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NUMERICAL METHOD FOR SOLVING PSEUDO-HYPERBOLIC EQUATIONS WITH PURELY INTEGRAL CONDITIONS IN REPRODUCING KERNEL HILBERT SPACE

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Abstract This paper studies a pseudo-hyperbolic equation with purely integral conditions using the reproducing kernel Hilbert space method (RKHSM). By leveraging the properties of reproducing kernel functions (RKFs), we derive exact and approximate solutions to the equation. We present three numerical examples to assess our approach's efficiency and accuracy. The results demonstrate that the RKHSM yields highly accurate approximations, underscoring its effectiveness as a reliable method for solving pseudo-hyperbolic equations with integral constraints. Our findings contribute to the growing research on analytical and numerical techniques for solving such equations.

Keywords Partial differential equations, approximate solutions, Gram-Schmidt orthogonalization process, computational techniques.

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1. Introduction

Differential equations serve as a fundamental mathematical framework for modeling various dynamic processes across multiple disciplines. These equations describe relationships involving unknown functions and their derivatives, making them essential for capturing the evolution of physical, biological, and financial systems over time. Among them, ordinary differential equations (ODEs) are widely used to represent processes where changes depend on a single independent variable [29], such as population dynamics, mechanical vibrations, and economic systems. Notably, systems of ODEs play a crucial role in financial modeling, where they help analyze the stability and dynamics of economic variables [21].

On the other hand, partial differential equations (PDEs) extend this concept by involving multiple independent variables, making them indispensable for modeling complex physical and engineering phenomena. PDEs are particularly useful for describing heat diffusion, wave propagation, and fluid and gas dynamics, including water flow and air movement. Many mathematicians have extensively explored these topics, particularly in the context of modern physics and technology. For example, in [17], Bouziani examined a class of nonclassical hyperbolic equations with nonlocal conditions. In [36], Merad and Bouziani applied the Laplace transform to solve pseudo-parabolic equations with nonlocal conditions. Finally, in [37], Merad and Bouziani investigated the solvability of the telegraph equation with purely integral conditions. Recent studies on nonlocal conditions have addressed boundary data that cannot be directly measured.

Pseudo-hyperbolic equations play a crucial role in physics by describing a wide range of physical phenomena. For example, in [25], Guo and Rui developed least-squares Galerkin procedures for pseudo-hyperbolic equations. In [33], Liu et al. proposed splitting positive definite mixed element methods for solving pseudo-hyperbolic equations. Finally, in [34], Liu et al. introduced a new splitting 1-Galerkin mixed method for pseudo-hyperbolic equations.. Recent research highlights their capability to model the dynamics of propagating waves in striated and rotating fluids, nerve conduction and reaction-diffusion processes, as well as applications in heat and mass transfer, engineering, and mathematical biology. In [8], Aronszajn introduced the theory of reproducing kernels, which is fundamental to many areas of mathematical analysis. In [40], Nagumo et al. developed an active pulse transmission line simulating nerve axons. In [7], Arima and Hasegawa studied global solutions for mixed problems of a semi-linear differential equation. In [42], Pao investigated a mixed initial boundary value problem arising in neurophysiology. In [43], Ponce studied the global existence of small solutions to a class of nonlinear evolution equations. Finally, in [18], Bouziani and Benouar addressed a mixed problem with integral conditions for a third-order parabolic equation.

Pseudo-hyperbolic equations with purely integral conditions are essential for understanding and modeling heat distribution, wave motion, and fluid dynamics. The study of such equations has gained significant attention in recent years [38, 44]. This study focuses on obtaining approximate solutions for pseudo-hyperbolic equations with purely integral conditions.

The general form of a pseudo-hyperbolic equation with purely integral conditions is given as follows:

Let $\tau > 0$, and define the domain as $\Theta = \{(z, l) \in \mathbb{R}^2 \mid 0 < z < 1,\ 0 < l \leq \tau\}$.

We seek a function $u : \Theta \to \mathbb{R}$ that satisfies the equation [38]:

$$\frac{\partial^2 u}{\partial l^2} - \alpha \frac{\partial^2 u}{\partial z^2} - \beta \frac{\partial^3 u}{\partial l \partial z^2} = g(z,l), \quad 0 < z < 1,\ 0 < l \leq \tau, \tag{1.1}$$

subject to the initial conditions:

$$u(z,0) = \varphi(z), \quad 0 < z < 1, \tag{1.2}$$

$$\frac{\partial u(z,0)}{\partial l} = \psi(z), \quad 0 < z < 1, \tag{1.3}$$

and the purely integral conditions:

$$\int_0^1 u(z,l)\, dz = E(l), \quad 0 < l \leq \tau, \tag{1.4}$$

$$\int_0^1 zu(z,l)\, dz = G(l), \quad 0 < l \leq \tau, \tag{1.5}$$

where $u : \Theta \to \mathbb{R}$ is the unknown function, and $g : \Theta \to \mathbb{R}$ is a sufficiently smooth function. The functions $\varphi, \psi : [0,1] \to \mathbb{R}$ and $E, G : [0,\tau] \to \mathbb{R}$ are given, while α and β are positive constants.

To handle the nonhomogeneous conditions, we transform the problem (1.1)-(1.5) into an equivalent one with homogeneous conditions. We introduce the transformation:

$$v(z,l) = u(z,l) + \mathfrak{S}(z,l), \tag{1.6}$$

where

$$\begin{aligned}\mathfrak{S}(z,l) = &- 6z(E_l(0)l - 2G_l(0)l - E(l) + E(0) + 2G(l) - 2G(0)) \\ &+ 2(2E_l(0)l - 3G_l(0)l - 2E(l) + 2E(0) + 3G(l) - 3G(0)) - l\psi(z) - \varphi(z).\end{aligned} \tag{1.7}$$

Consequently, problem (1.1)-(1.5) can be equivalently transformed into determining the function v that satisfies the following conditions:

$$\begin{cases} \dfrac{\partial^2 v}{\partial l^2} - \alpha \dfrac{\partial^2 v}{\partial z^2} - \beta \dfrac{\partial^3 v}{\partial l \partial z^2} = \mathcal{H}(z,l), & 0 < z < 1,\ 0 < l < \tau, \\ v(z,0) = 0, & 0 < z < 1, \\ \dfrac{\partial v(z,0)}{\partial l} = 0, & 0 < z < 1, \\ \displaystyle\int_0^1 v(z,l) dz = 0, & 0 < l < \tau, \\ \displaystyle\int_0^1 zv(z,t) dz = 0, & 0 < l < \tau, \end{cases} \tag{1.8}$$

where

$$\mathcal{H}(z,l) = g(z,l) + \frac{\partial^2 \mathfrak{S}(z,l)}{\partial l^2} - \alpha \frac{\partial^2 \mathfrak{S}(z,l)}{\partial z^2} - \beta \frac{\partial^3 \mathfrak{S}(z,l)}{\partial l \partial z^2}. \tag{1.9}$$

Numerical methods are widely applied to solve a variety of differential equations, including ordinary and partial differential equations, both classical and fractional derivatives. These methods are essential for understanding complex systems in fields such as physics, biology, and finance, providing approximations where analytical solutions are not feasible. Techniques like Tikhonov regularization, Chebyshev series, and B-spline methods have been effectively used to address different types of equations in these domains [5, 11, 22, 30, 35, 45, 47]. This study applies the RKHSM to approximate solutions for pseudo-hyperbolic equations with purely integral conditions. The RKHSM is a powerful numerical and analytical technique for solving a wide range of ordinary and partial differential equations involving various orders of derivatives. One of its key strengths is its ability to generate solutions in the form of rapidly convergent series, with efficiently computable components. Its advantages include (1) the ability to produce globally smooth numerical solutions; (2) uniform convergence of the numerical solutions and their derivatives to the exact solutions; (3) applicability of the numerical solutions and all their derivatives at any arbitrary point within the defined domain; and (4) its mesh-free nature, eliminating the need for time discretization while ensuring ease of implementation. The concept of reproducing kernels originated in the early 20th century with Zaremba [49] and Bergman [15] and has since been widely applied to various types of differential equations and numerical analysis. For instance, in [27], Inc and Akgül applied the reproducing kernel Hilbert space method to solve Troesch's problem. In [4], Akgül et al. presented numerical solutions of fractional differential equations of Lane-Emden type using an accurate technique. In [23], Fardi and Ghasemi solved nonlocal initial-boundary value problems for parabolic and hyperbolic integro-differential equations using the reproducing kernel Hilbert space method. In [31], Li and Wu introduced a new algorithm for solving nonclassical parabolic problems based on the reproducing kernel. In [2], Akgül and Bonyah applied the reproducing kernel Hilbert space method to solve the generalized Kuramoto-Sivashinsky equation. In [28], Jiang and Cui focused on solving nonlinear singular pseudo-parabolic equations with nonlocal mixed conditions in the reproducing kernel space. Finally, in [48], Yang and Lin used the reproducing kernel Hilbert space method to solve linear initial-boundary value problems.

For a better understanding of the RKHSM, including its theoretical background, historical development, modifications, fundamental characteristics, kernel functions, and its orthogonal and orthonormal basis properties, interested readers may refer to [20]. This work provides a comprehensive exploration of the methodology, properties, and practical applications of the RKHS approach.

Several authors have recently addressed problems involving integral conditions using the RKHS approach. For instance, L. Yingzhen and Z. Yongfang [32] applied RKHS to solve nonlinear pseudo-parabolic equations with nonlocal boundary conditions. Additionally, M. Cui, along with F. Geng, developed a method based on RKHSM for solving forced Duffing equations with integral boundary conditions [24]. In [26], Hemati et al. proposed a numerical solution for the multiterm time-fractional diffusion equation using reproducing kernel theory. In [46], Sakar et al. introduced a novel technique for solving the fractional Bagley-Torvik equation. In [10], Abu Arqub et al. applied the reproducing kernel approach to solve fuzzy fractional initial value problems under the Mittag-Leffler kernel differential operator. In [1], Akgül presented a new method for fractional derivatives with a nonlocal and non-singular kernel. In [3], Akgül et al. solved the fractional gas dynamics equation using a new technique. In [6], Allahviranloo and Sahihi used the reproducing kernel method to solve fractional delay differential equations. In [12], Attia et al. studied solutions for the time-fractional advection-diffusion equation using numerical methods. In [13], Azarnavid used Bernoulli polynomials to solve nonlinear Volterra integro-differential equations of fractional order. In [14], Babolian et al. applied the reproducing kernel method to solve Bratu-type fractional order differential equations. In [16], Beyrami and Lotfi introduced a method with error analysis for solving a logarithmic singular Fredholm integral equation. In [19], Chellouf et al. solved fractional differential equations with temporal two-point boundary value problems using the reproducing kernel Hilbert space method. In [24], Geng and Cui developed a new method for solving forced Duffing equations with integral boundary conditions. In [39], Momani et al. investigated Caputo-Fabrizio fractional Riccati and Bernoulli equations using the iterative reproducing kernel method. Finally, in [9], Abu Arqub et al. developed the reproducing kernel Hilbert space algorithm for solving time-fractional nonlocal reaction-diffusion equations.

2. Preliminaries

2.1. Reproducing kernel Hilbert spaces $S_1[0,1], S_2[0,\tau]$ and $\aleph(\Theta)$

To solve equations (1.1)-(1.5), we introduce the following reproducing kernel Hilbert spaces (RKHSs).

1. The function $S_1[0,1]$ is defined as [20]:

$$\begin{aligned} S_1[0,1] =&\{v|\, v, v', v^{(2)} \in AC[0,1], v^{(3)} \in L^2[0,1], \\ &\text{and} \int_0^1 v(z)dz = \int_0^1 zv(z)dz = 0\}, \end{aligned}$$

where AC denotes the space of absolutely continuous functions.
The inner product and norm of this space are defined as follows:

$$\langle v_1, v_2\rangle_{S_1[0,1]} = \sum_{i=0}^{2} v_1^{(i)}(0)v_2^{(i)}(0) + \int_0^1 v_1^{(3)}(z)v_2^{(3)}(z)dz, \tag{2.1}$$

and

$$\|v_1\|_{S_1[0,1]} = \sqrt{\langle v_1, v_1\rangle_{S_1[0,1]}}. \tag{2.2}$$

Theorem 2.1. *The Hilbert space $S_1[0,1]$ is a RKHS with the RKF $m_y(z)$. For each fixed $y \in [0,1]$ and any $v(z) \in S_1[0,1]$,there exists $m_y(z) \in S_1[0,1]$ and $z \in [0,1]$, such that* [20]

$$\langle v(.), m_y(.)\rangle = v(y).$$

The RKF $m_y(z)$ is given by:

$$m_y(z) = \begin{cases} \displaystyle\sum_{i=1}^{6} c_i(y)z^{i-1} + \frac{a_1(y)}{6!}z^6 + \frac{a_2(y)}{7!}z^7, & \text{if } z \le y, \\ \displaystyle\sum_{i=1}^{6} d_i(y)z^{i-1} + \frac{a_1(y)}{6!}z^6 + \frac{a_2(y)}{7!}z^7, & \text{if } y < z, \end{cases} \tag{2.3}$$

with

$$
\begin{aligned}
d_1(y) =& -\frac{28872}{26510995}y^7+\frac{3932}{757457}y^6+\frac{5201}{90894840}y^5+\frac{1236}{757457}y^4+\frac{7976}{757457}y^3\\
&+\frac{23928}{757457}y^2-\frac{29664}{757457}y+\frac{5201}{757457},\\
d_2(y) =&\frac{9516}{5302199}y^7-\frac{23584}{3787285}y^6-\frac{1236}{3787285}y^5-\frac{171665}{18178968}y^4-\frac{48404}{757457}y^3\\
&-\frac{145212}{757457}y^2+\frac{171665}{757457}y-\frac{29664}{757457},\\
d_3(y) =&\frac{13098}{26510995}y^7-\frac{40291}{22723710}y^6+\frac{997}{3787285}y^5+\frac{12101}{1514914}y^4+\frac{608257}{9089484}y^3\\
&+\frac{608257}{3029828}y^2-\frac{145212}{757457}y+\frac{23928}{757457},\\
d_4(y) =&\frac{4366}{26510995}y^7-\frac{40291}{68171130}y^6+\frac{997}{11361855}y^5+\frac{12101}{4544742}y^4-\frac{37300}{6817113}y^3\\
&-\frac{37300}{2272371}y^2-\frac{48404}{757457}y+\frac{7976}{757457},\\
d_5(y) =& -\frac{793}{10604398}y^7+\frac{2948}{11361855}y^6+\frac{103}{7574570}y^5-\frac{1017}{757457}s^4+\frac{12101}{4544742}y^3\\
&+\frac{12101}{1514914}y^2+\frac{24408}{757457}y+\frac{1236}{757457},\\
d_6(y) =& -\frac{1203}{132554975}y^7+\frac{983}{22723710}y^6-\frac{1306}{18936425}y^5+\frac{103}{7574570}y^4\\
&+\frac{997}{11361855}y^3+\frac{997}{3787285}y^2-\frac{1236}{(3787285}y-\frac{31344}{3787285},\\
a_1(y) =&\frac{400968}{26510995}y^7-\frac{220144}{3787285}y^6+\frac{23592}{757457}y^5+\frac{141504}{757457}y^4-\frac{322328}{757457}y^3\\
&-\frac{966984}{757457}y^2-\frac{3396096}{757457}y+\frac{2831040}{757457},\\
a_2(y) =& -\frac{744192}{26510995}y^7+\frac{400968}{3787285}y^6-\frac{173232}{3787285}y^5-\frac{285480}{757457}y^4+\frac{628704}{757457}y^3\\
&+\frac{1886112}{757457}y^2+\frac{6851520}{757457}y-\frac{4157568}{757457},\\
c_1(y) =& -\frac{28872}{26510995}y^7+\frac{3932}{757457}y^6-\frac{31344}{3787285}y^5+\frac{1236}{757457}y^4+\frac{7976}{757457}y^3\\
&+\frac{23928}{757457}y^2-\frac{29664}{757457}y+\frac{5201}{757457},\\
c_2(y) =&\frac{9516}{5302199}y^7-\frac{23584}{3787285}y^6-\frac{1236}{3787285}y^5+\frac{24408}{757457}y^4-\frac{48404}{757457}y^3\\
&-\frac{145212}{757457}y^2+\frac{171665}{757457}y-\frac{29664}{757457},\\
c_3(y) =&\frac{13098}{26510995}y^7-\frac{40291}{22723710}y^6+\frac{997}{3787285}y^5+\frac{12101}{1514914}y^4-\frac{37300}{2272371}y^3\\
&+\frac{608257}{3029828}y^2-\frac{145212}{757457}y+\frac{23928}{757457},\\
c_4(y) =&\frac{4366}{26510995}y^7-\frac{40291}{68171130}y^6+\frac{997}{11361855}y^5+\frac{12101}{4544742}y^4-\frac{37300}{6817113}y^3\\
&+\frac{608257}{9089484}y^2-\frac{48404}{757457}y+\frac{7976}{757457},
\end{aligned}
$$

$$c_5(y) = -\frac{793}{10604398}y^7 + \frac{2948}{11361855}y^6 + \frac{103}{7574570}y^5 - \frac{1017}{757457}y^4 + \frac{12101}{4544742}y^3 + \frac{12101}{1514914}y^2 - \frac{171665}{18178968}y + \frac{1236}{757457},$$

$$c_6(y) = -\frac{1203}{132554975}y^7 + \frac{983}{22723710}y^6 - \frac{1306}{18936425}y^5 + \frac{103}{7574570}y^4 + \frac{997}{11361855}y^3 + \frac{997}{3787285}y^2 - \frac{1236}{3787285}y + \frac{5201}{90894840}.$$

Proof. We need to prove the existence of $m_y(z) \in S_1[0,1]$. For any $v(z) \in S_1[0,1]$, we aim to show that

$$\langle v(.), m_y(.)\rangle_{S_1[0,1]} = v(y).$$

Let $m_y(z) \in S_1[0,1]$. Using equation (2.1) along with the conditions $\int_0^1 v(z)dz = \int_0^1 zv(z)dz = 0$, we obtain

$$\langle v(z), m_y(z)\rangle_{S_1[0,1]} = \sum_{i=0}^{2} v^{(i)}(0)\partial_z^i m_y(0) + \int_0^1 v^{(3)}(z)\partial_z^3 m_y(z)dz.$$

Applying integration by parts three times, we get:

$$\begin{aligned}
&\langle v(z), m_y(z)\rangle_{S_1[0,1]} \\
=&\sum_{i=0}^{2} v^{(i)}(0)[\partial_z^i m_y(0) - (-1)^{2-i}\partial_z^{5-i} m_y(0)] + \sum_{i=0}^{2}(-1)^{2-i}v^{(i)}(1)\partial_z^{5-i}m_y(1) \\
&- \int_0^1 v(z)\partial_z^6 m_y(z)dz \\
=&\sum_{i=0}^{2} v^{(i)}(0)[\partial_z^i m_y(0) - (-1)^{2-i}\partial_z^{5-i} m_y(0)] + \sum_{i=0}^{2}(-1)^{2-i}v^{(i)}(1)m_y^{(5-i)}(1) \\
&- \int_0^1 v(z)m_y^{(6)}(z)dz + a_1(y)\int_0^1 v(z)dz + a_2(y)\int_0^1 zv(z)dz \\
=&\sum_{i=0}^{2} v^{(i)}(0)[\partial_z^i m_y(0) - (-1)^{2-i}\partial_z^{3} m_y(0)] + \sum_{i=0}^{2}(-1)^{2-i}v^{(i)}(1)\partial_z^{5-i}m_y(1) \\
&- \int_0^1 v(z)[\partial_z^6 m_y(z) - a_1(y) - a_2(y)z]dz.
\end{aligned}$$

Now, define:

$$\begin{cases} \partial_z^i m_y(0) - (-1)^{2-i}\partial_z^{5-i}m_y(0) = 0, & i = 0,1,2, \\ \partial_z^{5-i}m_y(1) = 0, & i = 0,1,2. \end{cases} \tag{2.4}$$

Then, we obtain:

$$v(y) = -\int_0^1 v(z)[\partial_z^6 m_y(z) - a_1(y) - a_2(y)z]dz,$$

which implies that:

$$\partial_z^6 m_y(z) - a_1(y) - a_2(y)z = -\delta(z-y), \tag{2.5}$$

where $\delta(z-y)$ is the Dirac delta function defined as:

$$\delta(z-y) = \begin{cases} 1, \, z = y, \\ 0, \, z \neq y. \end{cases}$$

Thus, we confirm:

$$\langle v(.), m_y(.) \rangle_{S_1[0,1]} = v(y).$$

This proves that $S_1[0,1]$ is a RKHS and $m_y(z)$ is a RKF.
From equation (2.5), for $x \neq y$, $m_y(z)$ satisfies the following linear homogeneous differential equation of 6th order:

$$\partial_z^6 m_y(z) - a_1(y) - a_2(y)z = 0. \tag{2.6}$$

The boundary conditions are given by (2.4). The characteristic equation of (2.6) is $-\lambda^6 + a_1(y) + za_2(y) = 0$.
Solving equation (2.5), we obtain the general form of $m_y(z)$ (see formula (2.3)). The coefficients $c_i(y), d_i(y)$, (for $i = 1, \ldots, 6$) and $a_1(y), a_2(y)$ are determined using the following conditions:

$$\begin{cases} \partial_z^i m_y(0) - (-1)^{2-i} \partial_z^{5-i} m_y(0) = 0, & i = 0,1,2, \\ \partial_z^{5-i} m_y(1) = 0, & i = 0,1,2, \\ \lim_{x \to y^+} \dfrac{\partial^j m_y(z)}{\partial z^j} = \lim\limits_{z \to y^-} \dfrac{\partial^j m_y(z)}{\partial z^j}, & j = 0,1,2,3,4, \\ \lim_{z \to y^+} \dfrac{\partial^5 m_y(z)}{\partial z^5} - \lim\limits_{z \to y^-} \dfrac{\partial^5 m_y(z)}{\partial z^5} = -1, \\ \displaystyle\int_0^1 m_y(z) dz = 0, \\ \displaystyle\int_0^1 z m_y(z) dz = 0. \end{cases} \tag{2.7}$$

Thus, the proof is complete. □

2. The function space $S_2[0,\tau]$ is defined as [20]:

$$S_2[0,\tau] = \{v | \, v, v', v'' \in AC[0,\tau], v^{(3)} \in L^2[0,1], \text{ and } v(0) = v'(0) = 0\}. \tag{2.8}$$

The inner product and the norm in this space are defined as follows:

$$\langle v_1, v_2 \rangle_{S_2[0,\tau]} = \sum_{i=0}^{2} v_1^{(i)}(0) v_2^{(i)}(0) + \int_0^\tau v_1^{(3)}(l) v_2^{(3)}(l) dl, \tag{2.9}$$

and

$$\|v_1\|_{S_2[0,\tau]} = \sqrt{\langle v_1, v_1 \rangle_{S_2[0,\tau]}}, \tag{2.10}$$

where $v_1, v_2 \in S_2[0,\tau]$.

Theorem 2.2. *The function space* $S_2[0,\tau]$ *is a RKHS. The RKF* $p_s(l)$ *of* $S_2[0,\tau]$ *is given by* [4, 23]*:*

$$p_s(l) = \begin{cases} \dfrac{s^2}{4}l^2 + \dfrac{s^2}{12}l^3 - \dfrac{s}{24}l^4 + \dfrac{1}{120}l^5, & l \le s, \\ \dfrac{s^5}{120} - \dfrac{s^4}{24}l + \dfrac{s^3}{12}l^2 + \dfrac{s^2}{4}l^2, & s < l. \end{cases} \tag{2.11}$$

For the proof, refer to the references [4] and [23].

3. Let $\Theta = [0,1] \times [0,\tau]$. The binary function space $\aleph(\Theta)$ is defined as [20]: $\aleph(\Theta) = \{v(z,l) \,\big|\, \frac{\partial^4 v}{\partial z^2 \partial l^2}$ is completely continuous in Θ, $\frac{\partial^6 v}{\partial z^3 \partial l^3} \in L^2(\Theta), v(z,0) = \frac{\partial v(z,0)}{\partial l} = 0, \int_0^1 v(z,l)dz = \int_0^1 zv(z,l)dz = 0\}$. The inner product in $\aleph(\Theta)$ is defined as:

$$\begin{aligned} \langle v_1(z,l), v_2(z,l) \rangle_{\aleph(\Theta)} = & \sum_{i=0}^{2} \int_0^\tau \frac{\partial^3}{\partial l^3}\frac{\partial^i}{\partial z^i} v_1(0,l) \frac{\partial^3}{\partial l^3}\frac{\partial^i}{\partial z^i} v_2(0,l) dl \\ & + \sum_{j=0}^{2} \left\langle \frac{\partial^j}{\partial l^j} v_1(z,0), \frac{\partial^j}{\partial l^j} v_2(z,0) \right\rangle_{\aleph(\Theta)} \\ & + \int_0^\tau \int_0^1 \frac{\partial^3}{\partial z^3}\frac{\partial^3}{\partial l^3} v_1(z,l) \frac{\partial^3}{\partial z^3}\frac{\partial^3}{\partial l^3} v_2(z,l) dxdt. \end{aligned}$$

The norm is given by:

$$\|v_1\|_{\aleph(\Theta)} = \sqrt{\langle v_1, v_1 \rangle_{\aleph(\Theta)}}. \tag{2.12}$$

For further details on the inner product and norm, refer to [20, 23].

Theorem 2.3. *The Function space* $\aleph(\Theta)$ *is a Hilbert space. The RKF of* $\aleph(\Theta)$ *is given by* [20]*:*

$$K_{y,s}(z,l) = m_y(z)p_s(l), \tag{2.13}$$

where

- $m_y(z)$ *is the RKF in* $S_1[0,1]$.
- $p_s(l)$ *is the RKF in* $S_2[0,\tau]$.

Proof. We have

$$\begin{aligned} & \left\langle v(z,l), K_{(y,s)}(z,l) \right\rangle_{\aleph(\Theta)} \\ = & \langle v(z,l), m_y(z)p_s(l) \rangle_{\aleph(\Theta)} \\ = & \sum_{i=0}^{2} \int_0^\tau \frac{\partial^3}{\partial l^3}\frac{\partial^i}{\partial z^i} v(0,l) \frac{\partial^3}{\partial t^3} p_s(l) \frac{\partial^i}{\partial z^i} m_y(0) dl \\ & + \int_0^\tau \int_0^1 \frac{\partial^3}{\partial l^3}\frac{\partial^3}{\partial z^3} v(z,l) \frac{\partial^3}{\partial z^3} m_y(z) \frac{\partial^3}{\partial l^3} p_s(l) dzdl \\ & + \sum_{j=0}^{2} \left\langle \frac{\partial^j}{\partial l^j} v(z,0), \frac{\partial^j}{\partial l^j} m_y(z) P_s(0) \right\rangle_{S_1[0,1]} \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\tau} \left[\left\{ \sum_{i=0}^{2} \frac{\partial^3}{\partial l^3} \frac{\partial^i}{\partial z^i} v(0,l) \frac{\partial^3}{\partial l^3} p_s(l) \frac{\partial^i}{\partial z^i} m_y(0) \right.\right. \\
&\quad \left.\left. + \int_0^1 \frac{\partial^3}{\partial z^3} \frac{\partial^3}{\partial l^3} v(z,l) \frac{\partial^3}{\partial z^3} m_y(z) \frac{\partial^3}{\partial l^3} p_s(l) dz \right\} \right] dl \\
&\quad + \sum_{j=0}^{2} \frac{\partial^j}{\partial l^j} v(y,0) \frac{\partial^j}{\partial l^j} p_s(0) \\
&= \int_0^{\tau} \frac{\partial^3}{\partial l^3} p_s(l) \frac{\partial^3}{\partial l^3} \left[\int_0^1 \frac{\partial^3}{\partial z^3} v(z,l) \frac{\partial^3}{\partial z^3} m_y(z) dz \right. \\
&\quad \left. + \sum_{i=0}^{2} \frac{\partial^i}{\partial z^i} v(0,l) \frac{\partial^i}{\partial z^i} m_y(0) dz \right] dl \\
&\quad + \sum_{j=0}^{2} \frac{\partial^j}{\partial l^j} v(y,0) \frac{\partial^j}{\partial l^j} p_s(0) \\
&= \int_0^{\tau} \frac{\partial^3}{\partial l^3} p_s(l) \frac{\partial^3}{\partial l^3} \left\langle v(z,l), m_y(z) \right\rangle_{S_1[0,1]} dl \\
&\quad + \sum_{j=0}^{2} \frac{\partial^i}{\partial l^i} v(y,0) \frac{\partial^i}{\partial l^i} p_s(0) \\
&= \int_0^{\tau} \frac{\partial^3}{\partial l^3} p_s(l) \frac{\partial^3}{\partial l^3} v(y,l) dl + \sum_{j=0}^{2} \frac{\partial^j}{\partial l^j} v(y,0) \frac{\partial^j}{\partial l^j} p_s(0) \\
&= \langle p_s(l), v(y,l) \rangle_{V_2[0,\tau]} \\
&= v(y,s).
\end{aligned}
$$

Thus, the proof is complete. □

4. The function space $\mathtt{m}_0(\Theta)$ is defined as [20]:
   $\mathtt{m}_0(\Theta) = \{v(z,l) | v(z,l) \text{ is continoues in } \Theta \text{ and } v(z,l) \in L^2(\Theta)\}$.
   Here, $\mathtt{m}_0(\Theta)$ is a subspace of $L^2(\Theta)$.

3. Application of the RKHSM

We aim to determine the solution of equation (1.8) within the RKHS $\aleph(\Theta)$. To achieve this, we define a linear operator $\hbar : \aleph(\Theta) \longrightarrow \mathtt{m}_o(\Theta)$ as follows [38]:

$$
\hbar v(z,l) = \frac{\partial^2 v}{\partial l^2} - \alpha \frac{\partial^2 v}{\partial z^2} - \beta \frac{\partial^3 v}{\partial l \partial z^2}. \tag{3.1}
$$

Thus, problem (1.8) is transformed into the following operator form:

$$
\begin{cases}
\hbar v(z,l) = \mathcal{H}(z,l), & (z,l) \in (0,1) \times (0,\tau], \\
v_l(z,0) = 0, & 0 < z < 1, \\
v(z,0) = 0, & 0 < z < 1, \\
\int_0^1 v(z,l) dz = 0, & 0 < l \leq \tau, \\
\int_0^1 z v(z,l) dz = 0, & 0 < l \leq \tau.
\end{cases} \tag{3.2}
$$

Lemma 3.1. *The operator $\hbar$ is a bounded linear operator from $\aleph(\Theta)$ to $\mathfrak{m}_{\mathfrak{o}}(\Theta)$.*

Proof. To prove that $\hbar$ is a bounded operator, we show that

$$\|\hbar v\|_{\mathfrak{m}_{\mathfrak{o}}} \leq \mathcal{J} \parallel v \parallel_{\aleph(\Theta)},$$

where $\mathcal{J} > 0$ is a positive constant.

We have

$$\begin{aligned}
\|\hbar v\|^2_{\mathfrak{m}_{\mathfrak{o}}} &= \iint_\Theta |(\hbar v)(z,l)|^2 \, dzdl \\
&= \iint_\Theta \left| \frac{\partial^2 v}{\partial l^2} - \alpha \frac{\partial^2 v}{\partial z^2} - \beta \frac{\partial^3 v}{\partial l \partial z^2} \right|^2 dzdl \\
&\leq \iint_\Theta \left[\left| \frac{\partial^2 v}{\partial l^2} \right| + \alpha \left| \frac{\partial^2 v}{\partial z^2} \right| + \beta \left| \frac{\partial^3 v}{\partial l \partial z^2} \right| \right]^2 dzdl \\
&\leq \iint_\Theta \left(\left| \frac{\partial^2 v}{\partial l^2} \right|^2 + \alpha^2 \left| \frac{\partial^2 v}{\partial z^2} \right|^2 + \beta^2 \left| \frac{\partial^3 v}{\partial l \partial z^2} \right|^2 \right. \\
&\quad + 2\alpha \left| \frac{\partial^2 v}{\partial l^2} \right| \cdot \left| \frac{\partial^2 v}{\partial z^2} \right| + 2\beta \left| \frac{\partial^2 v}{\partial l^2} \right| \cdot \left| \frac{\partial^3 v}{\partial l \partial z^2} \right| \\
&\quad \left. +2\alpha\beta \left| \frac{\partial^2 v}{\partial z^2} \right| \cdot \left| \frac{\partial^3 v}{\partial l \partial z^2} \right| \right) dzdl.
\end{aligned}$$

Since

$$v(z,l) = \langle v(\xi,\varrho), \mathbb{E}_{(z,l)}(\xi,\varrho) \rangle_{\aleph(\Theta)},$$

we obtain:

$$\left| \frac{\partial^{m+n}}{\partial z^m \partial l^n} v(z,l) \right| = \langle v(\xi,\varrho), \frac{\partial^{m+n}}{\partial z^m \partial l^n} \mathbb{E}_{(z,l)}(\xi,\varrho) \rangle.$$

By the Cauchy-Schwarz inequality and the continuity of the RKF $\mathbb{E}_{(z,l)}(\xi,\varrho)$, we get:

$$\left| \frac{\partial^{m+n}}{\partial z^m \partial l^n} v(z,l) \right| \leq \mathfrak{B}_{m,n} \|v\|_{\aleph(\Theta)}, m = 0,1,2, \ n = 0,1,2.$$

Setting:

$$\mathfrak{B} = max\{\mathfrak{B}_{m,n}, m = 0,1,2, n = 0,1,2\},$$

we obtain:

$$\|\hbar v\|^2_{\mathfrak{m}_{\mathfrak{o}}} \leq (1 + (\alpha+\beta)^2 + 2(\alpha+\beta))\tau \mathfrak{B}^2 \|v\|^2_{\aleph(\Theta)}.$$

Thus,

$$\|\hbar v\|_{\mathfrak{m}_{\mathfrak{o}}} \leq \mathcal{J} \|v\|_{\aleph(\Theta)},$$

where

$$\mathcal{J} = \sqrt{(1 + (\alpha+\beta)^2 + 2(\alpha+\beta))\tau \mathfrak{B}^2},$$

and $\mathcal{J}$ is a positive real number .

This completes the proof. □

We select a countable dense subset $\mathcal{M} = \{(z_i, l_i)\}_{i=1}^{\infty} \subset \Theta$ and define $\Psi_i(z,l)$ as follows:

$$\Psi_i(z,l) = (\hbar_{(y,s)}) \mathbb{E}_{(y,s)}(z,l)|_{(y,s)=(z_i,l_i)}$$

$$= \frac{\partial^2 \mathbb{E}_{(y,s)}(z,l)}{\partial s^2} - \alpha \frac{\partial^2 \mathbb{E}_{(y,s)}(z,l)}{\partial y^2} - \beta \frac{\partial^3 \mathbb{E}_{(y,s)}(z,l)}{\partial s \partial y^2}\bigg|_{(y,s)=(z_i,l_i)}, i = 1, 2, \ldots, \tag{3.3}$$

where $\mathbb{E}_{(y,s)}(z,l)$ is the RKF in $\aleph(\Theta)$.

Lemma 3.2. $\Psi_i(z,l) \in \aleph(\Theta)$.

Proof. By the definition of $\aleph(\Theta)$, we need to prove the following conditions:

(i) $\frac{\partial^6 \Psi_i(z,l)}{\partial z^3 \partial l^3} \in L^2(\Theta)$.

(ii) $\frac{\partial^4 \Psi_i(z,l)}{\partial z^2 \partial l^2}$ is completely continuous in Θ.

(iii) $\Psi_i(z,0) = \frac{\partial}{\partial l}\Psi_i(z,0) = \int_0^1 \Psi_i(z,l)dz = \int_0^1 z\Psi_i(z,l)dz = 0$.

Using equations (2.13) and (3.3), we write:

$$\Psi_i(z,l) = \tfrac{\partial^2}{\partial s^2} p_s(l).m_y(z) - \alpha \tfrac{\partial^2}{\partial y^2} m_y(z).p_s(l) - \beta \tfrac{\partial^2}{\partial y^2} m_y(z).\tfrac{\partial}{\partial s} p_s(l).$$

Differentiating, we obtain:

$$\begin{aligned}\frac{\partial^6}{\partial z^3 \partial l^3}\Psi_i(z,l) =& \frac{\partial^5}{\partial s^2 \partial l^3} p_s(l).\frac{\partial^3}{\partial z^3} m_y(z) - \alpha \frac{\partial^5}{\partial y^2 \partial z^3} m_y(z).\frac{\partial^3}{\partial l^3} p_s(l) \\ &- \beta \frac{\partial^5}{\partial y^2 \partial z^3} m_y(z).\frac{\partial^4}{\partial s \partial l^3} p_s(l).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^4}{\partial z^2 \partial l^2}\Psi_i(z,l) =& \frac{\partial^4}{\partial s^2 \partial l^2} p_s(l).\frac{\partial^2}{\partial z^2} m_y(z) - \alpha \frac{\partial^4}{\partial y^2 \partial z^2} m_y(z).\frac{\partial^2}{\partial l^2} p_s(l) \\ &- \beta \frac{\partial^4}{\partial y^2 \partial z^2} m_y(z).\frac{\partial^3}{\partial s \partial l^2} p_s(l).\end{aligned}$$

1. Proof of $\frac{\partial^6 \Psi_i(z,l)}{\partial z^3 \partial l^3} \in L^2(\Theta)$:
   By the definition of $S_2[0,\tau]$ and the expression of $p_s(l)$, we have:

   $$\frac{\partial^3}{\partial l^3} p_s(l) \in L^2[0,\tau], \quad \text{with respect to } l.$$

   Since $\frac{\partial^5}{\partial l^3 \partial s^2} p_s(l)$ is piecewise continuous in $[0,\tau]$ with respect to l, we conclude:

   $$\frac{\partial^5}{\partial l^3 \partial s^2} p_s(l) \in L^2[0,\tau].$$

   Similarly,

   $$\frac{\partial^4}{\partial l^3 \partial s} p_s(l) \text{ is continuous with respect to } l, \Rightarrow \frac{\partial^4}{\partial l^3 \partial s} p_s(l) \in L^2[0,\tau].$$

   Also, using the expression of $m_y(z)$ and the definition of $S_1[0,1]$, we coclude:

   $$\frac{\partial^5}{\partial z^3 \partial y^2} m_y(z) \in L^2[0,1], \text{ and } \frac{\partial^3}{\partial y^3} m_y(z) \in L^2[0,1], \text{ with respect to } z.$$

   Therefore, $\frac{\partial^6}{\partial z^3 \partial l^3}\Psi_i(z,l) \in L^2(\Theta)$.

2. Proof that $\frac{\partial^4 \Psi_i(z,l)}{\partial z^2 \partial l^2}$ is completely continuous in Θ:
For $\mathbb{E}_{(y,s)}(z,l) \in \aleph(\Theta)$ according to the proof in [20]:

$$\frac{\partial^{i+f}}{\partial z^i \partial y^f} m_{(y)}(z). \frac{\partial^{j+t}}{\partial l^j \partial s^t} p_{(s)}(l) = \frac{\partial^{i+j+f+t}}{\partial z^i \partial l^j \partial y^f \partial s^t} \mathbb{E}_{(y,s)}(z,l),$$

which is completely continuous when $0 \leq i+f \leq 4$ and $0 \leq j+t \leq 4$. By (3.3), we conclude that $\Psi_i(z,l)$ is completely continuous in Θ.

3. Proof that $\Psi_i(z,0)$ satisfies the boundary conditions:
Since $m_y(z) \in S_1[0,1]$, we have:

$$\int_0^1 m_y(z)dz = \int_0^1 z m_y(z)dz = 0. \tag{3.4}$$

Since $p_s(l) \in S_2[0,\tau]$, we obtain:

$$p_s(0) = \frac{\partial}{\partial l} p_s(0) = 0. \tag{3.5}$$

Using the expression of $m_y(z)$, we also have:

$$\int_0^1 \frac{\partial^2}{\partial y^2} m_y(z)dz = \int_0^1 \frac{\partial^2}{\partial y^2} z m_y(z)dz = 0. \tag{3.6}$$

Similarly, using the expression of $p_s(l)$, we have:

$$\frac{\partial^2}{\partial s^2} p_s(0) = \frac{\partial}{\partial s} p_s(0) = 0, \quad \text{and} \quad \frac{\partial}{\partial l} p_s(0) = \frac{\partial^3}{\partial s^2 \partial l} p_s(0) = \frac{\partial^2}{\partial s \partial l} p_s(0) = 0. \tag{3.7}$$

From equations (3.4)-(3.7), it follows that:

$$\Psi_i(z,0) = \frac{\partial}{\partial l} \Psi_i(z,0) = \int_0^1 \Psi_i(z,l)dz = \int_0^1 z \Psi_i(z,l)dz = 0.$$

Thus, we conclude that:

$$\Psi_i(z,l) \in \aleph(\Theta).$$

This completes the proof.

□

Lemma 3.3. *The function system* $\{\Psi_1(z,l), \Psi_2(z,l), \Psi_3(z,l), \dots\}$ *forms a complete system in* $\aleph(\Theta)$ [12].

The orthonormal system $\{\tilde{\Psi}(z,l)\}_{i=1}^{\infty}$ in $\aleph(\Theta)$, obtained from the Gram-Schmidt orthogonalization process of $\{\Psi(z,l)\}_{i=1}^{\infty}$, is given by:

$$\tilde{\Psi}(z,l) = \sum_{k=1}^{i} \gamma_{ik} \Psi_i(z,l), \tag{3.8}$$

where γ_{ik} are the orthogonal coefficients [12].

Theorem 3.1. *If $v(z,l)$ is the exact solution of equation (3.2), then:*

$$v(z,l)=\sum_{i=1}^{+\infty}\sum_{k=1}^{i}\gamma_{ik}\mathcal{H}(z_k,l_k)\tilde{\Psi}(z,l). \tag{3.9}$$

An approximate solution of equation (3.2) is given by:

$$v_n(z,l)=\sum_{i=1}^{n}\sum_{k=1}^{i}\gamma_{ik}\mathcal{H}(z_k,l_k)\tilde{\Psi}(z,l). \tag{3.10}$$

The exact solution of equations (1.1)-(1.5) is

$$u(z,l)=v(z,l)-\mathfrak{S}(z,l), \tag{3.11}$$

where $\mathfrak{S}(z,l)$ is defined in equation (1.7).

Proof. We prove formulas (3.9)-(3.10). From equation (3.2) and the fact that $v(z,l)\in\aleph(\Theta)$, we have

$$\begin{aligned}
v(z,l)&=\sum_{i=1}^{+\infty}\langle v(z,l),\tilde{\Psi}_i(z,l)\rangle_{\aleph(\Theta)}\tilde{\Psi}_i(z,l)\\
&=\sum_{i=1}^{+\infty}\langle v(z,l),\sum_{k=1}^{i}\gamma_{ik}\Psi_i(z,l)\rangle_{\aleph(\Theta)}\tilde{\Psi}_i(z,l)\\
&=\sum_{i=1}^{+\infty}\sum_{k=1}^{i}\gamma_{ik}\langle v(z,l),\hbar_{(y,s)}\mathbb{E}_{(y,s)}(z,l)|_{(y,s)=(z_k,l_k)}\rangle_{\aleph(\Theta)}\tilde{\Psi}_i(z,l)\\
&=\sum_{i=1}^{+\infty}\sum_{k=1}^{i}\gamma_{ik}[\hbar_{(y,s)}\langle v(z,l),\mathbb{E}_{(y,s)}(z,l)\rangle_{\aleph(\Theta)}|_{(y,s)=(z_k,l_k)}\tilde{\Psi}_i(z,l)\\
&=\sum_{i=1}^{+\infty}\sum_{k=1}^{i}\gamma_{ik}\left[\hbar_{(y,s)}v(y,s)\right]|_{(y,s)=(z_k,l_k)}\tilde{\Psi}_i(z,l)\\
&=\sum_{i=1}^{+\infty}\sum_{k=1}^{i}\gamma_{ik}\mathcal{H}(z_k,l_k)\tilde{\Psi}_i(z,l).
\end{aligned}$$

□

4. Convergence in $\aleph(\Theta)$

Theorem 4.1. [32, 48]

1. *For each $v(z,l)\in\aleph(\Theta)$, let $\varepsilon_n^2=\|v_n(z,l)-v(z,l)\|^2$. Then, the sequence ε_n is monotonically decreasing, and $\varepsilon_n\to 0$ as $n\to\infty$.*
2. *The approximate solution $v_n(z,l)$ uniformly converges to the exact solution $v(z,l)$.*
3. *The derivatives $\partial_{z,l}^{i+j}v_n(z,l)$, for $i=0,1,2$ and $j=0,1$, uniformly converge to $\partial_{z,l}^{i+j}v(z,l)$, for $i=0,1,2$ and $j=0,1$.*

For the proof, refer to references [48] and [32].

5. Numerical experiments

This section demonstrates the effectiveness of the proposed method through numerical examples. We apply it to various pseudo-hyperbolic equations subject to purely integral conditions. hese examples illustrate the extensive applicability of the method in solving such differential equations.

Example 5.1. Considering the pseudo-hyperbolic equation [38]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial l^2} - \dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial^3 u}{\partial l \partial z^2} = g(z,l), & 0 < z < 1,\ 0 < l < 1, \\ u(z,0) = \operatorname{sech}^2(z),\ \dfrac{\partial u(z,0)}{\partial l} = -2\tanh(z)\operatorname{sech}^2(z), & 0 < z < 1, \\ \displaystyle\int_0^1 u(z,l)dz = \tanh(l+1) - \tanh(l), & 0 < l \le 1, \\ \displaystyle\int_0^1 zu(z,l)dz = \tanh(l) + \sinh(1)\operatorname{sech}(l)\operatorname{sech}(l+1) & \\ \qquad\qquad + \ln(\cosh(l)\operatorname{sech}(l+1)), & 0 < l \le 1, \end{cases} \tag{5.1}$$

where $g(z,l) = 4(\cosh(2(l+z)) - 5)\tanh(l+z)\operatorname{sech}^4(l+z)$.

The exact solution to (5.1) is:

$$u(z,l) = \frac{1}{\cosh^2(z+l)}. \tag{5.2}$$

In Example 5.1, we apply the RKHSM, as discussed earlier. By selecting $p \times q = n = 15 \times 15 = 225$ collocation points with $z_{\mathfrak{i}} = \frac{\mathfrak{i}}{p}$ for $\mathfrak{i} = 1, 2, \ldots, p$ and $l_{\mathfrak{j}} = \frac{\mathfrak{j}}{q}$ for $\mathfrak{j} = 1, 2, \ldots, q$, we obtain the approximate solution using the RKHSM. We compared the new solution with the exact solution. The results displayed in Table 1 illustrate the absolute error across the domain $[0,1] \times [0,1]$. Figure 1 shows the RKHSM solution alongside the exact solution at $l = 0.3$. Figure 2 presents the absolute error between the RKHSM solution and the exact solution at $l = 0.3$. Figure 3 is a 3D plot of the RKHSM solution, while Figure 4 depicts a 3D plot of the exact solution. Finally, Figure 5 illustrates the 3D plot of the absolute error between the RKHSM solution and the exact solution. These visualizations confirm that the RKHSM consistently produces results closely resembling the exact solution, demonstrating its effectiveness.

Example 5.2. Considering the pseudo-hyperbolic equation [41]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial l^2} - \dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial^3 u}{\partial l \partial z^2} = g(z,l), & 0 < z < 1,\ 0 < l < 1, \\ u(z,0) = e^z,\ \dfrac{\partial u(z,0)}{\partial l} = 0, & 0 < z < 1, \\ \displaystyle\int_0^1 u(z,l)dz = (e-1)\cosh(l), & 0 < l \le 1, \\ \displaystyle\int_0^1 zu(z,l)dz = \cosh(l), & 0 < l \le 1, \end{cases} \tag{5.3}$$

where $g(z,l) = -e^z \sinh(l)$.

Table 1. Numerical results of Example 5.1.

z	l	Exact Solution	RKHSM	Absolute Error
0.2	0.2	0.8556387858	0.8556281433	1.1×10^{-5}
	0.4	0.7115777629	0.7115533559	2.4×10^{-5}
	0.6	0.5590551680	0.5589741860	8.1×10^{-5}
	0.8	0.4199743415	0.4197860550	1.9×10^{-4}
	1.0	0.9610429822	0.9610429820	2.0×10^{-10}
0.4	0.2	0.7115777629	0.7116031528	2.5×10^{-5}
	0.4	0.5590551680	0.5589891901	6.6×10^{-5}
	0.6	0.4199743415	0.4198159681	1.6×10^{-4}
	0.8	0.3050199963	0.3048046490	2.2×10^{-4}
	1.0	0.8556387858	0.8556387859	1.0×10^{-10}
0.6	0.2	0.5590551680	0.5590904141	3.5×10^{-5}
	0.4	0.4199743415	0.4199959892	2.2×10^{-5}
	0.6	0.3050199963	0.3050961129	7.6×10^{-5}
	0.8	0.2161524591	0.2163801540	2.3×10^{-4}
	1.0	0.7115777629	0.7115777626	3.0×10^{-10}
0.8	0.2	0.4199743415	0.4199684418	5.9×10^{-6}
	0.4	0.3050199963	0.3050605976	4.1×10^{-5}
	0.6	0.2161524591	0.2162606510	1.1×10^{-4}
	0.8	0.1505270758	0.1507121439	1.9×10^{-4}
	1.0	0.5590551680	0.5590551680	0.0
1.0	0.2	0.3050199963	0.3049664090	5.4×10^{-5}
	0.4	0.2161524591	0.2160943939	5.8×10^{-5}
	0.6	0.1505270758	0.1503328150	1.9×10^{-4}
	0.8	0.1035583741	0.1030522690	5.1×10^{-4}
	1.0	0.4199743415	0.4199743410	5.0×10^{-10}

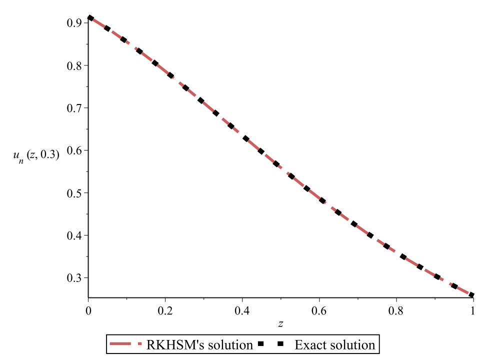

Figure 1. RKHSM and Exact solutions for Example 5.1 when $l = 0.3$.

The exact solution to (5.3) is:

$$u(z,l) = e^{z}\cosh(l). \tag{5.4}$$

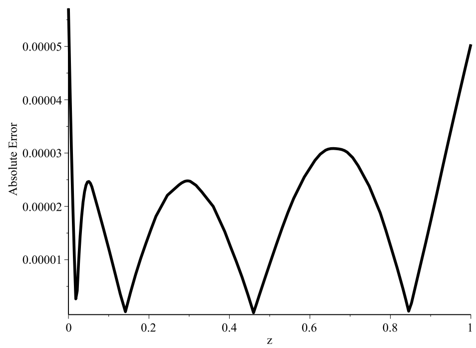

Figure 2. Absolute error of the RKHSM for Example 5.1 when $l = 0.3$.

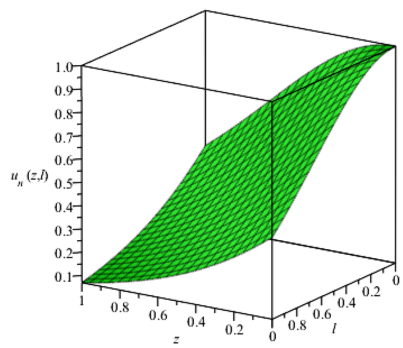

Figure 3. 3D Visualization of the RKHSM's solution for Example 5.1.

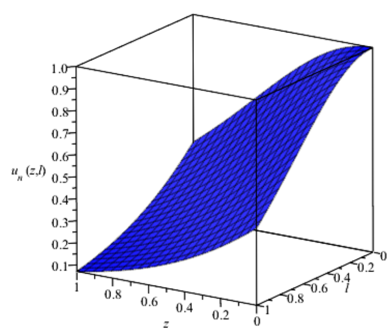

Figure 4. 3D Visualization of the exact solution solution for Example 5.1.

In Example 5.2, we apply the RKHSM, as discussed earlier. By selecting $p \times q = n = 15 \times 15 = 225$ collocation points with $z_{\mathfrak{i}} = \frac{\mathfrak{i}}{p}$ for $\mathfrak{i} = 1, 2, \ldots, p$ and $l_{\mathfrak{j}} = \frac{\mathfrak{j}}{q}$ for $\mathfrak{j} = 1, 2, \ldots, q$, we obtain the approximate solution using the RKHSM. We compared the new solution with the exact solution. The results displayed in Table 2 illustrate the absolute error across the domain $[0, 1] \times [0, 1]$. Figure 6 shows the RKHSM

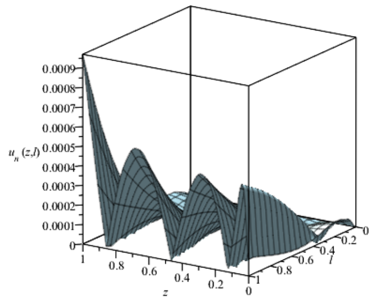

Figure 5. 3D Visualization of the absolute error for Example 5.1.

solution alongside the exact solution at $l = 0.1$. Figure 7 presents the absolute error between the RKHSM solution and the exact solution at $l = 0.1$. Figure 8 is a 3D plot of the RKHSM solution, while Figure 9 depicts a 3D plot of the exact solution. Finally, Figure 10 illustrates the 3D plot of the absolute error between the RKHSM solution and the exact solution. These visualizations confirm that the RKHSM consistently produces results closely resembling the exact solution, demonstrating its effectiveness.

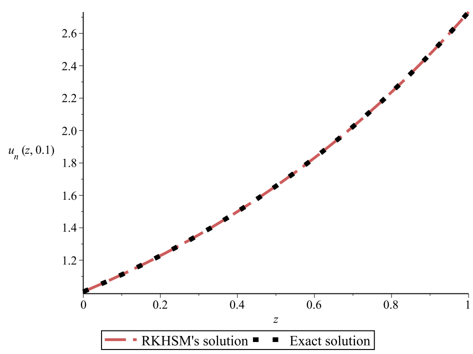

Figure 6. RKHSM and Exact solutions for Example 5.2 when $l = 0.1$.

Example 5.3. Considering the pseudo-hyperbolic equation [38]:

$$\begin{cases} \dfrac{\partial^2 u}{\partial l^2} - \dfrac{\partial^2 u}{\partial z^2} - \dfrac{\partial^3 u}{\partial l \partial z^2} = g(z,l), & 0 < z < 1,\ 0 < l < 1, \\ u(z,0) = \tanh^2(z),\ \dfrac{\partial u(z,0)}{\partial l} = 2\tanh(z)\operatorname{sech}^2(z), & 0 < z < 1, \\ \displaystyle\int_0^1 u(z,l)dz = \tanh(l) - \tanh(l+1) + 1, & 0 < l \le 1, \\ \displaystyle\int_0^1 zu(z,l)dz = -\tanh(l) + \ln(\operatorname{sech}(l)) + \ln(\cosh(l+1)) & \\ \qquad\qquad - \sinh(1)\operatorname{sech}(l)\operatorname{sech}(l+1) + \dfrac{1}{2}, & 0 < l \le 1, \end{cases} \tag{5.5}$$

Table 2. Numerical results of Example 5.2.

z	l	Exact Solution	RKHSM	Absolute Error
0.2	0.2	1.245912349	1.245912402	5.3×10^{-8}
	0.4	1.320424777	1.320424496	2.8×10^{-7}
	0.6	1.447930487	1.447929590	9.0×10^{-7}
	0.8	1.633546732	1.633544905	1.8×10^{-6}
	1.0	1.221402758	1.221402756	2.0×10^{-9}
0.4	0.2	1.521760780	1.521760588	1.9×10^{-7}
	0.4	1.612770465	1.612770801	3.4×10^{-7}
	0.6	1.768506291	1.768507574	1.3×10^{-6}
	0.8	1.995218484	1.995221356	2.9×10^{-6}
	1.0	1.491824698	1.491824698	0.0
0.6	0.2	1.858682813	1.858682776	3.7×10^{-8}
	0.4	1.969842293	1.969843208	9.2×10^{-7}
	0.6	2.160058460	2.160061014	2.6×10^{-6}
	0.8	2.436965359	2.436970457	5.1×10^{-6}
	1.0	1.822118800	1.822118800	0.0
0.8	0.2	2.270200315	2.270200528	2.1×10^{-7}
	0.4	2.405970810	2.405971061	2.5×10^{-7}
	0.6	2.638301361	2.638301603	2.4×10^{-7}
	0.8	2.976516211	2.976516369	1.6×10^{-7}
	1.0	2.225540928	2.225540928	0.0
1.0	0.2	2.772828926	2.772828489	4.4×10^{-7}
	0.4	2.938659384	2.938657212	2.2×10^{-6}
	0.6	3.222428560	3.222423442	5.1×10^{-6}
	0.8	3.635525110	3.635515629	9.5×10^{-6}
	1.0	2.718281828	2.718281828	0.0

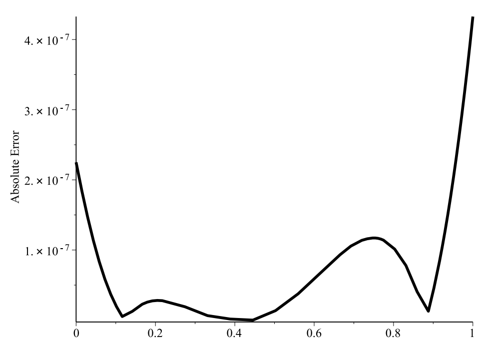

Figure 7. Absolute error of the RKHSM for Example 5.2 when $l = 0.1$.

where $g(z,l) = -2(\sinh(3(l+z)) - 11\sinh(l+z))\text{sech}^5(l+z)$.

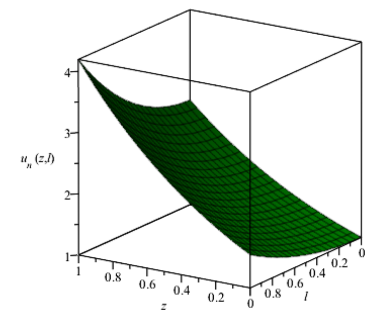

Figure 8. 3D Visualization of the RKHSM's solution for Example 5.2.

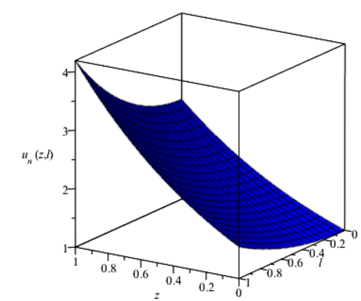

Figure 9. 3D Visualization of the exact solution solution for Example 5.2.

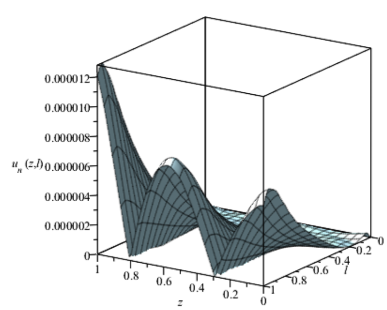

Figure 10. 3D Visualization of the absolute error for Example 5.2.

The exact solution to (5.5) is:

$$u(z,l) = \frac{1}{\coth^2(l+z)}. \tag{5.6}$$

In Example 5.3, we apply the RKHSM, as discussed earlier. By selecting $p \times q = n = 15 \times 15 = 225$ collocation points with $z_{\mathfrak{i}} = \frac{\mathfrak{i}}{p}$ for $\mathfrak{i} = 1, 2, \ldots, p$ and $l_{\mathfrak{j}} = \frac{\mathfrak{j}}{q}$ for $\mathfrak{j} = 1, 2, \ldots, q$, we obtain the approximate solution using the RKHSM. We compared the new solution with the exact solution. The results displayed in Table 3 illustrate the absolute error across the domain $[0, 1] \times [0, 1]$. Figure 11 shows the RKHSM solution alongside the exact solution at $l = 0.1$. Figure 12 presents the absolute error between the RKHSM solution and the exact solution at $l = 0.1$. Figure 13 is a 3D plot of the RKHSM solution, while Figure 14 depicts a 3D plot of the exact solution. Finally, Figure 15 illustrates the 3D plot of the absolute error between the RKHSM solution and the exact solution. These visualizations confirm that the RKHSM consistently produces results closely resembling the exact solution, demonstrating its effectiveness.

Table 3. Numerical results of Example 5.3.

z	l	Exact Solution	RKHSM	Absolute Error
0.2	0.2	0.6407986834	0.6408327447	3.4×10^{-5}
	0.4	0.7425668027	0.7425644140	2.4×10^{-6}
	0.6	0.8192933610	0.8193033003	9.9×10^{-6}
	0.8	0.8749901286	0.8750928380	1.0×10^{-4}
	1.0	0.5130826389	0.5130826389	0.0
0.4	0.2	0.2884222375	0.2883968450	2.5×10^{-5}
	0.4	0.4409448323	0.4410108100	6.6×10^{-5}
	0.6	0.5800256579	0.5801840714	1.6×10^{-4}
	0.8	0.6949800040	0.6951953661	2.2×10^{-4}
	1.0	0.1443612139	0.1443612140	1.0×10^{-10}
0.6	0.2	0.4409448323	0.4409095888	3.5×10^{-5}
	0.4	0.5800256579	0.5800040189	2.2×10^{-5}
	0.6	0.6949800040	0.6949039051	7.6×10^{-5}
	0.8	0.7838475416	0.7836199009	2.3×10^{-4}
	1.0	0.2884222375	0.2884222374	1.0×10^{-10}
0.8	0.2	0.5800256579	0.5800315519	5.9×10^{-6}
	0.4	0.6949800040	0.6949393916	4.1×10^{-5}
	0.6	0.7838475416	0.7837393425	1.1×10^{-4}
	0.8	0.8494729244	0.8492878571	1.9×10^{-4}
	1.0	0.4409448323	0.4409448321	2.0×10^{-10}
1.0	0.2	0.6949800040	0.6950335947	5.4×10^{-5}
	0.4	0.7838475416	0.7839055720	5.8×10^{-5}
	0.6	0.8494729244	0.8496671540	1.9×10^{-4}
	0.8	0.8964416253	0.8969477000	5.1×10^{-4}
	1.0	0.5800256579	0.5800256580	1.0×10^{-10}

6. Conclusion

This study employed the RKHS approach to solve pseudo-hyperbolic equations with purely integral conditions. The effectiveness of this method was demonstrated through three numerical experiments, with results presented in tables and figures. The use of highly effective reproducing kernel functions played a crucial role in achieving the desired outcomes. Our findings suggest that this method is capable of

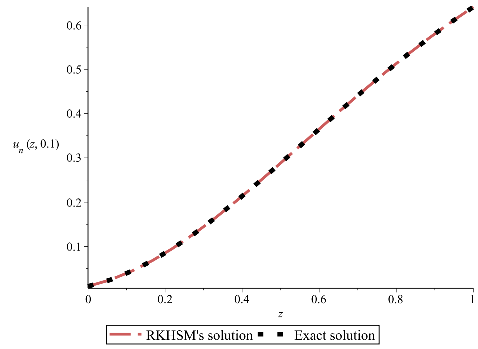

Figure 11. RKHSM and Exact solutions for Example 5.3 when $l = 0.1$.

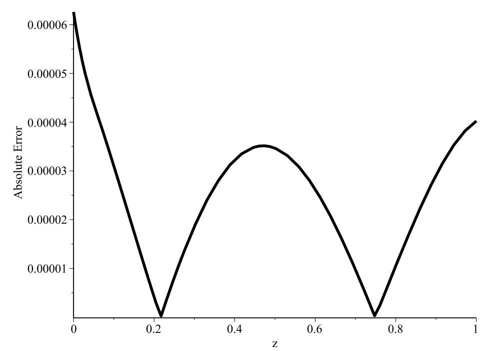

Figure 12. Absolute error of the RKHSM for Example 5.3 when $l = 0.1$.

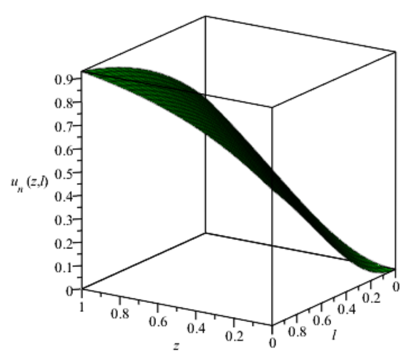

Figure 13. 3D Visualization of the RKHSM's solution for Example 5.3.

addressing even more complex problems, leading us to conclude that the proposed approach holds significant potential for application to more intricate challenges.

Regarding the applicability of the presented scheme to fractional derivatives, it is important to note that the current study focuses on classical pseudo-hyperbolic equations with integral conditions. To the best of our knowledge, this type of equation with fractional derivatives under purely integral conditions has not yet been

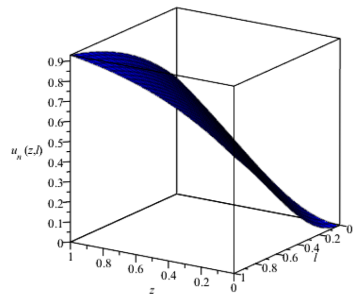

Figure 14. 3D Visualization of the exact solution solution for Example 5.3.

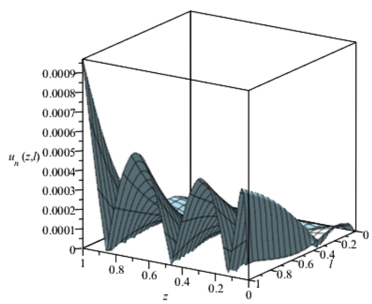

Figure 15. 3D Visualization of the absolute error for Example 5.3.

explored using RKHSM. Developing a fractional extension of this method would require further investigation into adapting the RKHS approach to fractional-order operators. Future studies could focus on extending this approach to fractional pseudo-hyperbolic equations, investigating its convergence properties, and comparing it with other numerical schemes.

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