0831632	7.2919060	7.3571430
3	7.0000531	7.0026030	7.0069161	7.0852090	7.1275510
4	**7**	7.0001330	7.0005751	7.0248730	7.0455540
5	7	7.0000070	7.0000480	7.0072605	7.0162690
6	7	**7**	7.0000040	7.0021190	7.0058100
7	6	7	7	7.0006186	7.0020750
8	7	7	7	7.0001810	7.0007410
9	7	7	7	7.0000530	7.0002650
10	7	7	7	7.0000150	7.0000950
11	7	7	7	7.0000040	7.0000340
12	7	7	7	7.0000010	7.0000120
13	7	7	7	**7**	7.0000040
14	7	7	7	7	7.0000020
14	7	7	7	7	7.0000010
15	7	7	7	7	**7**

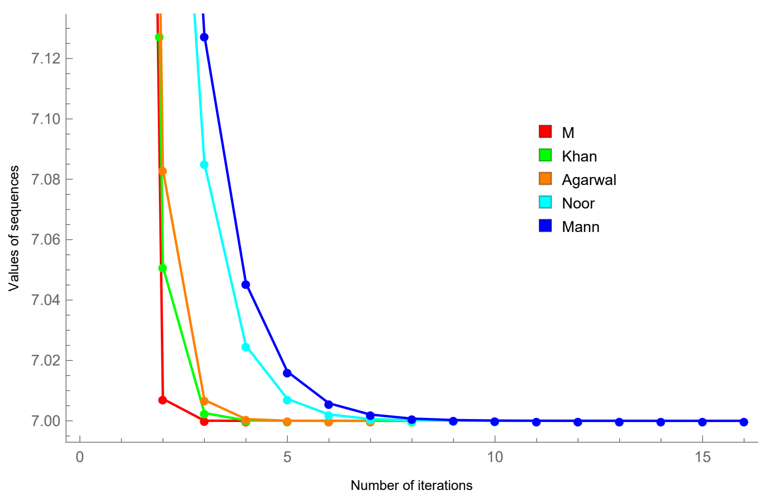


Figure 1. Convergence behavior of M (1.7), Khan (1.4), Agarwal (1.5), Noor (1.6), and Mann (1.3) iteration processes corresponding to Table 1.

results, we observe that for the initial starting point $\mathfrak{z}_1 = 7.3$, the M-iteration process approximates the fixed point in just 2 iterations. For different starting points, the number of iterations required by the M-iteration process varies between 2 and 3. The Khan iteration requires 3 iterations to reach the fixed point when $\mathfrak{z}_1 = 7.3$, and while the number of iterations fluctuates slightly for different starting points,

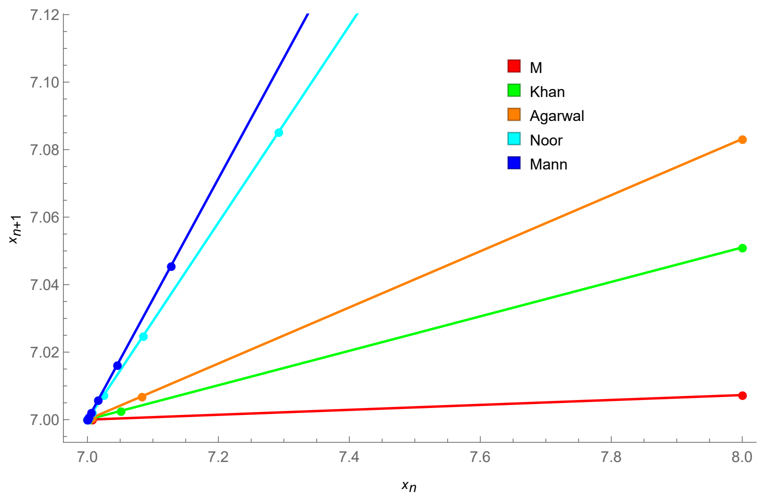


Figure 2. Graphical analysis of iterates of iteration processes corresponding to Table 1.

Table 2. Impact of the starting point $\mathfrak{z}_1$ on the number of performed iterations for different iteration processes.

$\mathfrak{z}_1$	M iteration	Khan iteration	Agarwal iteration	Noor iteration	Mann iteration
7.3	2	3	4	8	9
7.6	2	4	4	8	10
7.9	2	4	4	8	10
8.2	3	4	5	9	10
8.5	3	4	5	9	11
8.8	3	4	5	9	11

it remains stable for values between 7.6 and 8.8. The Agarwal iteration requires 4 iterations when $\mathfrak{z}_1$ but varies between 4 and 5 iterations for different starting points. The Noor and Mann iteration processes require 8 and 9 iterations, respectively, to find the fixed point when $\mathfrak{z}_1 = 7.3$. Notably, the Mann iteration process requires significantly more iterations compared to the other methods.

5. Comparison via polynomiography

Polynomiography is both a visual analysis technique for root-finding methods and a digital art form, introduced by mathematician and computer scientist Bahman Kalantari [15]. It focuses on the visualization of complex polynomials, often employing iterative algorithms and mathematical principles. The term polynomiography is derived from polynomial and graph, emphasizing its graphical representation of polynomial functions. Polynomiographic methods are widely utilized for comparing and analyzing various iterative processes (see, for example, [11, 25–27, 30, 39, 42]). This approach enables the graphical representation of convergence behavior in iterative root-finding methods. The roots of polynomials are approximated through

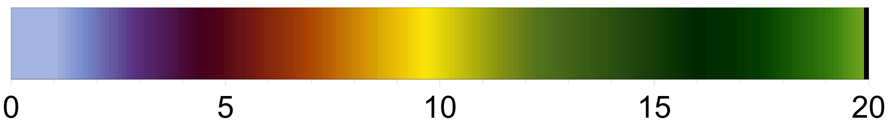


Figure 3. Color map used in the examples.

iteration functions, and polynomiography operates by employing an infinite number of such functions. A well-known example of a root-finding algorithm used in this context is Newton's method (also known as the Newton-Raphson method).

In this section, we embed the well-known Newton's iterative process [3] within the M-iteration framework, along with some classical iterative methods from the literature, to generate various basins of attraction. Let $p(t_n)$ be a complex polynomial. Note that for any initial value $t_0 \in \mathbb{C}$, Newton's iterative process is given by:

$$t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)} \quad \text{for} \quad (n = 0, 1, 2, 3...).$$

Here, $p'(t_n)$ denotes the first derivative of $p(t_n)$. Newton's iterative process can now be reformulated as a fixed-point iteration as follows:

$$t_{n+1} = N(t_n). \tag{5.1}$$

If the iterative process given in (5.1) converges to a fixed point t of N, then we have:

$$t = N(t) = t - \frac{p(t)}{p'(t)}. \tag{5.2}$$

If $\frac{p(t)}{p'(t)} = 0$, then $p(t) = 0$. Equation (5.2) implies that $t = N(t)$, which means that t is a root of $p(t)$. The set of all initial points t_0 that converge to the same root forms a basin of attraction. Instead of using the Picard iteration, we can apply other iterative processes, such as the Mann iteration introduced earlier or those defined in Sec. 1 for different values of α_n, β_n, and γ_n. We choose a grid of length $B = [-8.0, 8.0]^2$ and set $N = 20$, where N represents the number of iterations. By applying Newton's operator within the M, Mann, Khan, Agarwal, and Noor iterative processes, we generate a complex sequence $\{t_n\}$, starting from each grid point t_0. Suppose t_0 is the initial guess; if the sequence $\{t_n\}$ converges to a root with an accuracy of 0.001, we assign a specific color to t_0. If $\{t_n\}$ does not converge to any root, we assign it a green color. The set of allt_0 that converge to the same root forms a basin of attraction. We use the colormap presented in Figure 3.

To generate polynomiographs, we use the algorithm presented as a pseudocode in Algorithm 1.

In the considered example, polynomiographs were generated for the polynomial $p(t) = t^4 - 1$ using three different sets of iteration parameter values: (i) $\alpha = 0.03$, $\beta = 0.03$, $\gamma = 0.03$, (ii) $\alpha = 0.4$, $\beta = 0.4$, $\gamma = 0.4$, (iii) $\alpha = 0.7$, $\beta = 0.7$, $\gamma = 0.7$.

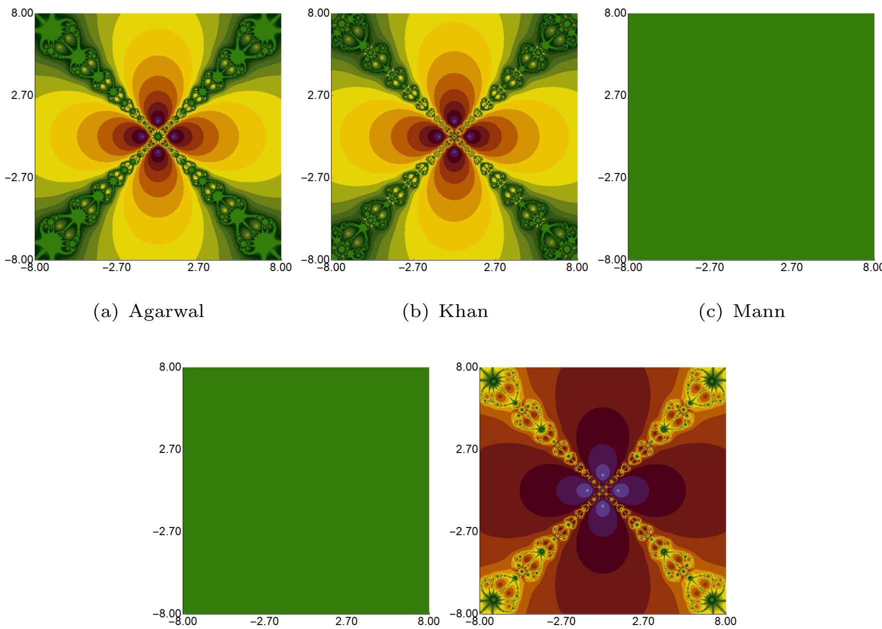
The generated polynomiographs are shown in Figs. 4–6. The obtained Average Number of Iterations (ANI) values from the polynomiographs are shown in Table 3. The Average Number of Iterations (ANI) is a key metric for evaluating the efficiency of iteration processes in reaching convergence. It represents the mean number of iterations required for a iteration process to approximate a root within a given

Algorithm 1: Generation of a polynomiograpgh.

Input: $p \in \mathbb{C}[Z]$, $\deg p \geq 2$ – polynomial; I – iteration process; $A \subset \mathbb{C}$ – area; N – the maximum number of iterations; ε – accuracy; *colours* – colour map.

Output: Polynomiograph for the complex-valued polynomial p within the area A.

for $t_0 \in A$ **do**
- $n = 0$
- **while** $|p(t_n)| > \varepsilon$ **and** $n < N$ **do**
  - $t_{n+1} = I(t_n, p)$
  - $n = n + 1$
- Map n to a colour from the colour map *colours* and colour t_0

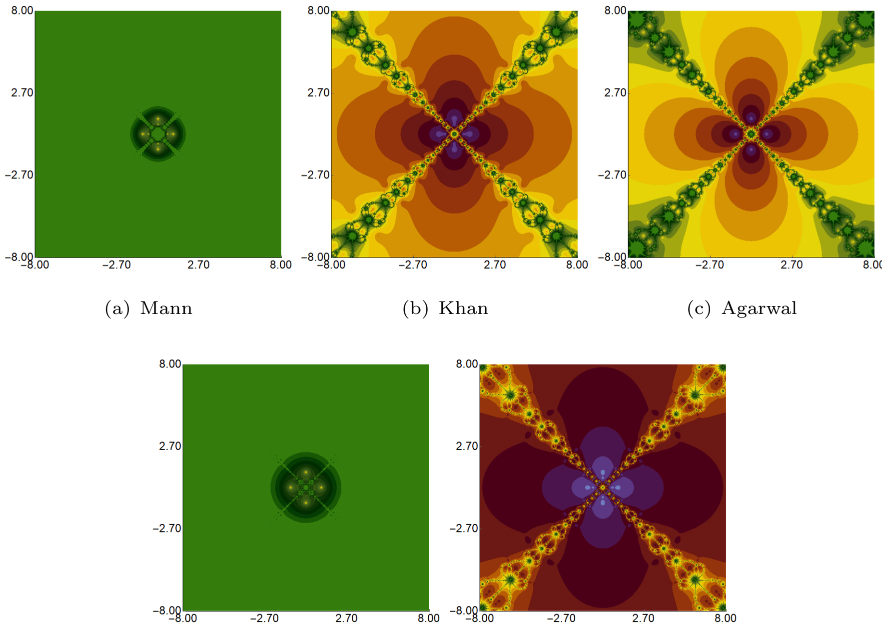


(a) Agarwal (b) Khan (c) Mann (d) Noor (e) M

Figure 4. Polynomiographs generated by various iteration processes with parameters $\alpha = \beta = \gamma = 0.03$.

tolerance. A lower ANI value indicates faster convergence, meaning the iterative process reaches a solution in fewer steps.

For low parameter values (Figure 4), we observe that two of the iterative processes fail to converge to any of the four roots of the polynomial $p(t)$. This is indicated by a uniform green color, which corresponds to the maximum iteration limit of 20. For the remaining three iterative processes, we notice different convergence speeds. Based on visual analysis, the fastest convergence is achieved by the M-iteration process, followed by the Khan and Agarwal iterative processes. These observations are confirmed by the ANI values in Table 3, where the lowest ANI



(a) Mann (b) Khan (c) Agarwal

(d) Noor (e) M

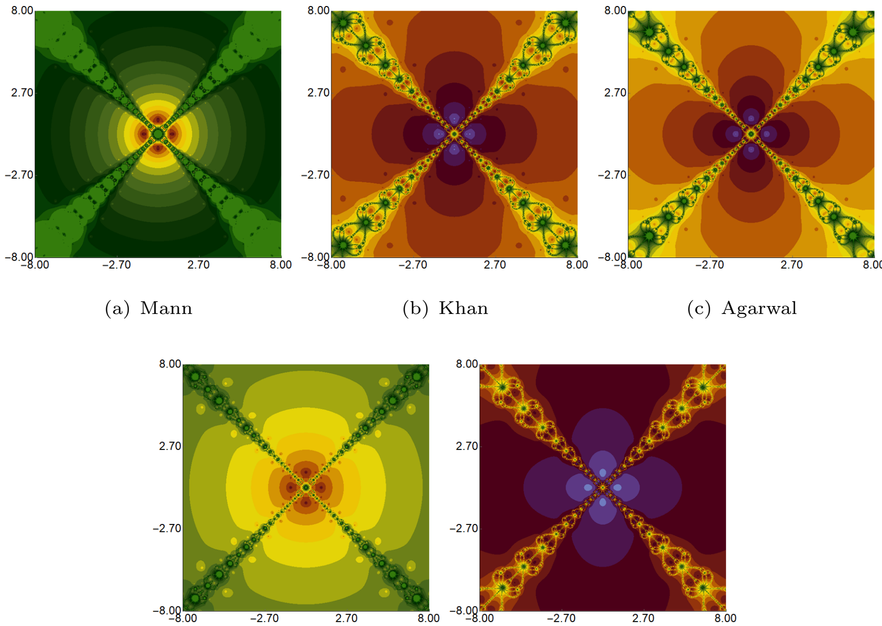
Figure 5. Polynomiographs generated by various iteration processes with parameters $\alpha = \beta = \gamma = 0.4$.

Table 3. ANI values calculated from polynomiographs presented in Figures 4, 5 and 6.

Iteration	$\alpha = \beta = \gamma = 0.03$	$\alpha = \beta = \gamma = 0.4$	$\alpha = \beta = \gamma = 0.7$
M	3.122	2.566	2.418
Khan	5.553	4.08	3.594
Noor	20	16.4	7.432
Agarwal	6.028	4.923	4.066
Mann	20	17.4	9.292

value of 3.122 is obtained by the M-iteration process, followed by Khan (5.553) and Agarwal (6.028). Similarly, in Figure 5, the lowest ANI value (2.556) is again observed for the M-iteration process, demonstrating its strong convergence compared to Khan (4.08), Agarwal (16.4), Noor (16.4), and Mann (17.4). In Figure 6, the M-iteration process once again proves to be the most efficient, with the lowest ANI value (2.418) compared to Khan (3.594), Agarwal (4.066), Noor (7.432), and Mann (9.292).

In the present analysis, the ANI values serve as a quantitative measure for comparing the efficiency of different iteration processes. The M-iteration process consistently yields the lowest ANI values across all cases, indicating its superior rate of convergence. In contrast, the higher ANI values associated with methods such as Noor and Mann suggest slower convergence or, in some instances, potential divergence. These numerical findings are consistent with the visual patterns observed in the polynomiographs, further validating ANI as a reliable metric for assessing the



(a) Mann (b) Khan (c) Agarwal

(d) Noor (e) M

Figure 6. Polynomiographs generated by various iteration processes with parameters $\alpha = \beta = \gamma = 0.7$.

performance of iterative methods.

6. Application to fractional delay differential equations

Fractional calculus plays an important role in physics, engineering, and control systems to analyze their working phenomena. Fractional calculus formulate models of engineering systems that are far better than developed by ordinary derivatives approaches. Fractional differential equations (FDEs) are used in electrical networks to model circuits containing capacitors, inductors, and resistors, particularly when non-integer order dynamics are present. In control systems, FDEs play a crucial role in robust control theory, helping design controllers that enhance stability and performance in uncertain or complex systems. Additionally, FDEs are applied in image and audio processing for edge detection and noise reduction. FDEs also have applications in fluid dynamics, where they are used to model anomalous diffusion and turbulence. In physics, they are employed in wave propagation models and dielectric material analysis. In biology, FDEs are utilized for modeling brain signal processing and memory-dependent neuronal activities. They also aid in understanding the spread of diseases with memory effects, such as COVID-19 dynamics. In chemistry, FDEs are used to model chemical reactions and diffusion processes in heterogeneous media. Furthermore, in fractal and chaos theory, FDEs are instrumental in modeling self-similar structures and complex patterns found in nature (see, e.g., [13, 14, 21, 23, 24, 31, 32]).

Various researchers have attempted to find numerical solutions for fractional differential equations (see, e.g., [4,5,41]). In this section, we approximate the solution of the following fractional delay differential equation in the Caputo sense using the M-iteration process (1.7).

Eventually, the DFDE is

$$^{c}Dg(\zeta) = h(\zeta, \zeta(g), g(\zeta - \sigma)),\ \zeta \in [e, B], \tag{6.1}$$

with initial conditions

$$g(\zeta) = \psi(\zeta),\ \ \zeta \in [e - \mu, e], \tag{6.2}$$

where the constant σ is stand for time delay, $\sigma > 0$, $B > 0$, $\mu > 0$, $g \in R^k$, $\psi \in C([e - \sigma, e] : R^k)$ and $h : [e, B] \times R^k \times R^k \to R^k$ is essentially a continuous function.

Some conditions are needed as follows that must be hold.

$(K1)$: One is able to select a real number $L_h > 0$ with

$$\|h(\zeta, x_1, y_1) - h(\zeta, x_2, y_2)\| \leq L_h(\|x_1 - y_1\| + \|x_2 - y_2\|),\ \forall\, x_1, x_2, y_1, y_2 \in R^k.$$

$(K2)$: One can find a constant, namely, $\delta_L > 0$ with the property that $\frac{2L}{\delta_L} < 1$ if $\mathfrak{M} \in (C([e - \sigma, B] : R^k) \bigcap C^1([e, B] : R^k))$ is a function satisfying (6.1) and (6.2), then $\mathfrak{M}$ is called a solution to problems (6.1) and (6.2).

In [18], the authors proved that the solution to the problem (6.1) and (6.2) is equivalent to the solution of the following integral equation:

$$t(\zeta) = \psi(e) + \frac{1}{\Gamma(\gamma)} \int_e^{\zeta} (\zeta - e)^{\gamma - 1} h(\mu, g(\mu), g(\mu - \sigma)) d\mu,\ \forall\, \zeta \in [e, B], \tag{6.3}$$

where $t(\zeta) = \psi(\zeta), \forall \zeta \in [e - \mu, e]$ and $\Gamma(\gamma) = \int_0^{\infty} e^{-x}\ x^{\gamma - 1} dx$ The solution set of (6.3) and problems (6.1) and (6.2) are same. Let us define the norm $\|.\|_{\delta_L}$ on $(C([e - \sigma, B] : R^k)$ as

$$\|\psi\|_{\delta_L} = \frac{sup\|\psi(\zeta)\|}{E_{\gamma}(\delta_L \zeta_{\gamma})}\ \forall \psi \in C([e - \sigma, e] : R^k), \tag{6.4}$$

where the notation E_{γ} stand for Mittag-Leffler function and it is reads as follows:

$$E_{\gamma}(\zeta) : \sum_{k=0}^{\infty} \frac{\zeta_k}{\Gamma(\gamma k + 1)}\ \forall \zeta \in R.$$

Clearly, $(C([e - \sigma, B] : R^k),\ \|.\|_{\delta_L})$ is a Banach space [1].

In [40], the authors from condition $(K1)$, proved that the solution of problem (6.1) and (6.2) exists and it is also unique. Now, we utilize the M-iteration process (1.7) to appromate the solution of problem (6.1) and (6.2). The main result in this section is given as follows:

Theorem 6.1. *Suppose that assumptions $(K1)$ and $(K2)$ hold true. Then the sequence defined by (1.7) converges to a unique solution $\mathfrak{M}$ of (6.3) in $G = C([e - \sigma, B] : R^k) \bigcap C^1([e, B] : R^k)$.*

Proof. Define an operator K as

$$Kg(\zeta) = \begin{cases} \psi(e) + \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_e^\zeta (\zeta - e)^{\gamma-1} h(\mu, g(\mu), g(\mu - \sigma)) d\mu, & \text{if } \zeta \in [e, B], \\ \psi(\zeta), \quad \text{if } \zeta \in [e - \mu, e]. \end{cases} \tag{6.5}$$

We need to show that $\mathfrak{z}_n \to \mathfrak{M}$ as $n \to \infty$. We must distinguish two cases:

Case 1. If $\zeta \in [e - \mu, e]$, then obviously $\mathfrak{z}_n \to \mathfrak{M}$ when $n \to \infty$.

Case 2. If $\zeta \in [e, B]$, then using (1.7), Lemma 3.1 and assumptions $(K1)$ and $(K2)$, we have

$$\begin{aligned} ||\mathfrak{z}_{n+1} - \mathfrak{M}|| &= ||(1 - \alpha_n) K\mathfrak{z}_n + \alpha_n K u_n - \mathfrak{M}|| \\ &\leq (1 - \alpha_n)||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n ||K\mathfrak{z}_n - \mathfrak{M}||. \end{aligned} \tag{6.6}$$

Taking the supremum on $[e - \sigma, B]$ on both sides, we have

$$\begin{aligned} \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_{n+1} - \mathfrak{M}|| &\leq \sup_{\zeta \in [e-\sigma, B]} ((1 - \alpha_n)||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n ||K\mathfrak{z}_n - K\mathfrak{M}||) \\ &\leq (1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n \sup_{\zeta \in [e-\sigma, B]} ||K\mathfrak{z}_n - K\mathfrak{M}||. \end{aligned}$$

Using (6.5), we get

$$\begin{aligned} \leq &(1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n \sup_{\zeta \in [e-\sigma, B]} (||\psi(e) \\ &+ \frac{1}{\Gamma(\gamma)} \int_e^\zeta (\zeta - e)^{\gamma-1} h(\mu, \mathfrak{z}_n(\mu), \mathfrak{z}_n(\mu - \sigma)) d\mu \; - \psi(e) \\ &- \frac{1}{\Gamma(\gamma)} \int_e^\zeta (\zeta - e)^{\gamma-1} h(\mu, \mathfrak{M}(\mu), \mathfrak{M}(\mu - \sigma)) d\mu||) \\ \leq &(1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - q|| + \alpha_n \sup_{\zeta \in [e-\sigma, B]} \frac{1}{\Gamma(\gamma)} \int_e^\zeta (\zeta - e)^{\gamma-1} \\ &\times ||h(\mu, \mathfrak{z}_n(\mu), \mathfrak{z}_n(\mu - \sigma)) - h(\mu, \mathfrak{M}(\mu), \mathfrak{M}(\mu - \sigma))|| d\mu. \end{aligned}$$

Using assumption $K1$, we obtained

$$\begin{aligned} \leq &(1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n \sup_{\zeta \in [e-\sigma, B]} \frac{1}{\Gamma(\gamma)} \int_e^\zeta (\zeta - e)^{\gamma-1} d\mu \\ &\times L_h(\|\mathfrak{z}_n - \mathfrak{M}(\mu)\| + \|\mathfrak{z}_n(\mu - \sigma) - \mathfrak{M}(\mu - \sigma)\|) \\ \leq &(1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - \mathfrak{M}|| + \alpha_n \frac{1}{\Gamma(\gamma)} \int_e^w (w - e)^{\gamma-1} d\mu \\ &\times L_h(\sup_{w \in [e-\sigma, B]} \|\mathfrak{z}_n - \mathfrak{M}(\mu)\| + \sup_{\zeta \in [e-\sigma, B]} \|\mathfrak{z}_n(\mu - \sigma) - \mathfrak{M}(\mu - \sigma)\|). \end{aligned}$$

Dividing by $E_\gamma(\delta_L \zeta_L)$

$$\begin{aligned} &\frac{\sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_{n+1} - \mathfrak{M}||}{E_\gamma(\delta_L \zeta_L)} \\ \leq &\frac{(1 - \alpha_n) \sup_{\zeta \in [e-\sigma, B]} ||\mathfrak{z}_n - \mathfrak{M}||}{E_\gamma(\delta_L \zeta_L)} \end{aligned}$$

$$+\alpha_n \frac{1}{\Gamma(\gamma)}(\int_e^{\zeta}(\zeta-e)^{\gamma-1}d\mu)\times L_h(\frac{\sup_{\zeta\in[e-\sigma,B]}\|\mathfrak{z}_n-\mathfrak{M}(\mu)\|}{E_\gamma(\delta_L\zeta_L)}$$
$$+\frac{\sup_{\zeta\in[e-\sigma,B]}\|\mathfrak{z}_n(\mu-\sigma)-\mathfrak{M}(\mu-\sigma)\|}{E_\gamma(\delta_L\zeta_L)}).$$

Using (6.4), we get

$$\begin{aligned}
&\|\mathfrak{z}_{n+1}-\mathfrak{M}\|_{\delta_L}\\
\leq&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\alpha_n\frac{1}{\Gamma(\gamma)}\int_e^{\zeta}(\zeta-e)^{\gamma-1}d\mu\\
&\times\ L_h(\|\mathfrak{z}_n-\mathfrak{M}(\mu)\|_{\delta_L}+\|\mathfrak{z}_n(\mu-\sigma)-\mathfrak{M}(\mu-\sigma)\|_{\delta_L})\\
\leq&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\alpha_n\frac{1}{\Gamma(\gamma)}\int_e^{\zeta}(\zeta-e)^{\gamma-1}d\mu\\
&\times\ 2L_h\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}\\
=&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\frac{\alpha_n 2L_h}{E_\gamma(\delta_L\zeta_L)}\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L},\\
&\frac{1}{\Gamma(\gamma)}\int_e^{\zeta}(\zeta-e)^{\gamma-1}E_\gamma(\delta_L\zeta_L)d\mu\\
=&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\frac{\alpha_n 2L_h}{E_\gamma(\delta_L\zeta_L)}\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L},\\
&{}^c\mathcal{I}^{\bigcirc}({}^cD\frac{E_\gamma(\delta_L\zeta_L}{\delta_L})\\
=&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\frac{\alpha_n 2L_h}{E_\gamma(\delta_L\zeta_L)}\frac{E_\gamma(\delta_L\zeta_L}{\delta_L}\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}\\
=&(1-\alpha_n)\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}+\frac{\alpha_n 2L_h}{\delta_L}\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}.
\end{aligned}$$

Using assumption $K2$, we get

$$\|\mathfrak{z}_{n+1}-\mathfrak{M}\|_{\delta_L}\leq\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}.$$

Put $\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}=\wp_n$, then

$$\wp_{n+1}\leq\wp_n\ \forall\ n\in\mathcal{N}.$$

$\Rightarrow\{\wp_n\}$ is a sequence of real numbers with monotone decreasing characteristics. Moreover it is bounded as well, so we can conclude that

$$\lim_{n\to\infty}\wp_n=\inf\{\wp_n\}=0$$
$$\Rightarrow\|\mathfrak{z}_n-\mathfrak{M}\|_{\delta_L}=0.$$

So $\{\mathfrak{z}_n\}$ converges to $\mathfrak{M}$. □

7. Conclusion

In this research article, the three-step M-iteration process is employed to estimate fixed points of mappings satisfying the (KSC)-condition. We establish weak and strong fixed-point convergence results for such mappings. Additionally, two new examples are provided to demonstrate that the (KSC)-condition is more general than condition (C). Several polynomiographs are generated for different settings of iteration parameters, and ANI values are computed to compare their convergence speed. Furthermore, a comparative numerical simulation is conducted to support our main findings. Finally, an application in a class of fractional differential equations is discussed to highlight the significance of the M-iteration process.

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