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BIFURCATIONS AND EXACT SOLUTIONS OF THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS DNLSI-DNLSIII: DYNAMICAL SYSTEM METHOD*

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Abstract For the derivative nonlinear Schrödinger equations DNLSI-DNLSIII, by using the dynamical system method, we investigate the exact explicit solutions with the form $q(x,t) = \phi(\xi)\exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \xi = x - ct$. In the given parameter regions, we present exact explicit parametric representations for more than 14 solutions.

Keywords Bifurcation, exact solution, Hamiltonian system, Kaup–Newell equation, Chen–Lee–Liu equation, Gerdjikov–Ivanov equation.

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1. Introduction

Integrable systems have a wide range of applications in many important physical fields, such as water waves, plasma physics, field theory, and nonlinear optics. In the study of integrable system, it is both fascinating and significant to identify certain equations with physical meaning and investigate their properties. Starting from a generalized Kaup–Newell spectral problem, a series of nonlinear evolution equations are derived. In [1], Abhinav et al. proposed the following coupled equations by using Lax pair:

$$q_t = iq_{xx} - (4\beta+1)q^2 r_x - 4\beta q q_x r + \frac{i}{2}(1+2\beta)(1+4\beta)q^3 r^2, \tag{1.1}$$

$$r_t = -ir_x x - (1+4\beta)r^2 q_x - 4\beta r r_x q - \frac{i}{2}(1+2\beta)(1+4\beta)q^2 r^3, \tag{1.2}$$

where q and r are two potential functions of Lax pair, x and t are the spatial variable and temporal variable, respectively.

Nonlinear Schrödinger (NLS) equation is a generic soliton equation used in a wide variety of fields, such as quantum field theory, weakly nonlinear dispersive

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water waves and nonlinear optics. The above coupled equation has three famous Schrödinger-type reductions. One is the Kaup–Newell equation (see [11, 17, 35]):

$$iq_t + q_{xx} + i(|q|^2 q)_x = 0. \tag{1.3}$$

The second type is the Chen–Lee–Liu equation [5]:

$$iq_t + q_{xx} + i|q|^2 q_x = 0. \tag{1.4}$$

The last one takes the form [13]:

$$iq_t + q_{xx} - iq^2 q_x^* + \frac{1}{2} q^3 (q^*)^2 = 0, \tag{1.5}$$

which is called the Gerjikov–Ivanov (GI) equation. q^* denotes the complex conjugation of q. These three Schrödinger equations possess derivative-type nonlinearities, which are called the derivative nonlinear Schrödinger equations (DNLSI, DNLSII, DNLSIII, respectively) .

The equations (1.3)-(1.5) have been studied by many authors (see [2, 3, 7–10, 12, 14–16, 18, 19, 21–33]). The purpose of this paper is to find the new exact travelling wave solutions in explicit form and dynamical properties shared by the three equations. Based on the research experience [6, 34, 36], we will employ the bifurcation theory of dynamical system, which can not only help us to study the dynamics of the solutions, but also help us to find the exact traveling wave solutions and understand the reason of their smoothness change.

This paper is organized as follows. In section 2, The traveling wave systems of the equations (1.3)-(1.5) are explored with traveling wave solution assumption. The phase portraits of these systems and their bifurcations are studied. In section 3, in given parameter regions, the solutions expressed in terms of parameters corresponding to all bounded orbits are presented. In section 4, all exact explicit parameter representations of equation (1.3)-(1.5) are derived.

2. The traveling wave systems and bifurcations of phase portraits for equations DNLSI, DNLSII and DNLSIII

For equations (1.3)-(1.5), in order to seek new traveling wave solutions, we set the exact explicit solutions of the three equations with the form:

$$q(x,t) = \phi(\xi) \exp\left[i(\kappa x - \omega t + \theta(\xi))\right], \quad \xi = x - ct, \tag{2.1}$$

where c is the wave velocity and $\phi(\xi), \theta(\xi)$ are two functions with variable ξ, and κ, ω are two constant parameters.

(i) Substituting (2.1) into equation (1.3) and separating the real part and imaginary part, we have

$$-\phi'' + \theta'\phi^3 + \kappa\phi^3 + (2\kappa - c)\theta'\phi + (\theta')^2\phi + (\kappa^2 - \omega)\phi = 0, \tag{2.2}$$

$$\theta''\phi + 2\theta'\phi' + (2\kappa - c)\phi' + 3\phi'\phi^2 = 0, \tag{2.3}$$

where "$'$" is the derivative with respect to ξ. Equation (2.3) can be transformed into a second-order nonhomogeneous linear differential equation concerning the function θ

$$\theta'' + \frac{2\phi'}{\phi}\theta' = \frac{(c-2\kappa)\phi'}{\phi} - 3\phi'\phi. \tag{2.4}$$

Solving the second-order differential equation (2.4), we have

$$\theta' = \phi^{-2}[\frac{1}{2}(c-2\kappa)\phi^2 - \frac{3}{4}\phi^4 + C_1]. \tag{2.5}$$

Taking integral constant $C_1 = 0$, we obtain

$$\theta' = \frac{1}{2}(c-2\kappa) - \frac{3}{4}\phi^2, \quad \theta(\xi) = \left(\frac{1}{2}c - \kappa\right)\xi - \frac{3}{4}\int \phi^2(\xi)d\xi. \tag{2.6}$$

Substituting (2.6) into (2.2), we obtain the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \left(c\kappa - \frac{1}{4}c^2 - \omega\right)\phi + \frac{1}{2}c\phi^3 - \frac{3}{16}\phi^5. \tag{2.7}$$

(ii) Substituting (2.1) into equation (1.4) and separating the real part and imaginary part, we obtain

$$-\phi'' + \theta'\phi^3 + \kappa\phi^3 + (2\kappa - c)\theta'\phi + (\theta')^2\phi + (\kappa^2 - \omega)\phi = 0, \tag{2.8}$$

$$\theta''\phi + 2\theta'\phi' + (2\kappa - c)\phi' + \phi'\phi^2 = 0. \tag{2.9}$$

Using the same method as solving (2.3) to solve the second-order differential equation (2.9), we can obtain the expression for θ in equation (2.9)

$$\theta' = \frac{1}{2}(c-2\kappa) - \frac{1}{4}\phi^2, \quad \theta(\xi) = \frac{1}{2}(c-2\kappa)\xi - \frac{1}{4}\int \phi^2(\xi)d\xi. \tag{2.10}$$

Substituting (2.10) into (2.8), we obtain the following planar dynamical system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \left(c\kappa - \frac{1}{4}c^2 - \omega\right)\phi + \frac{1}{2}c\phi^3 - \frac{3}{16}\phi^5. \tag{2.11}$$

(iii) Substituting (2.1) into equation (1.5) and separating the real part and imaginary part, we have

$$\phi'' + (c-2\kappa)\theta'\phi - (\kappa + \theta')\phi^3 + (\omega - \kappa^2)\phi - (\theta')^2\phi + \frac{1}{2}\phi^5 = 0, \tag{2.12}$$

$$\phi'\phi^2 - \theta''\phi - 2\phi'\theta' + (c-2\kappa)\phi' = 0. \tag{2.13}$$

Similarly, using the same method as solving (2.3) to solve the second-order differential equation (2.13), we can obtain the expression for θ in equation (2.13)

$$\theta' = \frac{1}{2}(c-2\kappa) + \frac{1}{4}\phi^2, \quad \theta(\xi) = \frac{1}{2}(c-2\kappa)\xi + \frac{1}{4}\int \phi^2(\xi)d\xi. \tag{2.14}$$

Substituting (2.14) into (2.12), we obtain the following planar dynamical system (2.11):

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \left(c\kappa - \frac{1}{4}c^2 - \omega\right)\phi + \frac{1}{2}c\phi^3 - \frac{3}{16}\phi^5.$$

Based on the above calculations and analyses, we found that, although equations (1.3)-(1.5) are three distinct mathematical and physical equations, their traveling wave solutions $\phi(\xi)$ have same dynamical properties. Then, we can draw the following conclusion:

Lemma 2.1. *The Schrödinger equations DNLSI, DNLSII, DNLSIII share the same type of traveling wave system:*

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \alpha\phi + \beta\phi^3 - \frac{3}{16}\phi^5, \tag{2.15}$$

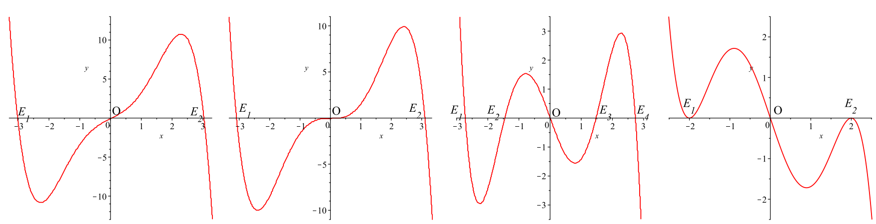
where $\alpha = c\kappa - \frac{1}{4}c^2 - \omega$, $\beta = \frac{1}{2}c$ *for equation* (1.3), (1.4), (1.5).

Remark 2.1. Systems (2.15) is a two-parameter planar dynamical system depending on the parameter group (α, β). Since the phase orbits defined by the vector fields of systems (2.15) give rise to all exact solutions with the form (2.1) of equation (1.3)-(1.5), we need to investigate the bifurcations of phase portraits for system (2.15) in the $(\phi, y)-$phase plane as the parameters change (see [20]).

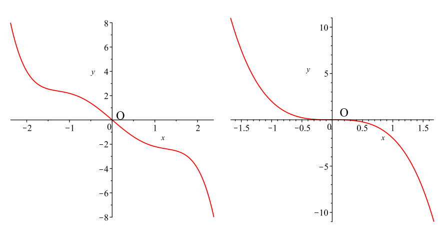
Set $f(\phi) = \alpha\phi + \beta\phi^3 - \frac{3}{16}\phi^5$, the Jacobian matrix of any equilibrium $E(\phi, y)$ of system (2.15) take the form

$$J(E) = \begin{pmatrix} 0 & 1 \\ f'(\phi) & 0 \end{pmatrix}.$$

The deter minant of $J(E)$ is $Det(J(E)) = -f'(\phi)$, and the trace always is $Tr(J(E))$ $= 0$. The real roots of $f(\phi) = 0$ are illustrated in Fig 1.



(a) $\alpha > 0, \beta \in (-\infty, \infty)$. (b) $\alpha = 0, \beta > 0$. (c) $\alpha < 0, \beta > \frac{\sqrt{3|\alpha|}}{2}$. (d) $\alpha < 0, \beta = \frac{\sqrt{3|\alpha|}}{2}$.



(e) $\alpha < 0, \beta < \frac{\sqrt{3|\alpha|}}{2}$. (f) $\alpha = 0, \beta < 0$.

Figure 1. The real roots of $f(\phi) = 0$.

Utilizing the qualitative theory of dynamical systems, we can easily get the following lemma:

Lemma 2.2. *Set* $\Delta = \beta^2 + \frac{3}{4}\alpha$, *then the following conclusions hold.*

(i) When $\alpha > 0$, *system* (2.15) *has three singular points* $O(0,0)$ *and* $E_{1,4}(\mp\phi_2, 0)$, $\phi_2 = \left(\frac{8}{3}\left(\beta + \sqrt{\Delta}\right)\right)^{\frac{1}{2}}$, *where* O *is saddle point and* $E_{1,4}$ *are center points.*

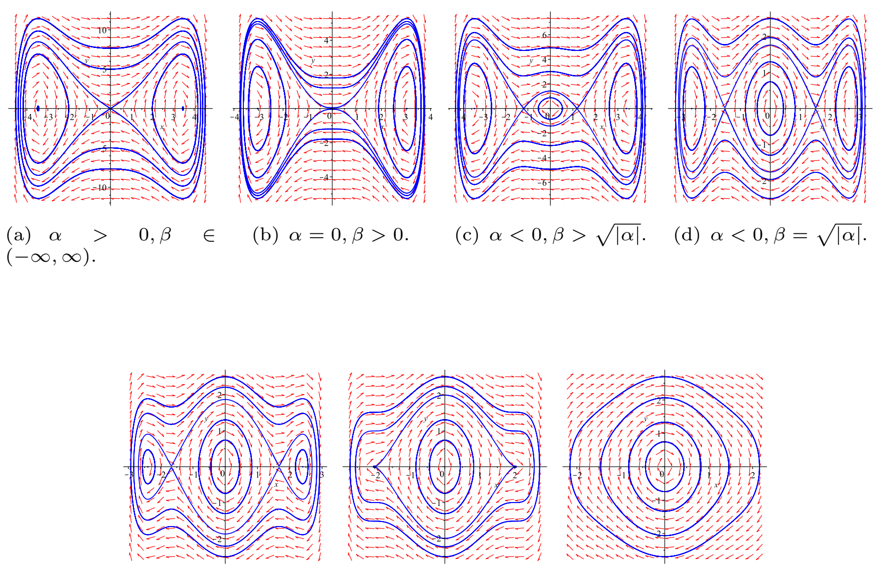
(ii) When $\alpha < 0, \beta < \frac{\sqrt{3|\alpha|}}{2}$, *system* (2.15) *has only one singular center point* $O(0,0)$.

(iii) When $\alpha < 0, \beta = \frac{\sqrt{3|\alpha|}}{2}$, *system* (2.15) *has one simple singular center point* $O(0,0)$ *and two double equilibrium points* $E_{1,4}\left(\mp\sqrt{\frac{8}{3}\beta}, 0\right)$, *where* $E_{1,4}$ *are degenerate equilibriums.*

(iv) When $\alpha < 0, \beta > \frac{\sqrt{3|\alpha|}}{2}$, *system* (2.15) *has five equilibrium points* $E_1(-\phi_2, 0)$, $E_2(-\phi_1, 0), O(0,0)$ *and* $E_3(\phi_1, 0), E_4(\phi_2, 0)$, $\phi_{1,2} = \left(\frac{8}{3}\left(\beta \mp \sqrt{\Delta}\right)\right)^{\frac{1}{2}}$, *where* $E_{1,4}$ *and* O *are center points,* $E_{2,3}$ *are saddle points.*

(v) When $\alpha = 0$, *system* (2.15) *has a degenerate equilibrium point* $O(0,0)$.

Meanwhile, the bifurcations of the phase portraits of traveling wave system (2.15) are shown in Figure 2 (a)-(g).



(a) $\alpha > 0, \beta \in (-\infty, \infty)$. (b) $\alpha = 0, \beta > 0$. (c) $\alpha < 0, \beta > \sqrt{|\alpha|}$. (d) $\alpha < 0, \beta = \sqrt{|\alpha|}$.

(e) $\alpha < 0, \frac{\sqrt{3|\alpha|}}{2} < \beta < \sqrt{|\alpha|}$. (f) $\alpha < 0, \beta = \frac{\sqrt{3|\alpha|}}{2}$ (g) $\alpha < 0, \beta < \frac{\sqrt{3|\alpha|}}{2}$.

Figure 2. The bifurcations of phase portraits of system (2.15).

Remark 2.2. From the phase portraits depicted in Figure 2, we observe that in the $(\phi, y)-$phase plane, homoclinic orbits, periodic orbits, and heteroclinic orbits coexist only when $\alpha < 0, \frac{\sqrt{3|\alpha|}}{2} < \beta$. When $\alpha < 0, \frac{\sqrt{3|\alpha|}}{2} \geq \beta$, homoclinic orbits do not exist and when $\alpha \geq 0$, heteroclinic orbits do not exist, respectively. This implies that for system (2.15), solitary waves, periodic waves and kink waves can simultaneously exhibit only under the condition $\alpha < 0, \frac{\sqrt{3|\alpha|}}{2} < \beta$. When $\alpha <$

0, $\frac{\sqrt{3|\alpha|}}{2} \geq \beta$ solitary waves do not exist and when $\alpha \geq 0$, kink waves do not exist, respectively.

3. Exact parametric representations of the solutions defined by all bounded orbits of system (2.15)

The Hamiltonian system (2.15) has the following first integral:

$$H(\phi, y) = y^2 - \alpha\phi^2 - \frac{1}{2}\beta\phi^4 + \frac{1}{16}\phi^6 = h. \tag{3.1}$$

Now, write that $h_0 = H(0,0) = 0$,

$$h_1 = H(\phi_1, 0) = \frac{64}{27}\left(\Delta^{\frac{3}{2}} - \beta\left(\beta^2 + \frac{9}{8}\alpha\right)\right),$$
$$h_2 = H(\phi_2, 0) = \frac{64}{27}\left(-\Delta^{\frac{3}{2}} - \beta\left(\beta^2 + \frac{9}{8}\alpha\right)\right).$$

Obviously, when $\beta = \sqrt{-\alpha}$, we have $h_2 = 0$.

We see from (3.1) that

$$y^2 = h + \alpha\phi^2 + \frac{1}{2}\beta\phi^4 - \frac{1}{16}\phi^6. \tag{3.2}$$

By using the first equation of (2.15), we obtain

$$\frac{1}{4}\xi = \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{16h + 16\alpha\phi^2 + 8\beta\phi^4 - \phi^6}} = \int_{\psi_0}^{\psi} \frac{d\psi}{2\sqrt{\psi(16h + 16\alpha\psi + 8\beta\psi^2 - \psi^3)}}, \tag{3.3}$$

where $\psi = \phi^2$. Thus, we can calculate the exact parametric representations defined by the bounded orbits of system (2.15).

In addition, we see from (2.6), (2.10) and (2.14) that the exact solutions of equation (1.3), (1.4) and (1.5) with the form (2.1) contain two function $\phi(\xi)$ and $\theta(\xi)$ which depends on $\int \phi^2(\xi)d\xi$. So that, we need to use the exact solutions of system (2.15) to derive the exact parametric representations of the function $\int \phi^2(\xi)d\xi$.

3.1. Exact periodic solutions and homoclinic solutions of system (2.15) when $\alpha > 0, \beta \in (-\infty, \infty)$ in Figure 2 (a).

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (2.15), enclosing $E_1(-\phi_2, 0)$ and $E_2(\phi_2, 0)$, respectively, (see Figure 2 (a)). Now, (3.3) can be written as $\frac{1}{2}\xi = \int_{\psi_b}^{\psi} \frac{d\psi}{\sqrt{(\psi_a - \psi)(\psi - \psi_b)\psi(\psi + \psi_d)}}$, where $\psi_a = \phi_a^2$ and $\psi_b = \phi_b^2$, ϕ_a and ϕ_b are two intersection points of the periodic orbit on the right with the ϕ-axis respectively, $\psi + \psi_d$ is one of the remaining factors after factorization. It gives rise to the following exact parametric representations of two families of periodic orbits of system (2.15):

$$\phi(\xi) = \mp\frac{\sqrt{\psi_b}}{(1 - \hat{\alpha}_1^2 \mathrm{sn}^2(\Omega_1\xi, k))^{\frac{1}{2}}}, \tag{3.4}$$

where $\hat{\alpha}_1^2 = 1 - \frac{\psi_b}{\psi_a}, k^2 = \frac{\hat{\alpha}_1^2\psi_d}{\psi_b+\psi_d}, \Omega_1 = \frac{1}{4}\sqrt{\psi_a(\psi_b+\psi_d)}, \mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$ are the Jacobian elliptic functions (see [4]).

By using (17), we have

$$\int \phi^2(\xi)d\xi = \int \left(\frac{\psi_b}{1-\hat{\alpha}_1^2\mathrm{sn}^2(\Omega_1\xi,k)}\right)d\xi = \psi_b\Pi\left(\arcsin(\mathrm{sn}(\Omega_1\xi,k)),\hat{\alpha}_1^2,k\right). \quad (3.5)$$

(ii) Corresponding the level curves defined by $H(\phi,y)=0$, there exist two homoclinic orbits of system (2.15), enclosing $E_1(-\phi_2,0)$ and $E_2(\phi_2,0)$, respectively, (see Figure 2 (a)). In this case, (3.3) becomes that $\frac{1}{2}\xi = \int_\psi^{\psi_M}\frac{d\psi}{\psi\sqrt{(\psi_M-\psi)(\psi+\psi_i)}}$, where $\psi_M = 4(\beta+\sqrt{\beta^2+\alpha}), \psi_i = 4(\sqrt{\beta^2+\alpha}-\beta)$. Hence, we obtain the exact parametric representations of two homoclinic orbits:

$$\phi(\xi) = \mp\left(\frac{2\psi_M\psi_i}{(\psi_M+\psi_i)\cosh(\omega_0\xi)-(\psi_M-\psi_i)}\right)^{\frac{1}{2}}, \quad (3.6)$$

where $\omega_0 = \frac{1}{2}\sqrt{\psi_M\psi_i}$.

By using (3.6), we obtain

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int\left(\frac{2\psi_M\psi_i}{(\psi_M+\psi_i)\cosh(\omega_0\xi)-(\psi_M-\psi_i)}\right)d\xi \\ &= 4\arctan\left(\sqrt{\frac{\psi_M}{\psi_i}}\tanh\left(\frac{1}{2}\omega_0\phi\right)\right).\end{aligned} \quad (3.7)$$

(iii) Corresponding the level curves defined by $H(\phi,y)=h, h\in(0,\infty)$, there exists a family of periodic orbits of system (2.15), enclosing three equilibrium points (see Figure 2 (a)). Now, (3.3) has the form $\frac{1}{2}\xi = \int_0^\psi \frac{d\psi}{\sqrt{(\psi_1-\psi)\psi[(\psi-b_1)^2+a_1^2]}}$. It follows the following exact parametric representations of the family of global periodic orbits:

$$\phi(\xi) = \left(\alpha_1+\frac{\beta_1}{1+\hat{\alpha}_2\mathrm{cn}(\Omega_2\xi,k)}\right)^{\frac{1}{2}}, \quad (3.8)$$

where $A_1^2 = (\psi_1-b_1)^2+a_1^2, B_1^2 = b_1^2+a_1^2, k^2 = \frac{(\psi_1^2-(A_1-B_1)^2}{4A_1B_1}, \alpha_1 = \frac{-\psi_1B_1}{A_1-B_1}, \beta_1 = \frac{2A_1B_1}{A_1^2-B_1^2}, \hat{\alpha}_2 = \frac{A_1-B_1}{A_1+B_1}, \Omega_2 = \frac{1}{2}\sqrt{A_1B_1}$.

(3.8) follows that

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int\left(\alpha_1+\frac{\beta_1}{1+\hat{\alpha}_2\mathrm{cn}(\Omega_2\xi,k)}\right)d\xi \\ &= \alpha_1\xi+\frac{\beta_1}{1-\hat{\alpha}_2}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_2\xi,k)),\frac{\hat{\alpha}_2}{\hat{\alpha}_2-1},k\right)-\hat{\alpha}_2f_1\right],\end{aligned} \quad (3.9)$$

where Π is the the third kind of elliptic integral, the function f_1 can be found in [4].

3.2. Exact periodic solutions and homoclinic solutions of system (2.15) when $\alpha=0,\beta>0$ in Figure 2 (b).

In this case, $\phi_2 = \frac{4\sqrt{3\beta}}{3}, h_2 = -\frac{128\beta^3}{27}$.

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (2.15), enclosing $E_1(-\phi_2, 0)$ and $E_2(\phi_2, 0)$, respectively, (see Figure 2 (b)). These periodic orbits have the same exact parametric representations as (3.4).

(ii) Corresponding the level curves defined by $H(\phi, y) = 0$, there exist two homoclinic orbits of system (2.15), enclosing $E_1(-\phi_2, 0)$ and $E_2(\phi_2, 0)$, respectively, (see Figure 2 (b)). In this case, (3.3) becomes that $\frac{1}{4}\xi = \int_{\psi}^{8\beta} \frac{d\psi}{\psi\sqrt{(8\beta-\psi)\psi}}$. Thus, we have the following exact parametric representations of two homoclinic orbits of system (2.15):

$$\phi(\xi) = \mp \frac{(8\beta)^{\frac{3}{2}}}{\sqrt{(8\beta)^2 + \xi^2}}. \tag{3.10}$$

(3.10) gives rise to

$$\int \phi^2(\xi) d\xi = \int \frac{(8\beta)^3 d\xi}{(8\beta)^2 + \xi^2} = (8\beta)^2 \arctan\left(\frac{\xi}{8\beta}\right). \tag{3.11}$$

(iii) Corresponding the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a global family of periodic orbits of system (2.15), enclosing three equilibrium points (see Figure 2 (b)). These periodic orbits have the same exact parametric representations as (3.8).

3.3. Exact periodic solutions, homoclinic and heteroclinic solutions of system (2.15) when $\alpha < 0, \beta > \sqrt{|\alpha|}$ in Figure 2 (c).

In this case, we have $h_2 < 0 < h_1$.

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (h_2, 0)$, there exist two families of periodic orbits of system (2.15), enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$, respectively, (see Figure 2 (c)). Now, (3.3) can be written as $\frac{1}{2}\xi = \int_{\psi_b}^{\psi} \frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)\psi(\psi+\psi_d)}}$. It gives rise to the same exact parametric representations of two families of periodic solutions of system (2.15) as (3.4).

Especially, the level curves defined by $H(\phi, y) = 0$ are two periodic orbits enclosing the singular points $E_1(-\phi_2, 0)$ and $E_2(\phi_2, 0)$, respectively. In this case, in (3.4), we have $\psi_a = 4(\beta + \sqrt{\beta^2 + \alpha}), \psi_b = 4(\beta - \sqrt{\beta^2 + \alpha}), \psi_d = 0, \hat{\alpha}_1^2 = 1 - \frac{|\alpha|}{\left(\beta+\sqrt{\beta^2+\alpha}\right)^2}, k^2 = 0, \Omega_1 = \sqrt{|\alpha|}$. Hence, the two periodic orbits have the following exact parametric representations:

$$\phi(\xi) = \mp \frac{\sqrt{\psi_b}}{\left(1 - \hat{\alpha}_1^2 \sin^2(\sqrt{|\alpha|}\xi)\right)^{\frac{1}{2}}}. \tag{3.12}$$

(3.12) gives that

$$\begin{aligned} \int \phi^2(\xi) d\xi &= \int \frac{\psi_b d\xi}{1 - \hat{\alpha}_1^2 \sin^2(\sqrt{|\alpha|}\xi)} \\ &= \frac{\psi_b}{\sqrt{|\alpha|(1 - \hat{\alpha}_1^2)}} \arctan\left(\sqrt{1 - \hat{\alpha}_1^2} \tan\left(\sqrt{|\alpha|}\xi\right)\right). \end{aligned} \tag{3.13}$$

(ii) Corresponding the level curves defined by $H(\phi,y)=h, h\in(0,h_1)$, there exist three families of periodic orbits of system (2.15), enclosing the origin $O(0,0)$ and $E_1(-\phi_2,0), E_4(\phi_2,0)$, respectively, (see Figure 2 (c)). Now, (3.3) can be written as $\frac{1}{2}\xi=\int_0^{\psi}\frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi_b-\psi)(\psi_c-\psi)\psi}}$ and $\frac{1}{2}\xi=\int_{\psi_b}^{\psi}\frac{d\psi}{\sqrt{(\psi_a-\psi)(\psi-\psi_b)(\psi-\psi_c)\psi}}$. The family of periodic orbits enclosing the origin has the exact parametric representation as follows:

$$\phi(\xi)=\frac{\sqrt{\psi_a}|\hat{\alpha}_3|\mathrm{sn}(\Omega_3\xi,k)}{(1-\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k))^{\frac{1}{2}}}, \tag{3.14}$$

where $\hat{\alpha}_3^2=\frac{-\psi_c}{\psi_a-\psi_c}, k^2=\frac{-\hat{\alpha}_3^2(\psi_a-\psi_b)}{\psi_b}, \Omega_3=\frac{1}{4}\sqrt{(\psi_a-\psi_c)\psi_b}$.

We see from (3.14) that

$$\begin{aligned}\int\phi^2(\xi)d\xi&=\int\frac{\psi_a\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)d\xi}{1-\hat{\alpha}_3^2\mathrm{sn}^2(\Omega_3\xi,k)}\\&=\frac{\psi_a}{\hat{\Omega}_3}\left[\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_3^2,k\right)-F(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),k)\right],\end{aligned} \tag{3.15}$$

where F is the first kind of elliptic integral, and Π is the third kind of elliptic integral.

The two families of periodic orbits enclosing the points E_1 an E_4 have the exact parametric representations as follows:

$$\phi(\xi)=\mp\left(\psi_c+\frac{\psi_b-\psi_c}{1-\hat{\alpha}_4^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)^{\frac{1}{2}}, \tag{3.16}$$

where $\hat{\alpha}_4^2=\frac{\psi_a-\psi_b}{\psi_a-\psi_c}, k^2=\frac{\hat{\alpha}_4^2\psi_c}{\psi_b}$.

(3.16) gives that

$$\begin{aligned}\int\phi^2(\xi)d\xi&=\int\left(\psi_c+\frac{\psi_b-\psi_c}{1-\hat{\alpha}_4^2\mathrm{sn}^2(\Omega_3\xi,k)}\right)d\xi\\&=\psi_c\xi+(\psi_b-\psi_c)\Pi\left(\arcsin(\mathrm{sn}(\Omega_3\xi,k)),\hat{\alpha}_4^2,k\right).\end{aligned} \tag{3.17}$$

(iii) Corresponding the level curves defined by $H(\phi,y)=h_1$, there exist two homoclinic orbits enclosing $E_1(-\phi_2,0)$ and $E_4(\phi_2,0)$, respectively, and two heteroclinic orbits enclosing the origin $O(0,0)$ (see Figure 2 (c)). In this case, (3.3) becomes that $\frac{1}{2}\xi=\int_0^{\psi}\frac{d\psi}{(\psi_1-\psi)\sqrt{(\psi_M-\psi)\psi}}$ and $\frac{1}{2}\xi=\int_{\psi}^{\psi_M}\frac{d\psi}{(\psi-\psi_1)\sqrt{(\psi_M-\psi)\psi}}$. Thus, the two heteroclinic orbits has the exact parametric representations:

$$\phi(\xi)=\mp\frac{\sqrt{\psi_M\psi_1}\tanh(\omega_1\xi)}{\left((\psi_M-\psi_1)+\psi_1\tanh^2(\omega_1\xi)\right)^{\frac{1}{2}}}, \tag{3.18}$$

where $\omega_1=\frac{1}{2}\sqrt{(\psi_M-\psi_1)\psi_1}$.

We see from (3.18) that

$$\begin{aligned}\int\phi^2(\xi)d\xi&=\int\frac{\psi_M\psi_1\tanh^2(\omega_1\xi)d\xi}{(\psi_M-\psi_1)+\psi_1\tanh^2(\omega_1\xi)}\\&=\psi_1\xi-2\mathrm{arctan}\left(\sqrt{\frac{\psi_1}{\psi_M-\psi_1}}\tanh\left(\frac{1}{2}\xi\right)\right).\end{aligned} \tag{3.19}$$

The two homoclinic orbits has the exact parametric representations:

$$\phi(\xi) = \mp\left(\psi_1 + \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(\omega_1\xi) + 2\psi_1 - \psi_M}\right)^{\frac{1}{2}}. \tag{3.20}$$

(3.20) follows that

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \left(\psi_1 + \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(\omega_1\xi) + 2\psi_1 - \psi_M}\right) d\xi \\ &= \psi_1\xi + 2\arctan\left(\sqrt{\frac{\psi_M - \psi_1}{\psi_1}}\tanh\left(\frac{1}{2}\omega_1\xi\right)\right).\end{aligned} \tag{3.21}$$

(iv) Corresponding the level curves defined by $H(\phi, y) = h, h \in (0, \infty)$, there exists a global family of periodic orbits of system (2.15), enclosing five equilibrium points (see Figure 2 (c)). These periodic orbits have the same exact parametric representations as (3.8).

3.4. Exact periodic solutions, homoclinic and heteroclinic solutions of system (2.15) when $\alpha < 0, \beta = \sqrt{|\alpha|}$ in Figure 2 (d).

In this case, we have $\phi_1 = \frac{2}{3}\sqrt{3}|\alpha|^{\frac{1}{4}}, \phi_2 = 2|\alpha|^{\frac{1}{4}}, \phi_M = \frac{4}{\sqrt{3}}|\alpha|^{\frac{1}{4}}, h_2 = 0, h_1 = \frac{16}{27}|\alpha|^{\frac{3}{2}}$.

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (0, h_1)$, there exist three families of periodic orbits of system (2.7), enclosing the origin $O(0, 0)$ and $E_1(-\phi_2, 0), E_4(\phi_2, 0)$, respectively, (see Figure 2 (d)). These periodic orbits have the same exact parametric representations as (3.9) and (3.10).

(ii) Corresponding the level curves defined by $H(\phi, y) = h_1$, there exist two homoclinic orbits enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$, respectively, and two heteroclinic orbits enclosing the origin $O(0, 0)$ (see Figure 2 (d)). In this case, (3.3) becomes that $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{(\psi_1 - \psi)\sqrt{(\psi_M - \psi)\psi}}$ and $\frac{1}{2}\xi = \int_{\psi}^{\psi_M} \frac{d\psi}{(\psi - \psi_1)\sqrt{(\psi_M - \psi)\psi}}$. Therefore, the two heteroclinic orbits has the exact parametric representations:

$$\phi(\xi) = \mp\left(\psi_1 - \frac{2\psi_1(\psi_M - \psi_1)}{\psi_M \cosh(\omega_1\xi) + (\psi_M - 2\psi_1)}\right)^{\frac{1}{2}}. \tag{3.22}$$

(3.22) follows that

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \left(\psi_1 - \frac{2(\psi_M - \psi_1)\psi_1}{\psi_M \cosh(\omega_1\xi) + 2\psi_1 - \psi_M}\right) d\xi \\ &= \psi_1\xi - 2\arctan\left(\sqrt{\frac{\psi_1}{\psi_M - \psi_1}}\tanh\left(\frac{1}{2}\omega_1\xi\right)\right).\end{aligned} \tag{3.23}$$

The two homoclinic orbits have the same exact parametric representations as (3.20).

(iii) Corresponding the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (2.15), enclosing five equilibrium points (see Figure 2 (d)). These periodic orbits have the same exact parametric representations as (3.8).

3.5. Exact periodic solutions, homoclinic and heteroclinic solutions of system (2.15) when $\alpha < 0, \frac{\sqrt{3|\alpha|}}{2} < \beta < \sqrt{|\alpha|}$ in Figure 2 (e).

In this case, we have $0 < h_2 < h_1$.

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (0, h_2]$, there exists a family of periodic orbits of system (2.15), enclosing the origin $O(0,0)$ (see Figure 2 (e)). When $0 < h < h_2$, these periodic orbits have the same exact parametric representations as (3.8).

(ii) Corresponding the level curves defined by $H(\phi, y) = h, h \in [h_2, h_1)$, there exist three families of periodic orbits of system (2.15), enclosing the origin $O(0,0)$ and $E_1(-\phi_2, 0), E_4(\phi_2, 0)$, respectively, (see Figure 2 (e)). The family of periodic orbits enclosing the origin has the exact parametric representations as (3.14). The two families of periodic orbits enclosing the points E_1 and E_4,respectively, have the exact parametric representations as (3.16).

Notice that the level curve defined by $H(\phi, y) = h_2$ contain a periodic orbit enclosing the origin $O(0,0)$ and two singular points $E_1(-\phi_2, 0), E_4(\phi_2, 0)$. In this case, in (3.16), $\psi_a = \psi_b = \phi_2^2, k^2 = 0, \hat{\alpha}_4^2 = \frac{-\psi_c}{\phi_2^2 - \psi_c}, \Omega_3 = \frac{1}{4}\sqrt{(\phi_2^2 - \psi_c)\phi_2^2}$. The periodic orbit has the exact parametric representation:

$$\phi(\xi) = \frac{\phi_2 |\hat{\alpha}_4| \sin(\Omega_3 \xi)}{(1 - \hat{\alpha}_4^2 \sin^2(\Omega_3 \xi))^{\frac{1}{2}}}. \tag{3.24}$$

We see from (3.24) that

$$\begin{aligned}\int \phi^2(\xi) d\xi &= \int \frac{\psi_2 \hat{\alpha}_4^2 \sin^2(\Omega_3 \xi) d\xi}{1 - \hat{\alpha}_4^2 \sin^2(\Omega_3 \xi)} \\ &= -\psi_2 \xi + \frac{\psi_2}{\Omega_3(1 - \hat{\alpha}_4^2)} \arctan\left(\sqrt{1 - \hat{\alpha}_4^2} \tan(\Omega_3 \xi)\right). \end{aligned} \tag{3.25}$$

(iii) Corresponding the level curves defined by $H(\phi, y) = h_1$, there exist two homoclinic orbits enclosing $E_1(-\phi_2, 0)$ and $E_4(\phi_2, 0)$, respectively, and two heteroclinic orbits enclosing the origin $O(0,0)$ (see Figure 2 (e)). These orbits have the same exact parametric representations as (3.18) and (3.20).

(iv) Corresponding the level curves defined by $H(\phi, y) = h, h \in (h_1, \infty)$, there exists a family of periodic orbits of system (2.7), enclosing six equilibrium points (see Figure 2 (e)). These periodic orbits have the same exact parametric representations as (3.8).

3.6. Exact heteroclinic solutions of system (2.15) when $\alpha < 0, \beta = \frac{\sqrt{3|\alpha|}}{2}$ in Figure 2 (f).

In this case, we have that $\phi_1 = \phi_2 = (\frac{4\sqrt{3|\alpha|}}{3})^{\frac{1}{2}}, h_1 = h_2 = -\frac{4\sqrt{3}|\alpha|^{\frac{3}{2}}}{9}$.

(i) Corresponding the level curves defined by $H(\phi, y) = h, h \in (0, h_1), (h_1, \infty)$, there exist two families of periodic orbits of system (2.15) (see Figure 2 (f)). These periodic orbits have the same exact parametric representations as (3.8).

(ii) Corresponding the level curves defined by $H(\phi, y) = h_1 = h_2$, there exist two heteroclinic orbits connecting two double equilibrium points of system (2.15) (see

Figure 2 (f)). Now, (3.3) can be written as $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{\sqrt{\psi(\psi_1-\psi)^3}}$. Thus, it follows the exact parametric representations:

$$\phi(\xi) = \mp\left(\frac{\psi_1^3\xi^2}{16+(\psi_1\xi)^2}\right)^{\frac{1}{2}}. \tag{3.26}$$

We see from (3.26) that

$$\int \phi^2(\xi)d\xi = \int \frac{\psi_1^3\xi^2 d\xi}{16+(\psi_1\xi)^2} = \psi_1\xi - 4\arctan\left(\frac{1}{4}\psi_1\xi\right). \tag{3.27}$$

3.7. Exact periodic solution family of system (2.15) when $\alpha < 0, \beta < \frac{\sqrt{3|\alpha|}}{2}$ in Figure 2 (g).

Corresponding the level curves defined by $H(\phi, y) = h, h \in (0,\infty)$, there exist a family of periodic orbits of system (2.15) (see Figure 2 (g)). These periodic orbits have the same exact parametric representation as (3.8).

3.8. Exact periodic solution family of system (2.15) when $\alpha = 0, \beta < 0$ in Figure 2 (h).

In this case, we have $y^2 = h + \frac{1}{2}\beta\phi^4 - \frac{1}{16}\phi^6 = h + \frac{1}{2}\beta\psi^2 - \frac{1}{16}\psi^3$. There exists a periodic orbit family of system (2.15) enclosing the origin $O(0,0)$ (see Figure 2 (h)).

(i) When $h \in \left(0, \frac{64}{27}|\beta|\right)$, (3.3) can be written as $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{\sqrt{(\psi_a-\psi)\psi(\psi-\psi_c)(\psi-\psi_d)}}$, where $\psi_d < -\frac{16}{3}|\beta| < \psi_c < 0 < \psi_a$. Thus, we have the exact parametric representation of the periodic family as follows:

$$\phi(\xi) = \left(\psi_c - \frac{\psi_c}{1-\hat{\alpha}_5^2\mathrm{sn}^2(\Omega_5\xi, k)}\right)^{\frac{1}{2}}, \tag{3.28}$$

where $\hat{\alpha}_5^2 = \frac{\psi_a}{\psi_a-\psi_c}, k^2 = \frac{\hat{\alpha}_5^2(\psi_c-\psi_d)}{-\psi_d}, \Omega_5 = \frac{1}{4}\sqrt{(-\psi_d)(\psi_a-\psi_c)}$.

(3.28) gives that

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \left(\psi_c - \frac{\psi_c}{1-\hat{\alpha}_5^2\mathrm{sn}^2(\Omega_5\xi, k)}\right)d\xi \\ &= \psi_c\xi - \psi_c\Pi\left(\arcsin(\mathrm{sn}(\Omega_5\xi, k)), \hat{\alpha}_5^2, k\right).\end{aligned} \tag{3.29}$$

(2) When $h = \frac{64}{27}|\beta|$, (3.3) can be written as $\frac{1}{2}\xi = \int_0^{\psi} \frac{d\psi}{\left(\psi+\frac{16}{3}|\beta|\right)\sqrt{\left(\frac{8}{3}|\beta|-\psi\right)\psi}}$. It gives rise to the following periodic solution:

$$\phi(\xi) = \left(-\frac{16}{3}|\beta| + \frac{32}{5-\sin\left(\sqrt{\frac{128}{3}|\beta|}\,(\xi+\xi_0)\right)}\right)^{\frac{1}{2}}, \tag{3.30}$$

where $\xi_0 = \sin^{-1}\left(\frac{5|\beta|-6}{|\beta|}\right)$. (3.30) implies that

$$\begin{aligned}\int \phi^2(\xi)d\xi &= \int \left(-\frac{16}{3}|\beta| + \frac{32}{5-\sin\left(\sqrt{\frac{128}{3}|\beta|}\,(\xi+\xi_0)\right)}\right)d\xi \\ &= -\frac{16}{3}|\beta|\xi + \frac{2}{\sqrt{|\beta|}}\arctan\left(\frac{5\tan\left(\sqrt{\frac{32}{3}|\beta|}(\xi+\xi_0)\right)-1}{2\sqrt{6}}\right).\end{aligned} \tag{3.31}$$

(iii) When $\frac{64}{27}|\beta| < h < \infty$, the periodic solution family has the same exact parametric representation as (3.8).

4. Conclusion

The main result in the present paper is summarized as follows.

Theorem 4.1. *Write that* $\alpha = c\kappa - \frac{1}{4}c^2 - \omega$, $\beta = \frac{1}{2}c$ *for equations* (1.3), (1.4) *and* (1.5). *Consider the solutions of equations* (1.3),(1.4) *and* (1.5) *with the form* $q(x,t) = \phi(\xi)\exp[i(\kappa x - \omega t + \theta(\xi))]$. *Then, the following conclusions hold.*

(i) The function $\phi(\xi)$ *is the solutions of the planar Hamiltonian system* (2.15) *which has the bifurcations of phase portraits given by Figure 2.*

(ii) For equation (1.3), *the function* $\theta(\xi) = \theta_1(\xi) = \left(\frac{1}{2}c - \kappa\right)\xi - \frac{3}{4}\int \phi^2(\xi)d\xi$; *For equation* (1.4), *the function* $\theta(\xi) = \theta_2(\xi) = \left(\frac{1}{2}c - \kappa\right)\xi - \frac{1}{4}\int \phi^2(\xi)d\xi$; *For equation* (1.5), *the function* $\theta(\xi) = \theta_3(\xi) = \left(\frac{1}{2}c - \kappa\right)\xi + \frac{1}{4}\int \phi^2(\xi)d\xi$.

(iii) System (2.15) *has 14 exact explicit solutions* $\phi(\xi)$ *with the parametric representations given by* (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.16), (3.18), (3.20), (3.22),(3.24), (3.26), (3.28) *and* (3.30).

(iv) Corresponding the above solutions, integrals $\int \phi^2(\xi)d\xi$ *have 14 exact explicit formulas with the parametric representations given by* (3.5), (3.7), (3.9), (3.11), (3.13), (3.15), (3.17), (3.19), (3.21), (3.23),(3.25), (3.27), (3.29) *and* (3.31).

References

[1] K. Abhinav, P. Guha and I. Mukherjee, *Study of quasi-integrable and non-holonomic deformation of equations in the NLS and DNLS hierarchy,* J. Math. Phys., 2018, 59, 101507.

[2] T. Aydemir, *New exact optical soliton solutions of the derivative nonlinear Schrödinger equation family,* Opt. Quant. Electron., 2024, 56(6). DOI: 10.1007/s11082-024-06822-9.

[3] H. M. Baskonus, M. Younis, M. Bilal, U. Younas, Shafqat-ur-Rehman and W. Gao, *Modulation instability analysis and perturbed optical soliton and other solutions to the Gerdjikov-Ivanov equation in nonlinear optics,* Modern Phys. Lett., 2020, 34(35), 2050404.

[4] P. F. Byrd and M. D. Fridman, *Handbook of Elliptic Integrals for Engineers and Sciensists,* Springer, Berlin, 1971.

[5] H. H. Chen, Y. C. Lee and C. S. Liu, *Integrability of nonlinear Hamiltonian systems by inverse scattering method,* Phys. Scr., 1979, 20, 490.

[6] M. Chen, *Bifurcations and exact solutions for the Kundu equation: Dynamical approach,* J. Appl. Anal. Comput., 2024, 14(6). DOI: 10.11948/JAAC-2024-0012.

[7] H. H. Dai and E. G. Fan, *Variable separation and algebro-geometric solutions of the Gerdjikov–Ivanov equation,* Chaos Soliton. Fract., 2004, 22, 93–101.

[8] M. Dong, L. Tian, J. Wei and Y. Wang, *Some localized wave solutions for the coupled Gerdjikov-Ivanov equation,* Appl. Math. Lett., 2021, 122, 107483.

[9] E. G. Fan, *Integrable systems of derivative nonlinear Schrödinger type and their multi-Hamiltonian structure,* J. Phys. A Math. Gen., 2001, 34, 513–519.

[10] E. G. Fan, *Bi-Hamiltonian structure and Liouville integrablity for a Gerdjikov–Ivanov equation hierarchy,* Chin. Phys. Lett., 2001, 18, 1–3.

[11] N. G. Fang, *Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation,* J. Phys. A: Math. Gen., 2000, 33, 6925–6933.

[12] S. Fromm, *Admissible boundary values for the Gerdjikov-Ivanov equation with asymptotically time-periodic boundary data,* SIGMA Symmetry Integrability Geom. Methods Appl., 2020, 16, 79.

[13] V. S. Gerdjikov and M. I. Ivanov, *A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures,* Bulg. J. Phys., 1983, 10(1), 13.

[14] S. C. A. Gómez, A. Jhangeer, H. Rezazadeh, R. A. Talarposhti and A. Bekir, *Closed form solutions of the perturbed Gerdjikov-Ivanov equation with variable coefficients,* East Asian J. Appl. Math., 2021, 11(1), 207–218.

[15] H. Indexed, M. Saleh and A. A. Altwaty, *Optical solitons of the extended Gerdjikov-Ivanov equation in DWDM system by extended simplest equation method,* Appl. Math. Inf. Sci., 2020, 14(5), 901–907.

[16] A. Kalbani, K. Kaltham, K. S. Al-Ghafri, E. V. Krishnan and A. Biswas, *Solitons and modulation instability of the perturbed Gerdjikov-Ivanov equation with spatio-temporal dispersion,* Chaos Soliton. Fract., 2021, 153, 111523.

[17] D. J. Kaup and A. C. Newell, *An exact solution for a derivative nonlinear Schrödinger equation,* J. Math. Phys., 1978, 19(4), 798.

[18] M. M. A. Khater, *Abundant wave solutions of the perturbed Gerdjikov-Ivanov equation in telecommunication industry,* Modern Phys. Lett., 2021, 35(26), 2150456.

[19] S. A. Khuri, *New optical solitons and traveling wave solutions for the Gerdjikov-Ivanov equation,* Optik, 2022, 268(39), 169784.

[20] J. Li, *Singular Nonlinear Traveling Wave Equations: Bifurcations and Exact Solutions,* Science Press, Beijing, 2013.

[21] M. Li, Y. Zhang, R. Ye and Y. Lou, *Exact solutions of the nonlocal Gerdjikov-Ivanov equation,* Commun. Theor. Phys., 2021, 73, 10.

[22] Y. Li, L. Zhang, B. Hu and R. Wang, *The initial-boundary value for the combined Schrödinger and Gerdjikov–Ivanov equation on the half-line via the Riemann–Hilbert approach,* Teoret. Mat. Fiz., 2021, 209(2), 258–273.

[23] Y. Lou, Y. Zhang, R. Ye and M. Li, *Modulation instability, higher-order rogue waves and dynamics of the Gerdjikov-Ivanov equation,* Wave Motion, 2021, 106, 102795.

[24] P. Lu, Y. Wang and C. Dai, *Abundant fractional soliton solutions of a space-time fractional perturbed Gerdjikov-Ivanov equation by a fractional mapping method,* Chinese J. Phys., 2021, 74, 96–105.

[25] J. Luo and E. Fan, *Dbar-dressing method for the Gerdjikov-Ivanov equation with nonzero boundary conditions,* Appl. Math. Lett., 2021, 120, 107297.

[26] J. Luo and E. Fan, *ϱ-dressing method for the coupled Gerdjikov-Ivanov equation,* Appl. Math. Lett., 2020, 110, 106589.

[27] S. T. R. Rizvi and S. Shabbir, *Optical soliton solution via complete discrimination system approach along with bifurcation and sensitivity analyses for the Gerjikov-Ivanov equation,* Optik, 2023, 294(5), 171456.

[28] T. Tsuchida, *Integrable discretizations of derivative nonlinear Schrödinger equations,* J. Phys. A Math. Gen. 2002, 35, 7827–7847.

[29] Vinita and S. Saha Ray, *Lie symmetry reductions, power series solutions and conservation laws of the coupled Gerdjikov-Ivanov equation using optimal system of Lie subalgebra,* Z. Angew. Math. Phys., 2021, 72(4), 133.

[30] X. Xiao and Z. Yin, *Exact single travelling wave solutions to the fractional perturbed Gerdjikov-Ivanov equation in nonlinear optics,* Modern Phys. Lett. 2021, 35, 22.

[31] S. Zhang, T. Xu, M. Li and X. Zhang, *Higher-order algebraic soliton solutions of the Gerdjikov-Ivanov equation: Asymptotic analysis and emergence of rogue waves,* Phys. D, 2022, 432, 133128.

[32] Z. Zhang and E. Fan, *Inverse scattering transform and multiple high-order pole solutions for the Gerdjikov-Ivanov equation under the zero/nonzero background,* Z. Angew. Math. Phys., 2021, 72(4), 153.

[33] Z. Zhang and E. Fan, *Inverse scattering transform for the Gerdjikov-Ivanov equation with nonzero boundary conditions,* Z. Angew. Math. Phys., 2020, 71(5), 149.

[34] Y. Zhou and J. Li, *Bifurcations of traveling wave solutions in the homogeneous Camassa-Holm type equations,* J. Appl. Anal. Comput., 2022, 12(1), 392–406.

[35] J. Zhu and Y. Chen, *A new form of general soliton solutions and multiple zeros solutions for a higher-order Kaup-Newell equation,* J. Math. Phys., 2021, 62, 12.

[36] J. Zhuang and Y. Zhou, *Bifurcations of solitary waves, periodic peakons and compactons of a coupled nonlinear wave equation,* J. Appl. Anal. Comput., 2022, 12(5), 1–7.

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